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Nodal lines of eigenfunctions of the fixed membrane problem in general convex domains

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Abstract. We describe the boundary behavior of the nodal lines of eigenfunctions of the fixed membrane problem in convex, possibly nonsmooth, domains. This result is applied to the proof of Payne's conjecture on the nodal line of second eigenfunctions [P1], by removing the C^∞ smoothness assumption which is present in the original proof of Melas [M].

Introduction

In 1967 Payne [P1] conjectured that, in any simply connected bounded domain Ω in the plane, any second eigenfunction of the Laplacian with Dirichlet boundary condition cannot have a closed nodal line. See also Yau [Y].

Quite recently, Melas [M] has proved that this conjecture is true for convex domains with C^∞ boundary. Previous significant results were obtained by Payne [P2], Lin [L], Pütter [Pü] and Jerison [J].

In the proof of Melas, the smoothness of the boundary is required in order to deduce information on the boundary behavior of the nodal line of eigenfunctions [M, Lemma 2.1]. Loosely speaking, the crucial step where the smoothness is needed is as follows. One performs a local flattening of the boundary and an odd reflection of the eigenfunction across the boundary. In this way, the study of the boundary behavior of the nodal line is reduced to the study of the behavior in the interior of the nodal line of the solution of an elliptic equation, which, as a consequence of the Hartman and Wintner formulas (see [H-W] and also Lemma 1.2 below) is, asymptotically, the same as the one of an harmonic function. For different purposes, this technique has been proven successful in domains with $C^{1,\alpha}$ boundary, by Alessandrini and Magnanini [A-M, Theorem 3.5]. However, it does not seem to be applicable when the regularity of the boundary is less than C^1.

Our present purpose is to describe the boundary behavior of the nodal lines of eigenfunctions in general convex domains, and to deduce from this study the validity of Payne's conjecture in any convex domain, with no smoothness assumption.

The main results are stated in the two theorems below. But first we need some notation.

We denote by Ω a bounded convex domain in the plane. For any $P \in \partial\Omega$, we denote by $\Gamma(P)$ the smallest open infinite sector with vertex at P which contains Ω. We denote by u an eigenfunction of the Laplacian in Ω with Dirichlet conditions, that is a nonzero $W_0^{1,2}(\Omega)$ function satisfying, for some positive constant λ,

$$\Delta u + \lambda u = 0 \qquad \text{in } \Omega. \tag{1}$$

The nodal line N of the eigenfunction u is given by

$$N = \overline{\{x \in \Omega \mid u(z) = 0\}}.$$

Here, and in the sequel, $z = x + iy$ will denote the complex coordinate in the plane.

THEOREM 1. *If the nodal line N intersects the boundary $\partial\Omega$ at a point P, then there exists $r_0 > 0$, such that $N \cap B_{r_0}(P)$ is composed by a finite number M, $M \geq 1$, of C^1 simple arcs which all end in P and whose tangent lines at P divide the sector $\Gamma(P)$ into $M + 1$ sectors of equal amplitude.*

THEOREM 2. *Let u be any eigenfunction corresponding to the second eigenvalue $\lambda = \lambda_2$. The nodal line N of u intersects the boundary at exactly two points.*

The main body of this paper is contained in Section 1, where a proof of Theorem 1 is given. The basic idea is to use a local change of coordinates near $P \in \partial\Omega$ which, rather than flattening the boundary to a line, transforms the boundary into the broken line which bounds $\Gamma(P)$. The appropriate way of doing such a transformation is by a conformal mapping, in fact we shall take advantage of the conformal invariance of elliptic equations in divergence form. On the other hand, we need somewhat delicate asymptotic estimates on such a mapping near the point P, see Lemma 1.1. Finally, we make use of the Hartman and Wintner formulas, in the improved form of Schulz [S], see Lemmas 1.2–1.3.

Section 2 contains the proof of Theorem 2. We will follow essentially the same track of the proof of Melas, and we will refer to it at various steps. However, some technical care, and a different viewpoint are necessary. This is mainly due to two facts: the first order derivatives of u may not be continuously defined up to the boundary, and the Hopf lemma may not be applicable on $\partial\Omega$.

1. The behavior at the boundary of nodal lines

With no loss of generality, we may set $P = 0$, $\Gamma(0) = \{re^{i\theta} \in \mathbb{C} \mid r > 0, 0 < \theta < \alpha\}$ where α is some number $0 < \alpha \leq \pi$. For any $R > 0$, let us define $D_{\alpha,R} = \Gamma(0) \cap B_R(0)$

and also $\Delta_{\alpha,R} = \Omega \cap D_{\alpha,R}$. Let R be small enough, in such a way that $\partial D_{\alpha,R} \cap \Omega \neq \emptyset$. Thus $\partial \Delta_{\alpha,R}$ can be decomposed into three arcs γ_0, γ_α and σ_R which, in polar coordinates, are parametrized as follows

$$\gamma_0 : \theta = \theta_0(r), \qquad 0 \le r \le R,$$

$$\gamma_\alpha : \theta = \theta_\alpha(r), \qquad 0 \le r \le R,$$

$$\sigma_R : r = R, \qquad \theta_0(R) \le \theta \le \theta_\alpha(R);$$

here the functions θ_0, θ_α are such that

$$0 \le \theta_0(r) < \theta_\alpha(r) \le \alpha, \qquad \text{for every } r,\ 0 \le r \le R,$$

θ_0 is nondecreasing, θ_α is nonincreasing and also we have

$$\lim_{r \to 0} \theta_0(r) = \lim_{r \to 0} [\alpha - \theta_\alpha(r)] = 0.$$

Let us recall that there exists a unique conformal mapping $z \to \zeta(z)$ from $\Delta_{\alpha,R}$ onto $D_{\alpha,R}$ which maps the corners 0, $Re^{i\theta_0(R)}$, $Re^{i\theta_\alpha(R)}$ into the corners 0, R, $Re^{i\alpha}$ respectively. Since $\Delta_{\alpha,R}$ and $D_{\alpha,R}$ are convex domains, they have Lipschitz boundary, and hence the mapping $\zeta(z)$ and its inverse are both uniformly Hölder continuous. It is convenient to introduce logarithmic coordinates

$$z = e^{t+i\theta} \qquad \text{in } \Delta_{\alpha,R},$$

$$\zeta = e^{\tau+i\phi} \qquad \text{in } D_{\alpha,R}.$$

We shall represent the mapping $\zeta = \zeta(z)$ by the transformation of coordinates $\tau = \tau(t, \theta)$, $\phi = \phi(t, \theta)$. An analogous representation will be used for the inverse mapping. We denote by $\partial(\tau, \phi)/\partial(t, \theta)$, $\partial(t, \theta)/\partial(\tau, \phi)$ the Jacobian matrices of such transformations and by I the 2×2 identity matrix. Notice that τ and ϕ are conjugate harmonic functions of the variables t, θ and viceversa.

The following Lemma says that, in an appropriate sense, the mapping $\zeta = \zeta(z)$ is nearly the identity as $z \to 0$ *nontangentially* in Ω.

LEMMA 1.1. *Let K be any compact subset of the interval* $(0, \alpha)$. *We have*

$$\lim_{t \to -\infty} [\phi(t, \theta) - \theta] = 0 \qquad \textit{uniformly when } \theta \in K, \tag{1.1}$$

$$\lim_{\tau \to -\infty} [\phi - \theta(\tau, \phi)] = 0 \qquad \textit{uniformly when } \phi \in K, \tag{1.2}$$

$$\lim_{t \to -\infty} \left[\frac{\partial(\tau, \phi)}{\partial(t, \theta)} - I \right] = 0 \qquad \textit{uniformly when } \theta \in K, \tag{1.3}$$

$$\lim_{\tau \to -\infty} \left[\frac{\partial(t, \theta)}{\partial(\tau, \phi)} - I \right] = 0 \qquad \textit{uniformly when } \phi \in K. \tag{1.4}$$

Proof. The function $\phi = \phi(t, \theta)$ satisfies the following conditions

$$\begin{aligned}
&\Delta\phi = 0 \quad &&\text{where } t < \log R \quad \text{and} \quad \theta_0(e^t) < \theta < \theta_\alpha(e^t),\\
&\phi = 0 \quad &&\text{where } t < \log R \quad \text{and} \quad \theta = \theta_0(e^t),\\
&\phi = \alpha \quad &&\text{where } t < \log R \quad \text{and} \quad \theta = \theta_\alpha(e^t),
\end{aligned}$$

moreover, we have $0 < \phi \le \alpha$ everywhere, and $\phi(\log R, \theta)$ is continuously increasing from 0 to α as θ ranges the interval $[\theta_0(R), \theta_\alpha(R)]$. For any $T < \log R$, let us define ϕ^-, ϕ^+ as the bounded solutions of the following Dirichlet problems

$$\begin{aligned}
&\Delta\phi^- = 0 \quad &&\text{where } t < T \quad \text{and} \quad \theta_0(T) < \theta < \alpha,\\
&\phi^- = 0 \quad &&\text{where } t < T \quad \text{and} \quad \theta = \theta_0(T),\\
&\phi^- = 0 \quad &&\text{where } t = T \quad \text{and} \quad \theta_0(T) \le \theta \le \alpha,\\
&\phi^- = \alpha \quad &&\text{where } t < T \quad \text{and} \quad \theta = \alpha,
\end{aligned}$$

$$\begin{aligned}
&\Delta\phi^+ = 0 \quad &&\text{where } t < T \quad \text{and} \quad 0 < \theta < \theta_\alpha(T),\\
&\phi^+ = 0 \quad &&\text{where } t < T \quad \text{and} \quad \theta = 0,\\
&\phi^+ = \alpha \quad &&\text{where } t = T \quad \text{and} \quad 0 \le \theta \le \theta_\alpha(T),\\
&\phi^+ = \alpha \quad &&\text{where } t < T \quad \text{and} \quad \theta = \theta_\alpha(T).
\end{aligned}$$

By the maximum principle, we have that, in the common domain of definition, $\phi^- \le \phi \le \phi^+$. We may choose T in such a way that $K \subset (\theta_0(T), \theta_\alpha(T))$, and by explicit estimation of the functions ϕ^-, ϕ^+, we may find a positive constant A such that the following inequalities hold

$$\alpha \frac{\theta - \theta_0(T)}{\alpha - \theta_0(T)} - A \exp\left[\frac{\pi t}{\alpha - \theta_0(T)}\right] \le \phi(t, \theta) \le \frac{\alpha\theta}{\theta_\alpha(T)} + A \exp\left[\frac{\pi t}{\theta_\alpha(T)}\right]$$

$$\text{for every } t < T \quad \text{and} \quad \theta \in K,$$

and (1.1) follows easily. By standard interior estimates for harmonic functions, (1.1) implies also

$$\lim_{t \to -\infty} \nabla[\phi(t, \theta) - \theta] = 0 \qquad \text{uniformly when } \theta \in K, \tag{1.5}$$

here the gradient ∇ is taken with respect to the (t, θ) coordinates. By conjugation, we also have

$$\lim_{t \to -\infty} \nabla[\tau(t, \theta) - t] = 0 \qquad \text{uniformly when } \theta \in K, \tag{1.6}$$

and hence (1.3) is proven.

By comparing $\phi(t, \theta)$ with $\phi(t, \theta + h)$, $h > 0$, in the common domain of definition, we obtain, again by the maximum principle, that ϕ is an increasing function of θ for any fixed t. Hence $\partial\phi/\partial\theta > 0$ in the interior and every level line $\{\phi(t, \theta) = \phi_0\}$, $0 < \phi_0 < \alpha$, is the graph of a function $\theta = \theta_{\phi_0}(t)$. Let a, b be numbers such that $0 < a < b < \alpha$ and also $K \subset [a, b]$. By (1.1) and by the monotonicity of ϕ with respect to θ, for any ϵ, $0 < \epsilon < \min\{a, \alpha - b\}$, there exists $T < \log R$, such that for every $t < T$ we have

$$\phi(\theta, t) \leq a \qquad \text{whenever } \theta \leq a - \epsilon,$$

$$\phi(\theta, t) \geq b \qquad \text{whenever } \theta \geq b + \epsilon.$$

Therefore, for every $\phi_0 \in K$, and for every $t < T$, if $\phi(\theta, t) = \phi_0$ then it must be $a - \epsilon \leq \theta \leq b + \epsilon$. Consequently, applying once more (1.1), we obtain

$$\lim_{t \to -\infty} \theta_{\phi_0}(t) = \phi_0 \qquad \text{uniformly when } \phi_0 \in K. \tag{1.7}$$

Notice also that $\theta_{\phi_0}(t(\tau, \phi_0)) = \theta(\tau, \phi_0)$ for every τ. Next, let us observe that the Hölder continuity of ζ and ζ^{-1} at 0 implies that

$$\lim_{\tau \to -\infty} t(\tau, \phi) = -\infty \qquad \text{uniformly in } \phi. \tag{1.8a}$$

$$\lim_{t \to -\infty} \tau(t, \theta) = -\infty \qquad \text{uniformly in } \theta. \tag{1.8b}$$

And now (1.2) follows easily from (1.7), (1.8a). Finally, (1.4) can be deduced from (1.2) by the same procedure used above to prove (1.3) from (1.1). □

The following lemma is an application of the Hartman and Wintner formulas. Quite similar statements are already well known, see for instance Cheng [C] and Alessandrini [A]. A proof is sketched, just for the sake of completeness.

LEMMA 1.2. *Let $a \in C^{\delta}(B_R(0))$, $0 < \delta \le 1$, be a positive function, and let v be a nontrivial $W^{1,2}$ solution of*

$$\operatorname{div}[a\nabla v] = 0 \qquad \text{in } B_R(0) \tag{1.9}$$

and let us set $v(0) = 0$. There exists numbers r_0, β, $0 < r_0 < R$, $0 \le \beta < \pi$, and a nonnegative integer M, such that the nodal line

$$\{z \in B_{r_0}(0) \mid u(z) = 0\}$$

is given by the union of $2(M+1)$ C^1 arcs $\gamma_1, \ldots, \gamma_{2(M+1)}$, which can be parametrized in polar coordinates as follows

$$\gamma_j : \theta = \theta_j(r), \qquad 0 \le r \le r_0, \quad j = 1, \ldots, 2(M+1). \tag{1.10}$$

The functions $\theta_1, \ldots, \theta_{2(M+1)}$ satisfy

$$\theta_j(r) = \beta + \frac{\pi j}{(M+1)} + o(1), \qquad \text{as } r \to 0, \text{ for every } j = 1, \ldots, 2(M+1), \tag{1.11}$$

$$r\frac{d}{dr}\theta_j(r) = o(1), \qquad \text{as } r \to 0, \quad \text{for every } j = 1, \ldots, 2(M+1). \tag{1.12}$$

Proof. The Hartman and Wintner theorem, in the version of Schulz, [S, Theorem 7.4.1, Corollary 7.4.2], gives us that there exist numbers β and M as in the statement, and a positive number A, such that

$$v(re^{i\theta}) = \frac{A}{M+1} r^{M+1} \sin[(M+1)(\theta - \beta)] + o(r^{M+1}), \qquad \text{as } r \to 0, \tag{1.13}$$

$$r\frac{\partial}{\partial r} v(re^{i\theta}) = A r^{M+1} \sin[(M+1)(\theta - \beta)] + o(r^{M+1}), \qquad \text{as } r \to 0, \tag{1.14}$$

$$\frac{\partial}{\partial \theta} v(re^{i\theta}) = A r^{M+1} \cos[(M+1)(\theta - \beta)] + o(r^{M+1}), \qquad \text{as } r \to 0, \tag{1.15}$$

the remainders being uniform with respect to θ. Notice that, for sufficiently small r, if $v(re^{i\theta}) = 0$ then $(\partial/\partial\theta)v(re^{i\theta}) \neq 0$. Thus there exist $r_0 > 0$, such that, equating v to 0 in (1.13) we obtain (1.10)–(1.12). □

The next lemma is a variant of the previous one, suitable for solutions of (1.9), but which are defined in a finite sector and vanish on the flat parts of the boundary.

LEMMA 1.3. *Let a be as in Lemma 1.2, and let* $v \in W^{1,2}(D_{\alpha,R})$ *satisfy, in the weak sense,*

$$\operatorname{div}[a\nabla v] = 0 \qquad \text{in } D_{\alpha,R},$$

$$v(re^{i\theta}) = 0 \qquad \text{whenever } 0 \le r \le R \quad \text{and} \quad \theta = 0, \alpha.$$

There exist a nonnegative integer M and a number r_0, $0 < r_0 < R$ *such that the nodal line*

$$\{z \in D_{\alpha,r_0} \mid v(z) = 0\}$$

is empty when $M = 0$, *and, when* $M > 0$, *it is the union of M* C^1 *arcs* $\gamma_1, \ldots, \gamma_M$ *which can be parametrized*

$$\gamma_j : \theta = \theta_j(r), \qquad 0 \le r \le r_0, \quad j = 1, \ldots, M,$$

and the functions $\theta_1, \ldots, \theta_M$ *satisfy*

$$\theta_j(r) = \frac{\alpha j}{(M+1)} + o(1), \qquad \text{as } r \to 0, \quad \text{for every } j = 1, \ldots, M, \tag{1.11}$$

$$r\frac{d}{dr}\theta_j(r) = o(1), \qquad \text{as } r \to 0, \quad \text{for every } j = 1, \ldots, M. \tag{1.12}$$

Proof. Let us perform the conformal change of variable $z \to w$, defined by $w = z^{\pi/\alpha}$. Let us set $\tilde{a}(w) = a(z)$, $\tilde{v}(w) = v(z)$, these new functions are defined in a half disk centered at the origin and contained in the upper half plane. Let us continue such functions to $\operatorname{Im} w < 0$ by reflecting $\tilde{a}$ evenly and $\tilde{v}$ oddly, across the line $\operatorname{Im} w = 0$. It follows that Lemma 1.2 is applicable to the equation $\operatorname{div}[\tilde{a}\nabla\tilde{v}] = 0$ which holds in a full neighborhood of the origin and the proof is readily completed by pulling back to $D_{\alpha,R}$ the parametric representation of the nodal line given in Lemma 1.2. □

Proof of Theorem 1. Let $J_0(x)$ be the Bessel function of the first kind and of zero order, let j_1 be its first positive zero. For any fixed R, $0 \le \sqrt{\lambda}R < j_1$, we have that $\psi(z) = J_0(\sqrt{\lambda}|z|)$ is positive in $B_R(0)$ and it satisfies $\Delta\psi + \lambda\psi = 0$. Consequently, we may factor $u = \psi w$ in $\Delta_{\alpha,R}$ where $w \in W^{1,2}(\Delta_{\alpha,R})$ turns out to satisfy in the weak sense the equation

$$\operatorname{div}[\psi^2\nabla w] = 0,$$

and the boundary condition

$$w = 0 \qquad \text{on } \gamma_0 \cup \gamma_\alpha = \partial\Omega \cap \partial\Delta_{\alpha,R}.$$

Using now the previously constructed conformal mapping $z \to \zeta(z)$, we define $a(\zeta) = \psi^2(z)$, $v(\zeta) = w(z)$, $\zeta \in D_{\alpha,R}$. By the conformality, we have

$$\operatorname{div}[a\nabla v] = 0 \qquad \text{in } D_{\alpha,R},$$

and Lemma 1.3 is applicable. Consequently, choosing R sufficiently small, the nodal line of u in $\Delta_{\alpha,R}$, which is the same as the one of w, is given by the union of M arcs $\gamma_1, \dots, \gamma_M$ (here $M > 0$ since the origin is a limit point of the nodal line by hypothesis). Such arcs can be parametrized, in the ζ-logarithmic coordinates τ, ϕ as follows

$$\gamma_j : \phi = \phi_j(\tau), \qquad \tau \le \log R, \quad j = 1, \dots, M,$$

$$\phi_j(\tau) = \frac{\alpha j}{M+1} + o(1), \qquad \text{as } \tau \to -\infty, \quad \text{for every } j = 1, \dots, M,$$

$$\frac{d}{d\tau}\phi_j(\tau) = o(1), \qquad \text{as } \tau \to -\infty, \quad \text{for every } j = 1, \dots, M.$$

By (1.4), (1.1) the angle θ in the z coordinate is such that

$$\theta(\phi_j(\tau), \tau) = \theta\left(\frac{\alpha j}{M+1}, \tau\right) + o(1) = \frac{\alpha j}{M+1} + o(1) \qquad \text{as } \tau \to -\infty,$$

Finally, recalling (1.8), we have that the curves γ_j, $j = 1, \dots, M$, can be parametrized

$$\gamma_j : \theta = \theta_j(r), \qquad 0 < r \le R,$$

where

$$\theta_j(r) = \frac{\alpha j}{M+1} + o(1), \qquad \text{as } r \to 0,$$

$$r\frac{d}{dr}\theta_j(r) = o(1), \qquad \text{as } r \to 0.$$

These formulas show that the curves γ_j are C^1 up to the endpoint $P = 0$, and that their tangent lines at Q are equally spaced in the sector $\Gamma(P)$. This concludes the proof of Theorem 1. □

2. The nodal line of second eigenfunctions

In this section we restrict our attention to the second eigenvalue λ_2 and we consider u to be any corresponding eigenfunction.

As is well known, the Courant nodal domain theorem, [C-H], implies that $\Omega \backslash N$ is composed by exactly two connected open sets, Ω_1 and Ω_2. By Theorem 1, we obtain that N must have one of the following three configurations

(I) *N is a simple curve,*
(II) *N is a Jordan curve and is compactly contained in Ω,*
(III) *N is a Jordan curve and intersects $\partial\Omega$ at exactly one point P. N is piecewise C^1, and its one sided tangents at P divide the sector $\Gamma(P)$ in three sectors of equal amplitude.*

In order to prove Payne's conjecture, that is Theorem 2, one has to show that cases (II) and (III) cannot occur.

Without loss of generality we shall assume that $u > 0$ in Ω_1 $u < 0$ in Ω_2 and, in case (II) or (III) holds, that Ω_2 is the domain whose boundary is N. We shall denote by ν the exterior unit normal to $\partial\Omega$. Being Ω convex, ν is defined almost everywhere on $\partial\Omega$.

Let us observe that, as a consequence of the convexity of Ω, since $u \in W_0^{1,2}(\Omega)$ and $\Delta u \in L^2(\Omega)$ then in fact $u \in W^{2,2}(\Omega)$, see [G, Theorem 3.2.1.2.]. In particular, the gradient of u has an L^2 trace on $\partial\Omega$.

LEMMA 2.1. *If* (II) *or* (III) *occurs, then for any unit vector ξ, and for any arc $\sigma \subset \partial\Omega$ such that $\nu \cdot \xi \geq 0$ almost everywhere on σ, the following inequality holds in the weak sense*

$$\frac{\partial u}{\partial \xi} \leq 0 \qquad \text{on } \sigma. \tag{2.2}$$

Proof. Without loss of generality we may suppose that ξ is parallel to the x-axis and, if (III) occurs, P is not an interior point of σ. The arc σ can be represented as a concave graph $x = x_\sigma(y)$. We have $u(x_\sigma(y), y) = 0$, $u(x, y) > 0$ for $x < x_\sigma(y)$ and therefore $\partial u/\partial x(x_\sigma(y), y) \leq 0$ for almost every y. Recalling that $u \in W^{2,2}(\Omega)$ we have that $\partial u/\partial x$ is an absolutely continuous function on almost every line, hence $[\partial u/\partial x]^+$ can be continued to 0 outside Ω as an absolute continuous function on almost every line α which is transversal to σ. The first partials of $[\partial u/\partial x]^+$ exist almost everywhere and they belong to $L^2(\Omega)$, hence $[\partial u/\partial x]^+$ has zero trace on σ. □

The next two lemmas are due, in the smooth case, to Lin [L, Lemma 2.4, Proof of Theorem 2.2], see also [M, Theorem 2.1]. We omit the proofs, since, in view of the previous considerations, it is a straightforward matter to modify those in [L] and [M] to the present setting.

LEMMA 2.2. *If* (II) *or* (III) *holds, then the second eigenvalue λ_2 has multiplicity one.*

LEMMA 2.3. *If* (II) *holds for some convex domain Ω, then there exists a convex domain $\tilde{\Omega}$, for which* (III) *holds.*

Proof of Theorem 2. We argue by contradiction. In view of Lemma 2.3, let us assume, that case (III) occurs.

Let us choose the coordinates in such a way that $P = 0$, Ω lies in the upper half plane $y > 0$, and the y-axis bisects the sector $\Gamma(P)$. The boundary of Ω can be split into four arcs A, B, C and D. A and B are line segments parallel to the x-axis and might consist of single points, $0 \in A$. The arc C stands to the left hand side of D. If e_1 is the unit vector in the x direction, then we have $\nu \cdot e_1 \leq 0$ on C and $\nu \cdot e_1 \geq 0$ on D.

For any real number t, let us define the following function

$$v_t = \frac{\partial u}{\partial x} + tu \qquad \text{in } \Omega,$$

v_t belongs to $W^{1,2}(\Omega)$ and it satisfies in the weak sense

$$\Delta v_t + \lambda_2 vt = 0 \qquad \text{in } \Omega.$$

By Lemma 2.1 we have

$$v_t \geq 0 \text{ on } C, \quad v_t \leq 0 \text{ on } D, \quad \text{in the weak sense,}$$

while, in the interior of $A \cup B$, we have $v_t = 0$.

LEMMA 2.4. *There cannot exist two disjoint connected open subsets of Ω such that, on their boundaries, $v_t = 0$ in the weak sense.*

Proof. If there were two of such sets, by the variational characterization of λ_2, and by the analiticity of v_t in the interior of Ω, it follows that v_t is a second eigenfunction in Ω. By Lemma 2.2, we obtain that v_t is a constant multiple of u in Ω. Therefore $\partial u/\partial x = (\text{Const.})u$ in Ω, which contradicts the zero Dirichlet condition. □

LEMMA 2.5. *There exists exactly one connected open subset Ω_t of Ω such that $v_t = 0$ on $\partial\Omega_t$ in the weak sense. Ω_t is simply connected.*

Proof. By Theorem 1 we know that there exists $R > 0$, such that $N \cap B_R(0)$ is composed by two C^1 arcs γ^+, γ^-, whose intersection consists only of their endpoint at 0. Moreover, we may choose R sufficiently small, in such a way that their tangents lines are never parallel to the x-axis and γ^+, γ^- lie, respectively, in the half planes $x > 0$, $x < 0$. From now on we shall remove the endpoint 0 from γ^+ and γ^-.

By our previous settings, we have that $v_t > 0$ on γ^+ and $v_t < 0$ on γ^-. We define Ω_t^+ as the connected component of the level set $\{z \in \Omega \mid v_t(z) > 0\}$ which contains γ^+, and, analogously, we define Ω_t^-. Now let us suppose that $v_t|_{\Omega_t^-} \notin W_0^{1,2}(\Omega_t^-)$, since $v_t \geq 0$ on C and $v_t = 0$ on $A \cup B$, then $\partial\Omega_t^-$ must have a nonempty intersection with D. Therefore, Ω_t^- separates Ω_t^+ from $C \cup B$, consequently $\partial\Omega_t^+$ may intersect $\partial\Omega$ on points of $D \cup A$ only, where we have $v_t \leq 0$. Since $v_t > 0$ inside Ω_t^+ we obtain that $v_t = 0$ weakly on $\partial\Omega_t^+$. Hence we have obtained that either $\Omega_t = \Omega_t^-$ or $\Omega_t = \Omega_t^+$ satisfies our thesis. Uniqueness and simple connectedness are consequences of Lemma 2.4. □

Now we apply the above arguments to the special case $t = 0$, that is $v_t = \partial u/\partial x$.

LEMMA 2.6. *There exists a nontrivial arc δ in C and a connected component G of the level set $\{z \in \Omega \mid (\partial u/\partial x)(z) > 0\}$ such that $\partial G \cap C$ contains δ.*

Proof. First we observe that $C \subset \partial\{z \in \Omega \mid (\partial u/\partial x)(z) \neq 0\}$, otherwise $\partial u/\partial x = 0$ on an open subset of Ω, which is not possible by analytic continuation. Next, we see that the set $C \cap \partial\{z \in \Omega \mid (\partial u/\partial x)(z) < 0\}$ has empty interior in $\partial\Omega$. Were it not so, we would have $\partial u/\partial x = 0$ weakly on an open arc γ of C. Continuing u to 0 to the left of γ, we would obtain a $W^{1,2}$ function satisfying, in the weak sense, $\Delta u + \lambda_2 u = 0$ on a full neighborhood of γ, by unique continuation we would have that u should vanish identically.

Therefore any connected component δ of $C \backslash \partial\{z \in \Omega \mid (\partial u/\partial x)(z) < 0\}$ is open in C and it is contained in the boundary of one connected component G of the level set $\{z \in \Omega \mid \partial u/\partial x > 0\}$. □

Let us denote by $\lambda_1(Q)$ the first eigenvalue of the Laplacian with Dirichlet data in the domain Q.

LEMMA 2.7. *There exists $T > 0$, such that $v_t > 0$ in Ω_t for every $t < -T$ and $v_t < 0$ in Ω_t for every $t > T$.*

Proof. Let us consider the case when $t > 0$. Let δ, G be as in Lemma 2.6 and let K be a sufficiently small connected neighborhood of δ in G such that $u > 0$ in K. Therefore, $v_t > 0$ in K, and $K \cap \Omega_t = \emptyset$, otherwise, we would have $K \subset \Omega_t$ and then $\partial u / \partial x = 0$ on δ, which is not possible by the unique continuation property, as we already observed in the proof of Lemma 2.6.

By the strict monotonicity of the eigenvalues we have $\lambda_1(\Omega \backslash \overline{(\Omega_2 \cup K)}) = \lambda_1(\Omega_1 \backslash \bar{K}) > \lambda_1(\Omega_1) = \lambda_2$, and by continuity, we may find a connected open set $E \subset\subset \Omega_2$ such that

$$\lambda_1(\Omega \backslash \overline{(E \cup K)}) > \lambda_2. \tag{2.3}$$

We may find $T > 0$, such that $v_t < 0$ in E for every $t > T$. For $t > T$, we have that either $E \subset \Omega_t$ or $E \cap \Omega_t = \emptyset$. But the second case cannot occur, because it would imply $\Omega_t \subset \Omega \backslash \overline{(E \cup K)}$ and, therefore $\lambda_2 = \lambda_1(\Omega_t) > \lambda_1(\Omega \backslash \overline{(E \cup K)})$ contrary to (2.3). Therefore $v_t < 0$ in Ω_t for every $t > T$. The case $t < -T$ can be treated analogously. □

The proof of Theorem 2 will be completed by the use of the following Lemma, which obviously leads to a contradiction with Lemma 2.7. □

LEMMA 2.8. *The sign of v_t in Ω_t is a continuous function of $t \in \mathbb{R}$.*

Proof. Let us suppose that, for a given t, we have, for instance, $\Omega_t = \Omega_t^-$. We can find a smooth path η in Ω_t^+, which joins C with γ^+ (the arc constructed in the proof of Lemma 2.5) and such that

$$\inf_{\eta} v_t > 0.$$

Were it not so, we would have $v_t = 0$ weakly on $\partial \Omega_t^+$, which is not possible. Hence, there exists $\epsilon > 0$, such that, for every h, $|h| < \epsilon$, we have $v_{t+h} = v_t + hu > 0$ on η, that is $\eta \cup \gamma^+ \subset \Omega_{t+h}^+$. Therefore, for every h, $|h| < \epsilon$, $\eta \cup \gamma^+$ separates Ω_{t+h}^- from D, and, by the arguments on Lemma 2.5, this implies $\Omega_{t+h} = \Omega_{t+h}^-$. □

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