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Inflationary tilings with a similarity structure

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1. Introduction

An inflationary tiling is, roughly speaking, a tiling of the plane with a finite number of shapes of tiles, which has an associated expanding linear map which maps tiles over themselves. One reason for studying such tilings is that they arise naturally in Markov partitions for smooth hyperbolic dynamical systems. Finding "good" constructions of Markov partitions is in general not so easy, but explicit constructions of inflationary tilings have led to explicit constructions of Markov partitions [2, 8, 9, 14].

There are many varieties of tilings of $\mathbb{R}^n$, depending on the "structure group", or underlying geometry, one requires of the tilings: from a finite set of tile types one can consider tilings made by translates of the tiles, tilings using isometries, tilings using similarities, or more complicated mappings. For inflationary tilings the most natural structure groups (and the ones giving rise to the most interesting problems) known so far are subgroups of the affine group $A(2, \mathbb{R})$ (linear maps and translations).

The classical tiling problems deal of course with the group of translations or isometries of $\mathbb{R}^2$, and indeed the first inflationary tilings to be studied were translation-tilings, i.e. tilings with a finite number of tiles up to translation.

The simplest of these are known as **self-replicating** tilings, and arise from integer radix representations (representation of numbers in integer bases in $\mathbb{R}$, or integer-matrix bases in $\mathbb{R}^n$ [6, 9]) in which again the structure group is a group of translations. These are tilings with only one shape of tile.

Thurston [15] generalized self-replicating tilings to **self-similar** tilings of $\mathbb{C}$, which arise from representation of complex numbers in (certain) non-integer complex bases; these have a finite number of tiles up to translation. He proved that the expansion constant (the base of the representation) had to be a certain kind of

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algebraic integer, a "complex Perron number". In addition he indicated a construction for a tiling with any such expansion. Thurston's techniques were extended to arbitrary linear expansions and higher dimensions in [8], still with structure group a group of translations.

The techniques of [15, 8] do not readily extend to inflationary tilings with more general structure group (for example tilings with a finite number of tiles up to isometry or similarity). In particular, little is known about inflationary *isometry*-tilings. We deal here with the case of expansions of similarity-tilings of the plane, giving the following characterization:

THEOREM 1. *A complex number* γ, $|\gamma| > 1$ *is the expansion factor for an inflationary similarity-tiling if and only if* γ *is algebraic.*

Our proof of algebraicity resembles the proof (originally of A. Weil) of local rigidity for hyperbolic n-manifolds, $n > 2$, in that a small quasiconformal deformation of the tiling is shown to be trivial. Likewise we can also see that the proof breaks down in dimension one; in section 3 we construct an inflationary similarity tiling of the line for *any* real expansion constant.

One advantage in using a larger structure group is that there is more freedom in the construction of tilings: whereas for translation-tilings the tiles were often necessarily non-polygonal with non-rectifiable boundaries [15], here we find:

THEOREM 2. *For any algebraic number* γ, $|\gamma| > 1$, *there is an inflationary similarity-tiling of the plane by polygons with expansion factor* γ.

Section 2 defines similarity structures, the space of tilings, quasiperiodicity, and some definitions useful in the proofs that follow. These objects seem to be the natural extension of the original definitions in [15] (see also [8, 14]). Since many of the concepts here are unfamiliar we illustrate them with an example in section 3. Section 4 gives a proof of the algebraicity of the expansion of a quasiperiodic similarity-tiling of the plane, and section 5 gives the construction for any algebraic number γ of an inflationary similarity-tiling by polygons.

I would like to thank Dennis Sullivan and Curt McMullen for contributing important ideas in section 4, and Fréderic Paulin for a careful proofreading of this paper.

2. Tilings with similarities

By a **similarity** we will mean a complex linear map of $\mathbb{C}$ of the form $z \mapsto az + b$, where $a, b \in \mathbb{C}$, $a \neq 0$. Let G be a fixed group of similarities, with identity $e(z) = z$.

By a **tile type** we will mean a compact subset of $\mathbb{C}$ which is the closure of its interior.

Let T be a finite set of tile types, $T = \{T_1, T_2, \ldots, T_k\}$. By a **marked tile** we mean a pair (g, T_i) with $g \in G$ and $T_i \in T$. The **image** of a marked tile (g, T_i) is $g(T_i)$. We usually identify a marked tile (g, T_i) with its image $g(T_i)$ in $\mathbb{C}$, in which case we refer to $g(T_i)$ as a **tile**. It is important to make the distinction when the tile type T_i has symmetries under G, or several tile types are similar, so that (g, T_i) may not be the same marked tile as (g', T_j) even if $g(T_i) = g'(T_j)$.

To build a tiling, we need to specify the ways in which tiles fit together locally. For a tile type T_i, a **surrounding** of T_i is a collection of marked tiles whose images have pairwise disjoint interiors, intersect the image of the marked tile (e, T_i) and cover a neighborhood of the image of (e, T_i).

More generally a surrounding of a collection of (marked) tiles A is a collection of marked tiles whose images have pairwise disjoint interiors, intersect the union of the tiles in A and cover a neighborhood of the tiles of A.

Let S be a set of surroundings using tiles T. By definition, a (G, S)**-tiling** is a set of marked tiles whose images form a locally finite covering of $\mathbb{C}$ with pairwise disjoint interiors, and such that for each tile $t = g(T_i)$ the set of tiles intersecting t is the image under g of a surrounding of T_i in S. We often abuse this notation and identify a tiling with the set of images of its tiles. By a G**-tiling** we mean a (G, S)-tiling for some *finite* set S. In the rest of the paper we will only consider tilings with S finite. We refer the reader to [10] for remarks about the case S infinite.

Notationally, if $A = \{(g, T_i)\}$ is a tiling and α is a similarity then by αA we mean the tiling $\{(\alpha g, T_i)\}$. In particular $x + A$ is the tiling obtained from A by translating all the tiles by x.

The usual periodic tilings of the plane with fundamental domain consisting of one tile are *translation*-tilings, (that is, G-tilings where G is the group of translations). Also in these cases S consists of a single surrounding. The familiar Penrose tilings [13] are examples of isometry-tilings, in this case G being a group of isometries of the plane; there are two tile types, the kite and the dart, and a finite number of surroundings determined by the allowed adjacencies of tiles.

The above definitions can be extended to other structure groups, most notably $G = A(2, \mathbb{R})$, the group of affine transformations of $\mathbb{R}^2$. However, throughout the rest of the paper (except section 3, where we deal with tilings of $\mathbb{R}$), we assume G is a group of similarities of $\mathbb{C}$.

2.1. *A topology on the set of tilings*

Fix G to be the group of all similarities. Fix a finite set of tile types T and set of surroundings S. (Recall that S is finite.)

Let $\mathscr{T}$ be the set of all (G, S)-tilings modulo the equivalence of homothety: two tilings A and B are equivalent, $A \sim B$, if there is a homothety ψ (a similarity $z \mapsto az$, $a \in \mathbb{C} - \{0\}$) such that $\psi A = B$. Let $[A]$ denote the equivalence class of a tiling A. Each equivalence class has a representative in which the set of tiles touching the origin has total measure 1. Call such a tiling a **standard** tiling. If A is a tiling, let $\langle A \rangle$ be a standard tiling equivalent to A.

We topologize the set $\mathscr{T}$ of equivalence classes in the following way. A neighborhood of $[A] \in \mathscr{T}$ is the set of tilings which are equivalent to a small translation of $\langle A \rangle$ on a large neighborhood of the origin. That is, the basic neighborhood $U_{r,R}([A])$, where $r > 0$, $R > 0$, is the set of $[B]$ (with standard form $\langle B \rangle$) such that there exists $r_1, r_2 \in \mathbb{C}$, $|r_1| < r$ such that $r_2(r_1 + \langle A \rangle)$ and $\langle B \rangle$ agree on the ball of radius R around the origin. (Two tilings are said to **agree on a region** U if their subsets of marked tiles with images intersecting U are the same.)

One must keep in mind in this definition that $\mathscr{T}$ consists of homothety classes of tilings, not tilings themselves, so that a small translation of a tiling A is close to A even if the origin crosses the boundary of a tile (the *standard* tilings change but not the equivalence classes).

Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous non-decreasing function. Let $\mathscr{T}_f \subset \mathscr{T}$ be the set of $[A]$ for which for all $R > 0$ and $x \in \mathbb{C}$ the number of tiles in the tiling $\langle x + A \rangle$ intersecting $B_R(0)$ is less than $f(R)$.

LEMMA 3. *The sets $U_{r,R}$ form a base for a Hausdorff topology on $\mathscr{T}$. In its subspace topology, $\mathscr{T}_f$ is a compact Hausdorff space.*

Proof. We need to show that if two basic neighborhoods intersect, then there is a basic neighborhood in their intersection. Suppose $[C] \in U_{r,R}([A]) \cap U_{r',R'}([A'])$. That is, $\langle C \rangle$ agrees with $r_2(r_1 + \langle A \rangle)$ on $B_R(0)$ and with $r_2'(r_1' + \langle A' \rangle)$ on $B_{R'}(0)$. Assume the origin is in the interior of a tile in the tiling C. Then for ϵ small, $\langle C \rangle + \epsilon$ agrees with $r_2(\epsilon/r_2 + r_1 + \langle A \rangle)$ on $B_{R-\epsilon}(0)$ and with $r_2'(\epsilon/r_2' + r_1' + \langle A' \rangle)$ on $B_{R'-\epsilon}(0)$. So $U_{\epsilon,Q}([C]) \subset U_{r,R}([A]) \cap U_{r',R'}([A'])$ as long as

$$\left|\frac{\epsilon}{r_2} + r_1\right| < r, \quad \left|\frac{\epsilon}{r_2'} + r_1'\right| < r' \quad \text{and} \quad Q = \max\{R, R'\}.$$

A similar argument works in case the origin is on the boundary of a tile in C. Thus the $U_{r,R}$ do form a base for a topology.

To prove that $\mathscr{T}$ is Hausdorff, if two tilings A and B have different tiles containing the origin, or the same tiles containing the origin but the origin is in a different place in the tiles, then $U_{r,R}([A])$ and $U_{r,R}([B])$ are disjoint if r is sufficiently small. If they have the same tiles at the origin and the origin is at the same place in those tiles, then by taking R sufficiently large, the $U_{r,R}([A])$ and $U_{r,R}([B])$ are disjoint unless of course $A \sim B$.

To prove compactness of $\mathscr{T}_f$, let T_i be a tile type in T. For $x \in T_i$ let $R_i(x) \subset \mathscr{T}_f$ be the set of tilings which have a tile $g(T_i)$ containing the origin for some g for which $g(x) = 0$. The $R_i(x)$ are homeomorphic as x ranges over T_i. Furthermore $R_i(x)$ in its subspace topology is an inverse limit of finite sets:

$$R_i(x) \cong \varprojlim S_n^i = \{S_0^i \leftarrow S_1^i \leftarrow S_2^i \leftarrow \cdots\}$$

where $S_0^i = \{T_i\}$ is the set consisting of the single tile type T_i, S_1^i is the set of surroundings of T_i which can be extended to a tiling in $\mathscr{T}_f$, for each $n > 1$ the set S_n^i is the set of possible surroundings of elements of S_{n-1}^i which can be extended to a tiling in $\mathscr{T}_f$, and the map $S_n^i \to S_{n-1}^i$ is the restriction (ignoring the outer tiles). The existence of the function f insures that each point in the inverse limit corresponds to a sequence of surroundings filling out the whole plane as $n \to \infty$, hence a (G, S)-tiling with a tile of type T_i at the origin. In addition, two tilings in the inverse limit are close if they agree on a large neighborhood of the origin, so $R_i(x)$ has the topology of the inverse limit.

Since S, the allowed surroundings of a single tile, is finite, each of the S_n^i are finite, so $R_i(x)$ is compact and totally disconnected. Let $R_i \subset \mathscr{T}_f$ be the set

$$R_i = \bigcup_{x \in T_i} R_i(x).$$

Then R_i is almost the set $R_i(x) \times T_i$, except there may be some identifications along the boundary of T_i. In any case R_i is a (Hausdorff) quotient of the compact set $R_i(x) \times T_i$, and so is compact. Finally, $\mathscr{T}_f$ is the union of the finite number of compact sets R_i for each tile type $T_i \in T$, and so is compact itself. □

The space $\mathscr{T}$ itself is not necessarily compact; a limit of tilings may cease to be locally finite and so not be a tiling in our sense: consider for example the tiling A of Figure 1, with two tile types, a square and a square annulus. In the tiling shown the square is a unit square centered at the origin and is surrounded by an infinite sequence of square annuli, each similar to the first which has outer boundary a square of side 2. Let A_n for $n \geq 0$ be the sequence of tilings $A_n = \langle A + (3 \cdot 2^n, 0)\rangle$, that is, A_n is just the standard version of the translate of A by $(3 \cdot 2^n, 0)$. The "limit" of this sequence is not locally finite at a point near the origin. (Precisely, at the point $(\sqrt{3}/4, 0)$.)

In the future we shall deal only with the spaces $\mathscr{T}_f$. From now on we consider f fixed.

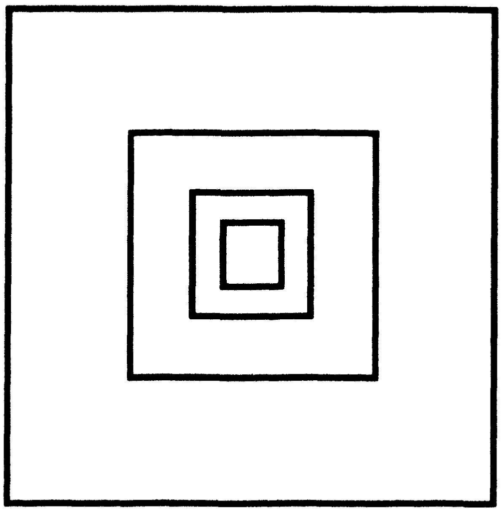

Figure 1. Tiling for which $\mathcal{T}$ is noncompact.

For each tiling A, $[A] \in \mathcal{T}_f$, there is a natural map π_A of $\mathbb{C}$ into $\mathcal{T}_f$, defined by $\pi_A(z) = [z + A]$. The map π_A is continuous, and since no tiling is equivalent to a small translation of itself, it is locally injective. Since for a similarity α we have $\pi(\alpha z) = [\alpha z + A]$ the map π induces a *similarity structure* on $\pi_A(\mathbb{C})$.

From this and the structure of the sets R_i from above proof we see that the space $\mathcal{T}_f$ is foliated by surfaces with similarity structures, so that *locally* $\mathcal{T}_f$ is a product $D \times K$ where D is a disk in $\mathbb{C}$ and K is a compact and totally disconnected set. Thus $\mathcal{T}_f$ is a (similarity-) **surface lamination**, in the sense of e.g. Candel [4].

Let $\mathcal{T}_A = closure(\pi_A(\mathbb{C})) \subset \mathcal{T}_f$. Then $\mathcal{T}_A$ with the subspace topology is again compact, and we call it the **orbit closure** of the tiling A.

If $y \in \mathcal{T}_f$ has a tile with the origin in its interior, then for small r the basic neighborhood $U_{r,R}(y)$ has itself a product structure. In particular r must be small enough so that each $x \in U_{r,R}(y)$ has the origin in the interior of a tile of the same type.

Referring to this product structure, we define for $x \in \mathcal{T}_f$ the **stable fiber of** x to be the set of tilings which have standard versions which agree with $\langle x \rangle$ in some neighborhood of the origin. We also refer to this set as

$$U_{0,0}(x) \stackrel{\text{def}}{=} \bigcup_{R \searrow 0} U_{0,R}(x).$$

The sets $U_{0,R}(x)$ are the *local* stable fibers of x.

Similarly we define the **unstable fiber of** x to be the leaf through x, alternatively the set of tilings obtained from x by translation. We also refer to this set as $U_{\infty,\infty}(x)$.

2.2. *Quasiperiodicity*

The tilings we are concerned with all have a certain regularity property, called quasiperiodicity, which we shall see is a very natural property to impose on a tiling, and in particular on an inflationary tiling (Lemma 6). They are in a sense **minimal** tilings, in the sense that all tilings in $\mathcal{T}_f$ accumulate on quasiperiodic tilings (Lemma 4).

The tiling A, $[A] \in \mathcal{T}_f$ is said to be **quasiperiodic** if for all $R > 0$ there is an $r > 0$ such that for every $x \in \mathbb{C}$, $\pi_A(\mathbb{C}) \subset U_{r,R}([x + A])$. Here one should think of R as large, and r even larger, so that $U_{r,R}([x + A])$ is contained in a product neighborhood about $[x + A]$ which is thin in the stable direction and fat in the unstable direction.

In other words, any local arrangement of tiles (of standard size R) occurring somewhere in the tiling A can be found in a "standard r-neighborhood" of any point x in the tiling A.

A simple characterization of quasiperiodicity is given by:

LEMMA 4. *A tiling A is quasiperiodic if and only if every leaf of $\mathcal{T}_A = \overline{\pi_A(\mathbb{C})}$ is dense in $\mathcal{T}_A$ (i.e. the foliation of $\mathcal{T}_A$ is minimal).*

Proof. Suppose A is quasiperiodic. Then for all $R > 0$ there exists an $r > 0$ such that $\pi_A(\mathbb{C}) \subset U_{r,R}(x)$ for any $x \in \pi_A(\mathbb{C})$, hence $\mathcal{T}_A \subset \text{closure}(U_{r,R}(x)) \subset U_{2r,R/2}(x)$.

For any $y \in \mathcal{T}_A$ take a sequence $\{x_i\}$ in $\pi_A(\mathbb{C})$ converging to y. If y has the origin in the interior of a tile, then for i sufficiently large we have $U_{2r,R/2}(x_i) \subset U_{3r,R/3}(y)$. If the origin in y is on the boundary of a tile, then for i sufficiently large $U_{2r,R/2}(x_i) \subset U_{mr,R/m}(y)$ for some constant m depending on the relative sizes of adjacent tiles. Assume $m > 3$ then in either case $\mathcal{T}_A \subset U_{mr,R/m}(y)$. Since a similar containment holds for any R, the leaf containing y is dense in $\mathcal{T}_A$.

Suppose A is not quasiperiodic. Then there exists R such that for all r, there are x_r, $y_r \in \pi_A(\mathbb{C})$ for which $y_r \notin U_{r,R}(x_r)$. Let $x, y \in \mathcal{T}_A$ be limit points of the x_r and y_r respectively, as r tends to infinity. They exist by compactness of $\mathcal{T}_A$. Then $y \notin U_{r,R}(x)$ for any r so the leaf containing x doesn't get near y, so $\mathcal{T}_A$ is not minimal. □

For two easy examples, the periodic tiling of the plane with unit squares and the tiling shown in figure 2 are both (G, S)-tilings, where G is the set of translations and S consists of three surroundings of a unit square. The first is quasiperiodic (as are all periodic tilings; one can in this case take $r = \sqrt{2}/2$ independently of R), but the second is not: a limit of tilings $x + A$ with x having horizontal coordinate tending to infinity is periodic, and hence not dense in $\mathcal{T}_A$.

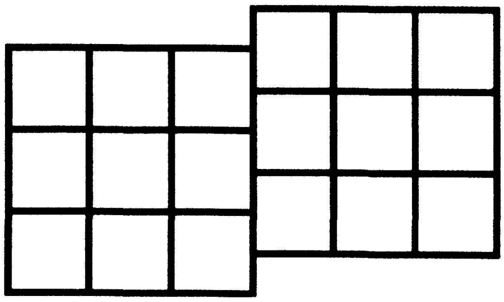

Figure 2. The tiling is obtained from the periodic tiling by shifting the tiles right of the y-axis up by a small amount $\delta < 1$.

2.3. *Markov expansions and inflationary tilings*

For each tile type T_i in T let $R_i \subset \mathcal{T}_f$ be the set of tilings for which the origin is contained in a tile of type T_i. The boundary of R_i is the set of tilings for which the origin is in the boundary of a tile of type T_i. The R_i form a finite partition of $\mathcal{T}_f$, that is, a covering by closed sets with non-overlapping interiors. In addition, as we saw in the proof of Lemma 3, R_i is a quotient of $T_i \times K_i$, where $T_i \subset \mathbb{C}$ is a copy of the tile T_i and K_i is compact and totally disconnected. The quotient map is injective on $\text{int}(T_i) \times K_i$, so by abuse of notation we refer to the image $\text{int}(T_i) \times K_i$ as $\text{int}(R_i)$. We will refer to points of R_i by giving the two coordinates of a preimage in $T_i \times K_i$ (some points in $R_i \backslash \text{int}(R_i)$ may have several such descriptions). For $(x, k) \in R_i$, we refer to the subset $T_i \times \{k\} \subset R_i$ as the **unstable fiber in** R_i through (x, k). The set $\{x\} \times K_i \subset R_i$ is just the stable fiber of (x, k), but in this context we also refer to it as the **stable fiber in** R_i through (x, k). The R_i are called **rectangles.**

A **Markov expansion**, or simply **expansion**, is a continuous map $\varphi : \mathcal{T}_f \to \mathcal{T}_f$ preserving the similarity structure on leaves, for which the partition $\bigcup R_i$ acts as a **Markov partition** in the following sense:

(1) for each $(x, k) \in R_i$, if $\varphi(x, k) \in \text{int}(R_j)$ then the entire stable fiber $\varphi(\{x\} \times K_i)$ is in $\text{int}(R_j)$, and
(2) if $(y, k') \in R_j \cap \varphi(R_i)$, then $\text{int}(T_j) \times \{k'\} \subset \varphi(\text{int}(R_i))$.

See figure 3. This is close to the usual definition of a Markov partition, from for example [3]: the unstable fibers of R_i map across the unstable fibers of R_j, and the stable fibers of R_i map into the stable fibers of R_j. Unlike [3] the R_i may have boundary identifications, and also $\varphi(R_i)$ may intersect R_j in more than one 'subrectangle'.

It is worth explaining what a Markov expansion means on the level of tilings. Since φ preserves the similarity structure on leaves of $\mathcal{T}_f$, the map $\pi^{-1}_{\varphi(A)} \circ \varphi \circ \pi_A$:

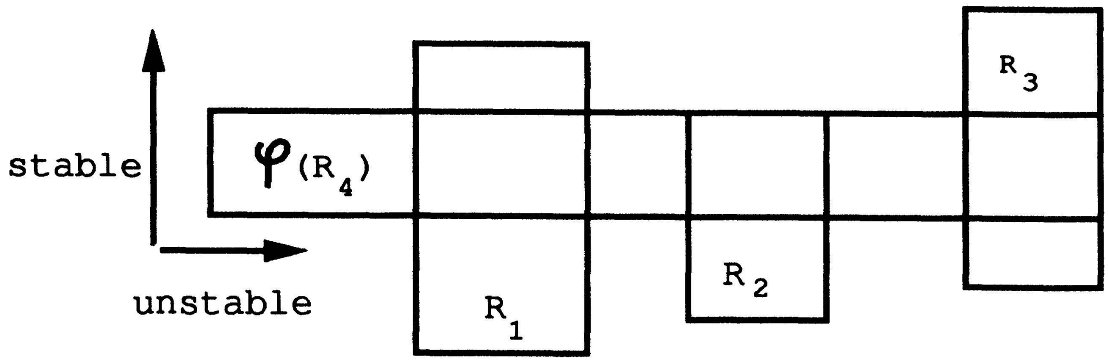

Figure 3. How an expansion maps rectangles.

$\mathbb{C} \to \mathbb{C}$ is a similarity, i.e. a map of the form $z \mapsto az + b$. The image of a tile $g(T_i)$ of A under this composition is a set $ag(T_i) + b \subset \mathbb{C}$ which must be a union of tiles of the tiling $\varphi(A)$ (by property 2 of a Markov expansion). Thus the tiling $\varphi(A)$ is obtained by applying a similarity $\psi z = az + b$ to A and **subdividing** the tiles $\psi g(T_i)$, that is, for each tile $g(T_i)$ of A replacing $\psi g(T_i)$ with a set of tiles whose union is $\psi g(T_i)$. Property 1 of the expansion implies that two tiles $g(T_i)$, $g'(T_i)$ in A of the same tile type have the *same* subdivision, that is, they are replaced with the same set of tiles, up to an element $g^{-1}g' \in G$.

An **inflationary** G**-tiling** is simply a quasiperiodic (G, S)-tiling A which is a fixed point of some Markov expansion φ of $\mathcal{T}$ and for which φ is expanding on A, that is, the similarity $\pi_A^{-1}\varphi\pi_A : \mathbb{C} \to \mathbb{C}$ is a homothety $z \mapsto \lambda z$ with $|\lambda| > 1$. The complex number λ is called the **expansion constant**.

There are three important properties of an inflationary tiling which we underline:

LEMMA 5. *Let A be an inflationary tiling.*
(1) *The image of a tile of A under φ exactly covers other tiles in A.*
(2) *Two tiles of the same type subdivide in the same way.*
(3) *Each tile eventually subdivides. For all $N \geq 1$ there is an n such that $\varphi^n(U_{0,R}(x)) \subset U_{0,NR}(\varphi^n(x))$ (i.e. φ eventually contracts stable fibers).*

Proof. The first and second of these three properties have already been shown. For the third, recall that we have $|\lambda| > 1$. Suppose T_i is a tile type which does not subdivide. Let $aT_i + b$ be an occurrence of T_i in the tiling A; then $\lambda^k(aT_i + b)$ is also an occurrence of T_i. Let $x \in \operatorname{int}(aT_i + b)$. If $R \geq |x|(1 + 1/|\lambda| + 1/|\lambda|^2 + \cdots)$ then for each k there are at least k tiles $\lambda^{k-l}(aT_i + b)$ for $l = 0, \ldots, k$ intersecting $B_R(0)$ in the tiling $\langle -\lambda^k x + A \rangle$. So $A \notin \mathcal{T}_f$. Thus each tile eventually subdivides. Furthermore, it does so after a finite number of applications of φ since there are a finite number of tile types. Each subdivision decreases the area of the tiles by a constant factor, so the second statement of part 3 follows. □

As an easy example of an inflationary tiling, if the set S consists of a single surrounding, the surrounding of a square by 8 squares in the usual way, then there is a single (G, S)-tiling: the periodic tiling. The space $\mathcal{T}$ in this case is the torus $\mathbb{R}^2$ mod 1, the rectangle R_0 is all of $\mathcal{T}$, and $\mathrm{int}(R_0)$ is homeomorphic to $(0, 1)^2 \times K$, where K consists of a single point. The map $(x, y) \mapsto (2x, 2y)$ mod 1 is a Markov expansion for $\mathcal{T}$, and the periodic tiling by squares having the origin at a vertex is an inflationary tiling with expansion factor 2. For maps $(x, y) \mapsto (nx, ny)$ mod 1, with $n > 2$, there are several inflationary tilings corresponding to different placements of the origin in the original square.

The Penrose tilings have the remarkable property that $\mathcal{T}$ itself is compact and minimal, yet has no closed leaves. They have a well-known subdivision rule, and there are countably many inflationary tilings (powers of φ have many fixed points).

Figure 4a shows a way to subdivide a square to yield an inflationary similarity-tiling with one tile type (Figure 4b; the picture is centered at the origin.) The expansion factor is 2. Each surrounding of a square consists of squares of the same size, twice the size, or one-half the size of the center square, located at the corners or midpoints of the center square. Hence the number of surroundings is finite. A square of side length 2^{-n} subdivides upon expansion by 2 into a square of size 2^{-n} surrounded by squares of size 2^{-n-1}, so in the tiling of the whole plane there are squares of size 2^{-k} for all $k \geq 0$, and so the tiling in particular is not an isometry tiling. (Note that there is only one tile of side 1, but an infinite number of side 2^{-n} for each $n \geq 1$.) Lemma 6 below shows that the tiling is quasiperiodic.

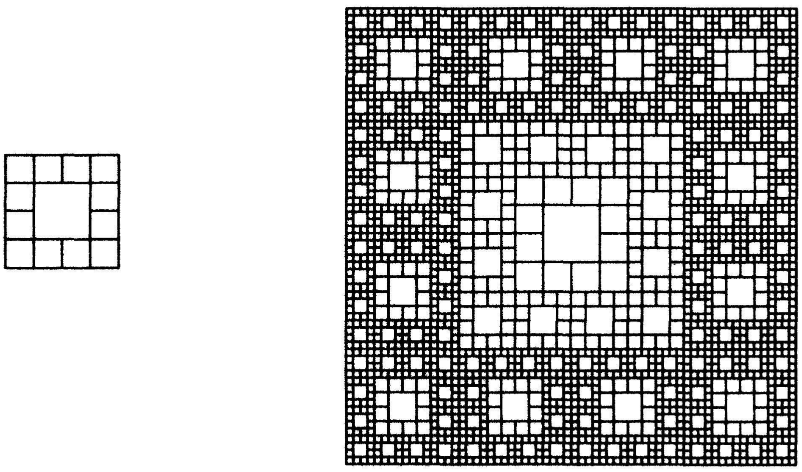

Figure 4. The subdivision of a tile, and part of the tiling near 0.

This example can be generalized to other surroundings of a square by squares, to give inflationary tilings with a single tile type and with any rational expansion constant $r > 1$.

2.4. *Mixing implies quasiperiodic*

Determining whether a given tiling is quasiperiodic is not always easy. In the case of a tiling which is a fixed point for a Markov expansion, however, there is a simple criterion. A Markov expansion φ of $\mathscr{T}_f$ with fixed point $[A]$ and expansion λ, $|\lambda| > 1$ is called **mixing** for $[A]$ if every tile type occurring in $[A]$ occurs in the eventual subdivision of any other tile type in $[A]$. That is, for any two rectangles $R_i, R_j \subset \mathscr{T}_f$ which intersect $\pi_A(\mathbb{C})$, there is an $n > 0$ such that $\varphi^n(R_i)$ intersects $\operatorname{int}(R_j)$. Since there are a finite number of tile types this condition is easily checked. We have

LEMMA 6. *If φ is mixing for $[A]$, and the origin is contained in the interior of a tile of A, then A is quasiperiodic, hence inflationary.*

This lemma can also be found in [14, 9] in the context of translation-tilings.

Proof. We will use Lemma 4.

Fix $r > 0$ and $R \geq 0$. By compactness of $\mathscr{T}_A$, there exists a finite number $k = k(r, R)$ of tilings $x_1, x_2, \ldots, x_k \in \pi_A(\mathbb{C})$, each having a tile containing the origin in its interior, such that

$$\mathscr{T}_A \subset \bigcup_{i=1}^{k} U_{r,R}(x_i).$$

Let $x \in \mathscr{T}_A$. We must show that the leaf through x intersects each $U_{r,R}(x_j)$. Since this will be true for each r and R, the leaf will be dense in $\mathscr{T}_A$, so by Lemma 4 the tiling A will be quasiperiodic.

Let R_0 be the rectangle in $\mathscr{T}_f$ containing the point $[A]$. Since the origin is in the interior of a tile in A and $|\lambda| > 1$, we have for all $N \geq 0$

$$\pi_A(\mathbb{C}) \subset \bigcup_{k=N}^{\infty} \varphi^k(R_0),$$

so there is an integer $n = n(R)$ so large that firstly for each $1 \leq i \leq k$, $\varphi^n(R_0)$ intersects each $U_{r,R}(x_i)$, and secondly there are $y_i \in R_0$ such that the entire stable fiber of y_i maps into the "local" stable fiber $U_{0,0}(\varphi^n(y_i)) \cap U_{r,R}(x_i)$ (this last by part 3 of Lemma 5).

Since φ is mixing for A, there is an integer n' so that for each i such that a tile of type T_i occurs in A, $\varphi^{n'}(R_i)$ intersects $\mathrm{int}(R_0)$.

Let $x \in \mathscr{T}_A$. Since $\varphi^{n+n'}$ is continuous on $\mathscr{T}_f$ and a homeomorphism (in fact a similarity) on $\pi_A(\mathbb{C})$, it extends to a continuous map from $\mathscr{T}_A$ onto itself. So there exists $y \in \mathscr{T}_A$ such that $\varphi^{n+n'}(y) = x$. Now $y \in R_i$ for some R_i occurring in A, so from the definition of n', $\varphi^{n'}(y) \in \varphi^{n'}(R_i)$ intersects $\mathrm{int}(R_0)$. Then $x = \varphi^{n+n'}(y) \in \varphi^{n+n'}(R_i)$ intersects $\varphi^n R_0$ which intersects each $U_{r,R}(x_j)$, and the stable fibers of R_i map into the local stable fibers of points in $U_{r,R}(x_j)$. This implies that the leaf through x intersects each $U_{r,R}(x_j)$. $\square$

For a tiling satisfying the hypothesis of Lemma 6, except that the origin is not in the interior of a tile, consider the set of tiles containing the origin. If this arrangement of tiles reoccurs in some repeated subdivision of any single tile (i.e. it is not an exceptional arrangement), then the tiling is quasiperiodic as before, by a similar argument.

In the example in figure 2, if the vertical displacement of tiles to the right of the origin is 1/3, then the expansion $x \to 4x$ leaves the tiling invariant and is mixing, since there is only one tile type, but the tiling is not quasiperiodic as we saw. This shows why the hypothesis about the arrangement at the origin (or the origin being in the interior of a tile) is important.

2.5. *Control points*

Given an inflationary G-tiling A with expansion φ, for each tile t of type T_i, the image $\varphi(t)$ subdivides into new tiles in the same way. Pick one of the tiles in the subdivision of $\varphi(t)$, and call it the successor of t, succ(t). Pick a single successor for each tile type in T. Define the **control point** of a tile t of type T_i to be a point $c(t) \in t$ such that for all $n \geq 1$, $\varphi^n(c)$ lies in the tile which is the successor of the tile containing $\varphi^{n-1}(c)$. Since there is an n such that for each tile type, $\varphi^n(t)$ does subdivide (Lemma 5) and each subdivision decreases the ratio of areas of new tiles to old tiles by a definite amount, there is a **unique** control point in every tile.

As an example, in the tiling of figure 4, we pick the successor of a square to be the central square in its subdivision. Then the control point of any tile is its barycenter: the preimages of the successors nest down to the barycenter of the square. If we had picked, say, the lower left square in the subdivision to be the successor, then the control point would instead be the lower left corner of a tile.

The set C of control points of all tiles in an inflationary G-tiling with expansion constant λ is invariant under $\lambda : \lambda \cdot c(t) = c(succ(t))$, so that $\lambda C \subset C$. The notion of control point is due to Thurston.

LEMMA 7. *Let A be an inflationary tiling with expansion φ and expansion constant λ. Let C be a set of control points for $\langle A \rangle$. For any $\epsilon > 0$ there is an $n > 0$ such that $\lambda^{-n}(C)$ is ϵ-dense in the ball of radius 1 around the origin. (That is, there is a point of $\varphi^{-n}C$ within ϵ of any point in $B_1(0)$.)*

Proof. By part 3 of Lemma 5, there is an N such that for each tile t intersecting $B_1(0)$ in $\langle A \rangle$, the tiles in $\varphi^N(t)$ have diameter at most ϵ diam$(\varphi^N(t))$.

Each tile contains a control point so $\lambda^{-N}C$ is ϵ-dense in t. Since there are a finite number of tiles intersecting $B_1(0)$, letting N' be the max of the N defined as above for each such tile t, $\lambda^{-N'}C$ is ϵ-dense in $B_1(0)$. □

One more definition: if A is a triangle in the complex plane with vertices a, b, c, in counterclockwise order, the **similarity class** of A is defined to be $s(A) = (c - a)/(b - a)$. It is a well-defined point in the upper half-plane, up to cyclic permutations of the vertices, which gives the equivalences

$$s(A) \sim \frac{1}{1 - s(A)} \sim 1 - \frac{1}{s(A)}.$$

Two triangles are similar iff they have the same similarity class up to this equivalence.

3. Tilings of $\mathbb{R}$

In this section only, we will consider (G, S)-tilings of $\mathbb{R}$, where G is a group of orientation-preserving similarities of $\mathbb{R}$, that is, maps of the form $x \mapsto ax + b$, with $a > 0$.

THEOREM 8. *For any real r, $|r| > 1$, there is an inflationary similarity-tiling of $\mathbb{R}$ with expansion r.*

Proof. Let $r \in \mathbb{R}$, $|r| > 1$, and let $\beta = (|r| - 1)/2$. We construct a (G, S)-tiling in which S has only one type of tile, the interval, and 4 surroundings, consisting of intervals in length ratios respectively $\beta:1:\beta$, $\beta:1:\beta^{-1}$, $\beta^{-1}:1:\beta^{-1}$, and $\beta^{-1}:1:\beta$. Thus adjacent tiles have length in ratio β or β^{-1}.

Let t_0 be the interval of length 2 centered at the origin. We subdivide the image $rt_0=[-|r|,|r|]$ into three intervals:

$$[-|r|,|r|]=[-|r|,-1]\cup[-1,1]\cup[1,|r|]$$

so that rt_0 consists of three tiles in ratios $\beta:1:\beta$. We then build the tiling by expansion and subdivision, subdividing an interval of length l into intervals of length $l\beta$, l and $l\beta$ in that order. Since the expansion-subdivision leaves larger and larger neighborhoods of 0 fixed, there is a unique limiting tiling of $\mathbb{R}$.

The surrounding of a tile in the image of one of the four surroundings is again one of the four surroundings, so the resulting tiling is a (G, S)-tiling. The expansion is mixing, and so is quasiperiodic by Lemma 6. □

What is the closure $\mathcal{T}_A$ in $\mathcal{T}$ of this tiling A? We suppose $\beta \neq 1$. A tiling in $\mathcal{T}$ can be described by a point in $[0, 1)$ (telling where the origin is located in the tile(s) containing the origin), and a biinfinite sequence of reals describing the ratio of tile lengths relative to the tile at the origin. In such a sequence the ratio of adjacent elements is β or β^{-1}.

It is easy to see that not all such sequences occur for tilings in $\pi_A(\mathbb{C})$. In fact a sequence of ratios for a tiling β in $\pi_A(\mathbb{C})$ can be grouped into groups of 3 consecutive ratios of the form β^n, β^{n-1}, β^n, coming from the tiles in the preimage $\varphi^{-1}B$. There is a unique way to do this grouping.

For a tiling $[x+A]$ the origin sits in the left, middle or right side of a group. Describe these possibilities with L, M, or R respectively. Then in the tiling $\varphi^{-1}([x+A])$ the origin again sits in the left, middle, or right of a group, and so on. Thus to a tiling $[x+A]$ we can associate an infinite sequence $\{x_0, x_1, \ldots\}$ of letters $\{L, M, R\}$, where x_i describes where the origin sits in the tiling $\varphi^{-i}([x+A])$. Since the tiling A is inflationary, for n large the origin of $\varphi^{-n}[x+A]$ is in the tile containing 0, so that the sequence $\{x_i\}$ terminates in $M, M, \ldots$.

Thus a tiling $[x+A]$ is described by a point in $[0,1)\times\Sigma^*$, where Σ^* is the set of sequences of $\{L, M, R\}$ terminating in $M, M, \ldots$. To obtain the right topology on $\pi_A(\mathbb{C})$ we need to identify some tilings having the origin at $0\in[0,1)$ with some limits of tilings as the origin goes to 1 in $[0, 1)$. So $\pi_A(\mathbb{C})$ is actually a quotient of $[0,1]\times\Sigma^*$.

Let $\Sigma=\{L, M, R\}^{\mathbb{N}}$, the set of all infinite sequences of letters $\{L, M, R\}$. The space $\mathcal{T}_A=\overline{\pi_A\mathbb{C}}$ is obtained from the space $[0,1]\times\Sigma$ by performing the following identifications:

$$(0, \{M, x_2, x_3, \ldots\}) \xrightarrow{\beta} (1, \{L, x_2, x_3, \ldots\})$$

$$(0, \{R, x_2, x_3, \ldots\}) \xrightarrow{1/\beta} (1, \{M, x_2, x_3, \ldots\})$$

$$(0, \{\underbrace{L, L, \ldots, L}_{k}, M, x_{k+2}, x_{k+3}, \ldots\}) \xrightarrow{\beta} (1, \{\underbrace{R, R, \ldots, R}_{k}, L, x_{k+2}, x_{k+3}, \ldots\})$$

$$(0, \{\underbrace{L, L, \ldots, L}_{k}, R, x_{k+2}, x_{k+3}, \ldots\}) \xrightarrow{1/\beta} (1, \{\underbrace{R, R, \ldots, R}_{k}, M, x_{k+2}, x_{k+3}, \ldots\})$$

$$(0, \{L, L, \ldots\}) \xrightarrow{\beta} (1, \{R, R, \ldots\})$$

$$(0, \{L, L, \ldots\}) \xrightarrow{1/\beta} (1, \{R, R, \ldots\})$$

The labels on the arrows show the gluing similarity (the ratio of the size of the left tile to the right tile). The last two show the two possible limits of the third and fourth kind of identifications as $k \to \infty$.

The space $\pi_A \mathbb{C}$ is the (dense) subset of $\mathcal{T}_A$ whose second coordinate ends in $M, M, \ldots$. The space $\mathcal{T}_A$ is almost the same as the 3-adic solenoid

$$\cdots \xrightarrow{\times 3} S^1 \xrightarrow{\times 3} S^1 \xrightarrow{\times 3} S^1,$$

except that the sequences $\{L^k, M, \ldots\}$ and $\{L^k, R, \ldots\}$ converge to different points of $\mathcal{T}_A$.

4. The expansion constant is algebraic

We begin with a brief sketch of some field theory involving transcendental numbers; our reference was [12].

Let K be a finitely generated field over $\mathbb{Q}$. An **embedding** of K into $\mathbb{C}$ is a field homomorphism from K into $\mathbb{C}$, which is necessarily an isomorphism onto its image. If K is transcendental let $\lambda_1, \ldots, \lambda_n$ be a transcendence basis for K, that is, $\lambda_1, \ldots, \lambda_n$ are algebraically independent and K is a finite extension of $\mathbb{Q}(\lambda_1, \ldots, \lambda_n)$. Any set of algebraically independent elements of K can be completed to a transcendence basis, which will have the same cardinality n. Since all transcendental numbers are the same from the point of view of the rationals we can embed K into $\mathbb{C}$ in many ways. In particular let ξ be a real number algebraically independent of $\lambda_1, \ldots, \lambda_n$. Then for any real $t > 0$ there is an embedding $\varphi_t : K \to \mathbb{C}$ sending λ_1 to $\lambda_1 + t + it\xi$ and λ_i to λ_i for $i > 1$. (Note that for each real $t \neq 0$ either

t or $t\xi$ is algebraically independent of $\lambda_1, \ldots \lambda_n$.) This gives an example of a continuous one-parameter family of embeddings $K \to \mathbb{C}$ starting at the identity. (We use this fact later.)

THEOREM 9. *If $\lambda \in \mathbb{C}$, $|\lambda| > 1$ is the expansion for an inflationary similarity-tiling of $\mathbb{C}$ then λ is algebraic. After an $\mathbb{R}$-linear change of coordinates commuting with the homothety λ, the similarities G generating the tiling are also algebraic, i.e. of the form $x \mapsto ax + b$, with a and b algebraic numbers.*

Proof. We begin with a sketch of the proof. If the field K generated by the control points is transcendental over $\mathbb{Q}$, a sufficiently small perturbation of the embedding of K doesn't mix up the tiles locally, and the inflationary property implies that it is not mixed up globally; thus one can define a new inflationary tiling with the same combinatorics as the original tiling. From Lemma 7, the union of all the preimages of the control points is dense, and the perturbation extends to this dense set to give a homeomorphism of the plane conjugating the dynamics of the old tiling to that of the new, which we show is quasiconformal. The expanding dynamics implies that this homeomorphism is linear, and complex linear unless λ is real. Thus the perturbation of K doesn't change λ, and so λ must be algebraic.

To begin, let A with surroundings S be an inflationary similarity-tiling of $\mathbb{C}$, with expansion $\varphi(z) = \lambda z$. We suppose A is a standard tiling, i.e. the union of tiles containing 0 has measure 1. Let C be a set of control points for the tiling.

Let K be the field generated by the points $C \subset \mathbb{C}$. Note that $\lambda \in K$, being the ratio of some two control points. Let π_0 be the given embedding of K as an abstract field into $\mathbb{C}$. Notationally, we will identify the abstract field K with $\pi_0 K$.

Let Δ be the Delauney triangulation of the plane with the set C as vertices: Δ is the triangulation which has the property that the disk bounded by the circumcircle of each triangle in Δ contains no point of C in its interior. This triangulation is uniquely defined unless a circle passes through more than 3 points of C. If this happens choose some triangulation in the circle which depends only on the arrangement of tiles near those points. By Lemma 7, this triangulation near a point x depends only on the surrounding of a certain standard radius near x (a triangle cannot have diameter larger than a constant times the diameter of tiles near x). The set D of similarity classes of triangles in Δ is thus a *finite subset of K.*

Let π be another embedding of K into $\mathbb{C}$. We construct a sequence of piece-wise linear maps $f_\pi^{(n)} : \mathbb{C} \to \mathbb{C}$, $n \geq 1$ as follows. For $c \in C$ we define $f_\pi^{(1)}(c) = \pi(c)$, and then extend $f_\pi^{(1)}$ linearly over each triangle $D_j \in \Delta$. The $f_\pi^{(1)}$ so defined is continuous.

We now define

$$f_\pi^{(n)}(x) = \pi(\lambda^{-n})f_\pi^{(1)}(\lambda^n x) \tag{1}$$

so that the $f_\pi^{(n)}$ are PL, and linear on smaller and smaller triangles as $n \to \infty$.

Now suppose that K is transcendental. Let $\{\pi_t\}$, $t \in [0, 1]$ be a continuous family of embeddings of K into $\mathbb{C}$, with π_0 being the original embedding; let $f_t^{(n)}$ be the PL maps constructed above associated to π_t. Since there are only a finite number of similarity classes of triangles, for t small no triangle changes its similarity class very much. Thus for a triangle $D_i \in \Delta$ and t small, the image $f_t^{(1)}(D_i)$ is another triangle of the same orientation, with similarity class close to that of D_i.

Near a given vertex of the triangulation, for t sufficiently small, $f_t^{(1)}$ is not only continuous but also injective, since none of the triangles reverse orientation.

So $f_t^{(1)}$ is locally injective. We show that for t sufficiently small $f_t^{(1)}$ is proper, hence a homeomorphism of $\mathbb{R}^2$.

LEMMA 10. *The map $f_t^{(1)}$ is proper for t sufficiently small.*

Proof. Let $\epsilon > 0$ be so small that $|\lambda|(1 - 2\epsilon) > 1 + 2\epsilon$. Let B_R be the ball of radius R around the origin, where R is large enough so that the edges of triangles in the triangulation intersecting B_R have length at most $R\epsilon/2$. Such an R exists by Lemma 7, since we can choose R so that the control points in $B_R(0)$ are $R\delta$-dense for any $\delta > 0$.

Let P_1 be a polygon contained in $B_R(0)$, surrounding the origin with edges contained in the edges of the triangulation Δ and vertices within $R\epsilon$ of ∂B_R.

Again by Lemma 7, there exists $n \geq 1$ such that the edges of triangles in $B_{|\lambda|^n R}$ have length at most $|\lambda|^n R\epsilon/4$. So each edge in the polygon $\lambda^n P_1$ can be replaced with a polygonal path having edges in Δ, running between the same endpoints and lying within $|\lambda|^n R\epsilon/2$ of the edge, giving a new polygon P_2 lying within $|\lambda|^n R\epsilon/2$ of $\lambda^n P_1$. The polygon P_2 is contained in an annulus about 0 of outer radius $|\lambda|^n R(1 + \epsilon + \epsilon/2)$ and inner radius $|\lambda|^n R(1 - \epsilon - \epsilon/2)$ which lies strictly outside P_1 by the choice of ϵ.

We repeat this construction starting with P_2, getting another polygon P_3 lying close to $\lambda^{n_2} P_2$, contained in an annulus of outer radius $|\lambda|^{n+n_2} R(1 + \epsilon + \epsilon/2 + \epsilon/4)$ and inner radius $|\lambda|^{n+n_2} R(1 - \epsilon - \epsilon/2 - \epsilon/4)$ and so on; so that in general ∂P_i lies far outside P_{i-1}, i.e. P_i are nested with boundaries going to infinity.

Now for t sufficiently small, the map $f_t^{(1)}$ is very near the identity on the region bounded by P_1. Also $\pi_t(\lambda)$ is close to λ, so they have nearly the same modulus. In particular $|\pi_t(\lambda)|(1 - 2\epsilon) > 1 + 2\epsilon$. If x is a control point, $f_t^{(1)}(\lambda^n x) = \pi_t(\lambda)^n f_t^{(1)}(x)$, and $f_t^{(1)}(\partial P_2)$ lies within $|\pi_t \lambda|^n R\epsilon'$ of $\pi_t(\lambda)^n f_t^{(1)}(\partial P_1)$, for some ϵ' slightly larger than ϵ, but close enough so that $|\pi_t(\lambda)|(1 - 2\epsilon') > 1 + 2\epsilon'$.

Hence by applying $f_t^{(1)}$ to the sequence P_i of polygons, we have that $f_t^{(1)}(\partial P_i)$ lies within $|\pi_t\lambda|^{n_i}R\epsilon'$ of $\pi_t(\lambda)^{n_i}f_t^{(1)}(\partial P_{i-1})$, and so is outside of $f_t^{(1)}(P_{i-1})$ by a wide margin. So the $f_t^{(1)}(P_i)$ form a nested sequence of polygons, with boundary going to infinity. Hence $f_t^{(1)}$ is proper. □

Aside. The remainder of the proof deals with properties of quasiconformal mappings. We refer the reader to [1] for the definition and relevant properties. Roughly a quasiconformal homeomorphism is a homeomorphism for which the image of a small disk is not too distorted: its image is contained in a disk S_1 and contains a disk S_2, with the ratio radius(S_1)/radius(S_2) bounded a.e. by a constant L as the size of the original disk shrinks to zero. The number L is called the **dilatation** of the mapping. Quasiconformal maps enjoy many of the properties of conformal maps. The two properties of quasiconformal mappings which we shall use are a compactness property (existence of normal families) and the existence of a derivative almost everywhere.

Continuation of proof of Theorem. So $f_t^{(1)}$ is a proper local homeomorphism, hence a covering map. Since the plane is simply connected, $f_t^{(1)}$ is a homeomorphism. From its definition, each $f_t^{(n)}$ is also a homeomorphism.

Let L be the maximum of the quasiconformal dilatations of the linear maps on triangles $D_i \to f_t^{(1)}D_i$. L is finite, since there are only a finite number of triangles up to (conformal) similarity. Each $f_t^{(n)}$ is L-quasiconformal, since it is L-quasiconformal on each triangle.

If a set of L-quasiconformal homeomorphisms of the plane is appropriately normalized, it forms a normal family. One possible normalization condition is that the maps coincide on three points $x_1 \neq x_2 \neq x_3$. Since the maps $f_t^{(i)}$ all agree on the set C, which has more than three points, and are each L-quasiconformal, they do form a normal family; hence, up to taking a subsequence, there is a limit homeomorphism f_t which is also L-quasiconformal. In fact there is only one limit since the $f_t^{(n)}$ for $n > n_0$ all agree on the increasingly dense sets $\lambda^{-n_0}(C)$.

Furthermore, the limit f_t is invariant: $\forall x \in \mathbb{C}$

$$f_t(\lambda x) = \pi_t(\lambda)f_t(x), \tag{2}$$

since this is true for each $x \in \bigcup_{n \geq 0} \lambda^{-n}C$, which is a dense set by Lemma 7.

A quasiconformal map has a non-singular derivative a.e.; let x be a point at which the derivative of f_t exists and is non-singular.

Now by (2), f_t is differentiable at $\lambda^k x$ for all k. Since $|\lambda| > 1$, f_t is very close to linear near $\lambda^n x$ for n large. We now show that the expanding dynamics and quasiperiodicity force f_t to be linear on the whole plane.

The map f_t is defined on any tiling $x + A$ by translation, and since at a point x the map f_t only depends on the local arrangement of tiles around x, f_t extends continuously to a map $f_t : \mathscr{T}_A \to \mathbb{C}$.

Fix R. Since φ is expanding on leaves of $\mathscr{T}_A$, f_t restricted to $B_R(0)$ in the tiling $\langle -\lambda^k x + A \rangle$ converges to a linear map as $k \to \infty$. Let $x_\infty \in \mathscr{T}_A$ be a limit of the tilings $[-\lambda^k x + A]$. Then f_t is linear on $B_R(0)$ in the tiling $\langle x_\infty \rangle$. Letting R tend to ∞, there is a tiling $x \in \mathscr{T}_A$ such that f_t is linear on the whole unstable fiber through x. By quasiperiodicity, the leaf through x is dense in $\mathscr{T}_A$, so by continuity f_t must be linear on each leaf of $\mathscr{T}_A$. In particular f_t is linear on $\pi_A(\mathbb{C})$.

But if f_t is (real)-linear, then (2) implies that $\pi_t(\lambda) = \lambda$ or $\bar{\lambda}$. Since t was small $\pi_t(\lambda) = \lambda$. Thus we see that any embedding of K which is sufficiently close to the identity fixes λ.This implies that λ is algebraic, by the comments at the beginning of this section.

Since the number $|S|$ of surroundings of tiles in A is finite, the positions of the tiles are generated by a finite set of similarities, that is, there is a finite set $\{\alpha\}$ of similarities such that if (g, T_i) is a tile in A then $g \in \mathscr{A}$, the group generated by $\{\alpha\}$. If two complex similarities $x \mapsto ax + b$ and $x \mapsto a'x + b'$ are conjugate by a real linear map close to the identity then $a = a'$. Thus a is algebraic as was the case for λ earlier. So the derivatives of the similarities in $\mathscr{A}$ must be algebraic. The b's may of course not be algebraic, since a homothetic change of coordinates $z \mapsto \gamma z$ conjugates $az + b$ to $az + \gamma b$.

Let $\psi_i(z) = a_i z + b_i$ for $i = 1, 2, 3$ be three similarities in $\mathscr{A}$. Then we have $\pi_t(b_i) = fb_i$, where $f \in GL_2(\mathbb{R})$ and on the righthand side b_i is thought of as an element of $\mathbb{R}^2$. Suppose b_1 and b_2 are independent over $\mathbb{R}$. If b_3 is not algebraic over the field generated by b_1 and b_2, then by choosing π_t to be an embedding moving b_3 but not b_1 or b_2, we see that f cannot be linear. So the group $\mathscr{A}$ is algebraic over the (possibly transcendental) field generated by b_1 and b_2.

Now if λ is not real, b_1 and λb_1 are independent over $\mathbb{R}$, and so in fact $\mathscr{A}$ is algebraic over the field generated by b_1. But now we can find a homothety commuting with λ and sending b_1 to 1. This change of coordinates renders $\mathscr{A}$ algebraic.

If λ is in fact real, then there is a map $m \in GL_2\mathbb{R}$ (which commutes with the scalar λ) sending b_1 to $(1, 0)$ and b_2 to $(0, 1)$, so that again $\mathscr{A}$ is algebraic. □

5. The construction

5.1. *Number fields and Pisot numbers*

Before we start the construction, we need to review a few facts and establish some notations regarding number fields. Our reference was [5].

Let α be an algebraic number. A **Galois conjugate** of α is a root of the minimal polynomial of α. Let K be the field generated by α over $\mathbb{Q}$, $K = \mathbb{Q}(\alpha) = \mathbb{Q}[\alpha]$. For each Galois conjugate α' of α, there is an embedding of K into $\mathbb{C}$ sending $\alpha \in K$ to $\alpha' \in \mathbb{C}$. If α' is real, the embedding has image in $\mathbb{R}$, and if α' is non-real, the embeddings corresponding to α' and $\overline{\alpha'}$ are complex conjugates of each other. Let r be the number of real embeddings and $2c$ the number of non-real embeddings. The product of the real embeddings with half the non-real embeddings (one embedding for each complex conjugate pair) is an embedding of K into $\mathbb{R}^r \times \mathbb{C}^c \cong \mathbb{R}^d$, where d is the degree of α, $d = r + 2c$.

Let $\mathcal{O}$ denote the lattice of algebraic integers in $K \subset \mathbb{C}$. Then $\mathcal{O} \subset \mathbb{C}$ embeds as a subring of K into $\mathbb{R}^d$. Let $\tilde{\mathcal{O}}$ be the image; $\tilde{\mathcal{O}}$ is a d-dimensional lattice in $\mathbb{R}^d$. For $x \in \mathcal{O}$, let $\tilde{x}$ denote the image of x in $\tilde{\mathcal{O}}$. For $x \in \mathcal{O}$ let $\pi_i(x)$ denote the ith Galois conjugate of x, that is the ith coordinate of $\tilde{x}$. We assume $\pi_1(x) = x$.

Multiplication by α in K extends to a linear map on $\mathbb{R}^d$, whose eigenvalues are the Galois conjugates of α. If α is an algebraic integer, this linear map preserves the lattice $\tilde{\mathcal{O}}$.

For $x \in \mathcal{O}$, define $|||x|||$ to be the (Euclidean) distance from $\tilde{x}$ to the x_1-axis; alternatively $|||x|||$ is the square root of the sum of the squares of $|\pi_i(x)|$ for $i \neq 1$.

A **Pisot number** is a real positive algebraic integer whose Galois conjugates except itself are all strictly inside the unit circle. Thus if λ is a Pisot number generating K, and $x \in K$, $x \neq 0$, we have $|||\lambda x||| < |||x|||$.

Two other facts about Pisot numbers which we shall use are Lemmas 11 and 12:

LEMMA 11. *Any number field K contained in $\mathbb{R}$ contains a Pisot number λ such that $K = \mathbb{Q}[\lambda]$. Moreover, for any integer n there is a Pisot number λ' generating K such that λ'/n is an algebraic integer.*

Proof. By Kronecker's theorem, if d numbers $a_1, a_2, \ldots, a_d$ are rationally independent, the ray in $\mathbb{R}^d$ from the origin through the point $(a_1, a_2, \ldots, a_d)$ passes arbitrarily close to a non-zero point of $\mathbb{Z}^d$.

Let $K = \mathbb{Q}[\alpha]$ be a number field contained in $\mathbb{R}$, with α of degree d. Let $\tilde{\mathcal{O}}$ be as above.

With respect to a basis for the lattice $\tilde{\mathcal{O}}$, the x_1-axis has rationally independent slopes: otherwise, $\Sigma_{i=1}^{k} n_i v_i(\alpha) = 0$ with $n_i \in \mathbb{Z} - \{0\}$ and $v_i(\alpha)$ integer polynomials in α of degree $\leq d - 1$ would give a polynomial equation for α of degree $\leq d - 1$, a contradiction.

Thus there is a nonzero point $\tilde{\lambda}$ of $\tilde{\mathcal{O}}$ at distance < 1 of the positive x_1-axis. Let $\lambda > 1$ be the projection of $\tilde{\lambda}$ to the x_1-axis; λ is an algebraic integer whose Galois conjugates are all less than 1 in modulus.

This number λ generates $\mathbb{Q}[\alpha]$: if not, $[\mathbb{Q}[\alpha]:\mathbb{Q}[\lambda]] \geq 2$ and there would be a Galois automorphism of $\mathbb{Q}[\alpha]$ moving α but fixing λ. But a Galois automorphism permutes the coordinates of points in $\tilde{\mathcal{O}}$. Thus $\tilde{\lambda}$ would have two coordinates with value $\lambda > 1$, a contradiction.

Given an integer n, choose an integer k so large that $|||\lambda^k||| < 1/n$; then $\lambda' \stackrel{\text{def}}{=} n\lambda^k$ is a Pisot number, generating $\mathbb{Q}[\alpha]$ for the same reason that λ generates $\mathbb{Q}[\alpha]$, and λ'/n is an algebraic integer. □

The next lemma is only used in the proof of Lemma 13.

LEMMA 12. *Let λ be any Pisot number and E be a finite subset of $\mathcal{O} = \mathcal{O}[\lambda]$, the algebraic integers in $\mathbb{Q}[\lambda]$. For any $R > 0$ and $k > 0$ the cardinality of the set $\{x \in E + \lambda E + \lambda^2 E + \cdots + \lambda^k E;\ |x| < R\}$ is bounded independently of k.*

Proof. Let $E_i = \max\{|\pi_i(e)|\,;\, e \in E\}$, the maximum absolute value of the ith Galois conjugate of an element of E, and $\lambda_i = \pi_i(\lambda)$. If $x \in \mathcal{O}$ is of the form $x = e_0 + \lambda e_1 + \cdots + \lambda^k e_k$ for some k with the $e_0, \ldots, e_k \in E$, then define $x' = (x - e_0)/\lambda = e_1 + \cdots + \lambda^{k-1} e_k$.

If for any coordinate $i \neq 1$ we have

$$|\pi_i(x)| > \frac{E_i}{1 - |\lambda_i|}$$

then

$$|\pi_i(x)|(1 - |\lambda_i|) > E_i$$

$$|\pi_i(x)| - E_i > |\lambda_i \pi_i(x)|$$

and so

$$|\pi_i(x')| = \left|\frac{\pi_i(x) - e_0}{\lambda_i}\right| \geq \frac{|\pi_i(x)| - E_i}{|\lambda_i|} > |\pi_i(x)|.$$

So the ith coordinate of x' is larger in absolute value than that of x. But then the same is true for $x^{(2)} \stackrel{\text{def}}{=} (x')'$, $x^{(3)} \stackrel{\text{def}}{=} (x^{(2)})'$, and so on until the ith coordinate of $x^{(k)}$ is larger than $\pi_i(x)$. But $x^{(k)} = 0$. This is a contradiction.

So we must have for each $i \neq 1$ that $|\pi_i(x)| \leq E_i/(1 - |\lambda_i|)$. But since by hypothesis we have in addition that the first coordinate of x is bounded in absolute value by R, the point $\tilde{x} \in \mathbb{R}^d$ is confined to a bounded region of space. Since

$\tilde{x} \in \tilde{\mathcal{O}}$, a discrete lattice in $\mathbb{R}^d$, there are a finite number of possible values for $\tilde{x}$; hence for x. □

5.2. *Example*

Since the construction has many details, we will start by giving a simplified example illustrating some of the ideas involved.

We will construct an inflationary similarity-tiling with expansion $\gamma = \sqrt{5}/2$. Let $K = \mathbb{Q}[\gamma]$. Let λ be the Pisot number $\lambda = 2 + \sqrt{5}$. Note that $K = \mathbb{Q}[\lambda]$. Let $\mathcal{O}$ be the algebraic integers in K.

Let T_0 be the square of side 4 centered at the origin. Then γT_0 has vertices in $\mathcal{O}$. Tile the region between γT_0 and T_0 with rectangles as shown in Figure 5: the two types of rectangles used (up to rotation) have sides $(\sqrt{5} - 2) \times (\sqrt{5} - 2)$ and $1 \times (\sqrt{5} - 2)$.

Let R_1, R_2, R_3, R_4 denote respectively the rectangles $(\sqrt{5} - 2) \times (\sqrt{5} - 2)$, 1×1, $1 \times (\sqrt{5} - 2)$, and $(2 + \sqrt{5}) \times (2 + \sqrt{5})$. When we multiply the rectangle R_1 by λ (not γ!) the result is an R_2. When we multiply the R_3 by λ we can tile the resulting $(2 + \sqrt{5}) \times 1$ rectangle as in Figure 6, into four R_2's and one R_1. The image of an R_2 gives an R_4, which we multiply again by λ to give a $(9 + 4\sqrt{5}) \times (9 + 4\sqrt{5})$ rectangle, which we subdivide into a $2\sqrt{5} \times 2\sqrt{5}$ rectangle

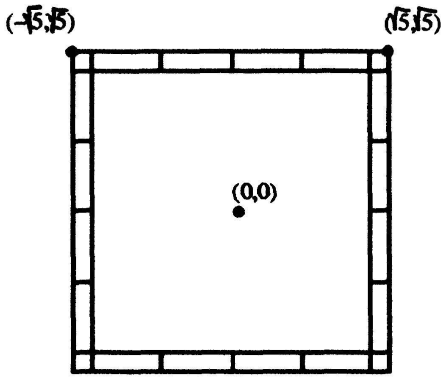

Figure 5

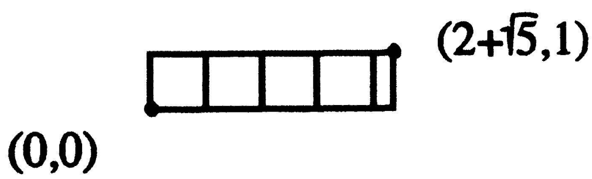

Figure 6. Subdivision of an R_3 rectangle.

and a bunch of rectangles of types R_1, R_2, R_3. In the $2\sqrt{5} \times 2\sqrt{5}$ rectangle we put a copy of Figure 5.

Now for each rectangle R_1, R_2, R_3, R_4 we have a way to subdivide λR_i into rectangles R_1, R_2, R_3, R_4 and T_0. The rectangle T_0 subdivides in the pattern as described above in γT_0.

To make the tiling, put T_0 centered at the origin. Now define the tiling of the whole plane using expansion by γ and the above-defined subdivisions; repeatedly multiply everything by γ, and subdivide all the tiles as above. Note that since $|\gamma| \neq |\lambda|$ the image of any tile of type R_i is only similar to λR_i, so the subdivision must be scaled appropriately.

Why does this give an inflationary similarity tiling? First, it does give a unique limiting tiling of the whole plane since larger and larger regions around the origin are fixed. Second, there are a finite number of tile types up to similarity. Also, the tiling is by construction invariant under multiplication by γ, and the subdivision is mixing.

It remains to check that there is a finite number of surroundings. This involves only a finite (but tedious) amount of checking. The real reason there are a finite number of surroundings is a bit subtle, and depends on the fact that λ was a Pisot number. We refer to the general construction for the proof (Lemma 13).

5.3. *General construction*

Let γ be an algebraic number, $|\gamma| > 1$, and γ_x, γ_y the real and imaginary parts of γ. We construct a tiling with expansion γ.

If γ is neither real nor pure imaginary define $l = \gamma_y/\gamma_x$, and choose an integer n_0 so that $n_0 l$ and n_0/l are both algebraic integers. If γ is real or pure imaginary take $n_0 = 1$. Let λ be a Pisot number in $\mathbb{Q}[\gamma_x, \gamma_y]$, generating it, such that λ/n_0 is an algebraic integer. The existence of such a λ is given by Lemma 11. Let d be the degree of λ.

Let $K = \mathbb{Q}[\lambda] \subset \mathbb{R}$ and $\mathcal{O} = \mathcal{O}[\lambda]$ be the algebraic integers in $\mathbb{Q}[\lambda]$.

Let $\tilde{\mathcal{O}} \subset \mathbb{R}^d$ be as before. Let $B_R(\tilde{x})$ denote the ball of radius R around $\tilde{x} \in \mathbb{R}^d$.

A polygon in the plane is said to be **rectilinear** if each of its sides is parallel to the x- or y-axis. Such a polygon is said to be **hv-convex** if its intersection with each horizontal or vertical line is convex. A triangle with two sides parallel to the axes is called an **hv-triangle**.

Let T be a hv-convex rectilinear polygon with vertices in $K^2 = K \times K$, containing the origin and sufficiently "round" so that T contains $\frac{1}{\gamma}T$ in its interior.

Note that the vertices of $\frac{1}{\gamma}T$ are in K^2. Tile the region $T \backslash \frac{1}{\gamma}T$ with rectangles having vertices in K^2 and hv-triangles having vertices in K^2 and their hypotenuses on the boundary of $\frac{1}{\gamma}T$. The "picture proof" of the existence of such a tiling is given in figure 7. (Note that the intersection of two line segments with vertices in $K^2 = K \times K$ is again in K^2. *If* γ is real or pure imaginary no triangles are needed.) Let $n > 0$ be an integer so that $nc \in \mathcal{O}$ for any coordinate c of any vertex in this tiling. Multiply everything by n so that all vertices are in $\mathcal{O}^2$. Now let $T_1 = nT$, $T_0 = \frac{n}{\gamma}T$ and forget about T. We have a tiling of $T_1 \setminus T_0$ with tiles with vertices in $\mathcal{O}^2$.

We now have a subdivision rule for a tile T_0. Each tile in the subdivision is either T_0, a rectangle, or an hv-triangle with hypotenuse of one of two possible slopes $-l$ or $1/l$. To define a subdivision rule for each rectangle and hv-triangle, the idea is to multiply one of these tiles by λ^N for some large N, and subdivide it into copies of T_0, other small rectangles, and hv-triangles with hypotenuses of the same slope.

Let S be the rectangle with sides parallel to the axes (a "rectilinear rectangle") circumscribed about T_1. Tile $S \setminus T_1$ using rectangles with vertices in $\mathcal{O}^2$. This is easily done since the edges of T_1 are rectilinear lines with coordinates in $\mathcal{O}^2$.

Let $H_x = \max |||c_1 - c_2|||$, where c_1 and c_2 range over the set of x-coordinates of vertices in the tiling of S. Let H_y be the same for the y-coordinates, and let $H = \max\{H_x, H_y\}$.

Let $V = \{x \in \mathcal{O} \mid \tilde{x} \in B_{|\lambda|R}(0)\}$, where R is chosen so large that

(1) $R \geq 2H$.
(2) V contains all horizontal and vertical edges between adjacent vertices in the tiling of S defined previously.
(3) For any $p \in \mathbb{R}^d$ the region $B_{R/4}(p)$ contains a point of $\tilde{\mathcal{O}}$.

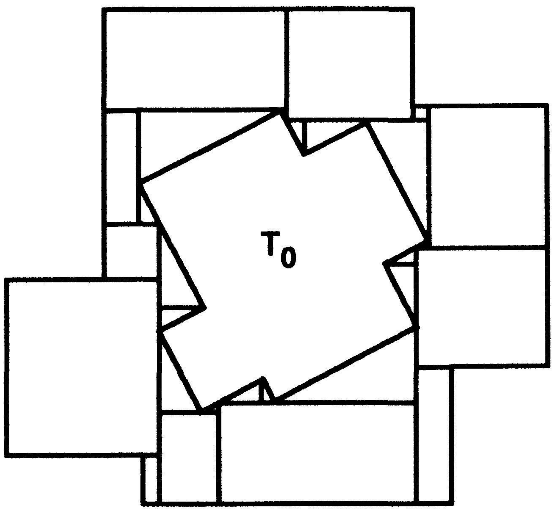

Figure 7. The tiling of T_1 with vertices in $\mathcal{O}^2$.

Let $\mathcal{R}$ be the union of the set of rectangles with edge lengths in V and the set of hv-triangles with horizontal and vertical edges having lengths in V and hypotenuses of slope l or $-1/l$. This is a finite set of tiles up to translation.

Let N be so large that $|||\lambda^{N-1}||| < \frac{1}{3|\lambda|}$ (here is the first place we use that λ is a Pisot number generating K) and for each tile $t \in \mathcal{R}$, $\lambda^{N-1}t$ has all edges long compared to R and the diameter of S, say longer than $5\max\{R, \operatorname{diam} S\}$.

For each rectangle t in $\mathcal{R}$, define the subdivision of $\lambda^N t$ as in figure 8, as follows: first put a copy of S in the lower left corner of $\lambda^N t$ (it fits because the edges of $\lambda^N t$ are long compared to the diameter of S). Let v be the upper right corner of S, and x and y the lengths of the horizontal and vertical segments extending from v to the right and upper sides of t, respectively. Let w and z be the endpoints of those segments on the right and upper boundaries of t, respectively.

Suppose the rectangle has edge lengths e_1 and e_2. For each edge e_i of the rectangle we have $|||e_i||| < R/3$ (since $\lambda^{-N}(e_i) \in V$ and $|||\lambda^N||| < |||\lambda^{N-1}||| < \frac{1}{3|\lambda|}$), and $|e_i| > 5R$ by choice of N. By the definition of H we have $|||x||| < |||e_1||| + H < R/3 + H$ and $|||y||| < |||e_2||| + H < R/3 + H$.

Subdivide the edge vw (which has length x) using elements of V as follows. Take the set of points $\{\tilde{u}\}$ of $\tilde{\mathcal{O}} \subset \mathbb{R}^d$ within $R/4$ of the line segment from $\tilde{0}$ to $\tilde{x}$ and project them to the x_1-axis. Keep only those u that lie between 0 and x (see figure 9). The differences $u - u'$ between adjacent points of this set are elements of V, for $|u - u'| < 2 \cdot R/4 = R/2$ (by condition (3) on R) and $|||u - u'|||$ is at most a little greater than $R/2$: it is in fact less than $R/2$ times

$$1 + \frac{|||x|||}{|x|} < 1 + \frac{H + R/3}{5R} < \frac{6}{5}.$$

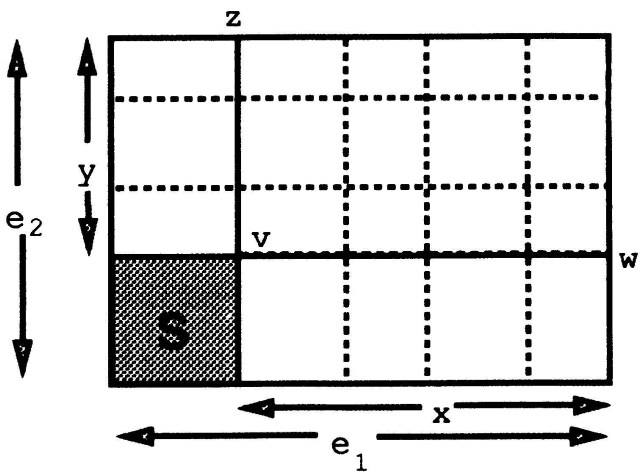

Figure 8. Sample subdivision of a rectangle.

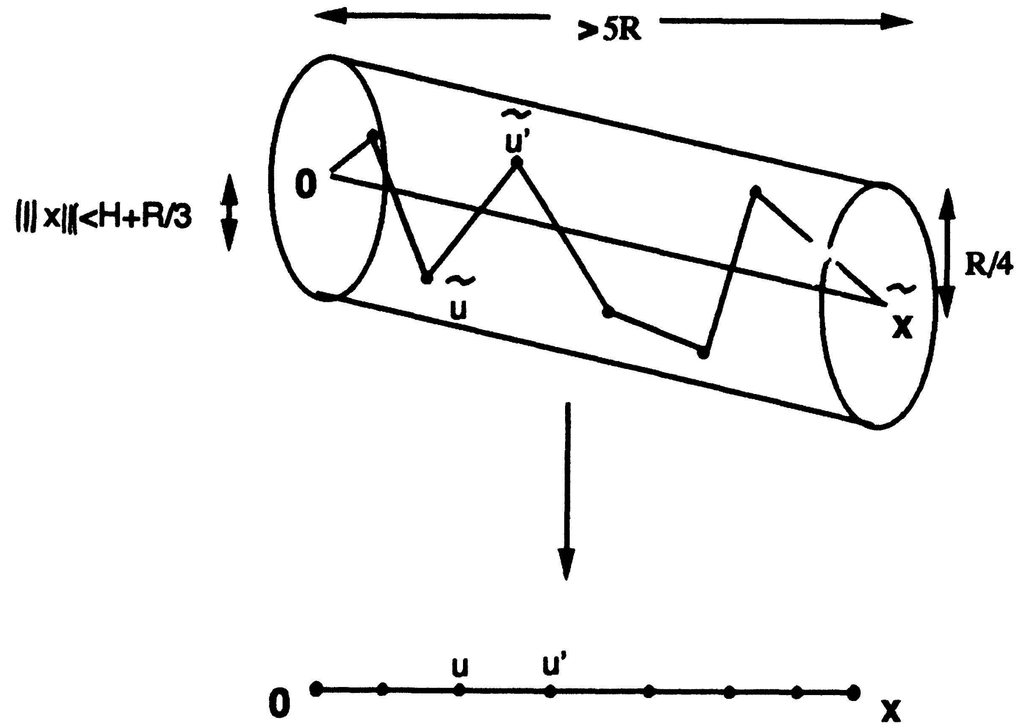

Figure 9. Defining the subdivision of $\overline{0x}$.

This subdivides the segment from 0 to x into edges in V (in fact, the edges have length $<R$, not just $<|\lambda|R$ as in the definition of V). The subdivision of vw is defined to be a translate of this subdivision. In a similar manner we define the subdivision of segment vz. Now subdivide the rectangle in a grid fashion using horizontal and vertical lines through the subdivision points of the segments vw, vz as shown in figure 8. This subdivides the rectangle $\lambda^N t$ into rectangles in $\mathscr{R}$ and a copy of the tiling in S.

To subdivide a triangle $t \in \mathscr{R}$, first consider $\lambda^{N-1}t$. Suppose that the horizontal edge is longer than the vertical edge. Subdivide the horizontal edge of $\lambda^{N-1}t$ using the projection of points in $\tilde{\mathcal{O}}$ as we did for the edge vw above. Draw vertical lines from these points to the hypotenuse and then from there draw horizontal lines to the other edge. This subdivides the triangle into rectangles and triangles (the triangles have hypotenuse lying along the hypotenuse of $\lambda^{N-1}t$). We now multiply everything by λ, and claim that this defines a subdivision of $\lambda^N t$ with tiles in $\mathscr{R}$. To see this, note that each horizontal edge of a tile in the original subdivision is in V. Furthermore the length of one of these edges is at most R (in fact $R/2$). A vertical edge is of length lu or u/l, for u some horizontal edge (recall that $-l$ or $1/l$ is the slope of the hypotenuse). When we multiply once more by λ, each subtile will have both horizontal and vertical edges in V (recall that λl and λ/l are algebraic integers, so each edge is now an algebraic integer of length $<|\lambda|R$ and small $|||\cdot|||$-value. Thus this defines the subdivision of $\lambda^N t$ into tiles in $\mathscr{R}$.

If in the original triangle t the vertical edge was longer than the horizontal, the same argument starting with the vertical edge defines the subdivision.

To define the inflationary tiling of the whole plane, start with a tile T_0 at the origin. Repeatedly multiply by γ and subdivide using the subdivision rules we defined above: T_0 subdivides into the tiling of $T_1 \setminus T_0$ and the tile T_0. Each tile in $\mathscr{R}$ subdivides after N multiplications by γ as we defined above (using expansion λ!). Since γ and λ are different in general the subdivisions have to be scaled (and rotated) to fit. (A good example to recall is that of Figure 4, where $\gamma = 2$ and $\lambda = 4$.)

Our set of tile types is a subset of

$$\{T_0\} \cup \mathscr{R}_0 \cup \mathscr{R}_1 \cup \cdots \cup \mathscr{R}_{N-1},$$

where the indices on the $\mathscr{R}$ correspond to the time left before subdivision. Thus for $i < N-1$ the image under γ of a tile in $\mathscr{R}_i$ is a tile in $\mathscr{R}_{i+1}$, and the image of a tile in $\mathscr{R}_{N-1}$ is a union of tiles in $\{T_0\} \cup \mathscr{R}_0$. Note that in the whole tiling of the plane, for $1 \leq i < N-1$ a tile in $\mathscr{R}_i$ is adjacent to tiles only in $\mathscr{R}_{i-1} \cup \mathscr{R}_i \cup \mathscr{R}_{i+1}$.

This defines a tiling of the whole plane as we saw in the example. There are a finite number of tiles up to similarity. Tiles of the same (indexed) type subdivide in the same way. Each tile eventually has a copy of T_0 in its subdivision, so that the subdivision is mixing. The tiling is invariant under multiplication by γ. We need only check that there is a finite number of surroundings.

LEMMA 13. *The tiling defined above has only a finite number of surroundings.*

Proof. Every tile occurring in the tiling is similar to one in $\{T_0\} \cup \mathscr{R}$. Call these the **unindexed** types. For each unindexed tile type $\tau \in \mathscr{R}$ fix a standard tile τ' which is a translate of τ with a vertex at the origin (and hence edges in V).

Let t be a tile in the tiling and define $\varphi_t : x \mapsto ax + b$ to be the similarity mapping t to the standard tile of the same unindexed tile type as t.

We need to show that for adjacent tiles t, t' in the tiling, with similarities φ_t and $\varphi_{t'}$ respectively, there are only a finite number of possibilities for the similarity $\varphi_{t'} \circ \varphi_t^{-1}$.

Let s be the tile which is the unique "parent" of t, i.e. $t \subset \gamma s$. Similarly let s' be the parent of t'. Note that s and s', if different, must be adjacent tiles. We will compute $\varphi_{t'} \circ \varphi_t^{-1}$ from $\varphi_{s'} \circ \varphi_s^{-1}$.

There are six cases.

Firstly, if the tiles s and s' are the same tile, so that t and t' occur in the subdivision of a single tile, then the set of possible $\varphi_{t'} \circ \varphi_t^{-1}$ is finite since there are a finite number of tile types (hence a finite number of subdivisions).

Secondly, if one of s or s' is a tile of type T_0, then there are a finite number of possibilities for $\varphi_{t'} \circ \varphi_t^{-1}$ since T_0 has only one type of surrounding in the tiling.

Thirdly if neither s nor s' subdivide under one multiplication of γ (that is, neither s nor s' is in $\mathscr{R}_{N-1}$), then $t = \gamma s$ and $t' = \gamma s'$, so that $\varphi_t = \varphi_s \circ 1/\gamma$ and $\varphi_{t'} = \varphi_{s'} \circ 1/\gamma$, and so $\varphi_{t'} \circ \varphi_t^{-1} = \varphi_{s'} \circ 1/\gamma \circ \gamma \circ \varphi_s^{-1} = \varphi_{s'} \circ \varphi_s^{-1}$.

Fourthly if both $s, s' \in \mathscr{R}_{N-1}$ then $\varphi_{s'} \circ \varphi_s^{-1}$ is a *translation* by our choice of standard tiles. Since the tiles t, t' in the subdivision are scaled by the same factor λ^N with respect to γs or $\gamma s'$, we have $\varphi_t = p \circ \varphi_s \circ 1/\gamma$ and $\varphi_{t'} = q \circ \varphi_{s'} \circ 1/\gamma$, where $p(z) = \lambda^N z + v$ and $q(z) = \lambda^N z + v'$ for some v, v'. Thus if $\varphi_{s'} \circ \varphi_s^{-1}(x) = x + c$ we have $\varphi_{t'} \circ \varphi_t^{-1} = q \circ (x + c) \circ p^{-1} = x + \lambda^N c + v_1$, where $v_1 = v' - v \in \mathcal{O}^2$ is the difference of two vectors taken from the finite set of translations defined by the positions of tiles in the subdivisions of all the tiles in $\mathscr{R}$. Thus v_1 is chosen from a finite set E.

Fifthly if only one of s, s' is in $\mathscr{R}_{N-1}$, say $s \in \mathscr{R}_{N-1}$, then s' is either in $\mathscr{R}_0$ or $\mathscr{R}_{N-2}$. Suppose $s' \in \mathscr{R}_0$. Let $r \in \mathscr{R}_{N-2}$ and $r' \in \mathscr{R}_{N-1}$ be the parents of s, s' respectively. Then $\varphi_{r'} \circ \varphi_r^{-1} = \gamma^{-1} x + c$ for some c. We see that $\varphi_s = \varphi_r \circ 1/\gamma$ and $\varphi_{s'} = q \circ \varphi_{r'} \circ 1/\gamma$ for some q, $q(z) = \lambda^N z + v'$. So we have

$$\varphi_t = p \circ \varphi_r \circ \frac{1}{\gamma^2}$$

for some p, $p(z) = \lambda^N z + v$, and

$$\varphi_{t'} = q \circ \varphi_{r'} \circ \frac{1}{\gamma^2}.$$

Hence

$$\varphi_{t'} \circ \varphi_t^{-1} = q \circ (\gamma^{-1} x + c) \circ p^{-1} = \gamma^{-1} x + \lambda^N c + v_1,$$

where $v_1 = v' - \gamma^{-1} v$ and v, v' are elements of a finite set of translations defined by the positions of tiles in the subdivisions of all the tiles as before.

Sixthly and lastly, in case $s \in \mathscr{R}_{N-1}$ and $s' \in \mathscr{R}_{N-2}$ we see that

$$\varphi_{t'} \circ \varphi_t^{-1} = \varphi_{s'} \circ \varphi_s^{-1} \circ p^{-1}$$

where $p(z) = \lambda^N z + v$ for v in a finite set as before. As we see below, the set of $\varphi_{s'} \circ \varphi_s^{-1}$ is finite and so this set is finite also. (The argument in case five allows us to skip over this case in the proof of finiteness for the other cases).

In all of the above cases if we look at the parents of s, s' and their parents, and so on, we eventually find that two ancestors came from a subdivision of the same tile. This implies that (in all but the sixth case) we can write $\varphi_{t'} \circ \varphi_t^{-1} = ax + b$ where $a \in \{1, \gamma, \gamma^{-1}\}$, $|b|$ is bounded since t and t' are adjacent, and b can be written:

$$b = v_1 + \lambda^N c = v_1 + \lambda^N(v_2 + \lambda^N c') = \cdots = v_1 + \lambda^N v_2 + \cdots + \lambda^{kN} v_k$$

where the v_i are all taken from a finite subset E of $\mathcal{O}^2$. By Lemma 12, the set of possible b arising this way is finite.

Thus for any adjacent tiles t, t' the similarity between them $ax + b$ is chosen from a finite set of possible similarities. This implies that there are a finite number of surroundings. □

6. Open questions

There are many interesting unanswered and unstudied questions in this area. Here are a few.

(1) What are the most general inflationary similarity-tilings with expansion γ? The above construction only provides some very special examples. Aside from some easy generalizations of the above construction, the reader familiar with the work of Thurston [15] may see how to generalize this construction to the case when λ is not necessarily a Pisot number but a **Perron number** (a real algebraic integer strictly larger in modulus than its Galois conjugates) generating $\mathbb{Q}[\gamma_x, \gamma_y]$. More generally, according to Thurston, one can do a similar construction when λ is a **complex Perron number**, that is, a complex algebraic integer strictly larger in modulus than its Galois conjugates except for $\bar{\lambda}$. In this last case λ should generate $\mathbb{Q}[\gamma]$ (not $\mathbb{Q}[\gamma_x, \gamma_y]$) and the tiles will not necessarily be polygons.

Even this is not the most general possibility; it is possible to have several intermediate expansions λ, lying in a finite extension field of $\mathbb{Q}[\gamma]$. In general the ratio of areas of tiles (of the same type) will be in a higher-rank multiplicative subgroup of $\mathbb{R}_+$ (in our example the area ratios are in a rank-2 subgroup generated by $|\gamma|^2$ and $|\lambda|^2$).

These more general constructions are of course more complicated. They are nevertheless relevant if one wishes to consider inflationary **isometry**-tilings, for example, since there our Pisot-number construction may not suffice.

(2) What are the expansions of inflationary isometry-tilings, where the structure group G is the group of isometries? What if G is the group of area-preserv-

ing linear maps? In both cases, the square modulus of the expansion must be a Perron number (leading eigenvalue of a nonnegative integer matrix, the subdivision matrix of the tiling [8, 14]). For the case of isometries, not every algebraic number with this condition works. For the area preserving linear maps, must the expansion be algebraic?

(3) Can one describe in some sense the inflationary G-tilings with only one equivalence class of tile? This has been done for G the group of translations, see [9]. A related question is, what tiles in the plane can be tiled by two similar copies of themselves? There are 17 such sets known to the author.

(4) More generally, can we describe all tilings having a certain subdivision rule? Again, for translation tilings and the simplest rule $T_0 \to \{n \text{ copies of } T_0\}$, quite a bit is known; see [9, 11, 7]. For rules with two tiles and non-integer expansion, say $T_0 \to T_1$, $T_1 \to \{T_0, T_1\}$ there are almost no known examples.

(5) What are the higher dimensional versions of these results?

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