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§0. Introduction

The Betchov–Da Rios equation, otherwise known as the "localized induction equation" (LIE), $\partial\gamma/\partial t = \partial\gamma/\partial s \times \partial^2\gamma/\partial s^2$, is a soliton equation for space curves $\gamma(s, t)$, best known as a model for the behavior of thin vortex tubes in an incompressible, inviscid, three-dimensional fluid ([2], [3], [6], [17], [27]). As a completely integrable Hamiltonian system, LIE possesses infinitely many conserved quantities, all of them integrals involving the curvature k and torsion τ of γ. The first five integrals in the LIE hierarchy are: $\mathcal{J}_1 = \int ds$, $\mathcal{J}_2 = \int \tau\, ds$, $\mathcal{J}_3 = \int \frac{1}{2}k^2\, ds$, $\mathcal{J}_4 = \int k^2\tau\, ds$, and $\mathcal{J}_5 = \int \frac{1}{2}(k')^2 + \frac{1}{2}k^2\tau^2 - \frac{1}{8}k^4\, ds$. These integrals may be used to define variational problems for curves $\gamma(s)$, and the resulting equilibria provide initial conditions $\gamma(s, 0) = \gamma(s)$ for soliton solutions to LIE.

Of course, equilibria for $\mathcal{J}_1$ are just geodesics; it turns out that equilibria for linear combinations of $\mathcal{J}_1$ and $\mathcal{J}_2$ are helices, and equilibria for combinations of $\mathcal{J}_1$, $\mathcal{J}_2$, and $\mathcal{J}_3$ are "Kirchhoff elastic rods" (as shown in [21]). The connection between elastic curves and solitons for LIE was the original discovery of Hasimoto [9, 10], which sparked a whole series of papers on LIE, including [5], [8], [12], [15], [16], [18], [25], and [28].

Here we derive classical Hamiltonian systems with $n = 6$, 7, or 8 degrees of freedom from variational problems for $\mathcal{J}_3$, $\mathcal{J}_4$, and $\mathcal{J}_5$, and show that these Hamiltonian systems are Liouville integrable in the sense that there exist n constants of motion in involution. Our results are for curves γ in M, any of the three-dimensional space forms, $\mathbb{R}^3$, S^3, or $\mathbb{H}^3$. In each case, five constants exist "automatically" due to the nature of the underlying spaces and the fact that each $\mathcal{J}_j$ is a "geometric" functional; the proof then comes down to exhibiting one, two, or three additional constants arising from the special nature of the integrals $\mathcal{J}_j$. (The full result for the elastic rod was first proved in [21].)

We conclude the introduction with a few comments:

(i) In the general theory of Hamiltonian systems, Liouville integrability tends to be regarded as the most satisfactory possible situation, since it implies the dynamics can be described simply as "linear flow on a torus" and – at least in principle – the trajectories can be represented explicitly in terms of quadratures and algebraic operations (see [1]).

However, the integrability results presented here are hardly the end of the story; in fact, we regard them more as "existence results" – invitations to actually integrate the equations for these problems using their special structure, not attempting to use the general theory. We believe this view is amply justified by numerous papers on elastic curves and rods, e.g., [4], [11], [13], [19], [20], [21], [23], [24].

(ii) Our results are related to some general phenomena in soliton theory. For example, it is shown in [26] that complete integrability of the KdV equation implies Liouville integrability of the associated 'stationary problems for higher order KdV' regarded as finite-dimensional Hamiltonian systems. A more abstract approach to this result is given in [29, 30], where the key ingredient is the existence of a one-parameter family of Bäcklund transformations. Our own approach is "from the bottom up", emphasizing the geometry of curves in space forms, rather than the general machinery of soliton theory, and it is not obvious how our concrete computations are related to the above-mentioned arguments. It should be noted that the geometric setting of LIE leads to some interesting special features not shared by KdV, including the fact that singular problems abound in the LIE hierarchy, and the fact that a fundamental role is played by non-canonical coordinates on the symplectic manifold T^*F.

(iii) We have tried to keep the "set-up" as brief as possible; more background is given in [22]. Following Jurdjevic [11], we have obtained our Hamiltonian systems with the aid of the Pontrjagin Maximum Principle. Another approach to such variational problems with non-holonomic constraints is the theory of exterior differential systems, as explained in [4].

§1. Geometric Hamiltonian systems

Though the constructions of this section are naturally formulated in the general Riemannian setting ([14], [31]), it will be convenient to specialize at once to the case of a three dimensional Riemannian manifold M. Since the orthonormal frame bundle F of M is six-dimensional, our 'core' symplectic manifold T^*F, the cotangent bundle of F, is twelve-dimensional. We will construct Hamiltonians on T^*F (or $T^*(F \times R^k)$) from geometric variational problems for curves $\gamma(t)$ in M.

Since $SO(3)$ acts on F by rotation of frames, the standard basis for $so(3)$ determines three 'fundamental vectorfields' A_j on F, the corresponding infinitesimal generators. Complementary to these are three 'basic vectorfields' B_i defined, at each point (frame) $f = (f_1, f_2, f_3) \in F$, using the Riemannian connection on M: B_i is the unique horizontal vector projecting to f_i. The vertical vectorfields A_j and horizontal vectorfields B_i display TF as a trivial bundle, and the corresponding 'linear Hamiltonians' provide half of a (non-canonical) coordinate system on T^*F: $\mathscr{A}_j(p) = p(A_j)$, $\mathscr{B}_i(p) = p(B_i)$, p any covector on F

The (generalized) Frenet System for lifts of curves in M to curves $f(t)$ in F may now be written:

$$\frac{df}{dt} = B_1(f) + k_1 A_1(f) - k_3 A_2(f) + k_2 A_3(f). \tag{FS}$$

A solution $f(t)$ to (FS) represents an 'adapted frame' along γ in the sense that $\gamma' = T = f_1$. To mention two useful special cases, $f(t)$ corresponds to the 'standard' (Frenet) frame $f = (T, N, B)$ in case $k_3 = 0$ (and consequently $k_1 = \tau$ and $k_2 = k$), while $f = (T, f_2, f_3)$ is a 'natural frame' if $k_1 = 0$. The relationship between standard and natural frames and invariants can be written: $k_2 + ik_3 = k\, e^{i\vartheta}$, $f_2 + if_3 = (N + iB)\, e^{i\vartheta}$, where $\vartheta' = \tau$. While the standard frame and invariants are uniquely determined along a given curve γ (with non-vanishing curvature), any γ has a circle's worth of natural frames and invariants associated to it.

In most of our applications of the Pontrjagin Maximum Principle, the invariants $\kappa = (k_1, k_2, k_3)$ will play the roles of the controls – so (FS) will be regarded as a control system – and the cost function will be a geometric Lagrangian of the form $\mathscr{L}(\kappa)$. It is then an optimal control problem is to minimize the cost, $\int \mathscr{L}(\kappa)\, ds$, among solutions to (FS) satisfying given endpoint conditions $f(0) = f_0, f(\ell) = f_\ell$.

In this context, the recipe of the maximum principle (for finding smooth, regular, optimal trajectories, in the non-singular case) can now be summarized roughly as follows:

(i) Define a time-dependent Hamiltonian on T^*F, depending on controls $\kappa(s)$, by subtracting the cost function $\mathscr{L}(\kappa(s))$ from the linear (time-dependent) Hamiltonian associated with the right hand side of (FS) (i.e., just replace A, B by $\mathscr{A}, \mathscr{B}$):

$$\mathscr{H}(p, \kappa) = \mathscr{B}_1(f) + k_1 \mathscr{A}_1(f) - k_3 \mathscr{A}_2(f) + k_2 \mathscr{A}_3(f) - \mathscr{L}(\kappa),$$

(ii) Obtain a time-independent Hamiltonian $\mathscr{H}(p) = \mathscr{H}(p, \kappa(p))$ on T^*F by maximizing $\mathscr{H}(p, \kappa)$ with respect to the controls; for our purposes, this will amount to solving the equations $\partial \mathscr{H}/\partial k_j = 0$ for uniquely determined $\kappa = \kappa(p)$.

(iii) Solve the Hamiltonian system on T^*F determined by $\mathscr{H}(p)$; the trajectories $p(t)$ of this system project to the solutions $f(t)$ to the optimal control problem (for various boundary conditions).

We illustrate the procedure using the Lagrangian for the Kirchhoff elastic rod: $\mathscr{L}(\kappa) = \alpha/2((k_2)^2 + (k_3)^2) + \beta/2(k_1)^2$, where α and β are positive constants. The first term here corresponds to the "bending energy" of the rod, while the second term gives the "twisting energy". (There is no "stretching energy" term, since the rod is assumed to be inextensible.) According to the Kirchhoff model, the rod in equilibrium solves the variational problem: minimize $\int \mathscr{L}(\kappa)\,ds$ among rods of length ℓ with clamped ends, $f(0) = f_0, f(\ell) = f_\ell$ (note that $f(s)$ includes the position $\gamma(s)$). Thus, we take as our time-dependent Hamiltonian

$$\mathscr{H}(p, \kappa) = \mathscr{B}_1 + k_1\mathscr{A}_1 - k_3\mathscr{A}_2 + k_2\mathscr{A}_3 - \left(\frac{\alpha}{2}((k_2)^2 + (k_3)^2) + \frac{\beta}{2}(k_1)^2\right).$$

Setting $0 = \partial\mathscr{H}/\partial k_i$, $i = 1, 2, 3$, we read off $k_1 = \mathscr{A}_1/\beta$, $k_2 = \mathscr{A}_3/\alpha$, and $k_3 = -\mathscr{A}_2/\alpha$. Substitution gives the time independent Hamiltonian

$$\mathscr{H}(p) = \mathscr{B}_1 + \frac{(\mathscr{A}_2)^2 + (\mathscr{A}_3)^2}{2\alpha} + \frac{(\mathscr{A}_1)^2}{2\beta}.$$

The integrability of this Hamiltonian system is discussed in the next section.

§2. Integrable Hamiltonian systems

To discuss integrability, we now specialize to three-dimensional space forms, $M = \mathbb{R}^3$, $M = S^3$, or $M = \mathbb{H}^3$, and let σ be the curvature of M: $\sigma = 0, 1$, or -1. In this case, the orthonormal frame bundle F may be identified with the isometry group G of M: $G = E(3)$, $G = SO(4)$, or $G = SO(3, 1)$.

Further, the A_j and B_i make up a basis of left invariant vectorfields on G satisfying the standard Lie bracket relations for the Lie algebra $\mathbf{g}$ of G: $[A_i, A_j] = \epsilon_{ijk}A_k$, $[A_i, B_j] = \epsilon_{ijk}B_k$, and $[B_i, B_j] = \sigma\epsilon_{ijk}A_k$, where $\epsilon_{123} = 1$, and ϵ_{ijk} is antisymmetric in i, j, k. The corresponding Poisson brackets of left-invariant linear Hamiltonians, $\mathscr{A}_j$, $\mathscr{B}_i$, can now be read off, using the following general fact: if V and W are two vectorfields on a Riemannian manifold G, and if $\{\ ,\ \}$ denotes the canonical Poisson bracket on T^*G, then the bracket of the corresponding linear Hamiltonians $\mathscr{H}_V$ and $\mathscr{H}_W$ is given by $\{\mathscr{H}_V, \mathscr{H}_W\} - \mathscr{H}_{[V,W]}$. Thus, part of the Poisson structure of T^*G – the only part we will ever need in our computations – is given by:

$$\{\mathscr{A}_i, \mathscr{A}_j\} = -\epsilon_{ijk}\mathscr{A}_k, \{\mathscr{A}_i, \mathscr{B}_j\} = -\epsilon_{ijk}\mathscr{B}_k, \quad \text{and} \quad \{\mathscr{B}_i, \mathscr{B}_j\} = -\sigma\epsilon_{ijk}\mathscr{A}_k.$$

Using these bracket formulas, it is an easy computation to verify the following key

LEMMA. *The quadratic Hamiltonians* $\mathscr{P} = \mathscr{A}_1\mathscr{B}_1 + \mathscr{A}_2\mathscr{B}_2 + \mathscr{A}_3\mathscr{B}_3$ *and* $\mathscr{Q} = \mathscr{B}_1^2 + \mathscr{B}_2^2 + \mathscr{B}_3^2 + \sigma(\mathscr{A}_1^2 + \mathscr{A}_2^2 + \mathscr{A}_3^2)$ *lie in the center of the Poisson–Lie algebra* $\mathscr{L}G$ *of left-invariant functions on* T^*G.

Recall that if $\mathscr{H}$ defines a Hamiltonian system, and if $\mathscr{K}$ is any other Hamiltonian, then the time evolution of $\mathscr{K}$ is given by $d\mathscr{K}/dt = \{\mathscr{K}, \mathscr{H}\}$. Using this, the above lemma will provide two "free" constants of motion, $\mathscr{P}$, $\mathscr{Q}$, for all of our Hamiltonian systems. Two more constants will come from the symmetry of M, a la Noether's Theorem. Specifically, if X is a right-invariant vectorfield on G (such a vectorfield projects to a Killing field on M), and if $\mathscr{R} = \mathscr{H}_X$ is the corresponding right-invariant linear Hamiltonian, then $\{\mathscr{R}, \mathscr{A}_j\} = -\mathscr{H}_{[X,A_j]} = 0$ and $\{\mathscr{R}, \mathscr{B}_j\} = 0$, $j = 1, 2, 3$. Therefore $\mathscr{R}$ Poisson-commutes with any element of $\mathscr{L}G$. Note that we can always choose two such right-invariant linear Hamiltonians $\mathscr{R}_1, \mathscr{R}_2$, satisfying $\{\mathscr{R}_1, \mathscr{R}_2\} = 0$. Thus, any $\mathscr{H} \in \mathscr{L}G$ which is independent of $\mathscr{P}$ and $\mathscr{Q}$ determines a Hamiltonian system on T^*G with five constants of motion in involution. This is precisely the situation in the first three of the following examples; for Liouville integrability, it will therefore suffice, in each case, to discover one additional constant of motion $\mathscr{C} \in \mathscr{L}G$.

EXAMPLE 1. Elastic rods and curves

For rods, we have already obtained the Hamiltonian

$$\mathscr{H}(p) = \mathscr{B}_1 + \frac{(\mathscr{A}_2)^2 + (\mathscr{A}_3)^2}{2\alpha} + \frac{(\mathscr{A}_1)^2}{2\beta}.$$

One readily checks that $\{\mathscr{A}_1, \mathscr{H}\} = 0$, so $\mathscr{C} = \mathscr{A}_1$ is a constant of motion (which reflects the symmetry of $\mathscr{H}(p)$ relative to the coordinates $\mathscr{A}_2$ and $\mathscr{A}_3$). Therefore the Kirchhoff system is Liouville integrable (for more details see [21], [22]).

The Lagrangian for elastic curves is defined by the "bending energy" $\mathscr{L} = k^2/2$; in other words, it is the special case $\beta = 0$ (and $\alpha = 1$) of the rod Lagrangian. However, the above optimal control problem for the rod is singular in the case $\beta = 0$ – one cannot solve for k_1 in the equations $\partial\mathscr{H}/\partial k_i = 0$. This technical difficulty can be finessed by working with natural frames, thus reducing the number of controls to two. Setting $k_1 = 0$ in (FS) leads to the time-dependent Hamiltonian

$$\mathscr{H}(p, \kappa) = \mathscr{B}_1 + k_2\mathscr{A}_3 - k_3\mathscr{A}_2 - \frac{(k_2)^2 + (k_3)^2}{2}.$$

The usual procedure now gives

$$k_2 = \mathscr{A}_3, \qquad k_3 = -\mathscr{A}_2, \qquad \mathscr{H}(p) = \mathscr{B}_1 + \frac{(\mathscr{A}_2)^2 + (\mathscr{A}_3)^2}{2},$$

and again $\mathscr{C} = \mathscr{A}_1$ is constant of motion. (Actually, it turns out that only the solutions with $\mathscr{C} = 0$ give elastic curves; solutions with $\mathscr{C} \neq 0$ are byproducts of the fact that $f(0) = f_0$, $f(\ell) = f_\ell$ "overdetermines" a natural framing.)

EXAMPLE 2. $\mathscr{I}_4 = \int k^2\tau\, ds$.

For this problem we work with standard frames, since general frames and natural frames both lead to singular problems (on a higher dimensional space). Thus, $\mathscr{H}(p, \kappa) = \mathscr{B}_1 + k\mathscr{A}_3 + \tau\mathscr{A}_1 - k^2\tau$. This time, maximizing $\mathscr{H}$ with respect to the controls, k and τ, gives $\mathscr{A}_1 = k^2$, $\tau = \mathscr{A}_3/2k$. Eliminating k and τ results in a rather surprising Hamiltonian: $\mathscr{H}(p) = \mathscr{B}_1 + \mathscr{A}_3\sqrt{\mathscr{A}_1}$. Using the Poisson bracket formulas, it can be checked that $\mathscr{C} = \mathscr{A}_1^2 + \mathscr{A}_2^2 + \mathscr{A}_3^2 - 4\sqrt{\mathscr{A}_1}\mathscr{B}_3 - 4\sigma\mathscr{A}_1$ is a constant of motion.

EXAMPLE 3. τ-elastic curves

Here we consider the Lagrangian $\mathscr{L} = k^2/2$ and the control system of example 2, except that we constrain the torsion to equal a fixed constant $\tau = c$ (so there is only one control k). This leads to the Hamiltonian $\mathscr{H}(p) = \mathscr{B}_1 + c\mathscr{A}_1 + (\mathscr{A}_3)^2/2$, and it can be checked that $\mathscr{C} = \mathscr{B}_3 - c\mathscr{A}_3$ is a constant of motion, in the special case $\sigma = 1$, $c = \pm 1$.

EXAMPLE 4. $\mathscr{I}_5 = \int \frac{1}{2}(k')^2 + \frac{1}{2}k^2\tau^2 - \frac{1}{8}k^4\, ds$ with natural frames.

In terms of the natural frame, this functional can be rewritten as

$$\mathscr{I}_5 = \int \frac{1}{2}\dot{k}_2^2 + \frac{1}{2}\dot{k}_3^2 - \frac{1}{8}(k_2^2 + k_3^2)^2\, ds.$$

In order to define a control problem, we enlarge our space to $T^*(E \times \mathbb{R}^2)$. The curvatures k_2 and k_3 are now space variables, with corresponding momenta $\mathscr{K}_2$ and $\mathscr{K}_3$. The controls, u_2 and u_3, correspond to the derivatives of the curvatures. The Poisson brackets in the new space are simple to compute; the old functions lift to functions on the product, which Poisson commute with the functions k_i and $\mathscr{K}_i$. We also need the brackets $\{k_i, \mathscr{K}_j\} = \delta_{ij}$.

Following the usual procedure, we write the Hamiltonian

$$\mathscr{H}(p,u)=\mathscr{B}_1+k_2\mathscr{A}_3-k_3\mathscr{A}_2+u_2\mathscr{K}_2+u_3\mathscr{K}_3-\frac{1}{2}(u_2^2+u_3^2)+\frac{1}{8}(k_2^2+k_3^2)^2$$

Set $\partial\mathscr{H}/\partial u_2=0=\mathscr{K}_2-u_2$, $\partial\mathscr{H}/\partial u_3=0=\mathscr{K}_3-u_3$, and get

$$\mathscr{H}(p)=\mathscr{B}_1+k_2\mathscr{A}_3-k_3\mathscr{A}_2+\frac{1}{2}(\mathscr{K}_2^2+\mathscr{K}_3^2)+\frac{1}{8}(k_2^2+k_3^2)^2.$$

Note that $\mathscr{H}$ is no longer a left-invariant Hamiltonian; however, it still commutes with $\mathscr{P}$, $\mathscr{Q}$, $\mathscr{R}_1$, and $\mathscr{R}_2$. Note also, $\mathscr{H}$ defines a system with eight degrees of freedom, so we must find three more first integrals, and they all must Poisson-commute with each other (and with the other functions).

One can check that the following functions satisfy these conditions:

$$\mathscr{C}_1=\mathscr{A}_1+k_3\mathscr{K}_2-k_2\mathscr{K}_3.$$

$$\mathscr{C}_2=k_3\mathscr{B}_2-k_2\mathscr{B}_3+\mathscr{A}_2\mathscr{K}_2+\mathscr{A}_3\mathscr{K}_3+\frac{1}{2}(k_2^2+k_3^2)\mathscr{A}_1-\sigma\mathscr{A}_1$$

$$\mathscr{C}_3=\mathscr{B}_2\mathscr{K}_2+\mathscr{B}_3\mathscr{K}_3+\frac{1}{2}(k_2^2+k_3^2-2\sigma)\mathscr{B}_1-\sigma(k_2\mathscr{A}_3-k_3\mathscr{A}_2)$$

The required computations can be organized using the notation, $\mathscr{B}_j=\varphi\mathscr{A}_j$, $\varphi^2=\sigma$, which allows us to regard the equations for the Poisson brackets $\{\mathscr{A}_j,\mathscr{B}_i\}$ and $\{\mathscr{B}_j,\mathscr{B}_i\}$ as formal consequences of the equations for $\{\mathscr{A}_i,\mathscr{A}_j\}$. In fact, we can write everything in terms of the $\mathscr{A}_j$'s (no $\mathscr{B}_j$'s), and then $\mathscr{C}_2$ and $\mathscr{C}_3$ can be written more compactly as $\mathscr{C}_2=\mathscr{G}_j\mathscr{A}_j$, $\mathscr{C}_3=\varphi\mathscr{C}_2$, where we have introduced "complex coefficients": $\mathscr{G}_1=k^2/2-\sigma$, $\mathscr{G}_2=\mathscr{K}_2+\varphi k_3$, and $\mathscr{G}_3=\mathscr{K}_3-\varphi k_2$. With this notation, e.g., one can make the trivial computation $\{\mathscr{C}_2,\mathscr{C}_3\}=\{\mathscr{C}_2,\varphi\mathscr{C}_2\}=\varphi\{\mathscr{C}_2,\mathscr{C}_2\}=0$; further, once one has checked that $\mathscr{C}_2$ is a constant of motion, it follows at once that $\mathscr{C}_3$ is also constant. We caution the reader that although φ may be treated as a scalar in the Lie algebra of linear Hamiltonians with "complex coefficients" (to which $\mathscr{C}_2$ and $\mathscr{C}_3$ belong), it may not be treated as such in the Poisson–Lie algebra $\mathscr{L}G$ of functions.

Finally, we note the relationship between the $\mathscr{G}_i$ and the vectorfields in the LIE hierarchy (see [18]). In terms of natural frames, the first and second vectorfields in LIE may be written:

$$X_1=-k_3f_2+k_2f_3,\quad\text{and}\quad X_2=(k^2/2-\sigma)T+\mathscr{K}_2f_2+\mathscr{K}_3f_3.$$

Then

$$X_2 - \varphi X_1 = \mathcal{G}_i f_i.$$

EXAMPLE 5. $\mathcal{I}_5 = \int \frac{1}{2}(k')^2 + \frac{1}{2}k^2\tau^2 - \frac{1}{8}k^4\, ds$ with standard frames.

The same variational problem may be set up as a nonsingular control problem using the standard frame. In this case we must introduce one new space variable k and the corresponding momentum $\mathcal{K}$, with $\{k, \mathcal{K}\} = 1$. This gives rise to a Hamiltonian system with seven degrees of freedom on the space $T^*(E \times \mathbb{R})$. The Hamiltonian is:

$$\mathcal{H}(p) = \mathcal{B}_1 + k\mathcal{A}_3 + \frac{\mathcal{A}_1^2}{2k^2} + \frac{1}{2}\mathcal{K}^2 + \frac{1}{8}\mathcal{K}^4$$

In this set-up we require two constants of motion in involution for integrability. One can check that the desired functions are:

$$\mathcal{C} = \frac{k^2}{2}\mathcal{A}_1 + \frac{1}{k}\mathcal{A}_1\mathcal{A}_3 - k\mathcal{B}_3 + \mathcal{A}_2\mathcal{K} - \sigma\mathcal{A}_1$$

$$\mathcal{D} = \frac{k^2}{2}\mathcal{B}_1 + \frac{1}{k}\mathcal{A}_1\mathcal{B}_3 - \sigma k\mathcal{A}_3 + \mathcal{B}_2\mathcal{K} - \sigma\mathcal{B}_1$$

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