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Lie algebras and coverings

CHRISTINE RIEDTMANN

Meinem Lehrer Peter Gabriel gewidmet

1. Introduction

1.1. Let Λ be an associative unitary finite dimensional $\mathbb{C}$-algebra which is representation finite. This means that the number of isomorphism classes of indecomposable finite dimensional Λ-left modules is finite. Let us fix a set $\mathscr{I}$ of representatives for these isomorphism classes.

We showed in [Rie] that the free $\mathbb{Z}$-module

$$L(\Lambda) = \bigoplus_{A \in \mathscr{I}} \mathbb{Z} v_A$$

generated by the symbols

$$\{v_A : A \in \mathscr{I}\}$$

can be made into a $\mathbb{Z}$-Lie algebra in the following way: set

$$[v_A, v_B] = \sum_{X \in \mathscr{I}} (\gamma^X_{A,B} - \gamma^X_{B,A}) v_X,$$

where

$$\gamma^X_{A,B} = \chi(V_\Lambda(A, B; X))$$

is the Euler-Poincaré characteristic of the algebraic variety

$$V_\Lambda(A, B; X) = \{0 \subseteq Y \subseteq X : Y \text{ is a } \Lambda\text{-submodule of } X \text{ isomorphic to } A \text{ with quotient module } X/Y \text{ isomorphic to } B\}.$$

This is the complex version of Ringel's construction of Lie algebras via Hall algebras over finite fields [Rin].

The construction of $L(\Lambda)$ carries over easily to the case where Λ is a locally representation finite $\mathbb{C}$-category. We will list the most important definitions and facts about locally representation finite categories and their coverings in chapter 3; the references for these results are [BG] and [Ga].

1.2. If the representation finite algebra – or more generally the locally representation finite $\mathbb{C}$-category – is simply connected [BG], the Lie algebra $L(\Lambda)$ has a particularly simple structure. Indeed, we proved in [Rie] that in this case one of the numbers

$$\gamma_{A,B}^{X} \quad \text{and} \quad \gamma_{B,A}^{X}$$

is zero for any choice of A, B, X in $\mathscr{I}$ and that for fixed A and B there is at most one X for which

$$\gamma_{A,B}^{X} \neq 0.$$

To any locally representation finite $\mathbb{C}$-category Λ one can associate a locally representation finite one which is simply connected: its universal cover $\tilde{\Lambda}$ ([BG], [Ga]). The reason why we consider $\mathbb{C}$-categories instead of $\mathbb{C}$-algebras in this paper is that $\tilde{\Lambda}$ is rarely a $\mathbb{C}$-algebra.

It is tempting to try and use the simple structure of $L(\tilde{\Lambda})$ in order to compute $L(\Lambda)$. The aim of this paper is to show that this is actually possible: if we choose a set $\tilde{\mathscr{I}}$ of representatives for the indecomposable $\tilde{\Lambda}$-modules which is stable under the fundamental group G (see chapter 3), the set of G-orbits $\{\bar{A} = GA : A \in \tilde{\mathscr{I}}\}$ is a set $\mathscr{I}$ of representatives for the indecomposable Λ-modules. Our goal is to prove:

$$\chi(V_{\Lambda}(\bar{A}, \bar{B}; \bar{X})) = \sum_{g,h \in G} \chi(V_{\tilde{\Lambda}}(A, g(B); h(X)))$$

for any $A, B, X \in \tilde{\mathscr{I}}$. Thus the structure constants for $L(\Lambda)$ are sums of – more easily accessible – structure constants of $L(\tilde{\Lambda})$.

In fact, we will define an "orbit" Lie algebra L/G in chapter 2 for an appropriate action of a group G on a Lie algebra L and show in chapter 4 that the action of the fundamental group G of Λ on $L(\tilde{\Lambda})$ is appropriate. Our aim is then:

THEOREM. *Let Λ be a locally representation finite $\mathbb{C}$-category with universal cover $\tilde{\Lambda}$ and fundamental group G. Then $L(\Lambda)$ is isomorphic to $L(\tilde{\Lambda})/G$.*

1.3. As a first application, let us prove again that for $\Lambda = \mathbb{C}[T]/(T^n)$ the bracket on $L(\Lambda)$ is trivial [Rie]. The universal cover $\tilde{\Lambda}$ of Λ is given by the quiver

$$\cdots - 1 \xrightarrow{\alpha_{-1}} 0 \xrightarrow{\alpha_0} 1 \xrightarrow{\alpha_1} 2 \cdots,$$

having $\mathbb{Z}$ as its vertex set and containing an arrow $\alpha_i : i \to i+1$ for $i \in \mathbb{Z}$, and the ideal of relations generated by all paths of length n. The fundamental group $G = \mathbb{Z}$ is generated by the shift $i \mapsto i+1$. For $\tilde{\mathscr{I}}$ we choose the set

$$\{(i, r) : i \in \mathbb{Z},\ 1 \le r \le n\},$$

where (i, r) is "the indecomposable with top i of length r" defined by

$$(i, r)(j) = \begin{cases} \mathbb{C} & \text{for } i \le j < i + r, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(i, r)(\alpha_j) = \begin{cases} \mathrm{id}_{\mathbb{C}} & \text{for } i \le j < i + r - 1, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that

$$V_{\tilde{A}}((i, r), (j, s); (k, t)) = \begin{cases} 1 \text{ point} & \text{for } i = j + s,\ k = j,\ t = r + s, \\ \varnothing & \text{otherwise,} \end{cases}$$

for any triple of indecomposables. Therefore

$$\chi(V_{\tilde{A}}((j + s, r), (j, s); (j, r + s))) = 1 = \chi(V_{\tilde{A}}((i + r, s), (i, r); (i, r + s)))$$

give the only non-trivial contributions to the bracket $[v_{\tilde{A}}, v_{\tilde{B}}]$ with $A = (i, r)$, $B = (j, s)$.

As a second example, consider the quotient A of the algebra of the quiver $\bullet \xrightarrow{\alpha} \bullet \circlearrowleft \beta$ by the ideal generated by β^3. In this case the quiver of $\tilde{A}$ is:

$$\begin{array}{ccccccccc}
\cdots & -1 & \xrightarrow{\beta_{-1}} & 0 & \xrightarrow{\beta_0} & 1 & \xrightarrow{\beta_1} & 2 & \cdots \\
 & \uparrow{\scriptstyle \alpha_{-1}} & & \uparrow{\scriptstyle \alpha_0} & & \uparrow{\scriptstyle \alpha_1} & & \uparrow{\scriptstyle \alpha_2} & \\
\cdots & -1' & & 0' & & 1' & & 2' & \cdots
\end{array}$$

and the ideal of relations is generated by $\{\beta_{i+2}\beta_{i+1}\beta_i : i \in \mathbb{Z}\}$. The fundamental group is generated by the shift again. Let us consider the indecomposables A, B, X

given by

$$\dim A(x) = \begin{cases} 1 & \text{for } x = 1, 2, 3, \\ 0 & \text{otherwise,} \end{cases}$$

$$\dim B(x) = \begin{cases} 1 & \text{for } x = 0, 1, 2, 1', 2', \\ 0 & \text{otherwise,} \end{cases}$$

$$\dim X(x) = \begin{cases} 2 & \text{for } x = 1, 2, \\ 1 & \text{for } x = 0, 3, 1', 2', \\ 0 & \text{otherwise,} \end{cases}$$

and

$$X(\beta_0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X(\beta_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X(\beta_2) = (0\ 1),$$

$$X(\alpha_1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad X(\alpha_2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The variety of embeddings of $\bar{A}$ into $\bar{X}$ is 6-dimensional, and possible quotients are quite hard to determine. Over $\tilde{\Lambda}$, however, it is easy to see that the only way to embed A into a translate of X with quotient a translate of B is to choose a map $f: A \to X$ of the form

$$f(1) = \begin{pmatrix} \lambda \\ \mu \end{pmatrix}, \quad f(2) = \begin{pmatrix} \lambda \\ \mu \end{pmatrix}, \quad f(3) = \mu$$

with $\mu \neq 0$. The quotient is isomorphic to B if and only if $\lambda \neq 0, \mu$. Hence

$$\chi(V_\Lambda(\bar{A}, \bar{B}; \bar{X})) = -1.$$

2. The "orbit" Lie algebra

2.1. Let L be a $\mathbb{Z}$-Lie algebra which is generated (as a $\mathbb{Z}$-module) by some basis $\mathscr{B}$:

$$L = \bigoplus_{b \in \mathscr{B}} \mathbb{Z}b.$$

Suppose the group G acts on L by Lie algebra automorphisms in such a way that it permutes the elements of $\mathscr{B}$ and that the following condition is satisfied:

$$\forall b, c \in \mathscr{B} : \#\{g \in G : [b, g(c)] \neq 0\} < \infty. \qquad (*)$$

This condition is obviously empty in case G is finite. The main example to have in mind here, however, is the fundamental group G of a locally representation finite $\mathbb{C}$-category Λ acting on ind $\tilde{\Lambda}$ for a G-stable set of representatives, and this is a free group [BG].

Set

$$\bar{\mathscr{B}} = \{\bar{b} = G \cdot b : b \in \mathscr{B}\}$$

and

$$L/G = \bigoplus_{\bar{b} \in \bar{\mathscr{B}}} \mathbb{Z}\bar{b}.$$

Let

$$\bar{?} : L \to L/G$$

be the $\mathbb{Z}$-linear map which takes b to $\bar{b}$ for $b \in \mathscr{B}$.

The following result is easy to prove:

PROPOSITION. *The bracket*

$$[\bar{b}, \bar{c}] = \overline{\sum_{g \in G} [b, g(c)]}, \qquad \bar{b}, \bar{c} \in \bar{\mathscr{B}},$$

defines a Lie algebra structure on L/G.

Note that the map $\bar{?}$ is *not* a Lie algebra homomorphism in general.

2.2. Comparing the structure constants of L/G with those of L, we find: if

$$[b, c] = \sum_{d \in \mathscr{B}} \gamma_{b,c}^{d} d$$

and

$$[\bar{b}, \bar{c}] = \sum_{\bar{d} \in \bar{\mathscr{B}}} \bar{\gamma}_{\bar{b},\bar{c}}^{\bar{d}} \bar{d},$$

then

$$\gamma_{\bar{b},\bar{c}}^{\bar{d}} = \sum_{d' \in \bar{d}} \sum_{g \in G} \gamma_{b,g(c)}^{d'}.$$

In case the action of G on $\mathscr{B}$ is free as it is in the case we are interested in this formula becomes

$$\gamma_{\bar{b},\bar{c}}^{\bar{d}} = \sum_{g,h \in G} \gamma_{b,g(c)}^{h(d)}.$$

2.3. Let G be a group acting on the Lie algebra

$$L = \bigoplus_{b \in \mathscr{B}} \mathbb{Z}b$$

in such a way that the hypotheses of 2.1 are satisfied. If $H \lhd G$ is a normal subgroup, they are satisfied for the action of H on L as well, so that we can consider the Lie algebra

$$L/H = \bigoplus_{\bar{b} \in \bar{\mathscr{B}}} \mathbb{Z}\bar{b}$$

with

$$\bar{\mathscr{B}} = \{\bar{b} = Hb : b \in \mathscr{B}\}.$$

Extending the action given by

$$\bar{g}(\bar{b}) = \overline{g(b)}$$

of $\bar{G} = G/H$ on $\bar{\mathscr{B}}$ by $\mathbb{Z}$-linearity, we obtain an action of $\bar{G}$ on L/H. It is easy to see that it satisfies again the hypotheses of 2.1 and that the following proposition holds:

PROPOSITION. *The $\mathbb{Z}$-linear map $L/G \to (L/H)/(G/H)$ sending the basis element Gb to $(G/H)\bar{b}$ is an isomorphism of Lie algebras.*

3. Locally representation finite categories and coverings

The references for this chapter are [BG], [Ga].

3.1. We begin by recalling some definitions:

A $\mathbb{C}$-category Λ is locally bounded if the following conditions are satisfied:

(i) *$\Lambda(x, x)$ is a local $\mathbb{C}$-algebra for all objects x of Λ.*

(ii) *Distinct objects of Λ are not isomorphic.*

(iii) *For all objects x we have*

$$\sum_y \dim_{\mathbb{C}} \Lambda(x, y) < \infty,$$

$$\sum_y \dim_{\mathbb{C}} \Lambda(y, x) < \infty,$$

where y ranges over the objects of Λ.

A finite dimensional Λ-left module is a covariant functor $B : \Lambda \to \operatorname{mod} \mathbb{C}$ with

$$\sum_x \dim_{\mathbb{C}} B(x) < \infty,$$

where x ranges over the objects of Λ.

We denote by mod Λ the category of finite dimensional Λ-modules and by ind Λ the full subcategory whose objects are a fixed set $\mathscr{I}$ of representatives for the isomorphism classes of indecomposables in mod Λ.

A locally bounded $\mathbb{C}$-category Λ is locally representation finite if, for every object x of Λ, the number of indecomposables B in $\mathscr{I}$ with $B(x) \neq 0$ is finite.

$\mathbb{C}$-algebras Λ which are sober, i.e. with $\Lambda/\operatorname{rad} \Lambda \xrightarrow{\sim} \mathbb{C} \times \cdots \times \mathbb{C}$, correspond to locally bounded $\mathbb{C}$-categories with finitely many objects and representation finite $\mathbb{C}$-algebras to locally representation finite $\mathbb{C}$-categories with finitely many objects.

3.2. Let $\tilde{\Lambda}$ be the universal cover of Λ, and choose a set $\tilde{\mathscr{I}}$ of representatives for the indecomposable $\tilde{\Lambda}$-modules which is stable under the action of the fundamental group G of Λ on mod $\tilde{\Lambda}$. Then G acts on the full subcategory ind $\tilde{\Lambda}$ of mod $\tilde{\Lambda}$ whose objects are the elements of $\tilde{\mathscr{I}}$ by $\mathbb{C}$-linear automorphisms. Moreover, we have

$$g(B) \neq B \qquad \text{for every } B \text{ in } \tilde{\mathscr{I}} \text{ and every } g \neq 1 \text{ in } G$$

and

$$\#\{g \in G : \mathrm{Hom}_{\tilde{\Lambda}}(A, g(B)) \neq 0\} < \infty \qquad \text{for every pair } A, B \text{ in } \tilde{\mathscr{I}}.$$

Under these circumstances ind $\tilde{\Lambda}$ has a quotient modulo G: its objects are the G-orbits of objects in $\tilde{\mathscr{I}}$, and the morphisms from the orbit of A to the orbit of B are families $(f_{g,h})_{g,h \in G}$,

$$f_{g,h} : g(A) \to h(B)$$

with

$$l(f_{g,h}) = f_{lg,lh}$$

for all g, h, l in G.

Similarly, there exists a quotient $\tilde{\Lambda}/G$, which is locally representation finite. The category (ind $\tilde{\Lambda}$)$/G$ is isomorphic to ind ($\tilde{\Lambda}/G$), which is in turn isomorphic to the so called mesh category $\mathbb{C}(\Gamma_\Lambda)$ associated with the Auslander-Reiten quiver of Λ. So $\tilde{\Lambda}/G$ is the "standard form" of Λ. But by [BGRS] non-standard algebras can exist only over ground fields of characteristic 2. Thus $\tilde{\Lambda}/G$ is isomorphic to Λ and (ind $\tilde{\Lambda}$)$/G$ to ind Λ. We fix the set $\mathscr{I}$ of G-orbits in $\tilde{\mathscr{I}}$ as a set of representatives of the isomorphism classes of indecomposable Λ-modules and identify ind Λ with (ind $\tilde{\Lambda}$)$/G$.

3.3. The $\mathbb{C}$-linear functor

$$F : \text{ind}\,\tilde{\Lambda} \to (\text{ind}\,\tilde{\Lambda})/G = \text{ind}\,\Lambda$$

defined by

$$F(B) = G \cdot B$$

for B in $\tilde{\mathscr{I}}$ and by associating to $f : A \to B$, $A, B \in \tilde{\mathscr{I}}$, the family

$$Ff = (f'_{g,h}), \qquad g, h \in G$$

with

$$f'_{g,h} = \begin{cases} g(f) & g = h \\ 0 & g \neq h \end{cases}$$

is a covering functor. This means that, for all A, B in $\tilde{\mathscr{I}}$, F induces $\mathbb{C}$-linear isomorphisms

$$\bigoplus_{g \in G} \operatorname{Hom}_{\tilde{\Lambda}}(A, g(B)) \xrightarrow{\sim} \operatorname{Hom}_{\Lambda}(FA, FB),$$

$$\bigoplus_{g \in G} \operatorname{Hom}_{\tilde{\Lambda}}(g(A), B) \to \operatorname{Hom}_{\Lambda}(FA, FB).$$

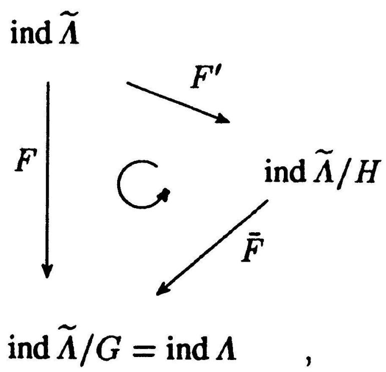
We will need also that F induces $\mathbb{C}$-linear isomorphisms:

$$\bigoplus_{g \in G} \operatorname{Ext}^1_{\tilde{\Lambda}}(A, g(B)) \xrightarrow{\sim} \operatorname{Ext}^1_{\Lambda}(FA, FB),$$

$$\bigoplus_{g \in G} \operatorname{Ext}^1_{\tilde{\Lambda}}(g(A), B) \xrightarrow{\sim} \operatorname{Ext}^1_{\Lambda}(FA, FB),$$

for all A, B in $\tilde{\mathscr{I}}$. This is an easy consequence of the isomorphisms for Hom-sets and the fact that F is an exact functor preserving projectivity.

3.4. If $H \lhd G$ is a normal subgroup, it is the fundamental group of $\tilde{\Lambda}/H$, and again we identify $(\operatorname{ind} \tilde{\Lambda})/H$ with $\operatorname{ind}(\tilde{\Lambda}/H)$ and note this quotient simply $\operatorname{ind} \tilde{\Lambda}/H$. There is a commutative triangle of covering functors:



,

where $\bar{F}$ sends HB to GB for B in $\tilde{\mathscr{I}}$ and a morphism

$$(f_{h_1,h_2} : h_1(A) \to h_1(B))_{h_1,h_2 \in H}$$

to

$$(f'_{g_1,g_2} : g_1(A) \to g_2(B))_{g_1,g_2 \in G}$$

with

$$f'_{g_1,g_2} = \begin{cases} g_2(f_{g_2^{-1}g_1,1}) & \text{if } g_2^{-1}g_1 \in H, \\ 0 & \text{if not.} \end{cases}$$

As $\bar{F}$ is a covering functor, $\bar{F}$ induces $\mathbb{C}$-linear isomorphisms:

$$\bigoplus_{\bar{g} \in G/H} \operatorname{Ext}^1_{\tilde{\Lambda}/H}(F'A, \bar{g}(F'B)) \xrightarrow{\sim} \operatorname{Ext}^1_{\Lambda}(FA, FB),$$

$$\bigoplus_{\bar{g} \in G/H} \operatorname{Ext}^1_{\tilde{\Lambda}/H}(\bar{g}(F'A), F'B) \xrightarrow{\sim} \operatorname{Ext}^1_{\Lambda}(FA, FB),$$

for any two elements A, B in $\tilde{\mathscr{I}}$.

4. The theorem – and its proof in some cases

4.1. Let Λ be a locally representation finite $\mathbb{C}$-category with universal cover $\tilde{\Lambda}$ and fundamental group G. Fix a G-stable set $\tilde{\mathscr{I}}$ of representatives for the isomorphism classes of indecomposable $\tilde{\Lambda}$-modules and identify ind Λ with ind $\tilde{\Lambda}/G$.

Extend the action of G on $\tilde{\mathscr{I}}$ to a $\mathbb{Z}$-linear action of G on $L(\tilde{\Lambda})$. Note that, for $A, B, X \in \tilde{\mathscr{I}}$ and $g \in G$, the varieties $V_{\tilde{\Lambda}}(A, B; X)$ and $V_{\tilde{\Lambda}}(g(A), g(B); g(X))$ are isomorphic and hence homeomorphic. Therefore G acts by Lie algebra automorphisms.

Moreover, the sets

$$\{h \in G : \operatorname{Hom}_{\tilde{\Lambda}}(A, h(X)) \neq 0\}$$

and, for any $h \in G$

$$\{g \in G : \operatorname{Hom}_{\tilde{\Lambda}}(h(X), g(B)) \neq 0\}$$

are finite for any A, B, X by 3.2. This implies that the action of G on $L(\tilde{\Lambda})$ satisfies the condition $(*)$ of 2.1 as well.

Now the statement of our theorem makes sense at least. In fact, both Lie algebras $L(\Lambda)$ and $L(\tilde{\Lambda})/G$ have as a basis the set of G-orbits in $\tilde{\mathscr{I}}$. The isomorphism is the identity on this basis.

4.2. We recall from [Rie] that there is another way to compute the structure constants of $L(\Lambda)$, which is more adapted to coverings: let A, B and X be indecomposable Λ-modules. Then the following Euler-Poincaré characteristics coincide:

$$\chi(V_{\Lambda}(B, A; X)) = \chi(\operatorname{Ext}^1_{\Lambda}(A, B)_X/\mathbb{C}^*).$$

The variety on the left hand side has been introduced in 1.1. As to the right hand side, $\operatorname{Ext}^1_{\Lambda}(A, B)_X$ is the algebraic subset of equivalence classes of short exact sequences in the $\mathbb{C}$-vector space $\mathrm{Ext}^1_\Lambda(A, B)$ whose middle term is isomorphic to X. It is stable under the action of $\mathbb{C}^*$ by homotheties on $\mathrm{Ext}^1_\Lambda(A, B)$.

4.3. PROPOSITION. *Let Λ be a locally representation finite $\mathbb{C}$-category with universal cover $\tilde{\Lambda}$ and fundamental group G, and suppose that the set*

$$\{g \in G : \mathrm{Ext}^1_\Lambda(A, g(B)) \neq 0\}$$

has at most one element for any pair A, B in $\tilde{\mathscr{I}}$. Then $L(\Lambda)$ is isomorphic to $L(\tilde{\Lambda})/G$.

Proof. Let

$$F : \mathrm{ind}\,\tilde{\Lambda} \to \mathrm{ind}\,\tilde{\Lambda}/G = \mathrm{ind}\,\Lambda$$

be the orbit covering functor. Choose A and B in $\tilde{\mathscr{I}}$ in such a way that

$$\mathrm{Ext}^1_\Lambda(FA, FB) \neq 0.$$

According to our hypothesis and 3.3 there is a unique element $g \in G$ such that $\mathrm{Ext}^1_{\tilde{\Lambda}}(A, g(B)) \neq 0$, and F induces a $\mathbb{C}$-linear isomorphism

$$\mathrm{Ext}^1_{\tilde{\Lambda}}(A, g(B)) \xrightarrow{\sim} \mathrm{Ext}^1_\Lambda(FA, FB).$$

Clearly the inverse image of $\mathrm{Ext}^1_\Lambda(FA, FB)_{FX}$ under this isomorphism is the disjoint union

$$\dot{\bigcup_{h \in G}} \mathrm{Ext}^1_{\tilde{\Lambda}}(A, g(B))_{h(X)}$$

for any X in $\tilde{\mathscr{I}}$.

As the characteristic $\chi(\mathscr{C})$ of a finite disjoint union $\mathscr{C} = \dot{\bigcup} \mathscr{C}_i$ of constructible subsets of a variery $\mathscr{C}$ is the sum $\Sigma\, \chi(\mathscr{C}_i)$, we conclude that

$$\chi(\mathrm{Ext}^1_\Lambda(FA, FB)_{FX}/\mathbb{C}^*) = \sum_{h \in G} \chi(\mathrm{Ext}^1_{\tilde{\Lambda}}(A, g(B))_{h(X)}/\mathbb{C}^*)$$

$$= \sum_{g,h \in G} \chi(\mathrm{Ext}^1_{\tilde{\Lambda}}(A, g(B))_{h(X)}/\mathbb{C}^*).$$

Remarks.

(i) As $\tilde{\Lambda}$ is simply connected, there is in fact at most one $h \in G$ with $\operatorname{Ext}^1_{\tilde{\Lambda}}(A, g(B))_{h(X)} \neq 0$, as indecomposables are determined by their composition factors [Ha].

(ii) Note also that in case

$$\{g \in G : \operatorname{Ext}^1_{\Lambda}(A, g(B)) \neq 0\}$$

contains at most one element for just a pair A, B in $\tilde{\mathscr{I}}$, it is still true that

$$\chi(\operatorname{Ext}^1_{\Lambda}(FA, FB)_{FX}/\mathbb{C}^*) = \sum_{g,h \in G} \chi(\operatorname{Ext}^1_{\tilde{\Lambda}}(A, g(B))_{h(X)}/\mathbb{C}^*).$$

5. $\mathbb{C}^*$-actions

We fix a locally representation finite $\mathbb{C}$-category Λ with universal cover $\tilde{\Lambda}$ and fundamental group G as well as a G-stable set $\tilde{\mathscr{I}}$ of representatives for the isomorphism classes of indecomposable $\tilde{\Lambda}$-modules. We denote by F the orbit covering functor

$$F : \operatorname{ind} \tilde{\Lambda} \to \operatorname{ind} \tilde{\Lambda}/G = \operatorname{ind} \Lambda.$$

5.1. Any map $\lambda : G \to \mathbb{Z}$ gives rise to a $\mathbb{C}$-linear $\mathbb{C}^*$-action on the $\mathbb{C}$-vector space

$$\bigoplus_{g \in G} \operatorname{Ext}^1_{\tilde{\Lambda}}(A, g(B))$$

by

$$t \cdot (\varepsilon_g)_{g \in G} = (t^{\lambda(g)}\varepsilon_g)_{g \in G}$$

for $t \in \mathbb{C}^*$, $\varepsilon_g \in \operatorname{Ext}^1_{\tilde{\Lambda}}(A, g(B))$ and for any $A, B \in \tilde{\mathscr{I}}$. A line through the origin in this vector space is stable under $\mathbb{C}^*$ if and only if there exists an integer n such that the line lies in

$$\bigoplus_{\substack{g \in G \\ \lambda(g) = n}} \operatorname{Ext}^1_{\tilde{\Lambda}}(A, g(B)).$$

Using the $\mathbb{C}$-isomorphism

$$\bigoplus_{g \in G} \operatorname{Ext}^1_{\tilde{\Lambda}}(A, g(B)) \to \operatorname{Ext}^1_{\Lambda}(FA, FB),$$

induced by F (see 3.3) we obtain a $\mathbb{C}^*$-action on

$$\mathrm{Ext}^1_\Lambda(FA, FB)/\mathbb{C}^*,$$

whose fixed points are the disjoint union

$$\dot{\bigcup_{n\in\mathbb{Z}}}\left[F\left[\bigoplus_{\substack{g\in G\\ \lambda(g)=n}} \mathrm{Ext}^1_{\tilde\Lambda}(A, g(B))/\mathbb{C}^*\right]\right]$$

Recall that the Euler-Poincaré characteristic of a variety Z admitting an algebraic action of $\mathbb{C}^*$ equals the characteristic of the fixed point set $Z^{\mathbb{C}^*}$.

Therefore our theorem would be proved if we could exhibit a map $\lambda : G \to \mathbb{Z}$ satisfying:

(i) the middle term of a short exact sequence in $\mathrm{Ext}^1_\Lambda(FA, FB)$ changes only up to isomorphism under the $\mathbb{C}^*$-action defined by λ.
(ii) for each integer n there is at most one $g \in G$ with $\lambda(g) = n$.

Indeed, such a λ would give rise to a $\mathbb{C}^*$-action stabilizing $\mathrm{Ext}^1_\Lambda(FA, FB)_{FX}$ for any $X \in \tilde{\mathscr{I}}$ by (i), and we could write

$$\begin{aligned}\chi(\mathrm{Ext}^1_\Lambda(FA, FB)_{FX}/\mathbb{C}^*) &= \chi((\mathrm{Ext}^1_\Lambda(FA, FB)_{FX}/\mathbb{C}^*)^{\mathbb{C}^*})\\ &= \chi\left[\dot{\bigcup_{\substack{n\in\mathbb{Z}\\ \exists g_n : \lambda(g_n)=n}}}\ \dot{\bigcup_{h\in G}} \mathrm{Ext}^1_{\tilde\Lambda}(A, g_n(B))_{h(X)}/\mathbb{C}^*\right]\\ &= \sum_{g,h\in G} \chi(\mathrm{Ext}^1_{\tilde\Lambda}(A, g(B))_{h(X)}/\mathbb{C}^*).\end{aligned}$$

Here we used again that the inverse image of $\mathrm{Ext}^1_\Lambda(FA, FB)_{FX}$ in $\mathrm{Ext}^1_{\tilde\Lambda}(A, g_n(B))$ is the disjoint union

$$\dot{\bigcup_{h\in G}} \mathrm{Ext}^1_{\tilde\Lambda}(A, g_n(B))_{h(X)}.$$

Unfortunately, such λ's need not exist. We will concentrate first on the condition (i), which is indispensable.

5.2. For $A, B, U \in \tilde{\mathscr{I}}$ we consider the pull-back map

$$\pi : \mathrm{Ext}^1_\Lambda(FA, FB) \times \mathrm{Hom}_\Lambda(FU, FA) \to \mathrm{Ext}^1_\Lambda(FU, FB)$$

which associates to an exact sequence

$$\varepsilon : 0 \to FB \to Z \to FA \to 0$$

with Z in mod Λ and a homomorphism $f : FB \to FU$ the pull-back sequence $\pi(\varepsilon, f)$ in $\mathrm{Ext}^1_\Lambda(FU, FB)$:

$$\begin{array}{lccccccccccc} \varepsilon & : & 0 & \to & FB & \to & Z & \to & FA & \to & 0 \\ & & & & \| & & \uparrow & & \uparrow f & & \\ \pi(\varepsilon, f) & : & 0 & \to & FB & \to & Z' & \to & FU & \to & 0. \end{array}$$

By Auslander's criterion [AR] two Λ-modules Z_1 and Z_2 are isomorphic if and only if

$$\dim_{\mathbb{C}} \mathrm{Hom}_\Lambda(FU, Z_1) = \dim_{\mathbb{C}} \mathrm{Hom}_\Lambda(FU, Z_2)$$

for all indecomposables FU over Λ. Thus two exact sequences

$$\begin{array}{lcccccccccc} \varepsilon_1 & : & 0 & \to & FB & \to & Z_1 & \to & FA & \to & 0 \\ \varepsilon_2 & : & 0 & \to & FB & \to & Z_2 & \to & FA & \to & 0 \end{array}$$

have isomorphic middle terms if and only if

$$\dim_{\mathbb{C}} \ker \pi(\varepsilon_1, ?) = \dim_{\mathbb{C}} \ker \pi(\varepsilon_2, ?)$$

for the two maps $\pi(\varepsilon_1, ?)$ and $\pi(\varepsilon_2, ?)$ from $\mathrm{Hom}_\Lambda(FU, FA)$ to $\mathrm{Ext}^1_\Lambda(FU, FB)$ and all indecomposables FU.

Let $\lambda : G \to \mathbb{Z}$ be a map and consider the $\mathbb{C}^*$-action on

$$\bigoplus_{l \in G} \mathrm{Hom}_{\tilde{\Lambda}}(l(U), A)$$

given by

$$t \cdot (f_l)_{l \in G} = (t^{-\lambda(l)} f_l)_{l \in G},$$

for B, U in $\tilde{\mathscr{I}}$. The $\mathbb{C}$-isomorphism

$$\bigoplus_{l \in G} \mathrm{Hom}_{\tilde{\Lambda}}(l(U), A) \xrightarrow{\sim} \mathrm{Hom}_\Lambda(FU, FA)$$

allows us to transfer this action of $\mathbb{C}^*$ to $\mathrm{Hom}_\Lambda(FU, FA)$.

LEMMA. *Let $\lambda : G \to \mathbb{Z}$ be a group homomorphism. Then the map*

$$\pi : \mathrm{Ext}^1_\Lambda (FA, FB) \times \mathrm{Hom}_\Lambda (FU, FA) \to \mathrm{Ext}^1_\Lambda (FU, FB)$$

is $\mathbb{C}^*$-equivariant, where on the left $\mathbb{C}^*$ acts diagonally.

Proof. It suffices to check that, for g, l in G, the pull-back map

$$\pi : \mathrm{Ext}^1_{\tilde{\Lambda}} (A, g(B)) \times \mathrm{Hom}_{\tilde{\Lambda}} (l(U), A) \to \mathrm{Ext}^1_{\tilde{\Lambda}} (l(U), g(B))$$

has the property

$$\pi(t^{\lambda(g)}\varepsilon_g, t^{-\lambda(l)}f_l) = t^{\lambda(l^{-1}g)}\pi(\varepsilon_g, f_l),$$

for $t \in \mathbb{C}^*$. This is clear, as $\lambda(l^{-1}g) = \lambda(g) - \lambda(l)$.

COROLLARY. *If $\lambda : G \to \mathbb{Z}$ is a group homomorphism, the $\mathbb{C}^*$-action on $\mathrm{Ext}^1_\Lambda (FA, FB)$ associated with λ stabilizes $\mathrm{Ext}^1_\Lambda (FA, FB)_{FX}$, for all A, B, X in $\tilde{\mathscr{I}}$.*

Thus our first condition is satisfied. But it is clear that a group homomorphism will rarely satisfy the second one.

5.3. PROPOSITION. *Let $\lambda : G \to \mathbb{Z}$ be a group homomorphism. Then $L(\Lambda)$ is isomorphic to $L(\tilde{\Lambda}/\ker \lambda)/(G/\ker \lambda)$.*

COROLLARY. *If the fundamental group G is $\mathbb{Z}$, $L(\Lambda)$ is isomorphic to $L(\tilde{\Lambda})/G$.*

Proof of the proposition. We have to show that, for any A, B, X in $\tilde{\mathscr{I}}$,

$$\chi(\mathrm{Ext}^1_\Lambda (FA, FB)_{FX}/\mathbb{C}^*) = \sum_{\bar{g},\bar{h} \in G/\ker \lambda} \chi(\mathrm{Ext}^1_{\tilde{\Lambda}/\ker \lambda} (F'A, \bar{g}(F'B))_{\bar{h}(F'X)}/\mathbb{C}^*)$$

where $F' : \mathrm{ind}\, \tilde{\Lambda} \to \mathrm{ind}\, \tilde{\Lambda}/\ker \lambda$ is the orbit functor. This follows easily from the formula for fixed points in 5.1, as the inverse image of $\mathrm{Ext}^1_\Lambda (FA, FB)_{FX}$ in $\mathrm{Ext}^1_{\tilde{\Lambda}/\ker \lambda} (F'A, \bar{g}(F'B))$ is the disjoint union

$$\dot{\bigcup_{\bar{h} \in G/\ker \lambda}} \mathrm{Ext}^1_{\tilde{\Lambda}/\ker \lambda} (F'A, \bar{g}(F'B))_{\bar{h}(F'B)}.$$

6. The proof

6.1. The last ingredient for our proof is the following:

PROPOSITION. *$L(\Lambda)$ is isomorphic to $L(\tilde{\Lambda}/G')/(G/G')$, where G' is the commutator subgroup of G.*

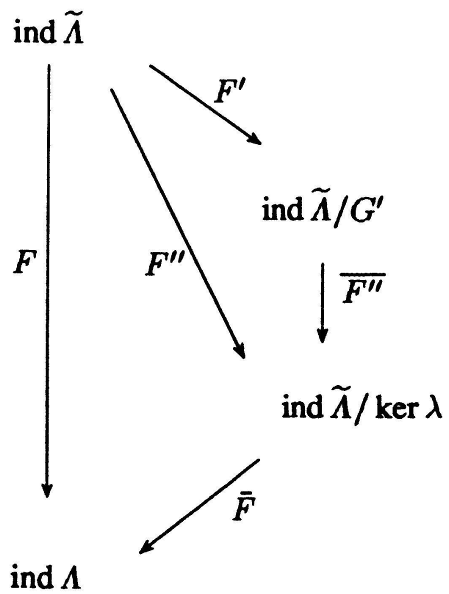
Proof. As G is free [BG], the quotient G/G' is free abelian. Let $\rho : G \to G/G'$ be the projection.

Fix A and B in $\tilde{\mathscr{J}}$, and let S be the finite subset

$$S = \{g \in G : \mathrm{Ext}^1_{\tilde{\Lambda}}(A, g(B)) \neq 0\}$$

of G. As G/G' is free abelian, there exists a group homomorphism $\bar{\lambda} : G/G' \to \mathbb{Z}$ whose restriction to $\rho(S)$ is injective. Choose for $\lambda : G \to \mathbb{Z}$ the composition $\bar{\lambda} \circ \rho$.

The following picture explains the notations we choose for orbit functors related to the groups $G' \subseteq \ker \lambda \subseteq G$:



We denote the residue class of an element g in G modulo G' by $\bar{g}$ and modulo $\ker \lambda$ by $\bar{\bar{g}}$.

We know from 5.3 that, for any X in $\tilde{\mathscr{J}}$,

$$\chi(\mathrm{Ext}^1_{\Lambda}(FA, FB)_{FX}/\mathbb{C}^*) = \sum_{\bar{\bar{g}}, \bar{\bar{h}} \in G/\ker\lambda} \chi(\mathrm{Ext}^1_{\tilde{\Lambda}/\ker\lambda}(F''A, \bar{\bar{g}}(F''B))_{\bar{\bar{h}}(F''X)}/\mathbb{C}^*).$$

Applying 3.4 to $\tilde{\Lambda}/\ker\lambda$, $G' \subseteq \ker\lambda$ and the elements $A, g(B) \in \tilde{\mathscr{I}}$, we find that $\overline{F''}$ induces an isomorphism

$$\bigoplus_{\bar{l} \in \ker\lambda/G'} \operatorname{Ext}^1_{\tilde{\Lambda}/G'}(F'A, \bar{l}(F'(gB))) \to \operatorname{Ext}^1_{\tilde{\Lambda}/\ker\lambda}(F''A, F''(gB)).$$

We claim that, by our choice of λ, there is a unique $\bar{l} \in \ker\lambda/G'$ for which

$$\operatorname{Ext}^1_{\tilde{\Lambda}/G'}(F'A, \bar{l}(F'(gB))) \neq 0$$

provided that

$$\operatorname{Ext}^1_{\tilde{\Lambda}/\ker\lambda}(F''A, F''(gB)) \neq 0.$$

Indeed, for $l \in \ker\lambda$, the space

$$\operatorname{Ext}^1_{\tilde{\Lambda}/G'}(F'A, F'(lgB))$$

is isomorphic to

$$\bigoplus_{g' \in G'} \operatorname{Ext}^1_{\tilde{\Lambda}}(A, g'lgB).$$

If now for $l_1, l_2 \in \ker\lambda$ there exists $g'_1, g'_2 \in G'$ such that

$$\operatorname{Ext}^1_{\tilde{\Lambda}}(A, g'_i l_i gB) \neq 0, \qquad i = 1, 2,$$

the elements $g'_i l_i g$ both belong to S, and their residue class modulo $\ker\lambda$ is $\bar{g}$. Thus their classes modulo G' coincide, and therefore $\bar{l}_1 = \bar{l}_2$.

Suppose now that

$$\operatorname{Ext}^1_{\tilde{\Lambda}/\ker\lambda}(F''A, F''(gB)) \neq 0,$$

and fix l in $\ker\lambda$ with

$$\operatorname{Ext}^1_{\tilde{\Lambda}/G'}(F'A, F'(lgB)) \neq 0.$$

Then $\overline{F''}$ induces an isomorphism

$$\operatorname{Ext}^1_{\tilde{\Lambda}/G'}(F'A, F'(lgB)) \to \operatorname{Ext}^1_{\tilde{\Lambda}/\ker\lambda}(F''A, F''(gB)).$$

The inverse image of

$$\operatorname{Ext}^1_{\tilde{\Lambda}/\ker\lambda}(F''A, F''(gB))_{F''(hX)}$$

under this isomorphism is the disjoint union

$$\dot{\bigcup_{\bar{k}\in\ker\lambda/G'}} \operatorname{Ext}^1_{\tilde{\Lambda}/G'}(F'A, F'(lgB))_{\bar{k}F'(hX)}.$$

Summing up we find

$$\chi(\operatorname{Ext}^1_{\Lambda}(FA, FB)_{FX}/\mathbb{C}^*) = \sum_{\bar{g},\bar{h}\in G/G'} \chi(\operatorname{Ext}^1_{\tilde{\Lambda}/G'}(F'A, \bar{g}(F'B))_{\bar{h}(F'X)}/\mathbb{C}^*).$$

6.2. The higher commutator subgroups $G^{(i)}$ of G, $i \in \mathbb{N}$, are defined inductively by

$$G^{(i+1)} = [G^{(i)}, G^{(i)}].$$

They are normal subgroups of G. As a consequence of Magnus' theorem on the lower central series, they intersect in the neutral element of G, since G is free.

COROLLARY. *For any $i \in \mathbb{N}$, $L(\Lambda)$ is isomorphic to $L(\tilde{\Lambda}/G^{(i)})/(G/G^{(i)})$.*

Proof. Indeed, proposition 6.1 applied to $\tilde{\Lambda}/G^{(i)}$ tells us that

$$L(\tilde{\Lambda}/G^{(i)}) \tilde{\rightarrow} L(\tilde{\Lambda}/G^{(i+1)})/(G^{(i)}/G^{(i+1)})$$

for all i. We conclude by induction applying 2.3.

6.3. PROPOSITION. *If Λ has finitely many objects there exists a natural number such that $L(\tilde{\Lambda}/G^{(t)})$ is isomorphic to $L(\tilde{\Lambda})/G^{(t)}$.*

Proof. In view of proposition 4.3 (applied to $\tilde{\Lambda}/G^{(t)}$) we only need to find $t \in \mathbb{N}$ such that

$$\{g \in G^{(t)} : \operatorname{Ext}^1_{\tilde{\Lambda}}(A, g(B)) \neq 0\}$$

has at most one element for all $A, B \in \tilde{\mathscr{J}}$. For $A, B \in \tilde{\mathscr{J}}$ we set

$$S(A, B) = \{g \in G : \operatorname{Ext}^1_{\tilde{\Lambda}}(A, g(B)) \neq 0\}$$

and

$$T(A, B) = \{g_1 g_2^{-1} : g_1, g_2 \in S(A, B)\}.$$

Clearly we have

$$S(h(A), l(B)) = hS(A, B)l^{-1}$$

and

$$T(h(A), l(B)) = hT(A, B)h^{-1}.$$

Since Λ has finitely many objects, $\tilde{\mathscr{I}}$ contains only finitely many G-orbits, and therefore the set

$$T = \bigcup_{A,B \in \tilde{\mathscr{I}}} T(A, B)$$

is a finite union of conjugacy classes in G.

Use now that the intersection $\bigcap_{i \in \mathbb{N}} G^{(i)}$ is reduced to $\{1\}$. Fix an integer t with

$$T \cap G^{(t)} = \{1\}.$$

Then the set

$$S(A, B) \cap G^{(t)} = \{g \in G^{(t)} : \operatorname{Ext}^1_{\tilde{\Lambda}}(A, g(B)) \neq 0\}$$

contains at most one element for all $A, B \in \tilde{\mathscr{I}}$, and our proof is complete.

6.4. In case Λ is finite the preceding proposition proves our theorem. Indeed, we have a chain of isomorphisms

$$L(\Lambda) \tilde{\to} L(\tilde{\Lambda}/G^{(t)})/(G/G^{(t)}) \tilde{\to} (L(\tilde{\Lambda})/G^{(t)})/(G/G^{(t)}) \tilde{\to} L(\tilde{\Lambda})/G.$$

In general, there is no reason why proposition 6.3 should hold. But "t exists locally", and this suffices to prove our theorem: for $A, B \in \tilde{\mathscr{I}}$ there exists $t = t(A, B)$ such that

$$T(A, B) \cap G^{(t)} = \{1\},$$

as $T(A, B)$ is finite.

Again this implies that

$$\{g \in G^{(t)} : \operatorname{Ext}^1_{\tilde{\Lambda}}(A, g(B)) \neq 0\}$$

contains at most one element. By the second remark in 4.2 the Lie bracket of $v_{G^{(t)}\cdot A}$ with $v_{G^{(t)}\cdot B}$ is "the same" in $L(\tilde{\Lambda}/G^{(t)})$ as in $L(\tilde{\Lambda})/G^{(t)}$. We finish the proof as in case Λ is finite, comparing the brackets of $v_{G\cdot A}$ and $v_{G\cdot B}$ in $L(\Lambda)$, $L(\tilde{\Lambda}/G^{(t)})/(G/G^{(t)})$, $(L(\tilde{\Lambda})/G^{(t)})/(G/G^{(t)})$ and $L(\tilde{\Lambda})/G$.

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Mathematisches Institut
Sidlerstr. 5
CH-3012 BERN
Switzerland

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