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On minimal annuli in a slab

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One of three beautiful theorems about a minimal annulus in a slab proved by Max Shiffman, [11], says that,

THEOREM 1. *If A is a minimal annulus in a slab $S \subset \mathbb{R}^3$, and $\Gamma = \partial A$ is a pair of circles lying in the boundary planes of the slab, separated by the interior of A, then for every plane P which is contained in the slab S, $A \cap P$ is a circle. In particular, A is embedded.*

In [4], it is proved that the same conclusion is true if we replace the boundary circles in Theorem 1 by parallel straight lines and assume A is properly embedded. Furthermore, Toubiana has proved that if two straight lines lying in distinct parallel planes, but the two straight lines are not parallel to each other, then they cannot bound any proper minimal annulus in the slab bounded by the two parallel planes.

In this article we will give generalizations of the results stated above, with a unified proof.

We will denote $P_t = \{(x, y, z): z = t\}$, and assume that the boundary planes of the slab are parallel to P_t; and we can further normalize the slab S such that $S = S(-1, 1) = \{(x, y, z): -1 \leq t \leq 1\}$. We will prove

THEOREM 2. *Suppose $A \subset S(-1, 1)$ is a proper minimal annulus in a slab and $A(1) = A \cap P_1$, $A(-1) = A \cap P_{-1}$ are straight lines or circles.*

(1) *If both $A(1)$ and $A(-1)$ are circles, then $A(t) = A \cap P_t$ is a circle, $-1 < t < 1$. In particular, A is embedded.*

(2) *If $A(1)$ or $A(-1)$ is a straight line and the other one is a circle, and A is embedded, then $A(t) = A \cap P_t$ is a circle, $-1 < t < 1$.*

(3) *If $A(1)$ or $A(-1)$ are both straight lines, A is embedded, then $A(t) = A \cap P_t$ is a circle, $-1 < t < 1$.*

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REMARK 1. The first part of Theorem 2 is exactly the Theorem 1, we will give it a simpler, straightforward proof, essentially in the same way as the proof of the other cases. We will see that the third part of Theorem 2 implies the result in [4]. The second part of Theorem 2 is new.

Let $A \subset S(-1, 1)$ be a proper minimal annulus such that $A(1) = A \cap P_1$ and $A(-1) = A \cap P_{-1}$ are straight lines or circles, $\partial A = A(1) \cup A(-1)$. In the case that there is only one straight line, we will always assume that $A(1)$ is a straight line. Then the conformal structure of the interior of A is equivalent to the interior of

$$A_R = \{z \in \mathbb{C}: 1/R \le |z| \le R\},$$

for some $1 < R < \infty$. In fact the interior of A is conformally equivalent to

$$\{z \in \mathbb{C}: \rho < |z| < P, 0 \le \rho < P \le \infty\},$$

for some ρ and P. Since A has 1-dimensional boundary ∂A which is separated by the interior of A, so $0 < \rho$ and $P < \infty$. Hence let $R = \sqrt{P/\rho} > 1$, then $\mathrm{Int}(A) \cong \mathrm{Int}(A_R)$.

There is a conformal harmonic immersion

$$X: A_R - C \hookrightarrow S(-1, 1),$$

where C is a subset of ∂A_R and $X(\{|z| = R\} - C) = A(1)$, $X(\{|z| = 1/R\} - C) = A(-1)$. If $A(1)$ and $A(-1)$ are both circles, then $C = \varnothing$; if only $A(1)$ is a straight line, then $C \subset \{|z| = R\}$; if $A(1)$ and $A(-1)$ are both straight lines, then $C \cap \{|z| = R\} \neq \varnothing$, $C \cap \{|z| = 1/R\} \neq \varnothing$. When $C \neq \varnothing$ we assume that X is an embedding. The Enneper–Weierstrass representation of A is

$$X(z) = \mathrm{Re} \int_1^z (\omega_1, \omega_2, \omega_3) + V,$$

where $V = (a, b, 0) \in \mathbb{R}^3$, and

$$\begin{aligned} \omega_1 &= \frac{1}{2}(1 - g^2(z))f(z)\,dz, \\ \omega_2 &= \frac{i}{2}(1 + g^2(z))f(z)\,dz, \\ \omega_3 &= g(z)f(z)\,dz, \end{aligned} \tag{1}$$

where g is the Gauss map and f is a holomorphic function. Since X is proper, the third coordinate function X^3, which is harmonic, $X^3 \mid (\{|z| = 1/R\} - C) = -1$ and $X^3 \mid (\{|z| = R\} - C) = 1$, and $-1 < X^3 \mid \mathrm{Int}(A_R) < 1$, can be extended to whole A_R such that $X^3 \mid \{|z| = 1/R\} = -1$ and $X^3 \mid \{|z| = R\} = 1$. By the uniqueness of the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathrm{Int}(A_R) \\ u \mid \{|z| = 1/R\} = -1, & u \mid \{|z| = R\} = 1, \end{cases}$$

where $\mathrm{Int}(A_R)$ is the interior of A_R, we have

$$X^3 = \frac{1}{\log R} \log |z|,$$

and

$$\omega_3 = f(z)g(z)\,dz = \frac{d}{dz}\left(\frac{1}{\log R}\log z\right)dz = \frac{1}{\log R}\frac{1}{z}\,dz.$$

Hence

$$f(z) = \frac{1}{\log R}\frac{1}{zg(z)}.$$

Thus by (1) we have

$$\omega_1 = \frac{1}{\log R}\frac{1}{2z}\left(\frac{1}{g} - g\right)dz$$

$$\omega_2 = \frac{1}{\log R}\frac{i}{2z}\left(\frac{1}{g} + g\right)dz$$

$$\omega_3 = \frac{1}{\log R}\frac{1}{z}\,dz,$$

and X can be represented as

$$X(p) = \frac{1}{\log R}\,\mathrm{Re}\int_1^p \left(\frac{1}{2z}(1/g - g), \frac{i}{2z}(1/g + g), \frac{1}{z}\right)dz + V, \tag{2}$$

Let

$$g(z)=\sum_{n=-\infty}^{\infty} a_n z^n, \qquad \frac{1}{g(z)}=\sum_{n=-\infty}^{\infty} b_n z^n, \tag{3}$$

then (2) gives a minimal annulus if and only if

$$\operatorname{Im}(b_0)=\operatorname{Im}(a_0), \qquad \operatorname{Re}(b_0)=-\operatorname{Re}(a_0). \tag{4}$$

The conformal factor of A is

$$A^2=\frac{1}{4(\log R)^2|z|^2}\left(\frac{1}{|g|}+|g|\right)^2 \tag{5}$$

and the Gauss curvature is

$$K=-\left[\frac{4\log R|z||g||g'|}{(1+|g|^2)^2}\right]^2. \tag{6}$$

The Gauss map g of the minimal annulus A has the features as stated in the following lemma.

LEMMA 1. *Let $X: A(R)-C\to S(-1,1)$ be a properly immersed minimal annulus, $A=X(A(R)-C)$. Then the Gauss map g of A has neither zero nor pole in the interior of A_R.*

If $A(1)$ and $A(-1)$ are both straight lines, then $C=\{p,q\}$, $|p|=R$, $|q|=1/R$; if only $A(1)$ is a straight line then $C=\{p\}$; In both cases, g can be extended to a neighborhood of A_R as a meromorphic function.

Furthermore, the extended Gauss map $\tilde{g}: A_R\to\mathbb{C}\cup\{\infty\}$ takes zero or ∞ on C, and C is the only point set in A_R, such that at which $\tilde{g}$ takes zero or ∞.

Proof. If g achieves a zero or pole in the interior point z of A_R, then the tangent plane of A at $X(p)$ is P_t, $t=\log|z|/\log R$. It is well known that the preimage $X^{-1}(A(t))$ contains an equiangular system at z, of at least order 4. This contradicts that $X^{-1}(A(t))=\{|z|=r=R^t\}$, as seen easily from the Enneper–Weierstrass representation. This contradiction proves that g has neither zero nor pole in the interior of A_R. Since ∂A is analytic, by Hopf boundary maximum principle, we known that at each point $p\in\partial A$, the co-normal of A has non-zero third component, hence the normal vector of A at p is not vertical, hence for any $z\in X^{-1}(p)$, $g(z)\neq 0$ or ∞.

If $A(1)$ is a straight line, we can assume that $A(1)$ is parallel to the y-axis in $\mathbb{R}^3$, then the unit normal vector of A along $A(1)$ is in the xz-plane. Let $C_1 = C \cap \{|z| = R\}$, we see that g and g^{-1} is real on $\{|z| = R\} - C_1$. Hence using (3), g and g^{-1} can be extended to $\{R < |z| < R^3\}$ by

$$\tilde{g}(z) = \overline{g(R^2/\bar{z})} = \sum_{-\infty}^{\infty} \overline{a_n} R^{2n} z^{-n} = \sum_{-\infty}^{\infty} a'_n z^n,$$

$$\frac{1}{\tilde{g}(z)} = \frac{1}{\overline{g(R^2/\bar{z})}} = \sum_{-\infty}^{\infty} \overline{b_n} R^{2n} z^{-n} = \sum_{-\infty}^{\infty} b'_n z^n,$$

for $R < |z| < R^3$, where $a'_n = a_{-n} R^{-2n}$, $b'_n = \overline{b_{-n}} R^{-2n}$. By (4), we see that we can use $\tilde{g}$ to substitute g in the Enneper–Weierstrass representation (2). Then we get a minimal surface

$$\tilde{X} : \{1/R < |z| < R^3\} - C_1 \to S(-1, 3).$$

Since X is properly immersed, the surface $\mathcal{S} = \tilde{X}(\{1/R < |z| < R^3\} - C_1)$ is properly immersed and contains a complete minimal annular end. By Cone Lemma of [5], this annular end has finite total curvature. By a theorem of Osserman, ([10], Theorem 9.1 and Lemma 9.5, pages 81–2) this annular end has the conformal structure of a punctured disk, and the Gauss map of $\mathcal{S}$ can be extended to the puncture. In particular, $C_1 = \{p\}$ is a single point, g can be extended to p, and $g(p)$ is either zero or ∞ by the Enneper–Weierstrass representation, otherwise $X(z)$ will remain bounded near p.

Similar treatment applies to the case that $A(-1)$ is also a straight line. The proof of the lemma is complete. □

REMARK 2. Let R be the rotation of π angle around the straight line $A(1)$ in $\mathbb{R}^3$, by Schwartz Reflection Principle, $A \cup R(A)$ is a properly embedded minimal surface. By uniqueness it must be $\mathcal{S} = A \cup R(A)$, where $\mathcal{S}$ is the surface we got in Lemma 1. In particular,

$$\int_A K\, dA = \frac{1}{2} \int_{\mathcal{S}} K\, dA, \tag{7}$$

where K is the Gauss curvature, and dA is the area element of A.

We would like to calculate the plane curvature of each $A(t) = A \cap P_t$, $-1 \le t \le 1$. At any point of $A(t)$, draw a tangent vector to the curve $A(t)$, and let

ψ be the angle made by this tangent vector with the positive x-axis. ψ maybe is a multivalued function, but we will see that ψ is a harmonic function. To see this, consider the normal vector of the curve $A \cap P_t$, and its angle with the positive x-axis, ϕ. If we orient the surface such that the normal is inward to the unbounded component of $S(-1+\epsilon, 1-\epsilon) - A$ in $S(-1+\epsilon, 1-\epsilon)$, for every $\epsilon > 0$ small enough, then we have $\psi = \phi + \pi/2$. By Lemma 1, $g \neq 0$ or ∞ in the interior of A_R, hence the normal vector must be $g/|g| \in \mathbb{R}^2$, and so $\phi = \arg g = \operatorname{Im} \log g$, thus ϕ is harmonic, so is ψ. Now suppose that s is the arc length parameter of the curve $A(t)$, notice that $X^{-1}(A(t)) = \{z \colon |z| = r = R^t\}$, write $z = r\,e^{i\theta}$, we can calculate the curvature of $A(t)$ as follows:

$$\kappa = \psi_s = \phi_s = \frac{d}{ds}(\operatorname{Im} \log g) = \operatorname{Im}\left(\frac{d}{ds} \log g\right) = \operatorname{Im}\left(\frac{d}{dz} \log g \frac{dz}{ds}\right)$$

$$= \operatorname{Im}\left(\frac{g'}{g}\frac{dz}{d\theta}\frac{d\theta}{ds}\right) = \operatorname{Im}\left(\frac{g'}{g} izr^{-1}\Lambda^{-1}\right) = r^{-1}\Lambda^{-1} \operatorname{Re}\left(z\frac{g'}{g}\right).$$

Here we have used the facts that on the curve $\{|z| = r = R^t\}$,

$$\frac{dz}{d\theta} = ir\,e^{i\theta} = iz, \quad \text{and} \quad ds = \Lambda|dz| = \Lambda r\,d\theta.$$

Calculation shows that

$$r\kappa_\theta = \operatorname{Im}\left[\frac{1}{2}\frac{|g|^2-1}{|g|^2+1}\left(z\frac{g'}{g}\right)^2 - z\frac{g'}{g} - z^2\frac{g''g-(g')^2}{g^2}\right].$$

Let

$$u = r\Lambda\kappa_\theta = \operatorname{Im}\left[\frac{1}{2}\frac{|g|^2-1}{|g|^2+1}\left(z\frac{g'}{g}\right)^2 - z\frac{g'}{g} - z^2\frac{g''g-(g')^2}{g^2}\right], \tag{8}$$

then simple calculation shows that

$$\Delta u = -\frac{8|g'|^2}{(1+|g|^2)^2}\,u. \tag{9}$$

By (5) and (6), we have

$$\frac{8|g'|^2}{(1+|g|^2)^2} = -\frac{1}{2}K\frac{(1+|g|^2)^2}{(\log R)^2|z|^2|g|^2} = -2K\Lambda^2.$$

Hence u is a Jacobi field.

REMARK 3. The Jacobi field u is the same as in [11], page 79, formulas (8), (9), and (10), where it is denoted by β in Ψ, H plane.

Remember that

$$\Delta_\Lambda = \Lambda^{-2}\Delta,$$

where Δ_Λ is the Lapalacian under the metric $ds^2 = \Lambda^2|dz|^2$. If $\Gamma = A(1)\cup A(-1)$ consists of straight lines or circles, then $\kappa_\theta \equiv 0$ on $\partial A_R - C$, hence on $A_R - C$ u satisfies

$$\begin{aligned} &\Delta_\Lambda u - 2Ku = 0, \\ &u \mid (\partial A_R - C) = 0. \end{aligned} \tag{10}$$

By the definition of u, we see that to prove Theorem 2 is equivalent to prove that $u \equiv 0$.

Now it is easy to prove the first part of Theorem 2. In fact since $\kappa = \Lambda^{-1}r^{-1}\,\mathrm{Re}(zg'/g)$ and $\mathrm{Re}(zg'/g) > 0$ on ∂A_R, $\mathrm{Re}(zg'/g)$ is harmonic, hence κ is positive on all of A_R, thus each $A(t)$ is strictly local convex. The total curvature of each $A(t)$ must be 2π by continuity, hence each $A(t)$ is strictly convex. In particular, A is embedded. Now $u \mid \partial A_R \equiv 0$, u is an eigenfunction corresponding to the eigenvalue 0 to the operator $L_A = \Delta_A - 2K$. And since κ must have at least one maximum and one minimum on each $A(t)$, κ_θ and so u changes sign, if $u \not\equiv 0$. Since the eigenfunction corresponding to the first eigenvalue never changes sign, 0 is not the first eigenvalue of L_A, hence $\lambda_2(\mathrm{Int}(A_R)) \le 0$. It is well known that g is a conformal diffeomorphism from the interior of A_R to C and $\Omega = g(\mathrm{Int}(A_R))$ is a proper domain of $S^2 = \mathbb{C}\cup\{\infty\}$ under the assumption that each $A(t)$ is a strictly convex Jordan curve. (In fact it is a local diffeomorphism by $\mathrm{Re}(zg'/g) > 0$ hence $g' \neq 0$, and a geometric, [8], or an analytic, [3], argument shows that g is one-to-one.) The eigenvalue problem of L_A is equivalent to the eigenvalue problem of the operator $\Delta_{S^2} + 2$, where Δ_{S^2} is the sphere Lapalacian. Therefore, $\lambda_2(\mathrm{Int}(A_R)) + 2 = \lambda_2(\Omega)$, the latter is the eigenvalue for Δ_{S^2}. Since Ω is a proper subdomain, $\lambda_2(\Omega) > 2$, which contradicts to $\lambda_2(\mathrm{Int}(A_R)) \leq 0$. This contradiction proves the first part of Theorem 2.

To complete the proof of Theorem 2, we need two more lemmas.

LEMMA 2. *Suppose that $A \subset S(-1, 1)$ is a proper minimal annulus, $A(1) = A \cap P_1$ is a straight line, $A(-1) = A \cap P_{-1}$ is a circle or straight line, and $\partial A = A(1) \cup A(-1)$. Then*

$$\int_A K\, dA > -4\pi,$$

if $A(-1)$ is a circle; and

$$\int_A K\, dA = -4\pi,$$

if $A(-1)$ is also a straight line.

Proof. We will use the extended surface $\mathscr{S}$ in the proof of Lemma 1 to calculate the total curvature of A. Notice that $\mathscr{S}$ has an embedded flat annular end corresponding to the point p. Take $Y_\rho = \mathscr{S} \cap S_\rho$, where S_ρ is the sphere in $\mathbb{R}^3$ centered at origin of radius ρ. Denote the compact ball bounded by S_ρ by B_ρ and let $O_\rho = \mathscr{S} \cap (\mathbb{R}^3 - B_\rho)$. If $A(-1)$ is a circle, take ρ large enough, such that $\partial\mathscr{S} \subset B_\rho$, then using Remark 2 and the Gauss–Bonnet theorem.

$$\int_{\mathscr{S} - O_\rho} K\, dA = -2\pi - \int_{R(A(-1))} \kappa_g\, ds - \int_{A(-1)} \kappa_g\, ds - \int_{Y_\rho} \kappa_g\, ds.$$

By a theorem of Jorge and Meeks, [7], when $\rho \to \infty$, $X_\rho = Y_\rho/\rho$ approaches to a geodesic of the unit sphere S^2 in a C^∞ way, thus

$$\int_{Y_\rho} \kappa_g\, ds \to 2\pi, \qquad \rho \to \infty.$$

Notice that the other two integrals are larger than -2π because $A(-1)$ and $R(A(-1))$ are circles. We have

$$\int_{\mathscr{S}} K\, dA > -8\pi.$$

By (7), we conclude that the total curvature of A is larger than -4π.

For the proof of the case that $A(-1)$ is also a straight line, we quote a lemma in [4] listed below, which implies that our surface has total curvature -4π, notice that A has genus 0. □

We only state the part of that lemma in [4] that is useful to us.

LEMMA 3. *Suppose A is a properly embedded minimal surface that is bounded by a pair of lines $L = L_0 \cup L_1$ and lies in a slab between parallel planes, $P = P_0 \cup P_1$ with $L_i \subset P_i$. Then A extends by Schwartz Reflection to a singly-periodic embedded minimal surface $\mathcal{S}$, invariant under a screw motion T, where T is $R \circ R_0$, R_i being rotation by π about L_i. If A has genus k, $\mathcal{S}/T$ has genus* $(2k+1)$, *two flat ends and total curvature* $-4\pi(2k+2)$.

The strategy of the proof of Theorem 2 is then as follows. The zero set of u divides A_R into subdomains, called nodal domains. If u can be continuously extended to A_R such that $u \mid \partial A_R = 0$, then u is an eigenfunction corresponding to the eigenvalue 0 for the operator L_A on these nodal domains. By a theorem of Barbosa and do Carmo, [1], the total curvature of each such nodal domain is less than or equal to -2π, hence the total curvature of A must be less than $-2k\pi$, if we have k nodal domains. If $k \geq 2$, we get a contradiction to Lemma 2. Thus we must have $u \equiv 0$, which is equivalent to each $A(t)$ is a circle, for $-1 < t < 1$.

By the four-vertex-theorem, see [6], which says that the zero set of κ_θ divides each $A(t)$ into at least four components, $-1 < t < 1$, if $\kappa_\theta \not\equiv 0$. Hence there are at least four nodal domains.

Lemma 4 below will prove that u is actually continuous on A_R, with $u \mid \partial A_R = 0$. Thus we will get the anticipated contradiction.

LEMMA 4. *Let A be as in Theorem* 2, *p, q be as in Lemma* 1, *and u be as defined in* (8). *Then u is continuous on A_R and $u \mid \partial A_R = 0$.*

Proof. Without loss of generality, we can assume that $p = R$. By Lemma 1, we can assume that the Gauss map g has limit zero at $p = R$, then g can be extended to the disk D_ρ centered at $z = R$, as a holomorphic function $\tilde{g}$, and let $\zeta = z - R$,

$$\tilde{g}(z) = (z - R)^n \tilde{h}(z) = \zeta^n h(\zeta),$$

where h is a holomorphic function and $h(0) \neq 0$. Since R corresponds to an embedded flat end, and that end intersects P_1 at a straight line, we have $n = 2$. Please see [12] for the argument or just see it from Lemma 3, since $\mathcal{S}$ has total curvature -8π, so the Guass map should be a degree 2 map.

For convenience, we will write g instead of $\tilde{g}$. Then

$$z\frac{g'(z)}{g(z)} = \frac{2R}{z-R} + 2 + z\frac{h'(z)}{h(z)},$$

and

$$z\frac{g'(z)}{g(z)} = \frac{a_{-1}}{\zeta} + \sum_{k=0}^{\infty} a_k \zeta^k,$$

$a_1 = 2R$. Then

$$\left(z\frac{g'(z)}{g(z)}\right)^2 = \frac{a_{-1}^2}{\zeta^2} + \frac{2a_{-1}a_0}{\zeta} + \sum_{k=0}^{\infty} b_k \zeta^k.$$

$$z\frac{d}{dz}\left(z\frac{g'(z)}{g(z)}\right) = -\frac{a_{-1}R}{\zeta^2} - \frac{a_{-1}}{\zeta} + (\zeta + R)\sum_{k=1}^{\infty} k a_k \zeta^{k-1}.$$

Since $a_{-1} = 2R$,

$$\frac{1}{2}a_{-1}^2 - a_{-1}R = 0.$$

Since

$$z\frac{h'(z)}{h(z)} = (\zeta + R)\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{h'(z)}{h(z)}\right)^{(k)}_{z=R}\zeta^k = R\frac{h'(R)}{h(R)} + \sum_{k=1}^{\infty} d_k \zeta^k,$$

$$a_0 = 2 + R\frac{h'(R)}{h(R)}.$$

We would like to calculate a_0. The Weierstrass representation for the extended surface $\mathcal{S}$ is

$$\begin{cases} \omega_1 = \dfrac{1}{\log R}\dfrac{1}{2z}\left(\dfrac{1}{\tilde{g}} - \tilde{g}\right)dz \\ \omega_2 = \dfrac{1}{\log R}\dfrac{i}{2z}\left(\dfrac{1}{\tilde{g}} + g\right)dz \\ \omega_3 = \dfrac{1}{\log R}\dfrac{1}{z}\,dz, \end{cases}$$

as commented in the proof of Lemma 1. Again we will write g instead $\tilde{g}$. Let C be a loop around $z = R$ in a small disk. Then since $X : \{z: 1/R < |z| < R^3\} - \{R\} \to \mathbb{R}^3$ is well defined and

$$X(z) = \operatorname{Re} \int_{p_0}^{z} (\omega_1, \omega_2, \omega_3),$$

we must have

$$\operatorname{Re} \int_C \frac{1}{2z} \left(\frac{1}{g(z)} - g(z) \right) dz = 0,$$

$$-\operatorname{Im} \int_C \frac{1}{2z} \left(\frac{1}{g(z)} + g(z) \right) dz = 0,$$

$$\int_C \frac{1}{zg}\, dz = \overline{\int_C \frac{d}{z}\, dz} = 0,$$

since $g(z)/z$ is holomorphic at $z = R$. Hence we know that the residue of $1/zg(z)$ at $z = R$ is zero. Then we have

$$0 = \lim_{z \to R} \left(\frac{(z-R)^2}{zg(z)} \right)' = \lim_{z \to R} \left(\frac{1}{zh(z)} \right)'$$

$$= \lim_{z \to R} \left(-\frac{1}{z^2 h(z)} - \frac{h'(z)}{zh^2(z)} \right)$$

$$= -\frac{1}{R^2 h(R)} - \frac{h'(R)}{Rh^2(R)}.$$

Hence

$$\frac{H'(R)}{h(R)} = -\frac{1}{R},$$

and

$$a_0 = 2 + R\frac{h'(R)}{h(R)} = 1.$$

Then $a_{-1}a_0 - a_{-1} = 0$ and

$$
\begin{aligned}
&-\frac{1}{2}\left(z\frac{g'(z)}{g(z)}\right)^2 - z\frac{d}{dz}\left(z\frac{g'(z)}{g(z)}\right) \\
&= -\frac{1}{2}\frac{a_{-1}^2 - 2a_{-1}R}{\zeta^2} - \frac{a_{-1}a_0 - a_{-1}}{\zeta} - \frac{1}{2}\sum_{k=0}^{\infty} b_k\zeta^k - (\zeta + R)\sum_{k=1}^{\infty} ka_k\zeta^{k-1} \\
&= \sum_{k=0}^{\infty} c_k\zeta^k,
\end{aligned}
$$

hence

$$
\Phi(z) = -\frac{1}{2}\left(z\frac{g'(z)}{g(z)}\right)^2 - z\frac{d}{dz}\left(z\frac{g'(z)}{g(z)}\right)
$$

is holomorphic near $z = R$. Now consider the function

$$
U(z) = \frac{|g|^2 - 1}{2(1 + |g|^2)}\left(z\frac{g'(z)}{g(z)}\right) - z\frac{d}{dz}\left(z\frac{g'(z)}{g(z)}\right),
$$

note that our function u is Im U. $U(z)$ can be rewritten as

$$
\begin{aligned}
U(z) &= -\frac{1}{2}\left(z\frac{g'(z)}{g(z)}\right)^2 - z\frac{d}{dz}\left(z\frac{g'(z)}{g(z)}\right) + \left(1 - \frac{1}{1 + |g|^2}\right)\left(z\frac{g'(z)}{g(z)}\right)^2 \\
&= \Phi(z) + \Psi(z).
\end{aligned}
$$

Note that

$$
\zeta^2\left(z\frac{g'(z)}{g(z)}\right)^2
$$

is holomorphic and since $|g|^2 = |z - R|^4|h(z)|^2 = |\zeta|^4|h(z)|^2$,

$$
\frac{1}{\zeta^2}\left(1 - \frac{1}{1 + |g|^2}\right) = \frac{1}{\zeta^2}\sum_{k=1}^{\infty}(-1)^{k+1}|g|^{2k} = \overline{\zeta^2}\sum_{k=1}^{\infty}(-1)^{k+1}|\zeta|^{4(k-1)}|h(z)|^{2k}
$$

is a C^∞ complex function in a neighborhood of R. Thus

$$
\Psi(z) = \frac{1}{\zeta^2}\left(1 - \frac{1}{1 + |g|^2}\right)\zeta^2\left(z\frac{g'(z)}{g(z)}\right)^2
$$

is a C^∞ complex function near $z = R$, so is $U(z)$. In particular, $u(z) = \operatorname{Im} U(z)$ is C^∞ near $z = R$. Hence $u(R) = 0$, since on $|z| = R$, if $z \neq R$, then $u(z) = 0$ by (10). Hence u can be continuously extended to $p = R$ such that $u(p) = 0$. The proof for q is exactly the same, if we note that either g or g^{-1} can be regarded as the Gauss map, therefore we can always assume g has a double zero at q. □

The proof of Theorem 2 is complete.

From the proof we see immediately that

COROLLARY 1. *Suppose that $A \subset S(-1, 1)$ is a proper minimal annulus. If $A(1) = A \cap P_1$ is a straight line and A is embedded in a neighborhood of $A(1)$, $A(-1) = A \cap P_{-1}$ is a circle, and the total curvature of A is larger than or equal to -8π. Then each $A(t) = A \cap P_t$, $-1 < t < 1$, is a circle. In particular, A is embedded.*

Proof. We only need to point out that we can still use the four-vertex theorem, even some level sets $A(t)$ may not be Jordan curves. But it is shown in [6], that all curves which have exactly two vertices are curves which have exactly two simple loops, on each loop the curvature is positive or negative, hence its total curvature must be 0. □

The minimal annuli which can be foliated by circles or straight lines, but not all straight lines, were classified by Enneper and B. Riemann in the 19th century, given by elliptic integrals. See [9], pages 85–90 for detail, or see [4] or [12] for a representation using elliptic functions. The facts are, all of those circles are in parallel planes; if any two circles are coaxial, the surface is a piece of a catenoid, the only rotational minimal surface; if the circles are not coaxial, then the surface has two parallel straight lines lying in parallel planes, as boundary. Based on these facts, we have proved that

COROLLARY 2. *Let $L_1 \subset P_1$, $L_{-1} \subset P_{-1}$ be two parallel straight lines. If $\Gamma = L_1 \cup L_{-1}$ is the boundary of a properly embedded minimal annulus A in $S(-1, 1)$, then A is one of the Riemann's examples.*

Finally, we have a non-existence theorem.

COROLLARY 3. *Let $L_1 \subset P_1$, $L_{-1} \subset P_{-1}$ be two straight lines, and they are not parallel to each other. Then $\Gamma = L_1 \cup L_{-1}$ cannot bound an properly embedded minimal annulus in $S(-1, 1)$.*

Corollary 2 is the main theorem of [4], in which it is proved via the elliptic function theory. Corollary 3 is a result of Toubiana [12].

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