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Superrigidity for the commensurability group of tree lattices

A. LUBOTZKY, S. MOZES AND R. J. ZIMMER

Introduction

Let $T = T_k$ be a k-regular tree ($k \geq 3$), $A = \mathrm{Aut}(T)$ the group of automorphisms of T. A is a locally compact group which has a number of properties in common with non-compact simple Lie groups G: e.g., A is essentially a simple group ([Ti]), has the Howe–Moore property ([LM]), and its representation theory is quite similar to that of rank one Lie groups (cf. [FTN]). In [BL1] rigidity properties of A were established which show that to a large extent T is determined by A. For example if A acts non-trivially on some tree T' then T' is nothing more than a decorated version of a barycentric subdivision of T. It is also shown there that every automorphism of A comes from an automorphism of T and hence it is inner. (This last result was proved earlier in [Zn].) Thus, the results of [BL1] can be considered as a tree analogue of the work of E. Cartan – relating simple Lie groups to their associated symmetric spaces.

In analogy to the well-developed theory of lattice subgroups of Lie groups, there is an ongoing program of investigating lattices in A. If Γ is such a lattice in A, we denote the commensurizer of Γ (or "the commensurability group of Γ") by $C = C(\Gamma) = \{g \in A \mid g\Gamma g^{-1} \cap \Gamma$ is of finite index in Γ and in $g\Gamma g^{-1}\}$. If we think of A as analogous to a simple real algebraic group $G(\mathbb{R})$ and Γ as an analogue of the arithmetic group $G(\mathbb{Z})$, then $C(\Gamma)$ plays the role of $G(\mathbb{Q})$, the $\mathbb{Q}$-rational points of G. The goal of this paper is to establish for C some superrigidity results of the same flavor as the work of Borel–Tits ([BT]) on $G(\mathbb{Q})$. (The methods, however, are entirely different).

In [BK] Bass and Kulkarni studied cocompact (=uniform) lattices in A. Among the many interesting results they proved are:

(a) Any two uniform lattices Γ_1 and Γ_2 of A are commensurable after conjugation; i.e., there exists $g \in A$ such that $g^{-1}\Gamma_1 g \cap \Gamma_2$ is of finite index in Γ_2 and in $g^{-1}\Gamma_1 g$. (This was proved earlier in a different language in [Le].) This shows that if Γ is a uniform lattice in A then the group $C = C(\Gamma)$ is independent of Γ up to conjugacy, and we denote it by $C = C_T$.

(b) C_T is dense in A. (This was later extended to arbitrary locally finite uniform trees in [Li].)

Here is the main result of this paper.

THEOREM 1. *Let $T = T_k$ be the k-regular tree, $A = \mathrm{Aut}(T)$, C the commensurability group of a uniform lattice Γ of A. Let $\rho : C \to \mathrm{Aut}(T')$ be a minimal action of C on a tree T'. Then either ρ can be extended to an action of A on T' or $l_{T'}(\rho(\gamma)) = 0$ for every $\gamma \in \Gamma$ where $l_{T'}$ is the length function of T' (i.e., for every $\gamma \in \Gamma$, $\rho(\gamma)$ has a fixed vertex or a fixed edge) in which case $\overline{\rho(\Gamma)}$ is compact.*

In Chapter 3 we construct for every k a uniform lattice Γ in $A = \mathrm{Aut}(T_k)$ which has no faithful embedding into a compact subgroup of A. Thus if ρ is an automorphism of C, Theorem 1 can be used to give:

COROLLARY 2. *If $\rho : C_{T_k} \to C_{T_m}$ is an isomorphism, then $k = m$ and ρ is induced by an automorphism of T_k. Hence* $\mathrm{Aut}(C) = N_A(C)$, *the normalizer of C in A.*

This answers a question raised in [BL1].

We leave open the question whether $N_A(C) = C$. This is equivalent to a question raised in [BK]: Let Γ_1 and Γ_2 be two uniform lattices with $C(\Gamma_1) = C(\Gamma_2)$. (Here we mean really equal not just conjugate!) Does this imply that Γ_1 and Γ_2 are commensurable?

Note in Corollary 2 that both groups C_{T_k} and C_{T_m} are subgroups of the "abstract commensurability group" of the free group on two generators (see [BK]).

We give two proofs of Theorem 1. The first in Chapter 2 is elementary. We describe the elements of $C = C(\Gamma)$ using periodic maps from T to $S_k = \mathrm{Perm}\{1, \dots, k\}$. This interpretation seems to be of independent interest. To illustrate its usefulness we give along the way a proof of the Bass–Kulkarni result [BK] that C is dense in A. We then show that two elements of Γ which have the same hyperbolic length are actually conjugate in C. This puts severe restrictions on the possible length functions defined on $\Gamma \subseteq C$ and hence on actions of C on trees. This will imply Theorem 1.

In Chapter 4 we present a different proof which is along the lines of Margulis superrigidity, i.e., uses methods of ergodic theory. The reader may recall that while Margulis superrigidity was proved for lattices in higher rank simple Lie groups (and A is analogous to a "rank-one group"), Margulis indeed proved a type of superrigidity for commensurability groups of lattices in simple Lie groups of arbitrary positive real rank when the commensurability group is not discrete. He used it to prove the theorem that a lattice in such a group is arithmetic if and only if its commensurability group is non-discrete. While our proof of Theorem 1 has a similar structure some additional issues arise which need to be overcome.

This more complicated proof has however some advantages as it proves a stronger theorem. In particular it is valid for non-uniform lattices Γ as well. (See Chapter 4 for the precise formulation.) A warning is in order: for non-uniform lattices, $C(\Gamma)$ is not necessarily dense, as shown by an example in [BL2].

Two more results are proved in Chapters 5 and 6 respectively:

THEOREM 3. *C is not a linear group over any field.*

THEOREM 4. *If $\Gamma \leq D \leq C$ with $[D : \Gamma] = \infty$, then D cannot be the fundamental group of any complete Riemannian manifold of curvature $-b^2 \leq K \leq -a^2 < 0$.*

A result similar to Theorem 4 holds also in the classical context if $\Gamma = G(\mathbb{Z})$ and $C = G(\mathbb{Q})$.

Finally we remark that the methods of this paper extend without difficulty to the case T is a bi-regular tree. Following this work, some of the results of this paper were extended to more general trees and spaces by M. Burger and S. Mozes in "Cat (-1) spaces, divergence groups and their commensurators" (a preprint).

§1. Preliminaries, notations and conventions

In this chapter we will present some definitions and notation to be used later and collect some basic results. Let T_k be a k-regular tree for some fixed $3 \leq k \in \mathbb{N}$. Let $A = \mathrm{Aut}\,(T_k)$ with its natural topology. Let A^+ be the index two subgroup of A generated by all the vertex stabilizers. This is the group of all automorphisms of T_k which preserve the two-coloring of T_k. By Tits [Ti], A^+ is a simple group.

Let G be a group acting on a tree T. We define the length function $l_T : G \to \mathbb{Z}$ by $l_T(g) = \min_{x \in |T|} d(x, gx)$ where $|T|$ is the geometric realization of T, and $d(\cdot, \cdot)$ is the distance function on T. Notice that if $g \in G$ is an involution then $l_T(g) = 0$. l_T satisfies $l_T(g^n) = |n| l_T(g)$. If $l_T(g) \neq 0$ then the set $\{x \in |T| \mid d(x, gx) = l_T(g)\}$ forms an infinite line on which g acts by translation. This line is called the axis of g. Two elements $g_1, g_2 \in A$ having $l_{T_k}(g_1) = l_{T_k}(g_2) \neq 0$ are conjugate in A. Let ∂T denote the boundary of the tree, i.e., its space of ends. Thus, $e \in \partial T$ is an equivalence class of rays in T where two rays are equivalent if they are within bounded distance of one another. $T \cup \partial T$ has a natural topology making it a compact space (cf. [FTN]).

An action of G on T extends in a natural way to an action on the space $T \cup \partial T$. An action of G on ∂T is called *minimal* if ∂T does not contain any nontrivial closed G invariant proper subset. An action of G on T is called *minimal* if

(i) ∂T contains more than two points;
(ii) G acts minimally on ∂T and
(iii) T is the minimal G invariant subtree of T.

Given G acting on a tree T the intersection of all the closed connected nonempty G-invariant subsets of $T \cup \partial T$, which we denote by $\bar{X}_G$, is a nonempty set whose intersection with T, $X_G = \bar{X}_G \cap T$ is either empty, a line or the unique G-invariant subtree on which G acts minimally. Recall [B, 7.7] that if $G_1 < G$ is of finite index then $X_{G_1} = X_G$. Notice that when the boundary $\partial \bar{X}_G = \bar{X}_G \cap \partial T$ contains at least three points the action of G on X_G may be reconstructed from its action on $\partial \bar{X}_G$. In particular Aut (T_k) acts faithfully on ∂T_k.

The stabilizer $B < A$ of a point $e \in \partial T_k$ is an extension by $\mathbb{Z}$ of an ascending union of compact subgroups and hence is amenable.

We shall also use the observation that if a group G acting on a tree T such that $l_T(G) = \{0\}$ then G stabilizes a vertex, an edge or an end. Moreover if G is finitely generated then it must stabilize either a vertex or an edge.

In section 2 we will use colored graphs. All of these graphs are k-regular. By a coloring we will always mean edge coloring by k colors such that the k edges meeting at a vertex have different colors. We remark that our graphs may, and will, contain multiple (differently colored) edges.

§2. Commensurizers as periodic recolorings

Let Γ be the free product of k cyclic groups of order two $Z_i = \langle a_i \rangle$, $i = 1, \dots, k$. Let T be the (right) Cayley graph of Γ with respect to $\{a_1, \dots, a_k\}$, i.e., $\gamma \in \Gamma$ is adjacent to γa_i via an edge "colored" by a_i, $i = 1, \dots, k$. T is a colored k-regular tree. Let A be the full group of automorphisms of T not necessarily preserving the coloring. Γ is identified via its left action on T with the subgroup of color preserving automorphisms. Let $K = \mathrm{Stab}_A(e)$ where e is the vertex of T corresponding to the identity of Γ. For every vertex t of T we denote by $\hat{t}$ the unique element of Γ taking e to t. Clearly $\Gamma \cdot K = A$ and $K \cap \Gamma = \{1\}$. Let $r : K \to \mathrm{Perm}\,\{a_1, \dots, a_k\} = S_k$ be the homomorphism sending each element of K to its induced action on the edges adjacent to e.

Every automorphism $g \in A$ induces a map $\sigma_g : T \to K$ defined as follows:

$$\sigma_g(t) = \widehat{g(t)}^{-1} \circ g \circ \hat{t} \tag{2.1}$$

and let $\bar{\sigma}_g : T \to S_k = \text{Perm}\{a_1, \ldots, a_k\}$

$$\bar{\sigma}_g = r \circ \sigma_g \tag{2.2}$$

The next proposition summarizes the properties of $\bar{\sigma}$ and σ:

PROPOSITION 2.1.

(i) *For every* $\gamma \in \Gamma$, *and* $t \in T$, $\sigma_\gamma(t) = 1$ *and* $\bar{\sigma}_\gamma(t) = 1$.

(ii) *Compatability*: *If* t_1 *and* t_2 *are adjacent via* a_i, *i.e.*, $\hat{t}_1 \cdot a_i = \hat{t}_2$, *then* $\bar{\sigma}_g(t_2)(a_i) = \bar{\sigma}_g(t_1)(a_i)$.

(iii) *The cocycle condition*: $\sigma_{g_1 g_2}(t) = \sigma_{g_1}(g_2 t) \circ \sigma_{g_2}(t)$.

(iv) *For every* $\gamma \in \Gamma$, $g \in A$, *and* $t \in T$, $\sigma_{\gamma g}(t) = \sigma_g(t)$.

(v) *For every* $g \in A$ *and* $t \in T$, $\bar{\sigma}_{g^{-1}}(t) = \bar{\sigma}_g(g^{-1}t)^{-1}$.

The proof is straight forward and we omit it. Instead, we will try to motivate the definition of $\bar{\sigma}$: The action of g on the tree takes every vertex t to some vertex $g(t)$ and hence the neighbors of t are taken to the neighbors of $g(t)$. Thus the colored edges coming out of t are mapped to the colored edges coming out of $g(t)$. This induces a permutation of colors which we denoted $\bar{\sigma}_g(t)$. Property (i) of the proposition expresses the fact that the action of Γ is color preserving. Property (ii) is just the assertion that the image of an edge is well defined.

It is clear from formula (2.1) that the pair $(g(e), \sigma_g(e))$ determines g completely. Moreover, $(g(e), \bar{\sigma}_g)$ also determines g, but note that we should know $\bar{\sigma}_g(t)$ for every $t \in T$. In fact one can easily visualize how g is "developed" from the local information $\bar{\sigma}_g(t)$, $t \in T$.

Conversely, let $(t, \bar{\sigma})$ be such that $t \in T$ and $\bar{\sigma}$ is a map from T to S_k satisfying the compatability condition (i.e., condition (ii) of Proposition 2.1). Then, $(t, \bar{\sigma})$ defines a unique automorphism g of T. Clearly $g \in K$ if and only if $t = e$.

Let now $C = C(\Gamma)$ be the commensurability group of Γ in A, i.e., $C(\Gamma) = \{g \in A \mid g^{-1}\Gamma g \cap \Gamma \text{ is of finite index in } \Gamma\}$.

The fact that $g \in A$ is actually in C can be recognized from $\bar{\sigma}_g$:

PROPOSITION 2.2. *Let g be an element of A. Then $g \in C$ if and only if $\bar{\sigma}_g$ is periodic as a function on T, i.e., there exists a finite index subgroup Γ_0 of Γ s.t. $\sigma_g(\delta t) = \sigma_g(t)$ for every $\delta \in \Gamma_0$.*

Proof. Assume $g \in C$. Then there exists $\Gamma_0 \leq \Gamma$ of finite index such that $g\Gamma_0 g^{-1} \leq \Gamma$. So for $\delta \in \Gamma_0$ there exists $\gamma \in \Gamma$ such $\delta = g^{-1}\gamma g$. We then have:

$$\begin{aligned} 1 &= \bar{\sigma}_\delta(t) = \bar{\sigma}_{g^{-1}\gamma g}(t) = \bar{\sigma}_{g^{-1}}(\gamma g t)\bar{\sigma}_\gamma(gt)\bar{\sigma}_g(t) \\ &= \bar{\sigma}_{g^{-1}}(\gamma g t)\bar{\sigma}_g(t) = \bar{\sigma}_g(g^{-1}\gamma g t)^{-1}\bar{\sigma}_g(t) = \bar{\sigma}_g(\delta t)^{-1}\bar{\sigma}_g(t) \end{aligned}$$

Conversely if $\bar{\sigma}_g(\delta t) = \bar{\sigma}_g(t)$ for every $\delta \in \Gamma_0$, then we want to show that $g\Gamma_0 g^{-1} \leq \Gamma$. Indeed we have for $\delta \in \Gamma_0$ and $\gamma = g\delta_0 g^{-1}$:

$$\begin{aligned} 1 &= \bar{\sigma}_\delta(t) = \bar{\sigma}_{g^{-1}\gamma g}(t) = \bar{\sigma}_{g^{-1}}(\gamma g t)\bar{\sigma}_\gamma(gt)\bar{\sigma}_g(t) = \bar{\sigma}_g(g^{-1}\gamma g t)^{-1}\bar{\sigma}_\gamma(gt)\bar{\sigma}_g(t) \\ &= \bar{\sigma}_g(\delta t)^{-1}\bar{\sigma}_\gamma(gt)\bar{\sigma}_g(t) \end{aligned}$$

Hence $\bar{\sigma}_g(\delta t)\bar{\sigma}_g(t)^{-1} = \bar{\sigma}_\gamma(gt)$ and from the assumptions we have $\bar{\sigma}_\gamma(gt) = 1$. Since t is arbitrary it follows that γ preserves the coloring and hence belongs to Γ. □

As we can always replace Γ_0 by a smaller sublattice, it will be convenient to assume henceforth that $\Gamma_0 \subseteq A^+$; thus $T \to \Gamma_0 \backslash T$ is a covering. Let $g \in C$ and $\Gamma_0 < \Gamma$ be as in the preceding proposition. Let $Y = \Gamma_0 \backslash T$ be the quotient graph. Let $y_0 \in Y$ be the image of $e \in T$. Y is a finite k-regular colored graph. The proposition, together with the compatability condition, implies that $\bar{\sigma}_g$ defines a recoloring of the graph Y, such that if we let $\rho(y) \in S_k$ be the permutation of the colors $\{a_1, \ldots, a_k\}$ of the k edges coming out of the vertex $y \in Y$ and $\pi : T \to Y$ be the natural covering map, then $\bar{\sigma}_g(t) = \rho(\pi(t))$.

Conversely, starting with a finite colored k-regular pointed graph (Y, y_0), then Y is covered by T with a unique covering map π which preserves colors and satisfies $\pi(e) = y_0$. Thus $Y = \Gamma_0 \backslash T$ for some finite index $\Gamma_0 \leq \Gamma$. Given a recoloring ρ of the edges of Y, we can define a map $\tau : T \to S_k$ by $\tau(t) = \rho(\pi(t))$. Clearly this τ is periodic and satisfies the compatability condition. Hence for every $t_0 \in T$, the pair (t_0, τ) determines an automorphism $g \in C$ such that $g(e) = t_0$ and $\bar{\sigma}_g = \tau$.

We have:

PROPOSITION 2.3. *There is a correspondence between elements of $\Gamma \backslash C$ and recolorings of k-regular finite pointed graphs.* □

The correspondence in Proposition 2.3 is not one to one. Namely, given an element of $\Gamma \backslash C$ we could have taken a smaller sublattice $\Gamma_1 \subset \Gamma_0 \subset \Gamma$ leading to a graph $Y' = \Gamma_1 \backslash T$ covering $Y = \Gamma_0 \backslash T$.

The following simple observation is singled out for future use:

PROPOSITION 2.4. *Let (Y, y_0) be a finite k-regular pointed graph colored by the colors $\{a_1, \ldots, a_k\}$, let ρ be a recoloring Y and h the corresponding element of $C \cap K$. If a closed path based at y_0 colored by $(a_{i_1}, a_{i_2}, \ldots, a_{i_l})$ is recolored by $(a_{j_1}, a_{j_2}, \ldots, a_{j_l})$ then h conjugates the elements $a_{i_1} \cdot a_{i_2} \cdot \cdots \cdot a_{i_l}$ of Γ to $a_{j_1} \cdot a_{j_2} \cdot \cdots \cdot a_{j_l}$, i.e., $ha_{i_1} \cdot \cdots \cdot a_{i_l} h^{-1} = a_{j_1} \cdot \cdots \cdot a_{j_l}$.* □

Another useful way of reconstructing $g \in C$ from the recoloring of the pointed graph $(Y, y_0) = (\Gamma_0 \backslash T, e)$ and a vertex $t_0 \in T$ is the following: The original coloring of Y defines a unique color preserving covering map π sending e to y_0. The new coloring and t_0 determine a unique covering map π' from T to Y, sending t_0 to y_0 and respecting the new coloring. For every $t \in T$, $\pi'([t_0, g(t)]) = \pi([e, t])$ where $[t_1, t_2]$ denotes the geodesic path going from t_1 to t_2. This last condition clearly determines g uniquely.

The following observation is of some interest.

PROPOSITION 2.5. *An element $g \in C \cap K$ is of finite order if and only if for some k-regular pointed graph (Y, y_0) and its recoloring ρ associated with g, the two colored pointed graphs are isomorphic.*

Proof. If $g \in C \cap K$ is of finite order m then we can find a sublattice $\Gamma_0 < \Gamma$ such that $g\Gamma_0 g^{-1} = \Gamma_0$. (Take $\Gamma_0 = \bigcap_{i=1}^{m} g^i \Gamma g^{-i}$). As the two colored pointed graphs associated with g are $(\Gamma_0 \backslash T, \Gamma_0 e)$ and $(g\Gamma_0 g^{-1} \backslash T, g\Gamma_0 g^{-1} e)$ it follows that for our g they are isomorphic. Conversely assume that the two colored graphs, (Y, y_0) and its recoloring, are isomorphic. Notice that the covering group of a pointed colored graph is determined by the colorings of all the closed paths based at the base point. Since the two colored graphs are isomorphic they have the same covering group, i.e., $g\Gamma_0 g^{-1} = \Gamma_0$. Hence powers of g correspond to recoloring of the same graph $\Gamma_0 \backslash T$. There are only finitely many such possible colorings so for some $n > 0$, g^n corresponds to the trivial (color preserving) recoloring which implies that $g^n \in \Gamma$. Since it is also in K it is the identity element 1. Alternatively, from $g\Gamma_0 g^{-1} = \Gamma_0$ it follows from [BK 6.4] that $\langle g, \Gamma_0 \rangle$ generates a lattice $\tilde{\Gamma}$ and so $g \in \tilde{\Gamma} \cap K$ which is finite. □

Proposition 2.3 can be used to give a new proof for the following result due to Bass and Kulkarni [BK, 4.25]. (A far reaching generalization of this result was proved by Liu [Li].)

PROPOSITION 2.6. *C is dense in A.*

Proof. It suffices to show that $C \cap K$ is dense in K, since $\Gamma K = A$ and $\Gamma \leq C$. To this end, let $g \in K$ and $r \in \mathbb{N}$. We want to prove that there exists $h \in C$ which induces the same action as that of g on the ball $B_r(e)$ of radius r around e. By Proposition 2.3, it suffices to find a finite k-regular pointed graph (Y, y_0) together with a recoloring such that the automorphism h it defines acts on $B_r(X)$ as g. Let Y_0 be the k-regular graph obtained by taking two copies $B_r^+(e)$ and $B_r^-(e)$ of the colored finite tree $B_r(e)$. Denote by $S_r^+(e)$ (resp: $S_r^-(e)$) the sphere of radius r around e in B_r^+ (resp: $B_r^-(e)$). The graph Y is obtained by connecting each vertex v in $S_r^+(e)$ to its "twin" in $S_r^-(e)$ by $k-1$ edges colored by the missing $k-1$ colors around v in $B_r^+(e)$. Let $y_0 \in Y$ be the vertex e of $B_r^+(e)$. The action of g on $B_r(e)$ gives automorphisms of $B_r^+(e)$ and $B_r^-(e)$ which can be combined together (and extended on the edges of $Y \backslash Y_0$) to give an automorphism $\tilde{g}$ of (Y, y_0). Note that the extension from Y_0 to Y is not unique. The automorphism $\tilde{g}$ induces a recoloring of (Y, y_0) by giving every edge the color of its image under $\tilde{g}$. By Proposition 2.3 the recoloring induces an automorphism $h \in C \cap K$. Clearly h acts on $B_r(e)$ in the same way as g. □

REMARKS 2.7. The element h constructed in the proof has finite order by Proposition 2.5. So as in [BK] we in fact prove that the set of elements of $C \cap K$ of finite order is dense in K. Moreover, the proof actually shows that given $r \in \mathbb{N}$ and $g_1, \dots, g_l$ in K, there exists $h_1, \dots, h_l \in C$ such that for every $1 \le i \le l$, h_i coincides with g_i on $B_r(e)$ and $h_1, \dots, h_l$ generate a finite subgroup of C.

We are ready to prove the following superrigidity result for C.

THEOREM 2.8. *Let Γ_0 be an arbitrary uniform lattice in A, $C = C(\Gamma_0)$ its commensurability group in A and $\rho : C \to \mathrm{Aut}(T')$ a minimal action of C on a tree T'. Then either $l_{T'}(\rho(\gamma)) = 0$ for every $\gamma \in \Gamma_0$ or T' is the tree obtained from T by subdividing each edge into m segments for some fixed m and ρ is conjugate to the original action. In particular ρ extends to an action of A on T'.*

We will break the proof into a few propositions which might have independent interest. We first notice that it suffices to prove the theorem for our original $\Gamma = \langle a_1 \rangle * \cdots * \langle a_k \rangle$ since Γ_0 is conjugate to a lattice commensurable to Γ so $C(\Gamma_0)$ is conjugate to $C(\Gamma)$. Moreover if Γ and Γ_0 are commensurable then $l_{T'}(\rho(\Gamma)) = \{0\}$ if and only if $l_{T'}(\rho(\Gamma_0)) = \{0\}$. From now on we will assume $\Gamma_0 = \Gamma$.

The crucial observation for the proof of the theorem is the following:

PROPOSITION 2.9. *Let x and y be non-trivial elements of Γ. Then the following three conditions are equivalent*:

(a) *x and y are conjugate in A.*

(b) $l_T(x) = l_T(y)$.

(c) *x and y are conjugate in C.*

Proof. Clearly (c) implies (a) and (a) implies (b). To prove (b) implies (c), assume first that $l_T(x) = l_T(y) = 0$. This implies that both x and y are of finite order, in fact of order two as this is the only possible torsion in Γ. As Γ is a free product of the order two groups $\langle a_i \rangle$, $i = 1, \ldots, k$, it follows that x is conjugate to, say, a_i and y to a_j. It is easy to see that a_i and a_j are conjugate in C. Assume now $l_T(x) = l_T(t) = l > 0$. First note that by replacing x and y by conjugates within Γ we can assume that $x = a_{i_1} \cdot \cdots \cdot a_{i_l}$ and $y = a_{j_1} \cdot \cdots \cdot a_{j_l}$. Indeed we can conjugate them in Γ to make their axes (see sec. 1) pass through e, in which case $l_T(x)$ is equal to the distance between e and x on the Cayley graph. Note that $a_{i_n} \neq a_{i_{n+1}}$, $a_{j_n} \neq a_{j_{n+1}}$ for $n = 1, \ldots, l-1$ and $a_{i_1} \neq a_{i_l}$, $a_{j_1} \neq a_{j_l}$. By Proposition 2.4 it suffices to find a k-regular pointed graph (Y, y_0) colored by $\{a_1, \ldots, a_k\}$ and a recoloring of it, such that there exists a closed path in Y based at y_0, colored by $(a_{i_1}, \ldots, a_{i_l})$ and recolored by $(a_{j_1}, \ldots, a_{j_l})$. Here is such a graph: Let Y_0 be a pair of twin cycles Z^+ and Z^- of length l, both colored $(a_{i_1}, a_{i_2}, \ldots, a_{i_l})$. Complete them to a k-regular graph Y by connecting each pair of twin vertices with $k-2$ edges colored by the $k-2$ missing colors. The recoloring of Y is given by coloring the twin cycles by $(a_{j_1}, \ldots, a_{j_l})$ and extending it to a legal coloring of Y (which clearly can be done). This completes the proof of the proposition. □

The length function l_T can be described explicitly on Γ: A word $a_{i_1} \cdot \cdots \cdot a_{i_r}$ of $\Gamma = \langle a_1 \rangle * \cdots * \langle a_k \rangle$ is *cyclically reduced* if $a_{i_j} \neq a_{i_{j+1}}$ for $j = 1, \ldots, r-1$ and $a_{i_1} \neq a_{i_r}$. (Note that words of length one are *not* cyclically reduced). Every element of Γ of infinite order is conjugate in Γ to a unique cyclically reduced word. A cyclically reduced word $\gamma = a_{i_1} \cdot \cdots \cdot a_{i_r}$ is a hyperbolic element whose axis passes through the origin e of T and $l_T(\gamma) = r$.

PROPOSITION 2.10. *Let $\rho : C \to \mathrm{Aut}(T')$ be an action of C on a tree T'. Then either $l_{T'}(\rho(\gamma)) = 0$ for every $\gamma \in \Gamma$ or there exists $m \in \mathbb{N}$ such that $l_{T'}(\rho(\gamma)) = m \cdot l_T(\gamma)$ for every $\gamma \in \Gamma$.*

Proof. Denote $m' = l_{T'}(\rho(a_1 a_2))$ and $m'' = l_{T'}(\rho(a_1 a_2 a_3))$. Now, $l_{T'}(\rho(\gamma^n)) = n l_{T'}(\rho(\gamma))$ for every $\gamma \in \Gamma$ and $n \in \mathbb{N}$, and Proposition 2.9 implies that $(a_1 a_2)^3$ is conjugate in C to $(a_1 a_2 a_3)^2$. Thus $3m' = 2m''$, showing that $m = m''/3 = m'/2$ is an integer. We claim that $l_{T'}(\rho(\gamma)) = m l_T(\gamma)$ for every γ in Γ. Indeed if $l_T(\gamma) = r \neq 0$ then γ is conjugate in Γ to a cyclically reduced word $a_{i_1} \cdot \cdots \cdot a_{i_r}$ where $r > 1$. By Proposition 2.9, γ^2 is conjugate in C to $(a_1 a_2)^r$. Hence

$$l_{T'}(\rho(\gamma)) = \frac{1}{2} l_{T'}(\rho(\gamma^2)) = \frac{1}{2} l_{T'}((a_1 a_2)^r)$$

$$= \frac{r}{2} m' = mr = m l_T(\gamma)$$

Now, if $l_T(\gamma) = 0$ then γ is conjugate in Γ to one of the a_i and it is of order 2. Its image under ρ must be also of finite order and so $l_T(\rho(\gamma)) = 0$. □

PROPOSITION 2.11. *Let* $\rho : C \to \mathrm{Aut}(T')$ *be a minimal action such that* $l_T(\rho(\Gamma)) \neq \{0\}$. *Then* $\rho(\Gamma)$ *acts minimally on* T'.

Proof. If Γ' is a finite index subgroup of Γ then $l_T(\rho(\Gamma)) \neq \{0\}$ if and only if $l_T(\rho(\Gamma')) \neq \{0\}$ in which case the minimal subtree $T'_{\rho(\Gamma')}$ is equal to $T'_{\rho(\Gamma)}$ (see [B2, 7.7]). Now, if $c \in C$ then there exists a finite index subgroup Γ' of Γ such that $c\Gamma'c^{-1}$ is of finite index in Γ. Thus, $\rho(c)T'_{\rho(\Gamma)} = \rho(c)T'_{\rho(\Gamma')} = T'_{\rho(c\Gamma'c^{-1})} = T'_{\rho(\Gamma)}$, i.e., $T'_{\rho(\Gamma)}$ is C-invariant and so $T'_{\rho(\Gamma)} = T'_{\rho(C)}$. □

We are ready now to prove Theorem 2.8:

Let $\rho : C \to \mathrm{Aut}(T')$ be a minimal action such that $l_T(\rho(\Gamma)) \neq \{0\}$. By Proposition 2.11 it defines a minimal Γ action whose length function, by Proposition 2.10, satisfies $l_T(\rho(\gamma)) = ml_T(\gamma)$ for some fixed $m \neq 0$ and every $\gamma \in \Gamma$. We know one such minimal action: this is the minimal action of Γ on the tree $T^{(m)}$ obtained from T by dividing every edge of T into m segments. But the length function of a minimal action completely determines the tree and the action up to conjugation [AB, 7.13 (b)]. So $T' = T^{(m)}$. As $\mathrm{Aut}(T^{(m)}) = \mathrm{Aut}(T) = A$ we can ignore the subdivision and further assume that $\rho(\gamma) = \gamma$ for every $\gamma \in \Gamma$. We want to prove now that $\rho(c) = c$ for every $c \in C$. Let $c \in C$ and say $\rho(c) = d \in A$. Since $\rho(\Gamma) = \Gamma$, $\rho(c)$ also commensurizes Γ and hence $\rho(C) \subseteq C$. Hence $d \in C$. There exists Γ' of finite index in Γ for which $c^{-1}\Gamma'c \leq \Gamma$ and $d^{-1}\Gamma'd \leq \Gamma$ and both are of finite index. Now for $\gamma \in \Gamma'$:

$$c^{-1}\gamma c = \rho(c^{-1}\gamma c) = \rho(c)^{-1}\rho(\gamma)\rho(c) = d^{-1}\gamma d$$

i.e., $d^{-1}c$ centralizes Γ'. By [BK, Proposition 6.1], $Z_A(\Gamma') = \{1\}$ and so $d = c$. This completes the proof of the theorem. □

REMARKS 2.12.

(i) We do not know whether Theorem 2.8 can be strengthened to conclude that either ρ can be extended to A or $l_T(\rho(c)) = 0$ for every $c \in C$. A somewhat related open problem is whether the index two subgroup $C^+ = C \cap A^+$ of C is a simple group. Proposition 5.1 might be useful in this direction. A simple application of it shows that if $N \lhd C^+$ with $N \cap \Gamma \neq \{1\}$ then N contains $\Gamma^+ = \Gamma \cap A^+$. (See proposition 5.1.)

(ii) The proof of Theorem 2.7 extends without difficulties to show that if $\rho : C^+ = C \cap A^+ \to \mathrm{Aut}(\Gamma')$ is a minimal action of C^+ on a tree T' then either $l_T(\rho(\Gamma^+)) = \{0\}$ or ρ can be extended to A^+. However, it does not apply to an arbitrary subgroup D with $\Gamma \leq D \leq C$ and dense in A. In Section 4 we will reprove Theorem 2.8 in a much more general form which contains this case.

Finally, we mention that the main idea of the proof of Theorem 2.8, i.e., the recovery of the length function, can also be used to prove a rigidity theorem for A. Of course, this also follows from the results in [BL], [Zn].

PROPOSITION 2.13. *Let T' be a tree on which A acts minimally and A^+ acts non-trivially. Then T' is an m-subdivision of T and the action is conjugate to the original action.*

Proof. Again, we just have to make sure that $l_{T'}(g) = ml_T(g)$ for some fixed $m \in \mathbb{N}$ and every $g \in A$. Note that A^+ is an infinite simple group so once it acts non-trivially, the action must be faithful and so $m \neq 0$. Now every two elements of the same positive translation length on T are conjugate in A. Elements for which $l_T(g) = 0$ lie inside pro-finite groups, so for them $l_{T'}(g) = 0$ as well ([B1, Theorem 5.2]). This implies $l_{T'}(g) = ml_T(g)$ for some fixed $m \in \mathbb{N}$ and all $g \in A$ and completes the proof of Proposition 2.13 in the same way as in the proof of Theorem 2.8. □

§3. Automorphisms of the commensurability group are geometric

We continue here with our notations from Chapter 2, i.e., T is the k-regular tree and $C = C(\Gamma)$ the commensurability group of a uniform lattice Γ in $A = \mathrm{Aut}\,(\Gamma)$. By [BK, Cor. 4.8] every two uniform lattices in A are commensurable after conjugation, so C is independent of Γ up to conjugation. The main goal of this chapter is to prove:

THEOREM 3.1. *Every automorphism of C is "geometric", i.e., if $\alpha : C \to C$ is an automorphism then there exists $g \in A$ such that $\alpha(c) = g^{-1}cg$ for every $c \in C$.*

COROLLARY 3.2. *$Ad : N_A(C) \to \mathrm{Aut}\,(C)$ is an isomorphism, where $N_A(C)$ is the normalizer of C in A and $Ad(g)(c) = g^{-1}cg$ for $c \in C$.*

This follows from the fact that the centralizer of C in A is trivial.

We do not know whether every automorphism of C is actually inner, i.e. whether $N_A(C) = C$. This is equivalent to the problem posed by Bass and Kulkarni

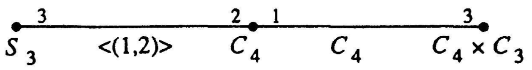
([BK, B.14 II]): Are two uniform lattices with *the same* commensurability groups, commensurable to each other?

Theorem 3.1 will follow immediately from Theorem 2.8 once we show that if α is an automorphism of $C = C(\Gamma)$ then $l_T(\alpha(\Gamma)) \neq \{0\}$. Recall that $l_T(\alpha(\Gamma)) = 0$ if and only if Γ stabilizes either a vertex, an edge or an end ([AB, (7.5)]). Since Γ is finitely generated once it stabilizes an end it stabilizes a vertex as well. So altogether we would like to construct a uniform lattice Γ of A which can not be embedded in a stabilizer of an edge or a vertex.

The reader is referred to [BK, (7.2)] for the method of construction of uniform lattices in Aut (T). We will use the notation and terminology of [BK]. We will deal separately with $k = 3$, $k = 4$ and $k \geq 5$, with $k = 4$ being the most difficult case.

3.3. *A construction of Γ for $k = 3$*

Let Γ be the lattice of Aut (T_3) given by the following graph of groups:



Here $C_4 = \langle x \rangle$ is the cyclic group of order 4 and the inclusion $\langle (1,2) \rangle \hookrightarrow C_4$ is given by $(1,2) \to x^2$. The rest of the inclusions are the obvious ones.

We claim that $\alpha(\Gamma)$ cannot stabilize an edge or a vertex. Indeed the stabilizer of an edge in Aut (T_3) is a pro-2 group and hence cannot contain $\alpha(S_3) < \alpha(\Gamma)$.

Assume now that $\alpha(\Gamma)$ is contained in the stabilizer K of a vertex. K is mapped via a homomorphism φ onto P, the permutation group of the three edges at this vertex, with a pro-2 kernel. Hence $\psi = \varphi \circ \alpha$ induces a homomorphism from Γ to S_3. The restriction of ψ to $C_4 < \Gamma$ must map x^2 to the identity. Hence the restriction of ψ to $S_3 < \Gamma$ also maps $(1,2)$ to the identity. As the conjugates of this element generate S_3 it follows that $\alpha(S_3)$ is contained in the kernel of φ, which is impossible.

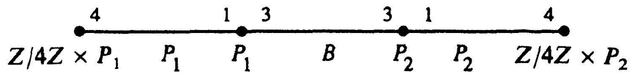
3.4. *A construction of Γ for $k = 4$*

The stabilizer of a vertex or an edge in T_4 is a pro-solvable group. Hence any group H embedded in either is necessarily residually solvable. In particular such a group H satisfies $[H, H] \neq H$. We will construct a uniform lattice Γ containing a non-trivial subgroup Γ_0 which satisfies $[\Gamma_0, \Gamma_0] = \Gamma_0$. This implies that Γ_0, and hence Γ, cannot be embedded in the stabilizer of an edge or a vertex.

Following D. Goldschmidt [G (3.6)] we define the following groups:

$$B = \langle a, b, s \mid a^4 = 1, b^4 = 1, [a, b] = 1, s^2 = 1, a^s = b, b^s = a\rangle$$
$$\cong (\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z}$$
$$P_1 = \langle B, x \mid x^3 = 1, a^x = b, b^x = a^{-1}b^{-1}, x^s = x^{-1}\rangle \cong (\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}) \rtimes S_3$$
$$P_2 = \langle B, y \mid y^3 = 1, (abs)^y = a^2 s, (sa^2)^y = ab^{-1}, y^{sa} = y^{-1}\rangle \simeq GU(2, 3)$$

P_1 and P_2 are both finite groups of order 96 containing the subgroup B of index 3.
We have the following graph of groups:



It defines a uniform lattice Γ in Aut (T_4).
Γ contains a subgroup $\Gamma_0 = P_1 \underset{B}{*} P_2$. In order to conclude that for any embedding $\alpha : C \to \text{Aut}\,(T_4)$, $\alpha(\Gamma)$ cannot stabilize a vertex or an edge it suffices to show:

PROPOSITION 3.5. $[\Gamma_0, \Gamma_0] = \Gamma_0$.

Proof. We compute first $P_1/[P_1, P_1]$. By examining the defining relations of P_1, we see that $P_1/[P_1, P_1] \cong \mathbb{Z}/2\mathbb{Z}$ and $a, b, x \in [P_1, P_1]$ whereas s has a nontrivial image in this quotient.

Similarly we have $P_2/[P_2, P_2] \cong \mathbb{Z}/4\mathbb{Z}$ and $y \in [P_2, P_2]$, $a[P_2, P_2] = b[P_2, P_2]$ a generator of $P_2/[P_2, P_2]$ and $s[P_2, P_2] = a^2[P_2, P_2]$.

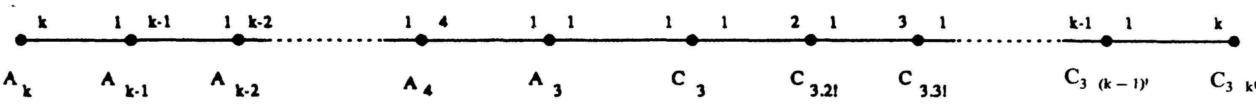
Let $\varphi : \Gamma_0 \to \Gamma_0/[\Gamma_0, \Gamma_0]$ be the natural map. The image of P_1 and P_2 generate $\bar{\Gamma}_0 = \Gamma_0/[\Gamma_0, \Gamma_0]$. The maps $\varphi : P_i \to \bar{\Gamma}_0$ factor through maps $\varphi : P_i/[P_i, P_i] \to \bar{\Gamma}_0$. The element $a \in B = P_1 \cap P_2$ has trivial image in $P_1/[P_1, P_1]$ hence also trivial image in $\bar{\Gamma}_0$. Since its image in $P_2/[P_2, P_2]$ generates the latter we conclude that $\varphi(P_2)$ is trivial. Moreover using $s[P_2, P_2] = a^2[P_2, P_2]$ it follows that $\varphi(s) = \text{identity}$. Since $s[P_1, P_1]$ is a generator of $P_1/[P_1, P_1]$ we conclude that $\varphi(P_1)$ is trivial and hence $\Gamma_0/[\Gamma_0, \Gamma_0]$ is the trivial group. □

We remark that Γ_0 may be realized as a uniform lattice in Aut (T_3). As stabilizers in Aut (T_3) are prosolvable this gives an alternative construction for $k = 3$.

3.6. *A construction of Γ for $k \geq 5$*

The stabilizer of an edge is a profinite group whose Jordan–Holder factors are simple groups of orders smaller than $|A_k|$. Hence a lattice Γ containing A_k cannot be embedded in the stabilizer of an edge. The stabilizer of a vertex is a profinite group K having a homomorphism φ to S_k whose kernel is a profinite group having Jordan–Holder factors which are simple groups of orders smaller than $|A_k|$. Γ will be constructed in such a way that it contains both A_k and a finite group H having an element $1 \neq z \in A_k \cap H$ and such that this element is in the kernel of any homomorphism from H to S_k. Since A_k is simple this implies that A_k is in the kernel of the homomorphism $\varphi \circ \alpha$, which is impossible.

Here is a method to construct a lattice with the above desirable properties: Consider the following graph of groups



where A_j is the alternating group operating on $\{1 \ldots j\}$ and fixing $\{j+1, \ldots, n\}$. $C_{3 \cdot i!} = \langle x_i \rangle$ is the cyclic group of order $3 \cdot i!$ and $C_{3 \cdot i!}$ is embedded in $C_{3 \cdot (i+1)!}$ by $x_i = x_{i+1}^{i+1}$. C_3 is identified with A_3 by $x_1 = (1, 2, 3)$. The sum of the indices at each vertex is $\leq k$ and one can easily add new edges and vertices in order to complete it to a graph of groups defining a uniform lattice of Aut (T_k). This enlargement of the graph can be carried out in many ways. Each way will give a different lattice Γ. Each such lattice Γ will have the following crucial property:

The element $(1, 2, 3) = x$, is embedded in $H = C_{3 \cdot k!} = \langle x_k \rangle$ and satisfies $(1, 2, 3) = x_k^{k!}$. Hence any homomorphism ψ of H to S_k maps $x_1 = (1, 2, 3)$ to the identity. Thus if $\psi = \varphi \circ \alpha$, where α is the embedding of Γ into Aut (T_k) mapping Γ into a stabilizer K of a vertex and φ is the homomorphism from K into the permutation group of the k adjacent edges, we conclude that $\psi((1, 2, 3)) = \text{identity}$ and since A_k is simple $\psi(A_k) = \{\text{identity}\}$. It follows that A_k is embedded in the kernel of φ which is a profinite group whose Jordan–Holder factors are finite simple groups of order $< |A_k|$, which is impossible.

REMARKS 3.7. (1) We have actually proved slightly more than promised: We showed that for every $k \geq 3$, a uniform lattice Γ in $A = \text{Aut}\,(T_k)$ can be found such that for every monomorphism $\alpha : \Gamma \to A$, $l_{T_k}(\alpha(\Gamma)) \neq \{0\}$. Thus if $\alpha : C = C(\Gamma) \to A$ is a minimal action of C on T_k, it is obtained, by (2.8), from a conjugation in A.

(2) However, there are non-minimal faithful actions of C on T_k. In fact there are even non-minimal faithful actions of A on T. This can be seen by the following: Replace T_k by a barycentric subdivision of T_k and then "graft" trees on the new vertices to make it k-regular. The resulting tree is a k-regular tree. The action of A on the original tree extends to an action on the larger tree which is non-minimal.

(3) Notice also that the proof shows that if C_{T_k} is isomorphic to C_{T_m} then $k = m$.

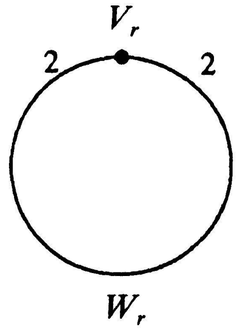
(4) What we do not know how to prove is the following: Given a monomorphism $\alpha : C \to \mathrm{Aut}\,(T')$ where T' is another tree, $l_{T'}(\alpha(\Gamma)) \neq \{0\}$. However we can show the following:

($*$) There exists an infinite ascending sequence of uniform lattices in Aut (T_k), $\Gamma(1) \leq \Gamma(2) \leq \cdots$ such that $\Delta = \bigcup_{i=1}^{\infty} \Gamma(i)$ is not a residually finite group.

As Δ is a subgroup of $C = C(\Gamma(1))$, it implies that for any faithful action α of C on a tree T', $\alpha(\Delta)$ cannot fix a vertex. We do not know how to show that $\alpha(\Delta)$ cannot fix an end. Having done so we could get a stronger rigidity result.

Here is a sketch of how ($*$) is proved for $k = 4$ (simple modifications will adapt the construction to $k = 3$ and to $k \geq 5$):

Recall the following construction of lattices from [BK (7.4)]: Let $M = \mathbb{Z}/2\mathbb{Z}$. For each $r > 0$, put $V_r = M^{\mathbb{Z}/r\mathbb{Z}}$, the group of set functions $X : \mathbb{Z}/r\mathbb{Z} \to M$. Define $\alpha_r \in \mathrm{Aut}\,(V_r)$ by $\alpha_r(X)(i) = X(i+1)$. Let $W_r = \{X \in V_r \mid X(0) = 0 \in M\}\,(\simeq M^{r-1})$. Let L^r be the graph of groups



where the edge group W_r is mapped into V_r in two ways: once by the identity map and once by the automorphism α_r. As in [BK (7.4)] if $r \mid r'$ the graph of groups L_r covers $L_{r'}$ and the corresponding lattice Γ_r embeds in $\Gamma_{r'}$. The group V_r embeds in the $V_{r'}$ by mapping $(x_1, \ldots, x_r) \to (x_1, \ldots, x_r, x_1, \ldots, x_r, \ldots, x_1, \ldots, x_r)$. Notice that this embedding is compatible with the embedding of W_r in $W_{r'}$. The group Γ_r is generated by V_r and t subject to the relations $t^{-1}(0, x_2, \ldots, x_r)t = (x_2, x_3, \ldots, x_r, 0)$ where $x_i \in \mathbb{Z}/2\mathbb{Z}$. Now let $\Gamma(i) = \Gamma_{(i+1)!}$ and $\Delta = \bigcup_{i=1}^{\infty} \Gamma(i)$. We claim that Δ is not residually finite. In fact the element $(1, 1) \in V_2 \subseteq \Gamma(1)$ has trivial image in any finite image $\varphi : \Gamma \to F$.

Let $\varphi : \Gamma \to F$ be given. Choose $i_0 > |F|$ and let $h_k = (1, 1, \ldots, 1, 0, \ldots, 0) \in V_{(i_0+1)!}$, $1 \leq k \leq (i_0+1)!$ the vector having k 1's and the rest 0. Clearly for some $1 \leq k < l \leq |F| + 1$ we have $\varphi(h_k) = \varphi(h_l)$. Hence $h_l - h_k$ belongs to the kernel of φ. Using conjugation by t^{-1} we conclude that $h_{l-k} \in \ker(\varphi)$. By repeated use of conjugation by t^{l-k} and the fact that $l - k$ divides $(i_0+1)!$ we deduce that

$h_{(i_0+1)!} \in \ker \varphi$. Since the embedding of $\Gamma(1)$ in $\Gamma_{(i_0+1)}$ is such that $(1, 1)$ is identified with $h_{(i_0+1)!}$ its image is trivial.

§4. Superrigidity via ergodic theoretic methods

Let T be a regular tree, with automorphism group Aut (T). Let $\Gamma \subset$ Aut (T) be a lattice and $C(\Gamma)$ the commensurizer of Γ in Aut (T). We emphasize that we do not assume that Γ is a uniform lattice. Let D be a group with $\Gamma \subset D \subset C(\Gamma)$. Let S be an arbitrary tree (for which ∂S has at least 3 points) and $\pi : D \to$ Aut (S) a homomorphism. The aim of this section is to prove:

THEOREM 4.1. *Assume $\pi(D)$ acts minimally on ∂S. Then either*:
(i) *$\pi(\Gamma)$ has compact closure in* Aut (S); *or*
(ii) *π extends to a continuous homomorphism $\pi : \bar{D} \to$* Aut (S) *where $\bar{D}$ is the closure of D in* Aut (T). *In particular, if D is dense, π extends to a homomorphism* Aut $(T) \to$ Aut (S).

REMARKS.
(1) The proof of this theorem is very much in the spirit of the proof of Margulis' superrigidity theorem for lattices in semisimple Lie groups with a dense commensurizer. (See [Zi, section 6.2].) There are of course a number of different features in our present situation, and we invite the reader to compare the proof below with that of [Zi, section 6.2].
(2) The condition (i) in the statement of the theorem cannot in general be replaced by the stronger assertion that $\pi(D)$ has compact closure. We do not know, however, whether this may be possible for the case in which D is the full commensurizer of a uniform lattice. An instructive example is the case of $D = H_1(\mathbb{Q})$, the rational norm one quaternions. This group commensurizes $H_1(\mathbb{Z}[1/p])$ for each p, which is a lattice in $SL_2(\mathbb{Q}_p)$ and hence a uniform lattice in Aut (T_{p+1}). However, if we fix one prime p, we see that by acting on the tree T_{q+1} for another prime q, that D does not act precompactly, that the homomorphism does not extend to $\bar{D}$ (by results of [Zi, chapter 10]), but that $H_1(\mathbb{Z}[1/p])$ does act pre-compactly.

We begin the proof of this theorem with some preliminaries. First, we consider some general properties of Aut (S) acting on ∂S. We recall the following:

LEMMA 4.2. *Suppose $a_n \in$* Aut (S) *and $a_n \to \infty$. Then there are*:
(i) *a subsequence a_{n_j}, and*
(ii) *points $x, y \in \partial S$ (not necessarily distinct) such that if $z \in \partial S$ and $z \neq x$, then $a_{n_j} z \to y$.*

Proof. Let $p \in S$ be any vertex. Since $S \cup \partial S$ is compact, for some subsequence $a_{m_j} p \to y$ for some $y \in S \cup \partial S$. Since $a_n \to \infty$ it follows that $y \in \partial S$. If for every $z \in \partial S$, $a_{m_j} z \to y$ then the lemma holds. Otherwise for some $x \in \partial S$ we have after passing to a further subsequence n_j, $a_{n_j} x \to x_0$ for some $x_0 \in \partial S$, $x_0 \neq y$.

Let $z \in \partial S$, $z \neq x$. We want to show that $a_{n_j} z \to y$. Indeed if for some subsequence a_{k_j} of a_{n_j} we had $a_{k_j} z \to z_0$ such that $z_0 \neq y$ then considering the images of the geodesic in S connecting x and z we conclude that $a_{k_j} p \to y$ would have been impossible. □

We shall be applying this result in a variety of ways. If X is a compact metrizable space, we let $M(X)$ be the space of probability measures on X. Thus, $M(X)$ is a compact convex separable set with the weak-$*$-topology. We let $M_2(X) \subset M(X)$ be the measures supported on at most 2 points, $M_1(X) \subset M_2(X)$ the measures supported on a singleton (so we can identify $M_1(X) \cong X$), and $M_2^0(X) \subset M_2(X)$ the set of measures which are either in $M_1(X)$ or assign measures $1/2$ to each point in the support. We have $M_1(X) \subset M_2^0(X) \subset M_2(X)$, and these are all closed subsets of $M(X)$. From Lemma 4.2, we immediately deduce, via the argument of [Zi, Lemma 3.2.1]:

LEMMA 4.3. *If* $a_n \in \mathrm{Aut}(S)$, $a_n \to \infty, \mu, \nu \in M(\partial S)$ *with* $a_n \cdot \mu \to \nu$, *then* $\nu \in M_2(\partial S)$. □

From this, we conclude:

LEMMA 4.4. *Consider the action of* Aut (S) *on* $M(\partial S)$. *Every orbit in (the open set)* $M(\partial S) - M_2(\partial S)$ *is closed in* $M(\partial S) - M_2(\partial S)$ *and the stabilizer of any point in this set is compact.* □

We shall need information not just on $M(\partial S)$ but on measurable functions into this space. Let (Y, ν) be a standard measure space. If X is a second countable metrizable space, we let $F(Y, X)$ be the space of measurable functions $Y \to X$ with two functions being identified if they agree $\nu - a.e.$ We give $F(Y, X)$ the topology of convergence in measure, which is a separable metrizable topology. If $X_0 \subset X$, we clearly have a natural inclusion $F(Y, X_0) \subset F(Y, X)$. If a group G acts on X, it also acts on $F(Y, X)$. For functions into $M(X)$, we let $F_2(Y, M(X))$ be those functions f for which there are two points $x_1, x_2 \in X$ (not necessarily distinct) such that for $a.e. t \in Y$, $f(t)$ has support in $\{x_1, x_2\}$.

LEMMA 4.5. *Let* (Y, ν) *be any standard measure space. Then*

(i) $F_2(Y, M_2(\partial S)) \subset F(Y, M(\partial S))$ *is closed and* Aut (S)*-invariant.*

(ii) *Let* $\mathcal{O} = F(Y, M(\partial S)) - F_2(Y, M_2(\partial S))$. *Then for any* $f \in \mathcal{O}$, Aut $(S) \cdot f$ *is closed in* $\mathcal{O}$.

Proof. (i) is clear. To see (ii), suppose $f \in F(Y, M(\partial S))$ and $a_n \cdot f \to g$. If a_n is bounded, then $g \in \mathrm{Aut}\,(S) \cdot f$. If not, by passing to a subsequence we can assume $a_n \to \infty$ and (by Lemma 4.2) that there are points $x, y, w \in \partial S$ such that for $z \neq x$, $a_n(z) \to y$, and that $a_n(x) \to w$. We can also assume (again by passing to a subsequence) that $a_n \cdot f(t) \to g(t)$ for *a.e.* $t \in Y$. It follows that for *a.e.* t, $g(t) \in M(\partial S)$ is supported on $\{y, w\}$, i.e. $g \notin \mathcal{O}$. □

We shall also need the following general fact from [Zi] about when a homomorphism of groups has a pre-compact image.

LEMMA 4.6. ([Zi, Proposition 5.1.9.]) *Suppose Y is an ergodic Γ-space (where Γ is any locally compact group) such that Γ is also ergodic on $Y \times Y$. (The measure on Y is only assumed to be quasi-invariant under Γ.) Suppose $\pi : \Gamma \to H$ is a homomorphism to a locally compact group H and that there is a measurable Γ-map $Y \to H/K$ where K is compact. Then $\overline{\pi(\Gamma)}$ is compact.* □

We now turn to the proof of Theorem 4.1, using many of the ideas of Margulis' superrigidity theorem, as described in [Zi].

We assume $\pi : D \to \mathrm{Aut}\,(S)$ and that $\overline{\pi(\Gamma)}$ is not compact.

LEMMA 4.7. *There is a measurable Γ-map $\varphi : \partial T \to M_2^0(\partial S)$.*

Proof. The Γ action on ∂T is amenable by [Zi, Corollary 4.3.7] and the fact that Aut (T) is transitive on ∂T with amenable stabilizer. It follows from [Zi, Proposition 4.3.9] that there is a measurable Γ-map $\varphi : \partial T \to M(\partial S)$. Since the Γ-action on ∂T is ergodic, it follows that either $\varphi(\partial T) \subset M_2(\partial S)(a.e.)$ or $\varphi(\partial T) \subset M(\partial S) - M_2(\partial S)$. The former case, we can clearly obtain such a map to $M_2^0(\partial S)$ which is just the assertion we want, so it suffices to consider the latter case. By Lemma 4.4, the action of Aut (S) on $M(\partial S) - M_2(\partial S)$ is tame (or "smooth" in the regrettable terminology of [Zi]) (Cf. [Zi, Theorem 2.1.14]) and it follows that the map $\bar{\varphi} : \partial T \to (M(\partial S) - M_2(\partial S))/\mathrm{Aut}\,(S)$ is essentially constant since Γ is ergodic on ∂T. (Cf. [Zi, Proposition 2.1.11].) In other words, $\varphi(\partial T)$ lies in a single Aut (S)-orbit in $M(\partial S) - M_2(\partial S)$. Letting K be the compact (by Lemma 4.2) stabilizer of a point in this orbit, we can view φ as a Γ-map $\partial T \to \mathrm{Aut}\,(S)/K$. Since Γ is ergodic on $\partial T \times \partial T$ (this is equivalent to the ergodicity of the "geodesic flow" on $\Gamma \backslash T$), Lemma 4.6 applies to show that $\overline{\pi(\Gamma)}$ is compact, contrary to our assumption. □

We wish next to assert that we can obtain a map as in Lemma 4.7 that is actually a D-map. We remark that if D is not discrete, it will not in general act amenably on ∂T, so that one cannot simply replace Γ by D in the above arguments (cf. [Zi2].) The key to obtaining a D-map is a suitable uniqueness assertion for φ as in Lemma 4.7.

LEMMA 4.8. *Suppose there is a measurable Γ-map $\varphi : \partial T \to M_2^0(\partial S) - M_1(\partial S)$. If $\Lambda \subset \Gamma$ is of finite index and $\psi : \partial T \to M_2^0(\partial S) - M_1(\partial S)$ is a measurable Λ-map, then $\varphi = \psi$ a.e.*

Proof. Suppose not. Then for a set of positive measure in ∂T, supp $(\varphi(t) + \psi(t))$ will consist of 3 or 4 distinct points. Since $\varphi + \psi$ is Λ-equivalent and Λ acts ergodically on ∂T (since Λ is itself a lattice) we have that for *a.e.* $t \in T$, supp $(\varphi(t) + \psi(t))$ consists of 3 or 4 distinct points. Let $\lambda(t)$ be the equidistributed measure on supp $(\varphi(t) + \psi(t))$. Then $\lambda : \partial T \to M(\partial S) - M_2(\partial S)$ is a Λ-map. However, the proof of Lemma 4.7 shows that this implies $\pi(\Lambda)$ is compact. Hence $\pi(\Gamma)$ is compact as well, yielding a contradiction. □

LEMMA 4.9. *Suppose there is a measurable Γ-map $\varphi : \partial T \to M_1(\partial S)$ but that there is no such Γ-map $\partial T \to M_2^0(\partial S) - M_1(\partial S)$. If $\Lambda \subset \Gamma$ is of finite index and $\psi : \partial T \to M_1(\partial S)$ is a measurable Λ-map, then $\varphi = \psi$ a.e.*

Proof. Suppose not. Then (using ergodicity of Λ on ∂T as in the proof of Lemma 4.8), $\frac{1}{2}(\varphi + \psi) : \partial T \to M_2^0(\partial S) - M_1(\partial S)$ is a Λ-map. By hypothesis, this cannot be a Γ-map. Let $\{\gamma_j\}$ be a finite set of representatives for Γ/Λ. Then $\Sigma_j \gamma_j \cdot (\varphi + \psi)$ will be a Γ-map (where $\lambda \in \Gamma$ acts on $F(\partial T, M(\partial S))$ by $(\lambda \cdot f)(t) = \pi(\lambda) f(\lambda^{-1} t))$. Letting $\mu(t)$ be the measure that is equidistributed on supp $((\Sigma_j \gamma_j \cdot (\varphi + \psi))(t))$, we deduce that $\mu : \partial T \to M(\partial S)$ is a Γ-map. Furthermore, since $\varphi + \psi$ is not Γ-equivariant, we have $\mu : \partial T \to M(\partial S) - M_2(\partial S)$. Once again, the proof of Lemma 4.7 would imply $\pi(\Gamma)$ is compact, a contradiction. □

COROLLARY 4.10. *There is a measurable D-map $\varphi : \partial T \to M_2^0(\partial S)$.*

Proof. (Cf. [Zi, Lemma 6.2.7) If $d \in D$, let d act on $F(\partial T, M_0^2(\partial S))$ by $(d \cdot f)(t) = \pi(d) f(d^{-1}(t))$. Let φ be as in either Lemma 4.8 or Lemma 4.9. If $d \in D$, choose $\Lambda \subset \Gamma$ of finite index such that $d\Lambda d^{-1} \subset \Gamma$. The map $d \cdot \varphi$ is then $d\Lambda d^{-1}$ equivariant, and hence by 4.8 and 4.9 we have $d \cdot \varphi = \varphi$, i.e. φ is a D-map. □

COROLLARY 4.11. *There is a measurable D-map $\varphi : \bar{D} \to M_2^0(\partial S)$.*

Proof. Since Aut (T) is transitive on ∂T, we can view φ as a D-map Aut $(T) \to M_2^0(\partial S)$. Since $\bar{D} \subset$ Aut (T) is closed, by Fubini φ defines a measurable D-map on almost every $\bar{D}$-orbit in Aut (T). Since each of these can be identified with $\bar{D}$ itself as a $\bar{D}$-space, the result follows. □

Set $G = \bar{D}$.

Now consider the map $\Phi : G \to F(G, M_2^0(\partial S))$ given by $\Phi(g)(a) = \varphi(ga)$. Then for $d \in D$, $\Phi(dg)(a) = \varphi(dga) = \pi(d)\Phi(g)(a)$, i.e., Φ is a D-map, where D acts on $F(G, M_2^0(\partial S))$ by $(d \cdot f)(g) = \pi(d)f(g)$. If $\Phi(g) \in F_2(G, M_2^0(\partial S))$ for some g, then $\varphi : G \to M_2^0(\partial S)$ takes on at most 2 values. This implies that there is a set in ∂S with at most 4 points that is D-invariant. Since D acts minimally on ∂S and ∂S has at least 3 points by hypothesis, this would imply that there is a set with 3 or 4 points that is D-invariant. However, this implies that D leaves a finite set in S itself invariant, and hence that $\overline{\pi(D)}$ is compact. This also contradicts our assumptions. Therefore we deduce that for all $g \in G$, $\Phi(g) \in F(G, M_2^0(\partial S)) - F_2(G, M_2^0(\partial S))$.

By Lemma 4.5, Aut (S) acts tamely on $\mathcal{O} = F(G, M_0^2(\partial S)) - F_2(G, M_2^0(\partial S))$. The map $\tilde{\Phi} : G \to \mathcal{O}/\text{Aut}\,(S)$ is D-invariant (since Φ is a D-map) and by ergodicity of D on G, we deduce that $\tilde{\Phi}$ is essentially constant. Summarizing, we have:

LEMMA 4.12. *There is a co-null set $G_0 \subset G$ such that all $\Phi(g)$, $g \in G_0$, lie in a single* Aut (S)*-orbit in $\mathcal{O}$.* □

We will need the following property of Φ.

LEMMA 4.13. *Fix $g_0 \in G_0$. Then the stabilizer of $\Phi(g_0)$ in* Aut (S) *is trivial.*

Proof. Let $K \subset \text{Aut}\,(S)$ be the stabilizer. Thus, $K = \{A \in \text{Aut}\,(S) \mid A(\Phi(g_0)(a)) = \Phi(g_0)(a)$ for *a.e.* $a \in G\}$. Thus we clearly have

$$K = \{A \in \text{Aut}\,(S) \mid A(\varphi(a)) = \varphi(a) \text{ for } a.e.\ a \in G\}.$$

Then $K \subset \text{Aut}\,(S)$ is closed, and since φ is a D-map the essential range of φ is D-invariant which implies that $\pi(D) \subset N_{\text{Aut}\,(S)}(K)$. To prove the lemma, it therefore suffices to see that K fixes a point in ∂S. For then the set of K-fixed points on ∂S will be closed and D-invariant, and since D acts minimally on ∂S, K fixes all points on ∂S. Since Aut (S) is faithful on ∂S, this would imply K is trivial.

We recall $\varphi : D \to M_2^0(\partial S)$, and that we have seen in the discussion preceding Lemma 4.12 that φ is not essentially constant. Since each $A \in K$ fixes almost all $\varphi(g)$, it follows from Fubini that there is a co-null set $G_1 \subset G$ such that for $g \in G_1$ almost every element of K fixes $\varphi(g)$. Therefore K fixes $\varphi(g)$ for all $g \in G_1$. If $\varphi(G) \subset M_1(\partial S)(a.e.)$, then in particular K fixes a point of ∂S. Therefore, we are reduced to considering the case $\varphi(G) \subset M_2^0(\partial S) - M_1(\partial S)(a.e.)$, where we deduce that there are distinct 2-point sets, say $\{x, y\}$ and $\{z, w\}$, in ∂S, that are fixed by K.

Let α (resp β) be the set of points in S on the geodesic in S from x to y (resp. from z to w). Then α and β are K-invariant. If x, y, z, w are all distinct points, then the set of points $s \in S$ minimizing $d(s, \alpha) + d(s, \beta)$ is finite and K-invariant. This implies (perhaps by the usual argument of passing to the barycentric subdivision of S) that there is a K-fixed point in S. On the other hand, if $\{x, y, z, w\}$ has only 3 distinct points (say $y = w$), then $\alpha \cap \beta$ is a geodesic ray from a uniquely defined point $s_0 \in S$ to $y \in \partial S$. This point s_0 is then clearly K-fixed. Thus, in either case, we may assume there is a K-fixed point $s_0 \in S$. Since $\pi(D)$ normalizes K, $\pi(d)s_0$ is K-fixed for all $d \in D$, and since $\pi(D)$ is not compact, we can find $d_n \in D$ such that $\pi(d_n)s_0$ converges in $\bar{S} = S \cup \partial S$ to a point in ∂S. This will also be K-fixed, and this completes the proof of the lemma. □

Completion of Proof of Theorem 4.1. If $g \in G_0 g_0^{-1}$, then

$$\Phi(gg_0) = \sigma(g)\Phi(g_0) \qquad \text{for some } \sigma(g) \in \text{Aut}\,(S).$$

(This is simply the assertion that $\Phi(G_0) \subset \text{Aut}\,(S)(\Phi(g_0))$.) Furthermore, $\sigma(g)$ is the uniquely defined element of Aut (S) with this property by Lemma 4.13. It is then easy to verify that σ is measurable and $\sigma(gh) = \sigma(g)\sigma(h)$ for almost all $(g, h) \in G \times G$. It follows [Zi, B.2] that there is a homomorphism $\sigma_0 : G \to \text{Aut}\,(S)$ such that $\sigma_0 = \sigma$ *a.e.* This means that for *a.e.* g, $\varphi(gg_0a) = \sigma_0(g)\varphi(g_0a)$ for *a.e.* a, which is clearly equivalent to the assertion that $\varphi(ga) = \sigma_0(g)\varphi(a)$ for almost every a. Fix $d \in D$. Then for *a.e.* g, this holds for dg, i.e. $\varphi(dga) = \sigma_0(dg)\varphi(a)$ for *a.e.* a, so that $\varphi(dga) = \sigma_0(d)\sigma_0(g)\varphi(a)$ for *a.e.* a. For *a.e.*g, we have $\sigma_0(g)\varphi(a) = \varphi(ga)$ for *a.e.* a, so we deduce that for *a.e.* g, $\varphi(dga) = \sigma_0(d)\varphi(ga)$ for *a.e.* a. This clearly implies $\varphi(da) = \sigma_0(d)\varphi(a)$ for *a.e.* a, and hence that $\pi(d)^{-1}\sigma_0(d)$ fixes $\varphi(a)$ for *a.e.* a. However, the essential range of φ is a D-invariant closed subset of ∂S, and by minimality of the D-action on ∂S, the essential range of φ is all of ∂S. This implies $\pi(d)^{-1}\sigma_0(d)$ fixes all ∂S, and hence $\pi(d) = \sigma_0(d)$. This completes the proof. □

The methods of this section may be used to obtain the following.

THEOREM 4.14. *Let k be a local field of characteristic* 0 *and $\pi : D \to H_k$ where H is a non-compact almost simple k-group. Then either $\pi(\Gamma)$ is compact or π extends to $\bar{D}$.* □

§5. Linear representations of C

In this section we show that C has no faithful linear representation. We begin with a corollary of proposition 2.9 which is of independent interest:

PROPOSITION 5.1. *Let $\Gamma = \langle a_1 \rangle * \langle a_2 \rangle * \cdots * \langle a_k \rangle$ be as in Section 2. Let N be a non-trivial normal subgroup of C. Then N contains $\Gamma^+ = \Gamma \cap A^+$.*

Proof. Let $1 \neq n \in N$. Then there exists $1 \neq \delta \in \Gamma$ such that $\delta \neq n^{-1}\delta n \in \Gamma$. It follows that $\gamma = \delta^{-1}n^{-1}\delta n \in \Gamma \cap N$ is a nontrivial element. If γ is an element of order 2 then it is conjugate (in Γ and hence in C) to one of the generators a_{i_0} and it follows that $a_i \in N\ \forall 1 \leq i \leq k$ and we have $\Gamma \subset N$. Hence we can assume γ is of infinite order. It follows that γ is conjugate to some cyclically reduced element $\gamma' = a_{i_1} \dots a_{i_r} \in N \cap \Gamma$ where $r = l_T(\gamma) \geq 2$.

By Proposition 2.9 there exists some $c \in C$ such that $c^{-1}\gamma' c = a_{i_1} \dots a_{i_{r-1}} b$ where $b = a_j$ for some $j \neq i_1, i_r$. Hence the element $\gamma'^{-1} c^{-1} \gamma' c = a_{i_r} b$ is in $N \cap \Gamma$. We have $l_T(a_{i_r} b) = 2$ and by Proposition 2.9 we conclude that all the elements $\lambda \in \Gamma$ with $l(\lambda) = 2$ are in $N \cap \Gamma$. Since these generate Γ^+ it follows that $\Gamma^+ \subset N$. □

THEOREM 5.2. *In any linear representation of C over any field F, Γ has finite image. In particular C does not have a faithful linear representation.*

Proof. Let $\rho : C \to GL(n, F)$ be a linear representation. Denote by $G \subset GL(n, F)$ the Zariski closure of the image $\rho(C)$ of C. Let $R \lhd G$ be the radical of G. Consider the linear representation $\bar{\rho} : C \to G/R$. (We may view G/R as embedded in $GL(n, F)$). We will show that $\bar{\rho}(\Gamma)$ is finite. This will imply that the image $\rho(\Gamma)$ is solvable by finite and hence that $\Gamma \cap \ker \varphi \neq \{1\}$. By Proposition 5.1, this would show that $\Gamma/(\ker \varphi \cap \Gamma)$ is finite. Assume that $\bar{\rho}(\Gamma)$ is infinite. Our goal is to show that we can realize the representation of C in a linear representation over a finitely generated field. Hence we can replace, when convenient, C by a subgroup of finite index. Let H be the connected component of the Zariski closure of $\bar{\rho}(\Gamma)$. It is normalized by $\bar{\rho}(C)$ (and hence by G/R). H is semisimple since its radical is a characteristic subgroup and hence gives a normal solvable subgroup of G/R and must be trivial. It follows that we can replace C by a subgroup of finite index C' so that for $c \in C'$ the map $h \to \bar{\rho}(c)^{-1} h \bar{\rho}(c)$ is an inner automorphism of H. Notice also that since the centralizer in C of Γ is trivial we did not change the kernel. We can decompose the connected semisimple group H as a product of simple groups. It is enough to consider one such simple factor at a time. Thus we have a homomorphism $C' \to S$ where S is a connected simple group which is the Zariski closure of the image of Γ. Choose some linear irreducible representation of S and let $\pi : C' \to GL(V)$ be the composed representation. Since Γ is finitely generated it follows that $\pi(\Gamma)$ is a linear representation over some finitely generated field $\tilde{F}$. The image of any finite index subgroup of Γ is irreducible and hence by Burnside's theorem contains a basis for End (V). Given an element $c \in C'$ it conjugates some finite index sublattice of Γ into Γ and hence its adjoint action on End (V), when written as a matrix with respect to some fixed basis of End (V), gives a matrix with entries in the finitely generated field $\tilde{F}$.

Recall that a finitely generated field may be embedded inside a local field. Thus we have shown that if we have a linear representation of C such that $\rho(\Gamma)$ is infinite then there exists a linear representation t of C over a totally disconnected local field (with $t(\Gamma)$ infinite). We shall need the following lemma.

LEMMA 5.3. *Let k be a totally disconnected local field. Then the orders of finite order elements in $GL(m, k)$ are bounded.*

We postpone the proof of the lemma. Using the arguments in the proof of 2.6 (see also remark 2.7) we can construct torsion elements in C of arbitrarily large order. In particular we obtain a non-trivial torsion element $c \in C$ such that $t(c) = 1$. There exists some $1 \neq \gamma \in \Gamma$ so that $\gamma \neq c^{-1}\gamma c \in \Gamma$. It follows that $1 \neq c^{-1}\gamma c\gamma^{-1} \subset \Gamma \cap \operatorname{Ker} t$. Using Proposition 5.1 it follows that $t(\Gamma)$ is finite and this completes the proof of Theorem 5.2. □

Proof of Lemma 5.3. Let $v : k \to \mathbb{Z}$ be a discrete valuation on k, $O \subset k$ its ring of integers and F_g its residue field, $g = p^r$. A finite order element in $GL(m, k)$ is conjugate to an element in $GL(m, O)$, which is a virtually pro-p group. If char $k = 0$ then, as is well known, it has a finite index torsion free subgroup. So we may assume char $k = p > 0$. Since $GL(m, O)$ is virtually pro-p the only possible high torsion may come from high powers of p. An element $g \in GL(m, O)$ such that $g^{p^k} = 1$ is necessarily unipotent and unipotent elements are of order p. □

We remark that Theorem 5.2 does not apply to arbitrary non-discrete subgroups D with $\Gamma \subset D \subset C$. For example consider the group D of rational quaternions as embedded in $PGL_2(\mathbb{Q}_p) \subset \operatorname{Aut}(T_{p+1})$. This group clearly admits faithful linear representations. However using Theorem 4.14 one can show that if $\Gamma \subset D \subset C$ and D is dense in A then there does not exist a faithful linear representation of D over a field of characteristic 0.

§6. Relation with manifolds of negative curvature

Let T be a k-regular tree and $\Gamma \subset \operatorname{Aut}(T)$ a lattice. Suppose $\Gamma \subset D \subset C(\Gamma)$. We wish to investigate when D may be the fundamental group of a complete Riemannian manifold of negative curvature. The main result of this section is the following.

THEOREM 6.1. *Suppose* $[D : \Gamma] = \infty$. *Then D is not isomorphic to the fundamental group of a complete Riemannian manifold of curvature K with* $-b^2 \leq K \leq -a^2$ *where* $a, b \in \mathbb{R}$, $a > 0$.

REMARKS.

(a) Since free groups are fundamental groups of negatively curved manifolds, clearly Γ itself may be such a fundamental group.

(b) It follows from Theorem 6.1 that if $\Gamma \subset D \subset C(\Gamma)$ with $[D : \Gamma] = \infty$, then D is not free. This can also be shown directly by purely group theoretic arguments.

(c) If D is a lattice in a higher rank semisimple group, then D is also not isomorphic to such a fundamental group. This can be shown in a number of ways, e.g. see [SZ].

The proof of Theorem 6.1 uses many of the same arguments as in section 4. Therefore, we shall only indicate here what additional argument is necessary. The basic step in proving 6.1 is the following analogue of Theorem 4.1.

LEMMA 6.2. *Let D be as in Theorem* 6.1 *and set* $G = \bar{D}$, *the closure in* Aut (T). *Let M be a complete simply connected Riemannian manifold with sectional curvature bounded away from* 0 *and* $-\infty$. *Let* $\pi : D \to \mathrm{Iso}(M)$ *be a homomorphism. Then either*:

(i) $\pi(D)$ *is compact*; *or*,

(ii) $\pi(D)$ *fixes a* 1 *or* 2 *point set in* ∂M; *or*,

(iii) π *extends to a continuous homomorphism* $G \to Q$ *where Q is a subquotient of* Iso (M) *defined as follows.*

There is a measurable D-map $\varphi : G \to M_2^0(\partial M)$ *(with D acting on* ∂M *via* π*). Let H be the subgroup of* Iso (M) *pointwise fixing the essential range of* φ, $N = N_{\mathrm{Iso}(M)}(H)$ *and* $Q = N/H$. *Then* $\pi(D) \subset N$ *and the projection* $D \to Q$ *of* π *extends to G.*

To prove Lemma 6.2, we need the following well-known analogue of Lemma 4.2.

LEMMA 6.3. *If* $h_n \in \mathrm{Iso}(M)$ *and* $h_n \to \infty$, *then by passing to a subsequence we can find* $x, y \in \partial M$ *such that for* $z \in \partial M$, $z \neq x$, *we have* $h_n(z) \to y$.

The proof of Lemma 6.2 now follows that of Theorem 4.1, using Lemma 6.3 in place of Lemma 4.2, with the exception that Lemma 4.13 is not available. (We recall that the proof of 4.13 used that $\pi(D)$ acted minimally on ∂S, which we do not assume here.) However, the argument following the proof of Lemma 4.13 yields conclusion (iii) above.

We now prove Theorem 6.1 from Lemma 6.2. If D is the fundamental group of a manifold of negative curvature, let $\pi : D \to \mathrm{Iso}(M)$ the discrete embedding as isometries of the universal covering. Since D is infinite, clearly $\overline{\pi(D)}$ cannot be compact. The subgroup of Iso (M) fixing a 1 or 2 point set is amenable and since D contains a free group and $\pi(D)$ is discrete, we clearly cannot have (ii) in Lemma 6.2. It therefore remains to show that (iii) is impossible. Since G is totally disconnected and not discrete (since D is not discrete owing to the fact that Γ is a lattice and $[D : \Gamma] = \infty$), we can choose a compact open infinite subgroup $K \subset G$. The group Q is a real Lie group and hence the homomorphism $K \to Q$ given by (iii) of Lemma 6.2 is trivial on a subgroup $K_0 \subset K$ of finite index. The group $D_0 = D \cap K_0$ is infinite and we have $\pi(D_0) \subset H$. Now consider the essential range of φ. This is a $\pi(D)$-invariant subset of $M_2^0(\partial M)$. If it intersects $\partial M \cong M_1(\partial M) \subset M_2^0(\partial M)$, it must contain at least 3 points in ∂M, otherwise we are in situation (ii) which we have shown is impossible. But if it contains at least 3 points in ∂M then H is compact by Lemma 6.3. Since $\pi(D_0) \subset H$ and $\pi(D_0)$ is infinite and discrete, this is also impossible. So we can assume the essential range is in $M_2^0(\partial M) - M_1(\partial M)$. Once again, if the essential range is a single point, we are in situation (ii), or if not, H permutes a 3 or 4 point set in ∂M. This implies H is compact, completing the proof as above.

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