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On the convergence of normalizations of real analytic surfaces near hyperbolic complex tangents

XIANGHONG GONG

1. Introduction

Let $M \subset \mathbb{C}^2$ be a real analytic surface with a non-degenerate complex tangent at p. After a change of coordinates, we may assume that $p = (0, 0)$ and M has the form

$$M : z_2 = z_1\bar{z}_1 + \gamma z_1^2 + \bar{\gamma}\bar{z}_1^2 + H(z_1, \bar{z}_1), \tag{1.1}$$

where $H(z_1, \bar{z}_1)$ is a convergent power series in z_1 and $\bar{z}_1$ starting with the third order terms, and $0 \leq \gamma < \infty$ is the Bishop invariant [3]. The complex tangent is said to be *elliptic* if $0 \leq \gamma < 1/2$, *parabolic* if $\gamma = 1/2$, or *hyperbolic* if $\gamma > 1/2$. Let λ be a root of $\gamma\lambda^2 - \lambda + \gamma = 0$. Then $|\lambda| = 1$ for $1/2 < \gamma < \infty$. We say that γ is *exceptional* if λ is a root of unity. When $0 < \gamma < 1/2$, or $\gamma > 1/2$ and non-exceptional, J. K. Moser and S. M. Webster [9] proved that, under the group of formal transformations of $\mathbb{C}^2$, M can be transformed into a surface given by

$$x_2 = z_1\bar{z}_1 + (1 + \epsilon x_2^s)(\gamma z_1^2 + \gamma\bar{z}_1^2), \qquad y_2 = 0, \tag{1.2}$$

in which $\epsilon = 0$, or $\epsilon = \pm 1$ with s a positive integer. The character of complex tangents plays an important role in the problem of convergence. They showed that the formal normal form (1.2) can be realized through biholomorphic mappings if $0 < \gamma < 1/2$, while the divergence of normalizations occurs, as a rule, near hyperbolic complex tangents.

In this paper, we shall first consider real analytic surfaces which are formally equivalent to a quadric

$$Q_\gamma : z_2 = z_1\bar{z}_1 + \gamma z_1^2 + \gamma\bar{z}_1^2.$$

This corresponds to the case $\epsilon = 0$ in (1.2). One says that λ satisfies the *Diophantine* *condition*, if λ is not a root of unity, and

$$|\lambda^n - 1| > \frac{1}{c_0 n^\delta}, \qquad n \in \mathbb{Z}_+, \tag{1.3}$$

in which δ and c_0 are positive constants with $c_0 > 1$. We have

THEOREM 1.1. *Let M, given by (1.1), be a real analytic surface with a hyperbolic complex tangent at 0. Assume that M is formally equivalent to a quadric Q_γ. Then M is actually equivalent to the quadric through biholomorphic mappings, provided that its corresponding λ is a root of unity, or satisfies the Diophantine condition.*

In [7], W. Klingenberg proved the existence of a pair of asymptotic curves on surfaces near a hyperbolic complex tangent, where only the Diophantine condition (1.3) is assumed. It is unknown whether Theorem 1.1 still holds without the Diophantine condition. Apparently, one encounters with a problem analogous to the normalization for area-preserving mappings [11].

A real analytic surface in $\mathbb{C}^2$ is said to be *holomorphically flat* if it can be transformed into a real hyperplane through biholomorphic mappings. A fundamental result in [9] is the discovery of a pair of involutions intrinsically attached to a real analytic surface near a non-degenerate complex tangent. The holomorphic flatness of a real analytic surface corresponds to the existence of non-constant holomorphic functions invariant under its pair of involutions. We notice that examples of real analytic surfaces with hyperbolic complex tangents, which cannot be holomorphically flattened, were constructed by E. Bedford [1], Moser and Webster [9]. However, our next result indicates that the holomorphic flatness is not equivalent to the convergence of normalizations for surfaces near hyperbolic complex tangents. This contrasts with the theory of Hamiltonian systems or symplectic mappings, where the convergence of normalizations is equivalent to the existence of invariant functions (see [2], [14], [6]).

We now state the following:

THEOREM 1.2. *Let ϵ and $\epsilon_{i,j}$, with $i+j \geq 3$, be positive numbers. Suppose that M is a holomorphically flat real analytic surface given by (1.1) with $H(z_1, \bar{z}_1) = \bar{H}(\bar{z}_1, z_1)$. Assume that $1/2 < \gamma < \infty$. Then there exists a holomorphically flat real analytic surface*

$$\hat{M} : z_2 = q_{\hat{\gamma}}(z_1, \bar{z}_1) + \hat{H}(z_1, \bar{z}_1), \qquad \hat{H}(z_1, \bar{z}_1) = \bar{\hat{H}}(\bar{z}_1, z_1) = O(3),$$

which cannot be transformed into the Moser–Webster normal form through any biholomorphic mapping, while $|\hat{\gamma} - \gamma| < \epsilon$, *and* $|\hat{H}_{i,j} - H_{i,j}| < \epsilon_{i,j}$ *for* $i + j \geq 3$.

The paper is arranged as follows. In section 2 we state an analogue of Theorem 1.1 in n dimensional case, which is reduced to the problem of the linearization for an elliptic pair of involutions. The convergence proof, for the latter, is given in sections 3 and 4, where a KAM argument is used to deal with the small divisors problem. In section 5, we discuss the divergence of normalizations for elliptic pairs of involutions with a fixed invariant function, where the small divisors are used in a similar way as in [4], [12]. The proof of Theorem 1.2 is given in section 6.

2. A pair of involutions

Throughout the discussion of the paper, a *formal* transformation of $\mathbb{C}^n$ is defined by $f(z) = (f_1(z), \ldots, f_n(z))$, where $f_k(z)$ is a formal power series in z without the constant term, and the linear part of $f(z)$ is an invertible linear transformation. Let $M \subset \mathbb{C}^n$ be a smooth real submanifold of dimension n. Assume that $p \in M$ is a complex tangent point of M, i.e. $T_pM \cap JT_pM \neq \{0\}$. For $\omega = dz_1 \wedge \cdots \wedge dz_n$, write

$$\omega\,|_M = \rho v,$$

where v is a real volume form of M, and ρ is a $\mathbb{C}$-valued smooth function. We have $\rho(p) = 0$. The non-degeneracy for the complex tangent of M at p is defined by

$$d\rho(p) \neq 0, \qquad \text{on } T_pM \cap JT_pM. \tag{2.1}$$

One can see that $T_pM \cap JT_pM$ is a two dimensional real space. Let $T_p^{(1,0)}$ be the space of vectors of type (1,0) in $T_pM \cap JT_pM \otimes \mathbb{C}$. Then the Bishop invariant (see [15]) is given by

$$\gamma = \frac{1}{2}\left|\frac{\bar{X}\rho}{X\rho}\right|,$$

where X spans $T_p^{(1,0)}M$.

Consider the quadric in $\mathbb{C}^n$ defined by

$$Q_\gamma : \begin{array}{l} z_n = q_\gamma(z_1, \bar{z}_1), \\ y_\alpha = 0, \qquad 2 \leq \alpha \leq n, \end{array} \tag{2.2}$$

where $q_\gamma(z_1, \bar z_1) = z_1\bar z_1 + \gamma z_1^2 + \gamma \bar z_1^2$ for $1/2 < \gamma < \infty$, and $q_\infty(z_1, \bar z_1) = z_1^2 + \bar z_1^2$. Put $p = 0$. We now assume that M is equivalent to (2.2) through a formal transformation. By truncating the formal transformation, we may further assume that, for a biholomorphic change of coordinates, M is given by

$$\begin{cases} z_n = q_\gamma(z_1, \bar z_1) + H(z_1, \bar z_1, x), & x = (x_2, \dots, x_{n-1}), \\ y_\alpha = f_\alpha(z_1, \bar z_1, x), & 2 \le \alpha \le n-1, \end{cases} \tag{2.3}$$

where each of $H(z_1, \bar z_1, x)$ and $f_\alpha(z_1, \bar z_1, x)$ is a convergent power series starting with the third order terms. For $1/2 < \gamma < \infty$, we let γ be a root of

$$\gamma\lambda^2 - \lambda + \gamma = 0.$$

It is easy to see that $|\lambda| = 1$. We assign $\lambda = \sqrt{-1}$ for $\gamma = \infty$.

An analogue of Theorem 1.1 in the higher dimensional case is the following.

THEOREM 2.1. *Let M be a real analytic submanifold given by* (2.3) *with $1/2 < \gamma \le \infty$. Suppose that the corresponding λ is a root of unity, or satisfies the Diophantine condition. Assume that M is formally equivalent to Q_γ. Then there exists a biholomorphic mapping which transforms M into Q_γ.*

One notices that, under the non-degeneracy assumption (2.1), the complex tangent may further satisfy one of the following degeneracy conditions:

(i) $d\rho = 0$ on $T_p^{(1,0)}M$,
(ii) $d\bar\rho = 0$ on $T_p^{(1,0)}M$,
(iii) $d\rho \wedge d\bar\rho = 0$ on T_pM,

which correspond to the cases $\gamma = \infty$, $\gamma = 0$ and $\gamma = 1/2$, respectively. Theorem 2.1 includes the first case. The case $\gamma = 0$ was discussed by J. K. Moser in [8], where an analogue of Theorem 2.1 for $\gamma = 0$ was proved. For the parabolic complex tangent, a typical case of real Lagrangian surfaces was also investigated by S. M. Webster in [16]. The real Lagrangian surfaces of complex tangents are always formally equivalent to the quadric $Q_{1/2}$ under unimodular transformations. However, one may not be able to transform the surfaces into $Q_{1/2}$ by any convergent mapping, as shown in [5].

The proof of Theorem 2.1 is based on the theory about a pair of involutions introduced in [9]. To describe the intrinsic involutions, we replace $(z, \bar z)$ by (z, w) in

(2.3) and consider the complexified submanifold in $\mathbb{C}^{2n}$ as follows

$$M^c : \begin{cases} z_n = q(z_1, w_1) + H(z_1, w_1, x), \\ w_n = q(z_1, w_1) + \bar{H}(w_1, z_1, x), \\ z_\alpha - w_\alpha = 2if_\alpha(z_1, w_1, x), \\ 2x_\alpha = z_\alpha + w_\alpha, \quad 2 \le \alpha \le n-1. \end{cases} \tag{2.4}$$

We shall use (z_1, w_1, x) as the coordinates to identify M^c with $\mathbb{C}^n$. The projections $\pi_1(z, w) = z$ and $\pi_2(z, w) = w$, restricted to M^c, induce two double-sheeted branched coverings. Let $(z_1', w_1', x') = \tau_1(z_1, w_1, x)$ be the covering transformations for π_2. Then we have

$$q(z_1', w_1) + \bar{H}(w_1, z_1', x') = q(z_1, w_1) + \bar{H}(w_1, z_1, x),$$
$$x_\alpha' - if_\alpha(z_1', w_1, x') = x_\alpha - if_\alpha(z_1, w_1, x), \qquad 2 \le \alpha \le n-1.$$

By the implicit function theorem, we get

$$\tau_1 : \begin{cases} z_1' = -z_1 - \dfrac{1}{\gamma} w_1 + O(2), \\ w_1' = w_1, \\ x_\alpha' = x_\alpha + O(2). \end{cases}$$

Let ρ be the restriction of the anti-holomorphic involution $(z, w) \mapsto (\bar{w}, \bar{z})$ to M^c. Then $\rho(z_1, w_1, x) = (\bar{w}_1, \bar{z}_1, \bar{x})$. Notice that M^c is invariant under ρ, and $\pi_2 = c \circ \pi_1 \circ \rho$ for $c(z) = \bar{z}$. Hence, the covering transformation τ_2 for π_1 satisfies the relation $\tau_2 = \rho\tau_1\rho$.

We change the notation and let

$$\xi = \frac{i\lambda^{1/2}}{1-\lambda^2}(\lambda z_1 + w_1), \qquad \eta = -\frac{i\lambda^{1/2}}{1-\lambda^2}(z_1 + \lambda w_1),$$
$$\zeta_\alpha = x_\alpha, \qquad 2 \le \alpha \le n-1. \tag{2.5}$$

Under the new coordinates, the pair of involutions is given by

$$\tau_j : \begin{cases} \xi' = \lambda_j \eta + f_j(\xi, \eta, \zeta), \qquad \lambda_1 = \lambda = \bar{\lambda}_2, \qquad |\lambda| = 1, \\ \eta' = \bar{\lambda}_j \xi + g_j(\xi, \eta, \zeta), \\ \zeta_\alpha' = \zeta_\alpha + h_{j;\alpha}(\xi, \eta, \zeta), \qquad 2 \le \alpha \le n-1, \end{cases} \tag{2.6}$$

in which f_j, g_j, $h_{j;\alpha}$ start with the second order terms. Meanwhile, the anti-holomorphic involution ρ and reality condition on τ_1 and τ_2 are given by

$$\rho(\xi, \eta, \zeta) = (\bar{\xi}, \bar{\eta}, \bar{\zeta}), \qquad \tau_2(\xi, \eta, \zeta) = \rho\tau_1\rho(\xi, \eta, \zeta). \tag{2.7}$$

We say that a biholomorphic mapping f satisfies the *reality condition* if $\rho f \rho = f$. The intrinsic property of the pair of involutions, generated by a real analytic submanifold in the form (2.3), implies that two submanifolds are equivalent through biholomorphic mappings in $\mathbb{C}^n$, if and only if their corresponding pairs of involutions are equivalent through a biholomorphic mapping which satisfies the reality condition (see [9], p. 263).

For a quadric Q_γ with $1/2 < \gamma \leq \infty$, it is easy to see that its pair of involutions is given by

$$\tau_j^* : \begin{cases} \xi' = \lambda_j \eta, & \lambda_1 = \lambda = \bar{\lambda}_2, \qquad |\lambda| = 1, \\ \eta' = \bar{\lambda}_j \xi, & \\ \zeta'_\alpha = \zeta_\alpha, & 2 \leq \alpha \leq n-1. \end{cases} \tag{2.8}$$

Thus, Theorem 2.1 is reduced to

THEOREM 2.2. *Let $\{\tau_1, \tau_2, \rho\}$ be a pair of involutions given by* (2.6) *and* (2.7). *Suppose that λ is a root of unity, or satisfies the Diophantine condition. Assume that there exists a formal transformation Ψ such that $\Psi \tau_j \Psi^{-1} = \tau_j^*$. Then there is a biholomorphic mapping which satisfies the reality condition and transforms $\{\tau_1, \tau_2\}$ into $\{\tau_1^*, \tau_2^*\}$.*

The proof of the theorem will be given in the following two sections.

3. Basic estimates

We need some notations. For a multi-index $I = (i_2, \ldots, i_{n-1})$, put

$$|I| = i_2 + \cdots + i_{n-1}, \qquad \zeta^I = \zeta_2^{i_2} \ldots \zeta_{n-1}^{i_{n-1}}.$$

If λ is not a root of unity, we let $m = 0$; otherwise, we choose m to be the smallest positive integer with $\lambda^{2m} = 1$. For a formal power series $a(\xi, \eta, \zeta_2, \ldots, \zeta_{n-1})$, we use the following notations

$$a(\xi, \eta, \zeta) = \sum a_{i,j,I}\xi^i\eta^j\zeta^I, \tag{3.1}$$

$$[a]_k(\xi, \eta, \zeta) = \sum_{i+j+|I|=k} a_{i,j,I}\xi^i\eta^j\zeta^I, \tag{3.2}$$

$$P_\epsilon a(\xi, \eta, \zeta) = \sum_{i-j=\epsilon \bmod (m)} a_{i,j,I}\xi^i\eta^j\zeta^I, \qquad \epsilon = 0, \pm 1. \tag{3.3}$$

Let ψ be a transformation given by

$$\psi: \begin{cases} \xi' = \xi + u(\xi, \eta, \zeta), \\ \eta' = \eta + v(\xi, \eta, \zeta), \\ \zeta' = \zeta + w(\xi, \eta, \zeta), \end{cases} \tag{3.4}$$

where u, v, w are power series starting with the second order terms. ψ is said to be *partially normalized* if

$$P_{-1}v = 0 = P_0 w_\alpha, \qquad 2 \le \alpha \le n-1. \tag{3.5}$$

We also call ψ a *normalized* transformation if it satisfies a further condition $P_{+1}u = 0$.

Let $\{\tau_1, \tau_2, \rho\}$ be as in Theorem 2.2. Assume that Ψ is a formal transformation such that $\Psi\tau_j\Psi^{-1} = \tau_j^*$. Then

$$d\Psi(0) \circ \tau_j^* \circ d\Psi(0)^{-1} = \tau_j^*.$$

Hence, $d\Psi(0)^{-1} \circ \Psi$ linearizes τ_1 and τ_2. From now on, we assume that $\{\tau_1, \tau_2\}$ can be linearized by a formal transformation

$$\Psi(\xi, \eta, \zeta) = (\xi, \eta, \zeta) + (U(\xi, \eta, \zeta), V(\xi, \eta, \zeta), W(\xi, \eta, \zeta)),$$

in which U, V, W start with the second order terms. By $\Psi\tau_j\Psi^{-1} = \tau_j^*$, we see that U, V, W satisfy

$$\begin{aligned} &\lambda_j V - U \circ \tau_j = f_j, \\ &\bar{\lambda}_j U - V \circ \tau_j = g_j, \\ &W - W \circ \tau_j = h_j, \qquad j = 1, 2. \end{aligned} \tag{3.6}$$

Following V. A. Pliss [10], we consider the following approximate equations

$$\begin{aligned} \lambda_j v - u \circ \tau_j^* &= [f_j]_d^{2d-2}, \\ \bar{\lambda}_j u - v \circ \tau_j^* &= [g_j]_d^{2d-2}, \\ w - w \circ \tau_j^* &= [h_j]_d^{2d-2}, \qquad j = 1, 2, \end{aligned} \tag{3.7}$$

in which d denotes the lowest order of non-vanishing homogeneous terms of f_j, g_j, h_j $(j = 1, 2)$, and

$$[f_j]_d^{2d-2} = [f_j]_d + \cdots + [f_j]_{2d-2}, \qquad \text{etc.}$$

The existence of a solution (u, v, w) to (3.7) will not follow directly from the theory of normal forms given in [9], because here λ can be a root of unity. The assumption that (3.6) has a formal solution (U, V, W) will be used to prove the existence of a solution (u, v, w) to (3.7). In fact, we shall prove that there is a unique solution (u, v, w) to (3.7), which satisfies the normalizing condition (3.5). Moreover, the corresponding transformation ψ defined by (3.4) satisfies the reality condition, whenever τ_1 and τ_2 satisfy the reality condition (2.7). Eventually, through the method of rapid iterations (i.e. KAM theory), we shall construct a sequence of transformations, whose formal limit transformation is actually convergent and linearizes the pair of involutions $\{\tau_1, \tau_2\}$.

There are $2n$ equations in (3.7) for n unknowns u, v and w_α $(2 \le \alpha \le n - 1)$. To solve for u, v and w_α, we need to find the compatibility conditions on the right hand side of (3.7). For simplicity, let us drop the subscript d and superscript $2d - 2$ for a moment. Obviously, the first two equations and the third one in (3.7) imply

$$g_j + \bar{\lambda}_j f_j \circ \tau_j^* = 0 = h_j + h_j \circ \tau_j^*, \qquad j = 1, 2. \tag{3.8}$$

Given (3.8), one can then get rid of the second equation in (3.7). Now it is easy to see that (3.7) implies

$$\begin{aligned} u \circ \tau_1^* \circ \tau_2^* - \lambda^2 u &= -f_1 \circ \tau_2^* + \lambda^2 f_2 \circ \tau_2^*, \\ v \circ \tau_1^* \circ \tau_2^* - \bar{\lambda}^2 v &= \bar{\lambda} f_1 \circ \tau_1^* \circ \tau_2^* - \bar{\lambda} f_2, \\ w \circ \tau_1^* \circ \tau_2^* - w &= -h_1 \circ \tau_2^* - h_2. \end{aligned} \tag{3.9}$$

Notice that for a power series f, we have $P_0(f \circ \tau_j) = P_0 f$. Thus, (3.8) implies that the last equation in (3.9) is solvable. The first two equations in (3.9) are solvable if and only if

$$P_1(-f_1 \circ \tau_2^* + \lambda^2 f_2 \circ \tau_2^*) = 0 = P_{-1}(\bar{\lambda} f_1 \circ \tau_1^* \circ \tau_2^* - \bar{\lambda} f_2).$$

Since $P_{\pm 1}(f \circ \tau_j^*) = (P_{\mp 1} f) \circ \tau_j^*$, the above is equivalent to

$$P_{-1}(f_1 - \lambda^2 f_2) = 0. \tag{3.10}$$

As an immediate consequence of (3.5) and (3.10), the first equation of (3.7) gives

$$P_1 u = -P_1(f_1 \circ \tau_1^*). \tag{3.11}$$

It is easy to see that, under the condition (3.5), $u - P_1 u$, v and w_α are uniquely determined by (3.9). Thus, (3.8) and (3.10) are the compatibility conditions to solve u, v, w from (3.7). Furthermore, u, v and w are uniquely determined by (3.5), (3.9) and (3.11).

LEMMA 3.1. *Let $\{\tau_j, \rho\}$ be as in Theorem 2.2. Then* (3.7) *has a unique solution* (u, v, w) *satisfying* (3.5). *Moreover, the transformation* $\psi(\xi, \eta, \zeta) = (\xi + u, \eta + v, \zeta + w)$ *satisfies the reality condition* $\psi \circ \rho = \rho \circ \psi$.

Proof. We need to verify (3.8) and (3.10). Assume that there is a formal transformation Ψ which linearizes $\{\tau_j\}$. Then

$$\Psi \circ \tau_j = \tau_j^* \circ \Psi.$$

Comparing the terms of both sides up to order $d - 1$, we get

$$\Psi_0 \circ \tau_j^* = \tau_j^* \circ \Psi_0,$$

in which $\Psi_0 = \Sigma_{k=1}^{d-1} [\Psi]_k$. Therefore, replacing Ψ by $\Psi_0^{-1} \circ \Psi$, we may assume that

$$\Psi(\xi, \eta, \zeta) = (\xi + U(\xi, \eta, \zeta), \eta + V(\xi, \eta, \zeta), \zeta + W(\xi, \eta, \zeta))$$

linearizes τ_1, τ_2, while U, V and W contain no terms of order less than d.

We now see that the linearized equation (3.7) is obtained from (3.6) by collecting terms up to order $2d - 2$. Hence, $u = [U]_d^{2d-2}$, $v = [V]_d^{2d-2}$ and $w = [W]_d^{2d-2}$ satisfy (3.7). Thus, we have verified the compatibility conditions (3.8) and (3.10).

Next, we show that $\psi(\xi,\eta,\zeta)=(\xi+u(\xi,\eta,\zeta),\eta+v(\xi,\eta,\zeta),\zeta+w(\xi,\eta,\zeta))$ satisfies the reality condition. The reality condition means that u, v, w have real coefficients. Conjugating the first equation in (3.7), we get

$$\bar{\lambda}_j\bar{v}-\bar{u}\circ\bar{\tau}_j=\bar{f}_j.$$

Notice that $\bar{\lambda}_1=\lambda_2$. Since $\tau_2=\rho\tau_1\rho$, we have $\bar{\tau}_2=\tau_1$ and $\bar{f}_2=f_1$. Therefore, we see that $\bar{u}$ and $\bar{v}$ satisfy the first equation in (3.7). Similarly, one can also show that $\bar{u}$, $\bar{v}$ and $\bar{w}$ satisfy the rest of equations in (3.7). Clearly, (3.5) implies that $P_{-1}\bar{v}=P_0\bar{w}_\alpha=0$. Now, the uniqueness of solutions to (3.7) implies that

$$\bar{u}=u,\qquad \bar{v}=v,\qquad \bar{w}=w.$$

Therefore, the transformation ψ satisfies the reality condition. □

Let us keep the notations in Lemma 3.1 and set

$$\hat{\tau}_j=\psi\circ\tau_j\circ\psi^{-1}=\tau_j^*+(\hat{f}_j,\hat{g}_j,\hat{h}_j),\qquad j=1,2. \tag{3.12}$$

Then we have

$$\begin{aligned}\hat{f}_j&=u\circ\tau_j\circ\psi^{-1}-\lambda_jv\circ\psi^{-1}+f_j\circ\psi^{-1},\\ \hat{g}_j&=v\circ\tau_j\circ\psi^{-1}-\bar{\lambda}_ju\circ\psi^{-1}+g_j\circ\psi^{-1},\\ \hat{h}_j&=w\circ\tau_j\circ\psi^{-1}-w\circ\psi^{-1}+h_j\circ\psi^{-1}.\end{aligned} \tag{3.13}$$

Notice that u, v, w start with terms of order d. One can obtain from (3.13) that

$$[\hat{f}_j]_d^{2d-2}=u\circ\tau_j^*-\lambda_jv+[f_j]_d^{2d-2}=0,$$

in which (3.7) is used. Similarly, one can show that $\hat{g}_j$ and $\hat{h}_j$ start with terms of order $2d-1$. Denote by $\operatorname{ord}(\tau)$ the lowest order of non-vanishing terms of f_j, g_j and $h_{j;\alpha}$. Then we obtain

$$\operatorname{ord}(\hat{\tau})\geq 2\operatorname{ord}(\tau)-1. \tag{3.14}$$

The above estimate is crucial to apply the KAM method, which means that the original functional equation is well approximated by the approximate equations (3.7) at least on the level of formal power series.

Let $\Delta_r = \{(\xi, \eta, \zeta) \in \mathbb{C}^n \mid |\xi|, |\eta|, |\zeta| \le r\}$. Given holomorphic functions $f_1, f_2, \dots, f_n$ defined near Δ_r, we denote

$$\|(f_1, \dots, f_n)\|_r = \max\{|f_j(z)|;\ z \in \Delta_r,\ 1 \le j \le n\}.$$

For τ_1 and τ_2 given by (2.6), we put

$$\|\tau - \tau^*\|_r = \max\{\|\tau_1 - \tau_1^*\|_r, \|\tau_2 - \tau_2^*\|_r\}.$$

We now give an estimate for the solution to (3.7) as follows.

LEMMA 3.2. *Let (u, v, w) be the solutions to* (3.7), *which satisfies the condition* (3.5). *Assume that λ is a root of unity, or satisfies the Diophantine condition* (1.3). *If τ_j is holomorphic near Δ_r, then we have*

$$\|(u, v, w)\|_{(1-\theta)r} \le c_0 c_1 \frac{\|\tau - \tau^*\|_r}{\theta^l}, \qquad l = [\delta] + 2 + n,$$

in which c_1 depends only on δ and n, and $c_1 > 1$.

Proof. From (3.9), we have

$$u_{i,j,I} = \frac{\lambda^{j-i}}{\lambda^{2(i-j)} - \lambda^2}\{\lambda^2 f_{2;j,i,I} - f_{1;j,i,I}\}, \qquad i - j \ne 1 \bmod (m),$$

$$v_{i,j,I} = \frac{\bar{\lambda}}{\lambda^{2(i-j)} - \bar{\lambda}^2}\{\lambda^{2(i-j)} f_{1;i,j,I} - f_{2;i,j,I}\}, \qquad i - j \ne -1 \bmod (m),$$

$$w_{\alpha;i,j,I} = \frac{1}{\lambda^{2(i-j)} - 1}\{\lambda^{i-j} h_{2,\alpha;j,i,I} + h_{1;\alpha;i,j,I}\}, \qquad i - j \ne 0 \bmod (m).$$

From (3.11), we get

$$u_{i,j,I} = -\bar{\lambda} f_{1;j,i,I}, \qquad i - j = 1 \bmod (m).$$

In view of the Diophantine condition on λ and Cauchy inequalities, we obtain

$$\begin{aligned} |u_{i,j,I}| &\le c_0 2^\delta (i + j + 1)^\delta (|f_{1;j,i,I}| + |f_{2;j,i,I}|) \\ &\le c_0 2^\delta (k+1)^\delta \frac{\|f_1\|_r + \|f_2\|_r}{r^k} \le c_0 2^{\delta+1} (k+1)^\delta \frac{\|\tau - \tau^*\|_r}{r^k}, \end{aligned}$$

in which $k = i + j + |I|$. Hence, for $(\xi, \eta, \zeta) \in \Delta_{(1-\theta)r}$, we have

$$\left| \sum_{i+j=|I|=d}^{2d-2} u_{i,j,I} \xi^i \eta^j \zeta^I \right| \leq c_0 2^{\delta+1} \|\tau - \tau^*\|_r \sum_{k=d}^{2d-2} (k+1)^{n+\delta} (1-\theta)^k$$

$$\leq c_0 2^{\delta+1} \|\tau - \tau^*\|_r \sum_{k=0}^{\infty} \frac{(k+l-1)!}{k!} (1-\theta)^k$$

$$= \frac{c_0 c_1}{\theta^l} \|\tau - \tau^*\|_r, \qquad l = n + [\delta] + 2,$$

where $c_1 = (l-1)! 2^{\delta+1}$. One can obtain similar estimates for $w_2, w_3, \ldots, w_{n-1}$ and v, so the lemma is proved. □

To control the involutions defined by (3.12), we need the following.

LEMMA 3.3. *Let $\psi(z)$ be holomorphic on $\Delta_{(1-\theta)r}$. Assume that*

$$\|\psi - \mathrm{Id}\|_{(1-\theta)r} \leq \frac{1}{n} \theta^2 r.$$

Then for $0 < r < 1$ and $0 < \theta < 1/3$, ψ has an inverse on $\Delta_{(1-3\theta)r}$, and

$$\psi^{-1}(\Delta_{(1-3\theta)r}) \subset \Delta_{(1-2\theta)r}, \quad \psi(\Delta_{(1-\theta)r}) \subset \Delta_r. \tag{3.15}$$

Proof. Clearly, we have $\psi(\Delta_{(1-\theta)r}) \subset \Delta_r$. Let $\psi(z) = (\psi_1(z), \ldots, \psi_n(z))$. We put

$$\|d\psi - \mathrm{Id}\|_{(1-2\theta)r} = \max_{1 \leq k \leq n} \sum_{j=1}^{n} \|\partial_{z_j}(\psi_k(z) - z_k)\|_{(1-2\theta)r}.$$

From Cauchy inequalities, we get

$$\|d\psi - \mathrm{Id}\|_{(1-2\theta)r} \leq n \frac{\|\psi - \mathrm{Id}\|_{(1-\theta)r}}{\theta r} \leq \theta. \tag{3.16}$$

Fixing $w \in \Delta_{(1-3\theta)r}$, we consider the mapping $T_w(z) = w + z - \psi(z)$. Clearly, we have $|\psi(0)| \leq \theta^2 r/n$. Hence, (3.16) gives

$$\|T_w\|_{(1-2\theta)r} \leq (1-3\theta)r + \frac{\theta^2 r}{n} + \|d\psi - \mathrm{Id}\|_{(1-2\theta)r}(1-2\theta)r \leq (1-2\theta)r.$$

From (3.16), we also know that $T_w : \Delta_{(1-2\theta)r} \to \Delta_{(1-2\theta)r}$ is a contraction mapping.

By the fixed-point theorem, one obtains the inverse mappings of ψ from $\Delta_{(1-3\theta)r}$ into $\Delta_{(1-2\theta)r}$. □

We now give estimates for the high order terms in $\hat{\tau}_j$.

LEMMA 3.4. *Let* $\{\tau_j, \rho\}$ *and* $\psi(\xi, \eta, \zeta) = (\xi + u, \eta + v, \zeta + w)$ *be as in Lemma 3.2. Assume that* τ_1 *and* τ_2 *satisfy*

$$\|\tau - \tau^*\|_r \le \frac{\theta^l}{c_0 c_1} \cdot \frac{1}{n} \theta^2 r. \tag{3.17}$$

Then for $0 < r < 1$ *and* $0 < \theta < 1/4$, *we have* $\hat{\tau}_j(\Delta_{(1-3\theta)r}) \subset \Delta_r$ *and*

$$\|\hat{\tau} - \tau^*\|_{(1-4\theta)r} \le \frac{3c_0c_1}{\theta^l} \|\tau - \tau^*\|_r (1-\theta)^{2\,\mathrm{ord}\,(\tau)-1}. \tag{3.18}$$

Proof. Clearly, one can get (3.15) from (3.17), Lemmas 3.2 and 3.3. Notice that $\tau_j^*(\Delta_{(1-2\theta)r}) = \Delta_{(1-2\theta)r}$. Then (3.17) gives $\tau_j(\Delta_{(1-2\theta)r}) \subset \Delta_{(1-\theta)r}$. Combining with (3.15), we have $\hat{\tau}_j(\Delta_{(1-3\theta)r}) \subset \Delta_r$.

The first identity in (3.13) gives

$$\begin{aligned} \|\hat{f}_j\|_{(1-3\theta)r} &\le \|v\|_{(1-\theta)r} + \|u\|_{(1-\theta)r} + \|f_j\|_{(1-\theta)r} \\ &\le \frac{3c_0c_1}{\theta^l} \|\tau - \tau^*\|_r. \end{aligned}$$

Since $\hat{f}_j$ vanishes at the origin with order $\ge 2\,\mathrm{ord}\,(\tau) - 1$, the Schwarz lemma gives

$$\begin{aligned} \|\hat{f}_j\|_{(1-4\theta)r} &\le \frac{3c_0c_1}{\theta^l}\left(\frac{1-4\theta}{1-3\theta}\right)^{2\,\mathrm{ord}\,(\tau)-1} \|\tau - \tau^*\|_r \\ &\le \frac{3c_0c_1}{\theta^l}(1-\theta)^{2\,\mathrm{ord}\,(\tau)-1}\|\tau - \tau^*\|_r. \end{aligned}$$

The rest estimates for $\hat{g}_j$ and $\hat{h}_{j,\alpha}$ can be obtained in a similar way from the last two identities in (3.13). □

4. Proof of Theorem 2.2

In this section, we shall prove Theorem 2.2 through a KAM argument.

Let $\tau_j^{(0)} \equiv \tau_j$ $(j = 1, 2)$. We consider the approximate equations (3.7) for the involutions $\{\tau_1^{(0)}, \tau_2^{(0)}\}$. The solution (u, v, w) to (3.7) define a transformation ψ_0 as in Lemma 3.4. Denote $\psi_0 \circ \tau_j^{(0)} \circ \psi_0^{-1}$ by $\tau_j^{(1)}$. Since ψ_0 satisfies the reality condition, $\{\tau_1^{(1)}, \tau_2^{(1)}\}$ is a pair of involutions satisfying the reality condition $\tau_2^{(1)} = \rho\tau_1^{(1)}\rho$. Obviously, $\{\tau_1^{(1)}, \tau_2^{(1)}\}$ is still linearizable by formal transformations. We repeat this process and obtain a sequence of pairs of formally linearizable involutions

$$\tau_j^{(k)} = \tau_j^* + (f_j^{(k)}, g_j^{(k)}, h_j^{(k)}), \qquad \tau_2^{(k)} = \rho\tau_1^{(k)}\rho.$$

Let d_k be the order of $\{\tau_1^{(k)}, \tau_2^{(k)}\}$. Then (3.14) gives that $d_{k+1} \geq 2d_k - 1$. This implies that $d_k > 2^{k+1}$. In particular, $\psi_\infty = \lim_{k\to\infty} \psi_k \circ \cdots \circ \psi_0$ is formally well-defined and satisfies the reality condition $\psi_\infty = \rho\psi_\infty\rho$. It is also clear that $\psi_\infty \circ \tau_j \circ \psi_\infty^{-1} = \tau_j^*$.

Next, we show that ψ_∞ converges near the origin. Let $0 < r < 1$. Set

$$r_k = \frac{1}{2}\left(1 + \frac{1}{k+1}\right)r, \qquad k = 0, 1, \ldots,$$

$$\theta_k = \frac{1}{4}\left(1 - \frac{r_{k+1}}{r_k}\right) = \frac{1}{4(k+2)^2}.$$

We need a numerical result.

LEMMA 4.1. *Let θ_k and r_k be as above. Then there exists an ϵ, $0 < \epsilon < 1$, such that for any sequence of non-negative numbers $\{\delta_k\}_{k=0}^N$, if $\delta_0 < \epsilon\delta_0^*$ and*

$$\delta_k \leq \frac{3c_0c_1}{\theta_{k-1}^l}(1 - \theta_{k-1})^{2k}\delta_{k-1}, \qquad 1 \leq k \leq N, \tag{4.1}$$

then we have

$$\delta_k \leq \delta_k^* \equiv \frac{\theta_k^l}{c_0c_1} \cdot \frac{1}{n}\theta_k^2 r_k, \qquad 0 \leq k \leq N. \tag{4.2}$$

Proof. Clearly, $\delta_{k+1}^*/\delta_k^* \to 1$ as $k \to \infty$. On the other hand, we have

$$\ln\frac{(1-\theta_k)^{2k}}{\theta_k^l} = \frac{2^k}{\theta_k}\left\{\ln(1-\theta_k)^{1/\theta_k} - \frac{l\theta_k}{2^k}\ln\theta_k\right\}.$$

Notice that the quantity in the brace tends to -1. Since $2^k/\theta_k \to +\infty$ as $k \to \infty$, we have

$$\frac{3c_0c_1}{\theta_k^l}(1-\theta_k)^{2k+1} \to 0.$$

Hence, it follows from (4.1) that there exists k_0, independent of δ_k, such that

$$\delta_{k+1} \leq \delta_k \delta^*_{k+1}/\delta^*_k, \qquad \text{for } k \geq \max\{k_0, N-1\}. \tag{4.3}$$

From (4.1), one can choose $\epsilon > 0$ so small that if $\delta_0 \leq \varepsilon\delta^*_0$, then

$$\delta_j \leq \delta^*_j, \qquad 0 < j \leq k_0.$$

Then (4.3) gives (4.2). □

To continue our proof for Theorem 2.2, we put

$$\delta_k = \|\tau^{(k)} - \tau^*\|_{r_k}, \qquad k = 0, 1, 2, \ldots.$$

Let r_k, δ^*_k and ϵ be as in Lemma 4.1. Since $f^{(0)}_j$ and $g^{(0)}_j$ vanish at the origin of order ≥ 2, then for a small r_0 and the ϵ given in Lemma 4.1, we have

$$\|\tau^{(0)} - \tau^*\|_{r_0} \leq \epsilon \frac{\theta^l_0}{c_0 c_1} \cdot \frac{1}{n} \theta^2_0 r_0 = \epsilon\delta^*_0.$$

Thus, Lemma 3.4 shows that (4.1) holds for $N = 1$. Now Lemma 4.1 gives (4.2) for $N = 1$. Inductively, one can prove that for all k

$$\|\tau^{(k)} - \tau^*\|_{r_k} \leq \frac{\theta^l_k}{c_0 c_1} \cdot \frac{1}{n} \theta^2_k r_k.$$

Now (3.15) implies that $\psi_k^{-1}(\Delta_{r_{k+1}}) \subset \Delta_{r_k}$. Hence, $\psi_0^{-1} \circ \cdots \circ \psi_k^{-1} : \Delta_{(1/2)r} \to \Delta_r$ is well defined. Notice that

$$\psi_k^{-1} = \mathrm{Id} + O(d_k).$$

Thus, we see that $\psi_0^{-1} \circ \cdots \circ \psi_k^{-1} - \psi_0^{-1} \circ \cdots \circ \psi_{k-1}^{-1}$ vanishes at the origin with order $\geq d_k$. Applying the Schwarz lemma, we obtain

$$\|\psi_0^{-1} \circ \cdots \circ \psi_k^{-1} - \psi_0^{-1} \circ \cdots \circ \psi_{k-1}^{-1}\|_{(1/4)r} \leq 2r\left(\frac{1}{2}\right)^{d_k}.$$

Since $0 < r < 1$ and $d_k \geq 2^{k+1}$, we see that $\{\psi_0^{-1} \circ \cdots \circ \psi_k^{-1}\}_{k=0}^{\infty}$ converges to ψ_∞ on $\Delta_{(1/4)r}$. The proof of Theorem 2.2 is complete.

5. Integrals of a pair of involutions

In this section, we shall first recall the Moser–Webster normal form for a pair of involutions. Then for a pair of involutions generated by a surface M in $\mathbb{C}^2$, we shall show that the existence of integrals for involutions is equivalent to the existence of a holomorphic function whose restriction to M is real-valued. Finally, we shall prove that there exist elliptic pairs of involutions, which have integrals, but cannot be transformed into the normal form.

Consider an elliptic pair of involutions $\{\tau_1, \tau_2\}$ given by

$$\tau_j(\xi, \eta) = (\lambda_j \eta + f_j(\xi, \eta), \lambda_j^{-1}\xi + g_j(\xi, \eta)), \qquad \lambda_1 = \lambda = \bar{\lambda}_2, \qquad |\lambda| = 1. \tag{5.1}$$

We assume that τ_1 and τ_2 satisfy the reality condition

$$\tau_2(\xi, \eta) = \rho\tau_1\rho(\xi, \eta), \qquad \rho(\xi, \eta) = (\bar{\xi}, \bar{\eta}). \tag{5.2}$$

We now state the Moser–Webster normal form as follows.

THEOREM 5.1 (Moser–Webster [9]). *Let τ_1 and τ_2 be a pair of involutions given by* (5.1) *and* (5.2). *Assume that λ is not a root of unity. Then there exists a unique normalized transformation ψ which satisfies the reality condition and transforms τ_j into*

$$\tau_j^*(\xi^*, \eta^*) = (\Lambda_j(\xi^*\eta^*)\eta^*, \Lambda_j^{-1}(\xi^*\eta^*)\xi^*), \qquad j = 1, 2, \tag{5.3}$$

where $\Lambda_1 = 1/\Lambda_2$ is a formal power series in $\xi^*\eta^*$, of which the constant term is λ.

We shall call (5.3) the *normal form* of pairs of involutions although one can further normalize (5.3) to get a full set of invariants. However, as shown in [9], the normal form (1.2) can be realized by convergent transformations, if and only if the normalized transformation ψ in (a) converges.

We now discuss the compatibility condition involved in (5.1) and (5.2). Since τ_2 is completely determined by τ_1 and ρ, we need only to find the compatability condition for $\tau = \tau_1$ to be an involution. Write

$$\tau: \begin{array}{l} \xi' = \lambda\eta + f(\xi, \eta), \\ \eta' = \bar{\lambda}\xi + g(\xi, \eta). \end{array} \tag{5.4}$$

From $\tau^2 = \mathrm{Id}$, we have

$$[g]_k(\xi, \eta) = -\bar{\lambda}[f]_k(\lambda\eta, \bar{\lambda}\xi) + e_1(\xi, \eta), \tag{5.5}$$

in which e_1 depends on the coefficients of f and g with order less than k. Inductively, one can show that τ is completely determined by f. Conversely, for each holomorphic function f without constant and linear terms, there exists an involution τ such that

$$\pi_1 \circ \tau(\xi, \eta) - \lambda\eta = f(\xi, \eta).$$

In fact, for the transformation $\psi(\xi, \eta) = (\xi, \eta + \bar{\lambda} f(\xi, \eta))$, we have

$$\psi^{-1} \circ \tau_0 \circ \psi(\xi, \eta) = (\lambda\eta + f(\xi, \eta), \bar{\lambda}\xi + g(\xi, \eta)),$$

in which τ_0 is the linear involution $(\xi, \eta) \mapsto (\lambda\eta, \bar{\lambda}\xi)$. This also shows that a single involution is always linearizable when its linear part is given by τ_0.

An *integral* of τ_1 and τ_2 is defined to be a holomorphic function K which satisfies

$$K \circ \tau_j(\xi, \eta) = K(\xi, \eta), \qquad \text{for } j = 1, 2.$$

One also defines a formal integral to be a formal power series satisfying the above relation. Assume that λ in (5.1) is not a root of unity. Then from the normal form (5.3), we see that $\{\tau_1, \tau_2\}$ has a formal integral $\kappa(\xi, \eta) = \xi^*\eta^*$.

Next, we want to show that any formal integral K can be written as a formal power series in κ. To see this, we need to find a power series $a(t) = \Sigma_{k=0}^{\infty} a_k t^k$ such that $K = a \circ \kappa$. Clearly, $a_0 = K(0)$. Assume that there exist $a_0, \ldots, a_{[(k-1)/2]}$ such that

$$K(\xi, \eta) = \sum_{j=0}^{[(k-1)/2]} a_j \kappa^j(\xi, \eta) + E(\xi, \eta),$$

where E is a formal power series in ξ and η of order $\geq k$. Obviously, E is a formal integral of τ_j $(j = 1, 2)$. Let

$$\varphi(\xi, \eta) = \tau_1\tau_2(\xi, \eta) = (\lambda^2\xi, \bar{\lambda}^2\eta) + O(2).$$

Then E is invariant by φ. This implies that

$$E(\lambda^2\xi, \bar{\lambda}^2\eta) = E(\xi, \eta) + O(k + 1).$$

Since λ is not a root of unity, it is easy to see that homogeneous terms of order k are given by

$$[E]_k(\xi, \eta) = \begin{cases} 0, & \text{for } k \text{ odd}, \\ a_k(\xi\eta)^{(1/2)k}, & \text{for } k \text{ even}. \end{cases}$$

Hence, we obtain

$$K(\xi, \eta) = \sum_{j=0}^{[k/2]} a_j \kappa^j(\xi, \eta) + O(k+1),$$

which implies that K is a formal power series in κ.

Now, if K is a non-constant convergent power series invariant under the pair of involutions, then there is a formal power series $\varphi(t)$ such that $K = \varphi \circ \kappa$. Assume that $\varphi(t) - \varphi(0)$ vanishes of order k. It is easy to see that

$$\tilde{K}(\xi, \eta) = (\varphi - \varphi(0))^{1/k} \circ \kappa(\xi, \eta)$$

is still a convergent power series invariant under the pair of involutions, while $\tilde{K}$ starts with the quadratic term. Certainly, one may also assume that K has real coefficients.

We now prove the following result which is contained in [9].

PROPOSITION 5.2. *Let M be a real analytic surface defined by* (1.1). *Assume that* $1/2 < \gamma < \infty$ *and its corresponding* λ *is not a root of unity. Then M is holomorphically flat near* 0, *if there is a non-constant holomorphic function f defined near* 0 *such that* $f|_M$ *is real.*

For the proof, assume that f is holomorphic near $0 \in \mathbb{C}^2$, and $f(z_1, z_2) = \bar{f}(\bar{z}_1, \bar{z}_2)$ on M. Through the totally real embedding $M \subset M^c$, f is extended to a holomorphic function which is invariant under τ_1 and τ_2. Denote the extended function by $K(\xi, \eta)$. From the above discussion, we may further assume that the extension of K has real coefficients and starts with the quadratic term in ξ and η. From $K(\xi, \eta) = O(2)$, we see that $f_{z_1}(0) = 0$. We now have

$$f(z_1, z_2) = az_2 + bz_1^2 + \cdots,$$

in which the terms omitted are in the form $z_1^i z_2^j$ with $i + 2j > 2$. Notice that $f(z_1, z_2) = \bar{f}(\bar{z}_1, \bar{z}_2)$ on M. Now (1.1) gives that $b = 0$, and $a = \bar{a} \neq 0$. This proves that M is holomorphically flat.

Let $\mathfrak{J}$ be the set of elliptic pairs of involutions $\{\tau_1, \tau_2\}$, given by (5.1) and (5.2), such that $\xi\eta$ is their integral. Denote by $\mathfrak{T}$ the set of biholomorphic transformations which have the form

$$\psi : (\xi, \eta) \mapsto (\mu\xi a(\xi), \bar{\mu}\eta a^{-1}(\xi)), \qquad |\mu| = 1 \quad \text{and} \quad a(0) = 1. \tag{5.6}$$

Then $\mathfrak{T}$ acts on $\mathfrak{J}$ by the conjugation.

THEOREM 5.3. *Let $\epsilon_1, \epsilon_2, \ldots$ be a sequence of positive numbers. Assume that $\{\tau_1, \tau_2\}$ is an elliptic pair of involutions in $\mathfrak{J}$. Then there exists a biholomorphic mapping $\psi \in \mathfrak{J}$ such that the pair of involutions $\hat{\tau}_1 = \psi\tau_1\psi^{-1}$ and $\hat{\tau}_2 = \rho\hat{\tau}_1\rho$ cannot be transformed into the normal form through any biholomorphic transformation, while $\hat{\lambda} = \lambda\mu^2$ is not a root of unity, and*

$$|\mu - 1| < \epsilon_1, \qquad |a_j| < \epsilon_j, \qquad \text{for } j > 0.$$

Proof. We shall recursively determine the conditions on μ and coefficients of $a(\xi)$ such that the unique normalized transformation Ψ normalizing the pair of involutions $\{\tau_1, \tau_2\}$ diverges. Let

$$\hat{\tau}(\xi, \eta) = \psi\tau\psi^{-1}(\xi, \eta) = (\hat{\lambda}\eta + r(\xi, \eta), \hat{\lambda}^{-1}\xi + s(\xi, \eta)).$$

Then we have

$$\pi_1\hat{\tau}\psi(\xi, \eta) = \mu\lambda\eta a(\xi)^{-1} + r(\mu\xi a(\xi), \bar{\mu}\eta(a(\xi))^{-1}).$$

Hence

$$[\pi_1\hat{\tau}\psi]_{k+1}(\xi, \eta) = [r]_{k+1}(\mu\xi, \bar{\mu}\eta) - \mu\lambda a_k\xi^k\eta + e_2(\xi, \eta),$$

in which e_2 depends only on the coefficients of $a(\xi)$ with order less than k. In the following discussion, we shall denote by $e_i(\xi, \eta)$ the error terms which are determined by the coefficients of $a(\xi)$ with order less than k. We also have

$$[\pi_1\psi\tau]_{k+1}(\xi, \eta) = \mu\lambda^{k+1}a_k\eta^{k+1} + \mu[f]_{k+1}(\xi, \eta) + e_3(\xi, \eta).$$

Thus, from the last two identities, we obtain

$$[r]_{k+1}(\mu\xi, \bar{\mu}\eta) = \mu\lambda a_k\xi^k\eta + \mu\lambda^{k+1}a_k\eta^{k+1} + \mu[f]_{k+1}(\xi, \eta) + e_4(\xi, \eta). \tag{5.7}$$

Let $\Psi(\xi, \eta) = (\xi + U(\xi, \eta), \eta + V(\xi, \eta))$ be the unique normalized transformation which normalizes $\{\hat{\tau}_1, \hat{\tau}_2\}$. Theorem 5.1 gives

$$\Psi\hat{\tau}_1\hat{\tau}_2\Psi^{-1}(\xi, \eta) = (\hat{\lambda}^2\xi\Lambda(\xi\eta), \hat{\lambda}^{-2}\eta\Lambda^{-1}(\xi\eta) \equiv \hat{\varphi}^*.$$

Hence

$$[\pi_1\hat{\varphi}^*\Psi]_{k+1}(\xi, \eta) = \hat{\lambda}^2[U]_{k+1}(\xi, \eta) + \hat{\lambda}^2\xi[\Lambda(\xi\eta)]_k + e_5(\xi, \eta),$$

$$[\pi_1\Psi\hat{\tau}_1\hat{\tau}_2]_{k+1}(\xi, \eta) = [U]_{k+1}(\hat{\lambda}^2\xi, \hat{\lambda}^{-2}\eta) + [r]_{k+1}(\hat{\lambda}^{-1}\eta, \hat{\lambda}\xi) + \hat{\lambda}[\bar{s}]_{k+1}(\xi, \eta) + e_6(\xi, \eta).$$

Now, for $i \neq j+1$, $i+j=k+1$, the compatibility condition (5.5) gives the following solution

$$U_{i,j} = \frac{\hat{\lambda}^{i-j+2}\bar{r}_{j,i} - \hat{\lambda}^{i-j} r_{j,i}}{\hat{\lambda}^{2(i-j)} - \hat{\lambda}^2} + e_{7;i,j} = \frac{\operatorname{Im}(\hat{\lambda}\bar{r}_{j,i})}{\operatorname{Im}(\hat{\lambda}^{i-j-1})} + e_{7;i,j}. \tag{5.8}$$

Substituting (5.7) into the above, we get

$$U_{k+1,0} = -\frac{\operatorname{Im}((\lambda\mu)^k a_k + \bar{\lambda}\mu^k f_{0,k+1})}{\operatorname{Im}\hat{\lambda}^k} + e_{7;k+1,0}. \tag{5.9}$$

We need the following.

LEMMA 5.4. *Let* $\{\delta_n\}_{n=0}^{\infty}$ *be a sequence of positive numbers. Given* λ *with* $|\lambda| = 1$, *there exists* $\hat{\lambda}$, $|\hat{\lambda}| = 1$, *such that* $\hat{\lambda}$ *is not a root of unity,* $|\hat{\lambda} - \lambda| < \delta_0$, *and*

$$|\hat{\lambda}^{n_k} - 1| < \delta_{n_k},$$

for a sequence of positive integers $n_k \to \infty$.

Let us assume the lemma and continue the proof. Put

$$\delta_0 = \frac{1}{2}\epsilon_1, \qquad \delta_n = \frac{1}{2}\epsilon_n n^{-n}.$$

By Lemma 5.4, there exist $\hat{\lambda}$ and a sequence $n_k \to \infty$ such that $\hat{\lambda}$ is not a root of unity, and

$$|\hat{\lambda} - \lambda| < \frac{1}{2}\epsilon_1, \qquad |\hat{\lambda}^{n_k} - 1| < \frac{1}{2}\epsilon_{n_k} n_k^{-n_k}.$$

Assume that $\epsilon_1 < 1$. Then for a suitable choice of root $\mu = (\hat{\lambda}\bar{\lambda})^{1/2}$, we have

$$|\mu - 1| < 1/2, \qquad |\mu + 1| > 1.$$

Hence

$$|\mu - 1| = \frac{|\hat{\lambda} - \lambda|}{|\mu + 1|} < \delta_0.$$

We now can choose the coefficients a_j. We may assume that $\epsilon_k < 1$. Put

$$a_j = 0, \qquad \text{for } j \neq n_k.$$

We define a_{n_k} recursively as follows: If

$$\left| -\frac{\operatorname{Im}(\bar{\lambda}\mu^{n_k} f_{0,n_k+1})}{\operatorname{Im}\hat{\lambda}^{n_k}} + e_{7;n_k+1,0} \right| \geq \frac{1}{4} n_k^{n_k},$$

we put $a_{n_k} = 0$; otherwise, we take

$$a_{n_k} = \frac{i}{2} \epsilon_{n_k} (\lambda\mu)^{-n_k}.$$

It is easy to see that in both cases, we have

$$|U_{n_k+1,0}| > \frac{1}{4} n_k^{n_k}, \qquad \text{for all } k.$$

Therefore, $U(\xi, \eta)$ diverges. This proves Theorem 5.3. □

We now follow Siegel [12] to give a proof for Lemma 5.4. One may assume that for all n, $\delta_n < 1$. Let $\lambda = e^{i2\pi\theta}$. Clearly, one may also assume that λ is not a root of unity. Choose integers p and q so large that

$$|p\theta - q| < 1, \qquad p > 8\pi\delta_0^{-1}.$$

Denote p by n_0. Let n_1 be the smallest positive integer satisfying

$$n_0 \mid n_1, \qquad n_1 > 8\pi n_0^3 \delta_1^{-1}.$$

Recursively, we define n_k to be the smallest positive integer such that

$$n_{k-1} \mid n_k, \qquad n_k > 8\pi n_{k-1}^3 \delta_{k-1}^{-1}. \tag{5.10}$$

Let $\hat{\theta} = q/p + \Sigma_{k=1}^{\infty} n_k^{-1}$ and $\hat{\lambda} = e^{i2\pi\hat{\theta}}$. We first show that $\hat{\lambda}$ is not a root of unity. To see this, we assume that there exists a positive integer x such that $x\hat{\theta} \in \mathbb{Z}$. Choose k so large that $n_{k-1} > 2x$. Then we get

$$n_{k-1} x\hat{\theta} - n_{k-1} x \frac{q}{p} - \sum_{j=1}^{k-1} \frac{x n_{k-1}}{n_j} = \sum_{j=k}^{\infty} \frac{x n_{k-1}}{n_j}. \tag{5.11}$$

The left side is an integer, because $n_j \mid n_{k-1}$ for $0 \le j \le k-1$. However, for the right side, we use (5.10) and obtain

$$0 < \sum_{j=k}^{\infty} \frac{xn_{k-1}}{n_j} < \frac{xn_{k-1}}{n_k} \sum_{j=0}^{\infty} n_k^{-j} < 2\frac{xn_{k-1}}{n_k} < 1,$$

which leads a contradiction. Thus, $\hat{\lambda}$ is not a root of unity.

To obtain the required estimate in the lemma, we write

$$n_k \hat{\theta} = n_k \frac{q}{p} + \sum_{j=1}^{k} \frac{n_k}{n_j} + \sum_{j=k+1}^{\infty} \frac{n_k}{n_j}.$$

Then $l_k = n_k q/p + \Sigma_{j=1}^{k} n_k/n_j \in \mathbb{Z}$. Using (5.10), we get

$$\sum_{j=k+1}^{\infty} \frac{n_k}{n_j} < 2\frac{n_k}{n_{k+1}} < \frac{1}{4\pi}\delta_{n_k}.$$

Hence

$$|\hat{\lambda}^{n_k} - 1| \le 4\pi|\hat{\theta} n_k - l_k| \le 4\pi \sum_{j=k+1}^{\infty} \frac{n_k}{n_j} < \delta_{n_k}.$$

We have proved Lemma 5.4.

6. Proof of Theorem 1.2

Theorem 5.3 provides examples of holomorphically flat real analytical surfaces which cannot be transformed into normal forms through any biholomorphic mapping. In fact, we shall show that such surfaces form a dense set, as stated in Theorem 1.2.

We first discuss the relation between the defining function of a real analytic surface and its pair of involutions. Denote by $\mathfrak{H}_{k+1}$ the set of homogeneous polynomials of degree $k+1$ in z_1, w_1, which satisfy the reality condition $h(z_1, w_1) = \bar{h}(w_1, z_1)$.

Fix a convergent power series $H(z_1, w_1)$ such that $H(z_1, w_1) = O(3)$ and $\bar{H}(w_1, z_1) = H(z_1, w_1)$. For each $h \in \mathfrak{H}_{k+1}$, we consider the holomorphically flat surface

$$M: z_2 = q_\gamma(z_1, \bar{z}_1) + H(z_1, \bar{z}_1) + h(z_1, \bar{z}_1).$$

Let $\{\tau_1, \tau_2\}$ be the pair of involutions generated by M. For $\tau = \tau_1$, we write

$$\tau(z_1, w_1) = \left(-z_1 - \frac{1}{\gamma} w_1 + p(z_1, w_1), w_1\right)$$

Let $\tilde{H}(z_1, w_1) = H(z_1, w_1) + h(z_1, w_1)$. By the definition of involution τ_1, we have

$$\begin{aligned} q_\gamma(z_1, w_1) + \tilde{H}(z_1, w_1) &= q_\gamma\left(-z_1 - \frac{1}{\gamma} w_1 + p(z_1, w_1), w_1\right) \\ &\quad + \tilde{H}\left(-z_1 - \frac{1}{\gamma} w_1 + p(z_1, w_1), w_1\right). \end{aligned}$$

Simplifying the above, we get

$$\begin{aligned} (2\gamma z_1 + w_1) p(z_1, w_1) &= \tilde{H}\left(-z_1 - \frac{1}{\gamma} w_1 + p(z_1, w_1), w_1\right) \\ &\quad - \tilde{H}(z_1, w_1) + \gamma p(z_1, w_1)^2. \end{aligned}$$

Hence

$$(2\gamma z_1 + w_1)[p]_k(z_1, w_1) = h\left(-z_1 - \frac{1}{\gamma} w_1, w_1\right) - h(z_1, w_1) + e_8(z_1, w_1), \tag{6.1}$$

where e_8 depends only on γ and the coefficients of H with order less than $k + 2$.

Let $(\xi, \eta) = T_\lambda(z_1, w_1)$ be the transformation defined by (2.5). Then in the new coordinates (ξ, η), τ_1 and τ_2 are given by (5.1) and (5.2). For a power series p in z_1 and w_1, let us denote by p^* the power series $p \circ T_\lambda^{-1}$. Then (6.1) becomes

$$\begin{aligned} &(\alpha_\lambda \xi - \beta_\lambda \eta) f(\xi, \eta) = h^*(\lambda\eta, \bar{\lambda}\xi) - h^*(\xi, \eta) + e_8^*(\xi, \eta), \\ &\alpha_\lambda = (1 - \bar{\lambda}^2)(1 - 2\gamma\lambda), \qquad \beta_\lambda = (1 - \bar{\lambda}^2)(2\gamma - \lambda). \end{aligned} \tag{6.2}$$

Obviously, T_λ transforms $\mathfrak{H}_{k+1}$ into the set $\mathfrak{H}_{k+1}^*$ of homogeneous polynomials of degree $k + 1$ with real coefficients. Let

$$a_{k+1}^*(\xi, \eta) = \eta^{k+1}.$$

In (6.2), substituting h^* by a_{k+1}^*, we get a solution $f(\xi, \eta)$ to (6.2), given by

$$r_{k,\lambda}^*(\xi, \eta) = \frac{1}{\beta_\lambda} \sum_{j=0}^{k} \left(\frac{\alpha_\lambda}{\beta_\lambda}\right)^j \xi^j \eta^{k-j} + e_9^*(\xi, \eta),$$

in which e_9^* is the solution f to (6.2), where h^* is replaced by 0. In particular, $e_9^*(\xi, \eta)$ depends only on γ and the coefficients of H with order less than $k+2$. Put

$$a_{k+1,\lambda}(z_1, w_1) = a_{k+1}^* \circ T_\lambda(z_1, w_1), \qquad c_{k+1,\lambda} = \max_{i+j=k+1} \{|a_{k+1,\lambda;i,j}|\},$$

in which $a_{k+1,\lambda;i,j}$ are coefficients of $a_{k+1,\lambda}(z_1, w_1)$. Notice that the binomial coefficients of $(\xi+\eta)^k$ are bounded by 2^k. Then from (2.5), we have

$$c_{k+1,\lambda} \le 2^{k+1}|\lambda^2 - 1|^{-k-1}.$$

Therefore, for $|\hat{\lambda} - \lambda| < |\hat{\lambda} - 1|/4$, one has

$$c_{k+1,\hat{\lambda}} \le 2^{2k+2}|\hat{\lambda} - 1|^{-k-1} = c_{k+1}^*. \tag{6.3}$$

We now can prove Theorem 1.2. Put $\epsilon_k = \min\{\epsilon_{i,j}; i+j=k\}$. Applying Lemma 5.4, we find $\hat{\lambda}$, $|\hat{\lambda}| = 1$, such that for a sequence $k_j \to \infty$

$$|\hat{\lambda}^{k_j} - 1| < \epsilon_{k_j+1} k_j^{-k_j}.$$

Clearly, we may assume that $\hat{\lambda}$ has been so chosen that the corresponding $\hat{\gamma}$ is non-exceptional and it satisfies $|\hat{\gamma} - \gamma| < \epsilon$. We also assume that (6.3) holds.

Next, we determine the coefficients of $\hat{H}$ recursively. Assume that $[\hat{H}]_j$ $(j \le k)$ has been given. Put $\hat{H}_k(z_1, \bar{z}_1) = \Sigma_3^k [\hat{H}]_j(z_1, \bar{z}_1)$. To find $[\hat{H}]_{k+1}$, we consider the involutions $\{\tau_1, \tau_2\}$ generated by the surface given by

$$z_2 = q_{\hat{\gamma}}(z_1, \bar{z}_1) + \hat{H}_k(z_1, \bar{z}_1) + \sum_{k+1}^{\infty} [H]_j(z_1, \bar{z}_1).$$

Write

$$\tau(\xi, \eta) = (\hat{\lambda}\eta + f_k(\xi, \eta), \bar{\hat{\lambda}}\xi + g_k(\xi, \eta)) + O(k+1),$$
$$f_k(\xi, \eta) = f_{k-1}(\xi, \eta) + [f]_k(\xi, \eta), \qquad g_k(\xi, \eta) = g_{k-1}(\xi, \eta) + [g]_k(\xi, \eta),$$

in which f_{k-1}, g_{k-1} depend only on $\hat{\gamma}$ and $\hat{H}_k$. Assume that the normalized transformation $\Psi(\xi, \eta) = (\xi + U(\xi, \eta), \eta + V(\xi, \eta))$ transforms $\{\tau_1, \tau_2\}$ into $\{\tau_1^*, \tau_2^*\}$ of the form (5.3). Using the formula (5.8), we get

$$U_{i,k-i} = -\frac{\operatorname{Im}(\bar{\hat{\lambda}} f_{k-i,i})}{\operatorname{Im}\hat{\lambda}^{2i-k-1}} + e_{10;i,k-j}, \qquad 2i \ne k+1, \tag{6.4}$$

in which $e_{10;i,k-j}$ depends only on $\hat{\gamma}$ and coefficients of $\hat{H}$ with order less than $k+1$.

We now determine $\hat{H}_{k+1}$ as follows. We put $[\hat{H}]_{k+1} = [H]_{k+1}$ in the case either $k \neq k_j + 1$, or $k = k_j + 1$ with

$$|U_{k_j+1,0}| = \left| -\frac{\operatorname{Im}(\bar{\hat{\lambda}} f_{0,k_j})}{\operatorname{Im} \hat{\lambda}^{k_j}} + e_{10;k_j+1,0} \right| > \frac{|\operatorname{Im} \beta_{\bar{\lambda}}^{-1}|}{4c^*_{k_j+1}} k_j^{k_j},$$

where f is the solution to (6.2) of which $h^* = 0$. Otherwise, for $k = k_j + 1$, we set

$$[\hat{H}]_{k+1}(z_1, \bar{z}_1) = [H]_{k+1}(z_1, \bar{z}_1) + \frac{1}{2c^*_{k+1}} \epsilon_{k+1} a_{k+1,\hat{\lambda}}(z_1, \bar{z}_1).$$

For the chosen $[\hat{H}]_{k+1}$, let us denote by

$$\hat{\tau}_1(\xi, \eta) = (\hat{\lambda}\eta + \hat{f}_k(\xi, \eta), \bar{\hat{\lambda}}\xi + \hat{g}_k(\xi, \eta)) + O(k+1)$$

the involution generated by $\hat{M}$. Assume that $\hat{\tau}_1$ and $\hat{\tau}_2$ are normalized by the transformation $\hat{\Psi}(\xi, \eta) = (\xi + \hat{U}(\xi, \eta), \eta + \hat{V}(\xi, \eta))$.

Recall that f_k is the solution f to (6.2), in which h^* is substituted by $[\hat{H} - H]_{k+1} \circ T_{\bar{\lambda}}^{-1}$. For the second choice of $[\hat{H}]_{k+1}$, we have

$$\hat{f}_k - f_k = \frac{1}{2c^*_{k+1}} \epsilon_{k+1} r^*_{k,\hat{\lambda}}.$$

Hence

$$\hat{f}_{0,k} - f_{0,k} = \frac{\epsilon_{k+1}}{2c^*_{k+1}} \beta_{\bar{\lambda}}^{-1}.$$

Thus, for $k = k_j + 1$, we obtain

$$\left| \frac{\operatorname{Im}(\bar{\hat{\lambda}} \hat{f}_{0,k_j})}{\operatorname{Im} \hat{\lambda}^{k_j}} + e_{10;k_j+1,0} \right| > \left| \frac{\operatorname{Im} \bar{\hat{\lambda}}(\hat{f}_{0,k_j} - f_{0,k_j})}{\operatorname{Im} \hat{\lambda}^{k_j}} \right| - \left| \frac{\operatorname{Im}(\bar{\hat{\lambda}} f_{0,k_j})}{\operatorname{Im} \hat{\lambda}^{k_j}} + e_{10;k_j+1,0} \right|$$
$$> \frac{|\operatorname{Im} \beta_{\bar{\lambda}}^{-1}|}{4c^*_{k_j+1}} k_j^{k_j}.$$

Now (6.4) gives

$$|\hat{U}_{k_j+1,0}| > \frac{|\operatorname{Im} \beta_{\bar{\lambda}}^{-1}|}{4c^*_{k_j+1}} k_j^{k_j}, \qquad j = 1, 2, \ldots.$$

This proves that the unique normalized transformation $\hat{\Psi}$ diverges. Therefore $\hat{M}$ cannot be transformed into the normal form through any biholomorphic transformation. □

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