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Root vectors in quantum groups

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The root vector defined in [L1-2] plays a fundamental role in quantum group theory. However even for some simple questions, such as the number of root vectors, the relations between root vectors, etc., we know little. There are several formulas concerned with the coproducts of root vectors in [AJS, KR, LS]. These formulas are important, but for many purposes it is inconvenient to use them, because these formulas in fact are not formulas in quantum group but in certain completions of quantum groups and are involved products of infinite sums. It seems also no explicit formula for the antipode of a root vector at hand. The arguments in the remarkable work [AJS] show that for a quantum group it is valuable to have formulas (in the quantum group) of coproducts and antipodes of root vectors. Therefore it is necessary to understand root vectors further. This paper is motivated by the work [AJS].

In this paper, we prove that for a root vector, certain presentation is unique (see Theorem 4.4 (ii) and Lemma 4.2). The uniqueness of the presentation is useful to prove that root vectors are linearly independent and can be used to get some explicit formulas concerned with root vectors, for example, coproduct formulas. The uniqueness of the presentation also can be used to count root vectors. Other known presentations of root vectors are not effective for these purposes. In this paper we also prove that for a root vector there exists a unique shortest element (in a reasonable sense) in the Weyl group attached to it (see Theorem 4.4 (iii) and Proposition 2.12 (i)). Using Theorem 4.4 (ii) and Proposition 4.8 we get an explicit formula for the coproduct of a root vector in a quantum group of type A. Unfortunately it is not easy to get such a formula for other types in general.

The contents of the paper are as follows. In section 1 we recall some basic definitions and fix notations. We also list some formulas for later uses. In section 2 we prove some results about root systems. Some of them are needed in sections 4 and 5. For the possible generalizations of the results in section 4, we also consider infinite root systems. In section 3 we give several lemmas which are important for our proof of the main result in technique. Lemma 3.2 is originally proved for type A, D, E by Lusztig in [L3] based on the relations between quantum groups and quivers. In this paper we prove the lemma in general by a simpler way. Lemma 3.5 is an essential ingredient for the proof of Lemma 4.2, which implies that root vectors are linearly independent. In section 4 we give the main result Theorem 4.4 which was explained before. We also get an explicit formula (Theorem 4.7) for the antipode of a root vector and give an upper bound for the number of root vectors (see Proposition 4.8). In section 5 we restrict ourselves to type A. We get an explicit formula (Theorem 5.5) for the coproduct of a root vector by using Theorem 4.4 (ii) and Proposition 4.8. (A very special case was treated in [R].) We finally list some commutation formulas for some root vectors (see 5.6), which are q-analogue of similar formulas in universal enveloping algebras.

We only discuss root vectors of positive roots since through the homomorphism Ω (see 1.3 (a)) all results can be transfered to those concerned with the root vectors of negative roots.

1. Introduction

We recall some basic concepts.

1.1. Let R be an irreducible root system with simple roots α_i $(1 \le i \le n)$, $R^\vee$ and $\alpha_i^\vee$ be the corresponding dual. Then $(a_{ij})_{1 \le i,j \le n}$ is a Carten matrix, where $a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$. Assume that we are given integers $d_i \in \{1, 2, 3\}$ $(1 \le i \le n)$ such that $d_i a_{ij} = d_j a_{ji}$. The quantum group U over $\mathbf{Q}(v)$ (v is an indeterminate) associated to (a_{ij}) is an associative algebra over $\mathbf{Q}(v)$ generated by E_i, F_i, K_i, K_i^{-1} $(1 \le i \le n)$ which satisfy the q-analogue of Serre relations (see for example, [L2]). The algebra U is in fact a Hopf algebra, the coproduct Δ, antipode S, counit ϵ are defined as follows:

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \qquad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \qquad \Delta(K_i) = K_i \otimes K_i,$$

$$S(E_i) = -K_i^{-1} E_i, \qquad S(F_i) = -F_i K_i, \qquad S(K_i) = K_i^{-1},$$

$$\epsilon(E_i) = \epsilon(F_i) = 0, \qquad \epsilon(K_i) = 1.$$

1.2. The root vectors in U are defined through elements of the Weyl group and some automorphisms of U (see [L2]). We recall the definition.

Let W be the Weyl group of R generated by simple reflections s_i $(1 \le i \le n)$ which are defined by $s_i(\alpha) = \alpha - \langle \alpha, \alpha_i^\vee \rangle \alpha_i$, $\alpha \in R$. For each i the automorphism

$T_{s_i} = T_i$ is defined by Lusztig as follows (see [L2]):

$$T_iE_i = -F_iK_i, \qquad T_iE_j = \sum_{r+s=-a_{ij}} (-1)^r v^{-d_i s} E_i^{(r)} E_j E_i^{(s)}, \qquad \text{if } i \neq j,$$

$$T_iF_i = -K_i^{-1}E_i, \qquad T_iF_j = \sum_{r+s=-a_{ij}} (-1)^r v^{d_i s} F_i^{(s)} F_j F_i^{(r)}, \qquad \text{if } i \neq j,$$

$$T_iK_j = K_iK_j^{-a_{ij}}.$$

where $E_i^{(N)} = E_i^N/[N]^!_{d_i}$, $F_i^{(N)} = F_i^N/[N]^!_{d_i}$, $[0]^!_{d_i} = 1$, $[N]^!_{d_i} = [1]_{d_i}[2]_{d_i}\cdots[N]_{d_i}$ if $N \geq 1$, and $[N]_{d_i} = (v^{Nd_i} - v^{-Nd_i})/(v^{d_i} - v^{-d_i})$, $N \geq 0$.

These automorphisms satisfy the braid relations, thus for each element $w \in W$ we can define the automorphism T_w of U as $T_{i_k} \ldots T_{i_2} T_{i_1}$ where $s_{i_k} \ldots s_{i_2} s_{i_1}$ is a reduced decomposition of w (see [L2, 3.1-2]).

1.3. The following are some simple properties about these automorphisms T_w (see [L2]): (a) Let Ω, $\Psi : U \to U^{opp}$ be the **Q**-algebra homomorphisms defined by

$$\Omega E_i = F_i, \qquad \Omega F_i = E_i, \qquad \Omega K_i = K_i^{-1}, \qquad \Omega v = v^{-1},$$

$$\Psi E_i = E_i, \qquad \Psi F_i = F_i, \qquad \Psi K_i = K_i^{-1}, \qquad \Psi v = v.$$

We have $\Omega T_i = T_i \Omega$ and $T_i' = T_i^{-1} = \Psi T_i \Psi$. So $\Omega T_w = T_w \Omega$ and $T_{w^{-1}}^{-1} = \Psi T_w \Psi$ for any $w \in W$.

(b) $T_wE_i = E_j$, if $w(\alpha_i) = \alpha_j$.

By (b) and the definition of T_w we get the following equalities.

(c) $T_iE_j = E_j, \quad T_iF_j = F_j, \quad T_iK_j = K_j, \quad$ if $a_{ij} = 0$.

(d) $T_i^{-1}E_j = T_jE_i, \quad T_i^{-1}F_j = T_jF_i, \quad T_i^{-1}K_j = T_jK_i, \quad$ if $a_{ij}a_{ji} = 1$.

(e) $T_i^{-1}E_j = T_jT_iE_j, \quad T_i^{-1}F_j = T_jT_iF_j, \quad T_i^{-1}K_j = T_jT_iK_j, \quad$ if $a_{ij}a_{ji} = 2$.

If $a_{ij}a_{ji} = 3$, then we have

(f) $T_i^{-1}E_j = T_jT_iT_jT_iE_j, \quad T_i^{-1}F_j = T_jT_iT_jT_iF_j, \quad T_i^{-1}K_j = T_jT_iT_jT_iK_j,$

(g) $T_j^{-1}T_i^{-1}E_j = T_iT_jT_iE_j, \quad T_j^{-1}T_i^{-1}F_j = T_iT_jT_iF_j, \quad T_j^{-1}T_i^{-1}K_j = T_iT_jT_iK_j,$

We also have

(h) $T_i^2 E_i = v^{2d_i} K_i^{-2} E_i$.

(i) $T_i^2 E_j = (1 - v^{-2d_i}) F_i K_i T_i(E_j) - v^{-d_i} E_j$ if $a_{ij} = -1$.

If $a_{ij} = -2$, then

(j) $$T_i^2 E_j = v^{-2}(1 - v^{-2})(1 - v^{-4}) F_i^{(2)} K_i^2 T_i(E_j) - v^{-1}(1 - v^{-2}) F_i K_i T_j^{-1}(E_i) + v^{-2} E_j.$$

If $a_{ij} = -3$, then

(k) $$T_i^2 E_j = v^{-6}(1 - v^{-2})(1 - v^{-4})(1 - v^{-6}) F_i^{(3)} K_i^3 T_i(E_j) - v^{-3}(1 - v^{-2})(1 - v^{-4}) F_i^{(2)} K_i^2 T_i T_j(E_i) + v^{-2}(1 - v^{-2}) F_i K_i T_j^{-1}(E_i) - v^{-3} E_j.$$

1.4. For any positive root $\alpha \in R^+$ (the set of positive roots in R), if $w^{-1}(\alpha) = \alpha_i (w \in W)$ is a simple root in R, then we set $E_{\alpha,w} = T_w(E_i)$ (resp. $E_{-\alpha,w} = F_{\alpha,w} = \Omega E_{\alpha,w} = T_w(F_i)$) and call it a root vector in U of root α (resp. $-\alpha$). The definition of root vectors looks simple.

2. Some facts on root system and Weyl group

2.1. To formulate the results in section 4 and section 5 we need some properties about root systems and Weyl groups. We are mainly interested in finite root systems. However, in view of the results in [L4, Chapters 39, 40], it is possible to generalize the main result of the paper to quantum analogue of the enveloping algebras of symmetrizable Kac-Moody algebras. Therefore, in this section we also consider infinite root systems.

Let Φ be the root system associated to a symmetrizable Kac-Moody algebra (for example, the root system of a semisimple Lie algebra), and denote by W the Weyl group of the root system. Let Φ^+ be the set of positive roots, denote by Π the basis of the root system, and let Φ^+_{real} be the set of positive real roots.

We shall define a function $h' : \Phi^+_{real} \to \mathbf{N}$ and prove some properties of the function. We also introduce the concept *shortable* and prove a result concerned with the concept.

We shall use the symbol "$\le$" for the Bruhat order in W as well as for the usual partial order in Φ^+. For positive roots α, β we also write $\beta < \alpha$ when $\beta \le \alpha$ and $\beta \ne \alpha$. The notation in this section has no relations with those in section 1. In particular, we allow α_i not to be a simple root.

LEMMA 2.2. *Let α be a positive real root and denoted by $s_\alpha \in W$ the corresponding reflection. Then the length $\ell(s_\alpha)$ of the reflection is an odd number, i.e. $\ell(s_\alpha) = 2m + 1$ for some $m \in \mathbf{N}$.*

Proof. The determinant of a reflection is -1, so a reflection can not be a product of an even number of reflections.

LEMMA 2.3. *Let α be a positive real root and suppose $\ell(s_\alpha) = 2m + 1$. Let β be a simple root and let $w \in W$ be such that $w(\beta) = \alpha$, and suppose $w = s_{\beta_1} \cdot \cdots \cdot s_{\beta_r}$ is a reduced decomposition.*

For $i = 1, \dots, r$ denote by w_i the word $s_{\beta_i} \cdot \cdots \cdot s_{\beta_r}$, denote by α_i the root $\alpha_i = w_i(\beta)$, let $\mathcal{S}_i := \{\delta > 0 \mid s_{\alpha_i}(\delta) < 0\}$ and set $\mathcal{S}_{r+1} := \{\beta\}$.

The following are equivalent:

(i) *$r = m$, i.e. $s_\alpha = s_{\beta_1} \cdot \cdots \cdot s_{\beta_r} s_\beta s_{\beta_r} \cdot \cdots \cdot s_{\beta_1}$ is a reduced decomposition.*

(ii) *$\beta := \alpha_{r+1} < s_{\beta_r}(\beta) = \alpha_r < s_{\beta_{r-1}} s_{\beta_r}(\beta) = \alpha_{r-1} < \dots < \alpha_1 = w(\beta) = \alpha$.*

(iii) *$\langle \alpha_{i+1}, \beta_i^\vee \rangle < 0$ for $i = 1, \dots, r$.*

(iv) *$\langle \alpha_i, \beta_i^\vee \rangle > 0$ for $i = 1, \dots, r$.*

(v) *$\beta_i \notin \mathcal{S}_{i+1}$ and $\langle \beta_i, \alpha_{i+1}^\vee \rangle \ne 0$ for $i = 1, \dots, r$.*

(vi) *$\beta_i \in \mathcal{S}_i$ for $i = 1, \dots, r$.*

REMARK. It is clear that the property (i) is independent of the choice of the reduced decomposition of w. So if one of the properties holds for some reduced decomposition of w, then it holds for all reduced decompositions of w.

Proof. The equivalence of (ii), (iii) and (iv) are obvious. Now $s_{\alpha_i}(\beta_i) = \beta_i - \langle \beta_i, \alpha_i^\vee \rangle \alpha_i < 0$ if and only if $\langle \beta_i, \alpha_i^\vee \rangle > 0$ and hence if and only if $\langle \alpha_i, \beta_i^\vee \rangle > 0$, which proves the equivalence of (iv) and (vi). The equivalence of (iii) and (v) follows in the same way.

Suppose now (i) holds, i.e. $2\ell(w) + 1 = \ell(s_\alpha)$. This implies obviously that $2\ell(w_i) + 1 = \ell(s_{\alpha_i})$, and hence $s_{\alpha_i} = s_{\beta_i} \cdot \cdots \cdot s_\beta \cdot \cdots \cdot s_{\beta_i}$ is a reduced decomposition, so $\beta_i \in \mathcal{S}_i$. To prove that (vi) implies (i), note that if $\gamma \in \mathcal{S}_i$, then $-s_{\alpha_i}(\gamma) \in \mathcal{S}_i$. Further, $w_{r+1} := s_\beta$ is a reduced decomposition. We prove now by decreasing induction that $2\ell(w_i) + 1 = \ell(s_{\alpha_i})$.

We may assume that $s_{\alpha_{i+1}} = s_{\beta_{i+1}} \cdot \cdots \cdot s_\alpha \cdot \cdots \cdot s_{\beta_{i+1}}$ is a reduced decomposition. Since $\beta_i \in \mathcal{S}_i$ we know that $\beta_i \notin \mathcal{S}_{i+1}$. So $\mathcal{S}_i \supseteq \{s_{\beta_i}(\delta) \mid \delta \in \mathcal{S}_{i+1}\} \cup \{\beta_i, -s_{\alpha_i}(\beta_i)\}$, and hence $|\mathscr{S}_i| \geq |\mathscr{S}_{i+1}| + 2$, this forces that $s_{\alpha_i} = s_{\beta_i} \cdot \cdots \cdot s_\beta \cdot \cdots \cdot s_{\beta_i}$ is a reduced decomposition.

DEFINITION 2.4. For a real positive root α set $h'(\alpha) := (\ell(s_\alpha) - 1)/2$.

REMARK. The relation between the function h' and the *depth* function is $h'(\alpha) = \mathrm{dp}(\alpha) - 1$, for the definition of $\mathrm{dp}(\alpha)$, see "A finiteness property and an automatic structure for Coxeter groups" (by B. Brink and R. B. Howlett, Math. Ann. *296*, 179–190 (1993), Definition 1.5 (i), p. 181).

LEMMA 2.5. *The function $h' : \Phi^+_{real} \to \mathbf{N}$ has the following properties*:

(i) *$h'(\alpha) = 0$ if and only if α is a simple root.*

(ii) *If β is a simple root such that $\alpha = w(\beta)$ for some $w \in W$, then $h'(\alpha) \leq \ell(w)$. Moreover, if $\beta \nleq \alpha$, then $h'(\alpha) < \ell(w)$.*

(iii) *If β is a simple root such that $0 < s_\beta(\alpha) < \alpha$, then $h'(s_\beta(\alpha)) + 1 = h'(\alpha)$.*

(iv) *Let α be a positive real root and suppose $h'(\alpha) = m$. There exists a reduced decomposition $s_\alpha = s_{\beta_1} \cdot \cdots \cdot s_{\beta_m} s_\beta s_{\beta_m} \cdot \cdots \cdot s_{\beta_1}$.*

(v) *Let α be a positive real root and suppose $h'(\alpha) = m$. There exists a simple root β and $w \in W$ such that $w(\beta) = \alpha$ and $\ell(w) = h'(\alpha)$.*

Proof. Now (i), (ii) and (iii) are simple consequences of Lemma 2.3, and (iv) and (v) are equivalent. We give a proof of (iv):

If α is a simple root, then nothing is to prove. So suppose α is not simple and let γ be a simple root such that $\langle \alpha, \gamma^\vee \rangle > 0$. (Such a γ exists since α is a positive root.) Then $\delta := s_\gamma(\alpha) < \alpha$ is a positive real root. By induction on the height we may assume that there exists a $\kappa \in W$ and a simple root β such that $\kappa(\beta) = \delta$ and $\ell(s_\delta) = 2\ell(\kappa) + 1$. Then $s_\gamma \kappa(\beta) = \alpha$, and by Lemma 2.3 we have in addition $2\ell(s_\gamma \kappa) + 1 = \ell(s_\alpha)$.

We shall now prove more properties of the function h'.

LEMMA 2.6. *Let α be a positive real root and suppose $h'(\alpha) = m$. Let $\beta_1, \ldots, \beta_m, \beta, \gamma_1, \ldots, \gamma_m, \gamma$ be simple roots such that $\alpha = s_{\beta_1} \cdot \cdots \cdot s_{\beta_m}(\beta) = s_{\gamma_1} \cdot \cdots \cdot s_{\gamma_m}(\gamma)$. We have*

(i) *Either $s_{\gamma_1} s_{\beta_1} \cdot \cdots \cdot s_{\beta_m} \leq s_{\beta_1} \cdot \cdots \cdot s_{\beta_m}$ or $s_{\gamma_1}(\alpha) = s_{\gamma_2} \cdot \cdots \cdot s_{\gamma_m}(\gamma) = s_{\beta_1} \cdot \cdots \cdot s_{\beta_{m-1}}(\beta_m)$.*

(ii) *Assume that the Dynkin diagram of Φ_α (the connected component of Φ containing α) includes no cycles. If $s_{\gamma_1} s_{\beta_1} \cdot \cdots \cdot s_{\beta_m} \geq s_{\beta_1} \cdot \cdots \cdot s_{\beta_m}$, then $s_{\gamma_1}(\alpha) \ngeq \beta$.*

(iii) *Assume that the Dynkin diagram of Φ_α includes no cycles. Let δ be a simple root. If $\beta < s_\delta(\beta) \leq \alpha$, then $s_{\beta_1} \cdot \cdots \cdot s_{\beta_m} s_\delta \leq s_{\beta_1} \cdot \cdots \cdot s_{\beta_m}$.*

Proof. Set $w := s_{\beta_1} \cdot \cdots \cdot s_{\beta_m}$, $u := s_{\gamma_2} \cdot \cdots \cdot s_{\gamma_m}$.

(i) When $s_{\gamma_1} w \le w$, nothing need to prove. Now assume that $s_{\gamma_1} w \ge w$. Then $w^{-1}(\gamma_1) > 0$. By Lemma 2.3 (iv), we have $\langle \alpha, \gamma_1^\vee \rangle > 0$. Since β is a simple root, we get $w^{-1} s_{\gamma_1} w(\beta) = w^{-1}(\alpha - \langle \alpha, \gamma_1^\vee \rangle \gamma_1) = \beta - \langle \alpha, \gamma_1^\vee \rangle w^{-1}(\gamma) < 0$. Thus $w^{-1} s_{\gamma_1} w s_\beta \le w^{-1} s_{\gamma_1} w$. Set $\gamma_1 = \beta_0$, since $w s_\beta \ge w$, we can find i in $[1, m-1]$ such that $s_{\beta_i} \cdot \cdots \cdot s_{\beta_1} s_{\gamma_1} w s_\beta = s_{\beta_{i+1}} \cdot \cdots \cdot s_{\beta_1} s_{\gamma_1} w$. That is, $s_{\gamma_1} w s_\beta w^{-1} s_{\gamma_1}$ and $s_{\beta_1} \cdot \cdots \cdot s_{\beta_i} s_{\beta_{i+1}} s_{\beta_i} \cdot \cdots \cdot s_{\beta_1}$ are equal. We also have $s_{\gamma_1} w s_\beta w^{-1} s_{\gamma_1} = u s_\gamma u^{-1}$ since $s_{\gamma_1} w(\beta) = u(\gamma)$. Thus we get $s_{\beta_1} \cdot \cdots \cdot s_{\beta_i} s_{\beta_{i+1}} s_{\beta_i} \cdot \cdots \cdot s_{\beta_1} = u s_\gamma u^{-1}$. According to Lemma 2.5 (iii) and Lemma 2.3 (ii) we must have $i = m-1$ for the reason of length. Hence $s_{\gamma_1}(\alpha) = s_{\gamma_2} \cdot \cdots \cdot s_{\gamma_m}(\gamma) = s_{\beta_1} \cdot \cdots \cdot s_{\beta_{m-1}}(\beta_m)$.

(ii) By (i) and the assumption in (ii) we have $\gamma' := s_{\gamma_1}(\alpha) = s_{\beta_1} \cdot \cdots \cdot s_{\beta_{m-1}}(\beta_m)$. Note that $h'(\gamma') = m-1$ (Lemma 2.5 (iii)). According to the definition of $h'(\alpha) (= m)$, $h'(\gamma')$, and using Lemma 2.3 (iii) we see

(a) $\langle \beta_m, \beta^\vee \rangle$ is negative and $\langle \beta_{m-1}, \beta_m^\vee \rangle$ is negative. In particular $\beta_m \ne \beta, \beta_{m-1}$. Since $s_{\gamma_1} w \ge w$, we get

(b) $\delta := w^{-1}(\gamma_1) > 0$. By (i) we see that $s_{\gamma_1} w(\beta) = -w(\beta_m)$, that is

(c) $s_\delta(\beta) = \beta - \langle \beta, \delta^\vee \rangle \delta = -\beta_m$. By (a), (b) and (c) we get

(d) $\langle \beta, \delta^\vee \rangle = 1$ and $\delta = \beta + \beta_m$.

Note that $\beta + \beta_m$ is a real positive root of height 2, by Lemma 2.5 (v) and Lemma 2.3 (ii) we see that δ is equal to $s_\beta(\beta_m)$ or $s_{\beta_m}(\beta)$. That is, s_δ is equal to $s_\beta s_{\beta_m} s_\beta$ or $s_{\beta_m} s_\beta s_{\beta_m}$. Using (c) we get

(e) $s_\beta s_{\beta_m}(\beta) = \beta_m$ and $s_{\beta_m} s_\beta(\beta_m) = \beta$, and hence by Lemma 2.3, $\beta_{m-1} \ne \beta$. Since $s_{\gamma_1} w \ge w$, by (i) we know that

(f) $s_{\gamma_1}(\alpha)$ is a linear combination of $\beta_1, \cdots\cdots, \beta_{m-1}, \beta_m$.

We also have

(g) Let τ be a simple root such that $s_{\beta_1} \cdot \cdots \cdot s_{\beta_{m-1}} s_\tau \le s_{\beta_1} \cdot \cdots \cdot s_{\beta_{m-1}}$. Since the properties of β_{m-1} do not depend on the chosen reduced decomposition of $s_{\beta_1} \cdot \cdots \cdot s_{\beta_{m-1}}$, τ has the same properties with β_{m-1}, i.e. $\langle \tau, \beta_m^\vee \rangle$ is negative and τ is not equal to β.

Assume that $s_{\gamma_1}(\alpha) \ge \beta$. By (f) we see that $\beta_i = \beta$ for some i in $[1, m]$. According to (a), (e) and Lemma 2.3 (ii) we know that $i < m-1$. We may choose i such that all $\beta_{i+1}, \ldots, \beta_m$ are not equal to β. According to (e) and (g) we can find indices $i = i_1 < i_2 < \ldots < i_p = m$ $(p \geq 3)$ such that $\langle \beta_{i_a}, \beta_{i_{a+1}}^\vee \rangle < 0$ for $a = 1, 2, \ldots, p-1$. But β_i is equal to β and $\langle \beta, \beta_m^\vee \rangle < 0$ (see (a)), so the Dynkin diagram of Φ_α includes a cycle. This contradicts to our assumption. Therefore $s_{\gamma_1}(\alpha) \not\geq \beta$.

(iii) Since $\alpha \geq s_\delta(\beta) > \beta$, we have

(a) $\langle \beta, \delta^\vee \rangle < 0$ and β is not equal to δ.

(b) For each reduced decomposition $s_{\tau_1} \cdot \cdots \cdot s_{\tau_m}$ of w ($\tau_1, \ldots, \tau_m$ are simple roots), there exists i in $[1, m]$ such that $\tau_i = \delta$.

We may choose a reduced decomposition $s_{\tau_1} \cdot \cdots \cdot s_{\tau_m}$ of w such that the index i in (b) is maximal in all possibilities. If i is not equal to m, then we can find indices $i = i_1 < i_2 < \ldots < i_p = m$ $(p \geq 2)$ such that

(c) $\langle \tau_{i_a}, \tau_{i_{a+1}}^\vee \rangle < 0$ for $a = 1, 2, \ldots, p-1$.

By Lemma 2.3 (ii) we know that

(d) $\langle \beta, \tau_m^\vee \rangle < 0$.

According to (a), (c) and (d), $\delta = \tau_i, \tau_{i_2}, \ldots, \tau_{i_p}, \beta$ generate a sub-root-system of Φ_α whose Dynkin diagram includes a cycle. This contradicts to our assumption. Therefore $i = m$, that is, $ws_\delta \leq w$.

The lemma is proved.

PROPOSITION 2.7. *Let α be a positive real root. Then*

(i) *The set $\Lambda_\alpha := \{\beta \in \Pi \mid w(\beta) = \alpha$ for some $w \in W$ with length $h'(\alpha)\}$ is connected. (That is, for any $\beta, \gamma \in \Lambda_\alpha$, we can find a sequence $\beta = \delta_1, \delta_2, \ldots, \delta_k = \gamma$ in Λ_α such that $\langle \delta_i, \delta_{i+1}^\vee \rangle \neq 0$ for $i = 1, 2, \ldots, k-1$. We also say that β, γ are connected in Λ_α).*

(ii) *Assume that the Dynkin diagram of Φ_α includes no cycles, then for each $\beta \in \Lambda_\alpha$, the element $w \in W$ such that $w(\beta) = \alpha$ and $\ell(w) = h'(\alpha)$ is unique.*

(iii) *Assume that the Dynkin diagram of Φ_α includes no cycles. If the set $\Pi_\alpha := \{\beta \in \Pi \mid \beta \leq \alpha$, and β, α are conjugate under $W\}$ is connected, then $\Lambda_\alpha = \Pi_\alpha$.*

(iv) *Let α, β, w be as in* (ii), *and let s_γ be a simple reflection, then $s_\gamma w \leq w$ if and only if $\beta \leq s_\gamma(\alpha) < \alpha$; and $ws_\gamma \leq w$ if and only if $\beta < s_\gamma(\beta) \leq \alpha$.*

(v) *Let $\alpha_1, \ldots, \alpha_k$ be simple roots. Assume that $s_{\alpha_i}(\alpha) < \alpha$ for $i = 1, 2, \ldots, k$. Then $s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_k}$ generate a finite group P. If further $\alpha, \alpha_1, \ldots, \alpha_k$ are* *linearly independent, then we can find a simple root β and an element $w \in W$ such that $w(\beta) = \alpha$ and $\ell(w) = \ell(u_0) + \ell(u_0 w) = h'(\alpha)$, where u_0 is the longest element of P.*

REMARK. When the Dynkin diagram of Φ_α includes cycles, the assertions (ii), (iii) and (iv) may be false. As an example we consider affine root system $A_2^{(1)}$. Let $\alpha_0, \alpha_1, \alpha_2$ be the simple roots, then $\langle \alpha_i, \alpha_j^\vee \rangle = -1$ when $i \neq j$. We have $\alpha_0 < s_{\alpha_1}(\alpha_0) < s_{\alpha_2}s_{\alpha_1}(\alpha_0) < s_{\alpha_0}s_{\alpha_2}s_{\alpha_1}(\alpha_0) < s_{\alpha_1}s_{\alpha_0}s_{\alpha_2}s_{\alpha_1}(\alpha_0) < s_{\alpha_2}s_{\alpha_1}s_{\alpha_0}s_{\alpha_2}s_{\alpha_1}(\alpha_0) = 2\alpha_0 + 3\alpha_1 + 3\alpha_2$ and $\alpha_0 < s_{\alpha_2}(\alpha_0) < s_{\alpha_1}s_{\alpha_2}(\alpha_0) < s_{\alpha_0}s_{\alpha_1}s_{\alpha_2}(\alpha_0) < s_{\alpha_2}s_{\alpha_0}s_{\alpha_1}s_{\alpha_2}(\alpha_0) < s_{\alpha_1}s_{\alpha_2}s_{\alpha_0}s_{\alpha_1}s_{\alpha_2}(\alpha_0) = 2\alpha_0 + 3\alpha_1 + 3\alpha_2 := \alpha$. But $s_{\alpha_2}s_{\alpha_1}s_{\alpha_0}s_{\alpha_2}s_{\alpha_1} \neq s_{\alpha_1}s_{\alpha_2}s_{\alpha_0}s_{\alpha_1}s_{\alpha_2}$, thus (ii) is not true for $A_2^{(1)}$. Moreover, $0 < s_{\alpha_1}(\alpha) < \alpha$ but $s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}s_{\alpha_0}s_{\alpha_2}s_{\alpha_1} \geq s_{\alpha_2}s_{\alpha_1}s_{\alpha_0}s_{\alpha_2}s_{\alpha_1}$, so (iv) is not true for $A_2^{(1)}$. Note that $\alpha_0 < s_{\alpha_2}(\alpha_0) < s_{\alpha_1}s_{\alpha_2}(\alpha_0) = \alpha_0 + 2\alpha_1 + \alpha_2$ and $\alpha_0 + 2\alpha_1 + \alpha_2 \neq s_{\alpha_0}s_{\alpha_2}(\alpha_1)$, $s_{\alpha_2}s_{\alpha_0}(\alpha_1)$ we see that (iii) is not true for $A_2^{(1)}$.

In addition, Π_α may be not connected. For example, consider affine root system $C_2^{(1)}$. Let $\alpha_0, \alpha_1, \alpha_2$ be the simple roots, such that $\langle \alpha_0, \alpha_1^\vee \rangle = -2 = \langle \alpha_2, \alpha_1^\vee \rangle$, and $\langle \alpha_0, \alpha_2^\vee \rangle = 0$, $\langle \alpha_1, \alpha_0^\vee \rangle = -1 = \langle \alpha_1, \alpha_2^\vee \rangle$. Then $\alpha := \alpha_0 + 2\alpha_1 + 2\alpha_2 \in \Phi^+_{real}$ and $\Pi_\alpha = \{\alpha_0, \alpha_2\}$ is not connected.

Proof. Suppose $h'(\alpha) = m$. In parts (i) and (ii), $\beta_1, \ldots, \beta_m, \beta, \gamma_1, \ldots, \gamma_m, \gamma$ are simple roots and $w = s_{\beta_1} \cdots\cdots s_{\beta_m}$, $u = s_{\gamma_1} \cdots\cdots s_{\gamma_m}$, and α' stands for $s_{\gamma_1}(\alpha)$.

(i) Suppose that $w(\beta) = u(\gamma)$. We need to prove that β, γ are connected in Λ_α. We use induction on m. When $m = 0, 1$, the assertion is obvious. Now assume that $m \geq 2$. By Lemma 2.5 (iii) and Lemma 2.3 (ii), $h'(\alpha') = m - 1$, this implies that $\Lambda_{\alpha'} \subseteq \Lambda_\alpha$. If $s_{\gamma_1}w \leq w$, then $\ell(s_{\gamma_1}w) = m - 1 = h'(\alpha')$. By induction hypothesis β, γ are connected in $\Lambda_{\alpha'}$. In particular, β, γ are connected in Λ_α. If $s_{\gamma_1}w \geq w$. By Lemma 2.6 (i) we have $\alpha' = s_{\gamma_2} \cdots\cdots s_{\gamma_m}(\gamma) = s_{\beta_1} \cdots\cdots s_{\beta_{m-1}}(\beta_m)$. By induction hypothesis β_m, γ are connected in $\Lambda_{\alpha'}$. Obviously $\langle \beta, \beta_m^\vee \rangle \neq 0$, so β, γ are connected in Λ_α.

(ii) Suppose that $w(\beta) = u(\beta)$. We use induction on m to prove that w and u are equal. When $m = 0, 1$, the assertion is obvious. Now assume that $m \geq 2$. Note that $h'(\alpha') = m - 1$. If $s_{\gamma_1}w \geq w$, by Lemma 2.6 (i) and Lemma 2.3 (ii) we have $\alpha' = s_{\gamma_2} \cdots\cdots s_{\gamma_m}(\beta) = s_{\beta_1} \cdots\cdots s_{\beta_{m-1}}(\beta_m) \geq \beta$. By Lemma 2.6 (ii), this is impossible. Therefore we must have $s_{\gamma_1}w \leq w$, then $\ell(s_{\gamma_1}w) = m - 1 = h'(\alpha')$. By induction hypothesis $s_{\gamma_1}w = s_{\gamma_2} \cdots\cdots s_{\gamma_m}$, hence $w = u$.

(iii) We first establish the following fact.

(a) Assume that the Dynkin diagram of Φ includes no cycles. Let β, γ be simple roots in Φ such that $\langle \beta, \gamma^\vee \rangle < 0$ and $\gamma = x(\beta)$ for some $x \in W$. Then $s_\beta s_\gamma(\beta) = \gamma$ and $s_\gamma s_\beta(\gamma) = \beta$.

(Remark: If the Dynkin diagram of Φ includes cycles the assertion (a) may be false. As an example we consider the root system generate by simple roots $\alpha_0, \alpha_1, \alpha_2$

with relations $\langle\alpha_i, \alpha_{i+1}^\vee\rangle = \langle\alpha_{i+1}, \alpha_i^\vee\rangle = -1$ for $i = 0, 1$, $\langle\alpha_0, \alpha_2^\vee\rangle = \langle\alpha_2, \alpha_0^\vee\rangle = -2$. Denote s_i for s_{α_i}, $i = 0, 1, 2$. Then $s_1 s_2 s_0 s_1(\alpha_0) = \alpha_2$ but $s_0 s_2(\alpha_0) = 3\alpha_0 + 4\alpha_2 \neq \alpha_2$.)

Let δ be a simple root such that $xs_\delta \leq x$ and let x_1 be the shortest element of the coset $x\langle s_\delta, s_\beta\rangle$ (we denote $\langle s_\delta, s_\beta\rangle$ the subgroup of W generated by s_δ, s_β). Let y be the element in $\langle s_\delta, s_\beta\rangle$ such that $x = x_1 y$. Since $x_1(\delta) > 0$ and $x_1(\beta) > 0$, so $y(\beta)$ is a simple root, denote by τ_1. We have

(*) $\tau_1 = \beta$ or $\langle\beta, \tau_1^\vee\rangle < 0$, and $x_1(\tau_1) = \gamma$.

Note that $\ell(x_1)$ is smaller than $\ell(x)$. We may continue this process, and finally we get a sequence of simple roots $\gamma = \tau_r, \tau_{r-1}, \ldots, \tau_1, \tau_0 = \beta$ such that

Either $\tau_i = \tau_{i+1}$ or $\langle\tau_i, \tau_{i+1}^\vee\rangle < 0$ for $i = 0, 1, \ldots, r-1$. (†)

Since $\langle\beta, \gamma^\vee\rangle < 0$ and the Dynkin diagram of Φ includes no cycles, by (†) we must have

Either $\tau_i = \beta$ or $\tau_i = \gamma$ for $i = 1, \ldots, r-1$. (•)

Therefore there exists an element x' in $\langle s_\beta, s_\gamma\rangle$ such that $x'(\beta) = \gamma$. It is easy to check that $x'(\beta) = \gamma$ implies that $\langle\beta, \gamma^\vee\rangle = \langle\gamma, \beta^\vee\rangle = -1$ and $x' = s_\beta s_\gamma$. This completes the argument for (a).

Now we argue for (iii). By Lemma 2.5 (v), the set Λ_α is non-empty. Obviously, $\Lambda_\alpha \subseteq \Pi_\alpha$. Let $\beta \in \Lambda_\alpha$ and let $\gamma \in \Pi_\alpha$. Assume that $\beta \neq \gamma$. Since Π_α is connected, we can find a sequence $\beta = \delta_1, \delta_2, \ldots, \delta_k = \gamma$ in Π_α such that $\langle\delta_i, \delta_{i+1}^\vee\rangle < 0$ for $i = 1, \ldots, k-1$. By (a) and the definition of Π_α we obtain

(b) $s_{\delta_{i+1}} s_{\delta_i}(\delta_{i+1}) = \delta_i$ and $s_{\delta_i} s_{\delta_{i+1}}(\delta_i) = \delta_{i+1}$ for $i = 1, 2, \ldots, k-1$.

Let $w \in W$ be such that $w(\beta) = \alpha$ and $\ell(w) = h'(\alpha)$. By (b) we get $ws_{\delta_2} s_\beta(\delta_2) = \alpha$. Note that $s_{\delta_2}(\beta) = \beta + \delta_2 \leq \alpha$, using Lemma 2.6 (iii) we see $ws_{\delta_2} \leq w$. Thus $\ell(ws_{\delta_2} s_\beta) = h'(\alpha)$ and $\delta_2 \in \Lambda_\alpha$. Continue this process, finally we see that $\ell(w') = h'(\alpha)$ and $w'(\gamma) = \alpha$, here $w' = ws_{\delta_2} s_{\delta_1} s_{\delta_3} s_{\delta_2} \cdots\cdots s_{\delta_k} s_{\delta_{k-1}}$. Hence $\gamma \in \Lambda_\alpha$.

(iv) The "only if" parts follow from Lemma 2.3 (ii). Assume $\beta \leq s_\gamma(\alpha) < \alpha$. By Lemma 2.5 (iii) there exists a simple root τ and an element $u \in W$ such that $u(\tau) = s_\gamma(\alpha)$ and $\ell(u) = h'(s_\gamma(\alpha)) = h'(\alpha) - 1$. Thus $s_\gamma u(\tau) = \alpha$. According to the definition of $h'(\alpha)$ we must have $h'(\alpha) = \ell(u) + 1$. Since $s_\gamma(\alpha) \geq \beta$, applying Lemma 2.6 (ii) we see $s_\gamma w \leq w$. If $\beta < s_\gamma(\beta) \leq \alpha$, by Lemma 2.6 (iii) we see that $ws_\gamma \leq w$.

(v) If $\alpha, \alpha_1, \ldots, \alpha_k$ are linearly dependent, then $\alpha_1, \ldots, \alpha_k$ span a finite root system [K, Corollary 4.3, p. 42]. In particular, its Weyl group P is finite. Now assume that $\alpha, \alpha_1, \ldots, \alpha_k$ are linearly independent. Let $\tau_r, \ldots, \tau_1$ be simple roots in $\{\alpha_1, \ldots, \alpha_k\}$ such that $s_{\tau_r} \cdot \cdots \cdot s_{\tau_1}$ is a reduced decomposition. Define $w_0 := e$ (the neutral element in P), $w_i := s_{\tau_i} \cdot \cdots \cdot s_{\tau_1}$ for $i = 1, \ldots, r$. Then

(a) For $i = 0, 1, \ldots, r-1$, we have $w_{i+1} = s_{\tau_{i+1}} w_i \geq w_i$. In particular $w_i^{-1}(\tau_{i+1})$ is a positive root and is a linear combination of $\alpha_1, \ldots, \alpha_k$.

Since $\langle \alpha, \alpha_j^\vee \rangle > 0$ for $j = 1, \ldots, k$, by (a) we obtain

(b) For $i = 0, 1, \ldots, r-1$, one has $\langle w_i \alpha, \tau_{i+1}^\vee \rangle > 0$.

Since $\alpha, \alpha_1, \ldots, \alpha_k$ are linearly independent, by (a) and (b) we get the following assertion.

(c) $\alpha > w_1(\alpha) > w_2(\alpha) > \cdots > w_r(\alpha) > 0$.

The height of α is finite, hence the group P must be finite and has a longest element u_0. We may take the element w_r in (c) to be the longest element u_0. Then $u_0(\alpha)$ is a positive real root. By Lemma 2.5 (iii) we see

(d) $h'(\alpha) = \ell(u_0) + h'(u_0(\alpha))$.

Let $\beta \in \Pi$ and $x \in W$ be such that $x(\beta) = u_0(\alpha)$ and $\ell(x) = h'(u_0(\alpha))$. Set $w := u_0 x$, then $w(\beta) = \alpha$ and $\ell(w) \leq \ell(u_0) + \ell(u_0 w) = h'(\alpha)$. By (d) and the definition of $h'(\alpha)$, these imply that $\ell(w) = \ell(u_0) + \ell(u_0 w) = h'(\alpha)$.

The proposition is proved.

2.8. Assume Φ is finite, then $\Phi^+ = \Phi^+_{real}$. We shall give another interpretation for the function $h' : \Phi^+ \to \mathbf{N}$. We recall some simple facts about finite root systems. Let $\Phi^\vee$ be the dual root system of Φ. For a root α in Φ, denote by $\alpha^\vee$ its corresponding root in $\Phi^\vee$. We identify the Weyl group of $\Phi^\vee$ with W, the Weyl group of Φ. Let α, β be positive roots.

(i) If α is a short root and $\alpha \neq \beta$, then $|\langle \alpha, \beta^\vee \rangle| \leq 1$.
(ii) For $w \in W$ we have $w(\beta) = \alpha$ if and only if $w(\beta^\vee) = \alpha^\vee$.
(iii) α is a long (resp. short) root in Φ if and only if $\alpha^\vee$ is a short (resp. long) root in $\Phi^\vee$.
(iv) Assume that both α, β are short (resp. long) roots, then $\beta \leq \alpha$ if and only if $\beta^\vee \leq \alpha^\vee$.

PROPOSITION 2.9. *Assume that Φ is of finite type. Let α be a positive root. Then*

(i) *$h'(\alpha) = h(\alpha) - 1$ when α is a short root, and $h'(\alpha) = h(\alpha^\vee) - 1$ when α is a long root, where h denotes the height function of Φ^+ or $\Phi^{\vee +}$.*

(ii) *$\Lambda_\alpha = \Pi_\alpha$, and for each $\beta \in \Pi_\alpha$, there exists a unique element $w \in W$ such that $w(\beta) = \alpha$ and $\ell(w) = h'(\alpha)$.*

(iii) *Let w be as in* (ii) *and let $s_{\beta_1} \cdot \cdots \cdot s_{\beta_m}$ be a reduced decomposition of w. Set $\beta_{m+1} := \beta$. Then for any $1 \le i \le j \le m$, we have $s_{\beta_i} \cdot \cdots \cdot s_{\beta_j}(\beta_{j+1}) \ge s_{\beta_{i+1}} \cdots s_{\beta_j}(\beta_{j+1})$.*

(iv) *Let w be as in* (ii) *and let $s_{\gamma_1}, s_{\gamma_2}$ be simple reflections in W such that $s_{\gamma_1} w \le w$, $s_{\gamma_2} w \le w$ (resp. $ws_{\gamma_1} \le w$, $ws_{\gamma_2} \le w$), then $s_{\gamma_1} s_{\gamma_2} = s_{\gamma_2} s_{\gamma_1}$.*

Proof. Using 2.8 and Lemma 2.3 we get (i).

(ii) Since Φ is finite, the Dynkin diagram of Φ includes no cycles and Π_α is always connected. By parts (ii) and (iii) of Proposition 2.7 we see that (ii) is true.

(iii) By 2.8 we may assume that α is a short root. If $j = m$, we always have $s_{\beta_i} \cdot \cdots \cdot s_{\beta_j}(\beta_{j+1}) > s_{\beta_{i+1}} \cdot \cdots \cdot s_{\beta_j}(\beta_{j+1})$ by Lemma 2.3. Now assume $j < m$. Using 2.8 and Lemma 2.3 we see $s_{\beta_i} \cdots s_{\beta_m}(\beta) = \beta_i + s_{\beta_{i+1}} \cdot \cdots \cdot s_{\beta_m}(\beta) = \beta_i + s_{\beta_{i+1}} \cdot \cdots \cdot s_{\beta_j}(\beta_{j+1} + s_{\beta_{j+2}} \cdot \cdots \cdot s_{\beta_m}(\beta)) = \beta_i + s_{\beta_{i+1}} \cdot \cdots \cdot s_{\beta_j}(\beta_{j+1}) + s_{\beta_{i+1}} \cdot \cdots \cdot s_{\beta_j} s_{\beta_{j+2}} \cdot \cdots \cdot s_{\beta_m}(\beta)$ and $s_{\beta_i} \cdot \cdots \cdot s_{\beta_m}(\beta) = s_{\beta_i} \cdot \cdots \cdot s_{\beta_j}(\beta_{j+1}) + s_{\beta_i} \cdot \cdots \cdot s_{\beta_j} s_{\beta_{j+2}} \cdot \cdots \cdot s_{\beta_m}(\beta) \le s_{\beta_i} \cdot \cdots \cdot s_{\beta_j}(\beta_{j+1}) + \beta_i + s_{\beta_{i+1}} \cdot \cdots \cdot s_{\beta_j} s_{\beta_{j+2}} \cdot \cdots \cdot s_{\beta_m}(\beta)$. Hence we must have $s_{\beta_i} \cdot \cdots \cdot s_{\beta_j}(\beta_{j+1}) \ge s_{\beta_{i+1}} \cdot \cdots \cdot s_{\beta_j}(\beta_{j+1})$.

(iv) Assume $s_{\gamma_1} w \le w$, $s_{\gamma_2} w \le w$. Let u be the longest element in the dihedral group generated by $s_{\gamma_1}, s_{\gamma_2}$, then $\ell(w) = \ell(u) + \ell(uw)$. Thus $\gamma_1, u(\alpha), \gamma_2$ are simple roots of the finite root system $(\mathbf{Z}\gamma_1 + \mathbf{Z}u(\alpha) + \mathbf{Z}\gamma_2) \cap \Phi$, whose Dynkin diagram is not a cycle. Therefore we have $\langle \gamma_1, \gamma_2^\vee \rangle = 0$, i.e. $s_{\gamma_1} s_{\gamma_2} = s_{\gamma_2} s_{\gamma_1}$.

If $ws_{\gamma_1} \le w$, $ws_{\gamma_2} \le w$, then $\langle \beta, \gamma_1^\vee \rangle < 0$, $\langle \beta, \gamma_2^\vee \rangle < 0$. But the Dynkin diagram of Φ includes no cycles, so $\langle \gamma_1, \gamma_2^\vee \rangle = 0$.

The proposition is proved.

REMARK. When Φ is infinite, the relations between the functions h', h are not so simple as in Proposition 2.9 (i). Also Proposition 2.9 (iii) and (iv) may be false. In fact, consider the affine root system $D_4^{(1)}$. Let $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the simple roots such that $\langle \alpha_2, \alpha_i^\vee \rangle = -1$ for $i = 0, 1, 3, 4$. Let $\alpha = \alpha_0 + 2\alpha_1 + 3\alpha_2 + \alpha_3 + \alpha_4$, then $h'(\alpha) = 6$. Let $w = s_{\alpha_1} s_{\alpha_2} s_{\alpha_1} s_{\alpha_0} s_{\alpha_3} s_{\alpha_4}$, then $w(\alpha_2) = \alpha$. We have $s_{\alpha_1} s_{\alpha_2}(\alpha_1) < s_{\alpha_2}(\alpha_1)$, and $s_{\alpha_1} w \le w$, $s_{\alpha_2} w \le w$, but $s_{\alpha_1} s_{\alpha_2} \ne s_{\alpha_2} s_{\alpha_1}$. So (iii) and (iv) are not true for $D_4^{(1)}$.

2.10. Set $\mathscr{H} = \{(w, \beta) \in W \times \Pi \mid w(\beta) \in \Phi^+_{real}\}$. We call an element $(w, \beta) \in \mathscr{H}$ *shortable* if there exist $w_1, u_1 \in W$ such that $w = w_1 \cdot u_1$ and $u_1(\beta) \in \Pi$, $\ell(u_1) \ge 1$, $u_1 \in \langle s, t \rangle$ for some simple reflections $s, t \in W$; we also call $\ell(w)$ the length of (w, β).

Here we use the convention: for $x, x_1, x_2, \ldots, x_m \in W$, we write $x = x_1 \cdot x_2 \cdot \cdots \cdot x_m$ if $x = x_1 x_2 \cdots x_m$, and $\ell(x) = \ell(x_1) + \ell(x_2) + \cdots + \ell(x_m)$.

Let $(w, \beta), (u, \gamma) \in \mathcal{H}$, we write $(w, \beta) \sim (u, \gamma)$ if there exists $u_1 \in W$ such that $w = u \cdot u_1$ and $u_1(\beta) = \gamma$. The relation $\sim$ generates an equivalence relation in $\mathcal{H}$, we denote it also by $\sim$. The equivalence class containing (w, β) is denoted by $(w, \beta)^{\sim}$. The set of all equivalence classes in $\mathcal{H}$ is denoted by $\tilde{\mathcal{H}}$.

LEMMA 2.11. *Let β, γ, δ be simple roots and $w \in W$. Assume $w(\beta) = \gamma$ and $\ell(w) \geq 1$.*

(i) *If $ws_\delta \leq w$, then s_δ, s_β generate a finite group, denote by $\langle s_\delta, s_\beta \rangle$. In particular, $|\langle \delta, \beta^\vee \rangle \langle \beta, \delta^\vee \rangle| < 4$.*

(ii) *(w, β) is shortable.*

Proof. (i) We apply the method in [L1, 1.8]. Let w_1 be an element of the coset $w\langle s_\delta, s_\beta \rangle$ of minimal length and let $w_2 \in \langle s_\delta, s_\beta \rangle$ be such that $w = w_1 \cdot w_2$. Then $w_1(\delta)$, $w_1(\beta)$ are positive roots, so $w_2(\beta)$ is a simple root. Obviously, $w_2(\delta) < 0$, and $w_2 s_\beta(\beta) < 0$. Since $w_2(\delta)$ and $w_2(\beta)$ are linearly independent, we have $w_2 s_\beta(\delta) = w_2(\delta - \langle \delta, \beta^\vee \rangle \beta) < 0$. Thus $w_2 s_\beta$ is an element in $\langle s_\delta, s_\beta \rangle$ of maximal length. So $\langle s_\delta, s_\beta \rangle$ is finite.

(ii) By the definition and the proof of (i) we get (ii).

PROPOSITION 2.12. (i) *For each equivalence class $(w, \beta)^{\sim}$ in $\mathcal{H}$, there exists a unique shortest element (u, γ) in $(w, \beta)^{\sim}$. Furthermore, we have $w = u \cdot u_1$ for some $u_1 \in W$.*

(ii) *Assume that Φ is finite. For two elements $(w, \beta), (u, \gamma) \in \mathcal{H}$, choose arbitrary $(x, \delta), (y, \varepsilon) \in \mathcal{H}$ such that $x^{-1}w = x^{-1} \cdot w$, $y^{-1}u = y^{-1} \cdot u$ and $w(\beta) = x(\delta)$, $u(\gamma) = y(\varepsilon)$, then $(w, \beta) \sim (u, \gamma)$ if and only if $(x, \delta) \sim (y, \varepsilon)$. In particular, if x is a shortest element such that $x^{-1}w = x^{-1} \cdot w$, and $x^{-1}w(\beta)$ is a simple root δ, then (x, δ) is the unique shortest element in $(x, \delta)^{\sim}$. We also denote $(x, \delta)^{\sim}$ by $(w, \beta)^{\sim}_{*}$.*

Proof. (i) Let (u, γ) be an element in $(w, \beta)^{\sim}$ with minimal length. We shall prove that $w = u \cdot u_1$ for some $u_1 \in W$, this forces that (u, γ) is the unique shortest element in $(w, \beta)^{\sim}$.

Let $(u', \gamma'), (w', \beta') \in (w, \beta)^{\sim}$ be such that $u' = u \cdot u_1'$, $u' = w' \cdot w_1'$, where $u_1' \in W$, and w_1' is one of the following elements (δ is a simple root): s_δ, $\langle \delta, \beta'^\vee \rangle = 0$; $s_{\beta'} s_\delta$, $\langle \delta, \beta'^\vee \rangle \langle \beta', \delta^\vee \rangle = 1$; $s_\delta s_{\beta'} s_\delta$, $\langle \delta, \beta^\vee \rangle \langle \beta', \delta^\vee \rangle = 2$; $s_\delta s_{\beta'} s_\delta s_{\beta'} s_\delta$, $\langle \delta, \beta'^\vee \rangle \langle \beta', \delta^\vee \rangle = 3$. Because (u, γ) is an element in $(w, \beta)^{\sim}$ of minimal length, using exchange condition [K, Lemma 3.11 (c), p. 33] we get $u_1' = u_2 \cdot w_1'$ for some $u_2 \in W$, thus $w' = u \cdot u_2$. According to the definition of $\sim$ and Lemma 2.11 we see that there exists $u_1 \in W$ such that $w = u \cdot u_1$.

(ii) Suppose that $(w, \beta) \sim (u, \gamma)$. It is no harm to assume that (u, γ) is the shortest element in $(w, \beta)^{\sim}$. By (i) we know that $x^{-1}u = x^{-1} \cdot u$, $y^{-1}u = y^{-1} \cdot u$ and $x(\delta) = u(\gamma) = y(\varepsilon)$. Let $u_0 \in W$ be such that $u_0 u s_\gamma = u_0 \cdot u s_\gamma = w_0$, the longest elements of W. Then $u_0 = x_1 \cdot x^{-1} = y_1 \cdot y^{-1}$ for some $x_1, y_1 \in W$. Since $\sigma := u_0 u(\gamma) \in \Pi$, we get $(x, \delta) \sim (u_0^{-1}, \sigma) \sim (y, \varepsilon)$. The "only if" part is similar when one notes that $w^{-1}x = w^{-1} \cdot x$, $u^{-1}y = u^{-1} \cdot y$.

The proposition is proved.

REMARK. Part (ii) gives a way to compute the shortest elements in $\mathscr{H}$ for finite root systems.

For infinite root systems, sometimes it is impossible to find an element such that $x^{-1}w = x^{-1} \cdot w$ and $x^{-1}w(\beta) \in \Pi$. As an example we pick up again $A_2^{(1)}$. Let $\alpha = 2\alpha_0 + 3\alpha_1 + 3\alpha_2$, $\beta = \alpha_0$, $w = s_{\alpha_1} s_{\alpha_2} s_{\alpha_0} s_{\alpha_1} s_{\alpha_2}$, then we can not find an element $x \in W$ such that $x^{-1}w = x^{-1} \cdot w$ and $x^{-1}w(\beta) \in \Pi$.

3. Several Lemmas

3.1. Keep the notation in section 1. In this section we give several lemmas concerned with the automorphisms T_i. We refer to [L3]. The Lemma 3.5 is an essential ingredient to the proof of Lemma 4.2, which implies that the root vectors are linearly independent.

Let $s_{k_1} s_{k_2} s_{k_3} \dots s_{k_{\nu-1}} s_{k_\nu}$ be a reduced expression of the longest element w_0 of W. For any $c = (c_1, c_2, \dots, c_\nu) \in \mathbf{N}^\nu$, $r = (r_1, \dots, r_n) \in \mathbf{Z}^n$, we set

$$E^c = E_{k_1}^{c_1} T_{k_1}(E_{k_2}^{c_2}) T_{k_1} T_{k_2}(E_{k_3}^{c_3}) \dots T_{k_1} T_{k_2} \dots T_{k_{\nu-1}}(E_{k_\nu}^{c_\nu}), \qquad F^c = \Omega(E^c).$$

$$G^c = E_{k_1}^{c_1} E_{k_2}^{c_2} T_{k_2}(E_{k_3}^{c_3}) T_{k_2} T_{k_3}(E_{k_4}^{c_4}) \dots T_{k_2} T_{k_3} \dots T_{k_{\nu-1}}(E_{k_\nu}^{c_\nu}),$$

$$H^c = \Omega(G^c),$$

$$K^r = K_1^{r_1} \dots K_n^{r_n}.$$

Let U^+ be the subalgebra of U generated by all E_i. The following two lemmas are due to Lusztig (see [L3, 2.4])

LEMMA 3.2. *We fix $i \in [1, n]$. Let $O_i = \{\xi \in U^+ \mid F_i\xi - \xi F_i \in K_i^{-1} U^+\}$. Let O_i' be the $\mathbf{Q}(v)$-subalgebra of U^+ generated by the elements $T_i(E_j)$, $T_i T_j(E_i)$, $T_i T_j T_i(E_j)$, $T_i T_j T_i T_j(E_i)$ for j such that $a_{ij}a_{ji} = 3$, the elements $T_i(E_j)$, $T_i T_j(E_i)$ for j such that $a_{ij}a_{ji} = 2$, the elements $T_i(E_j)$ for j such that $a_{ij}a_{ji} = 1$, and by E_j for $j \neq i$. Choose a reduced expression $s_{k_1} s_{k_2} s_{k_3} \dots s_{k_{\nu-1}} s_{k_\nu}$ of w_0 be such that $k_1 = i$. Let O_i''*

be the $\mathbf{Q}(v)$*-subspace of* U^+ *spanned by the elements* E^c *(defined in* 3.1) *for various* $c=(c_1,\ldots,c_\nu)\in\mathbf{N}^\nu$ *such that* $c_1=0$. *We have* $O_i=O_i'=O_i''=U^+\cap T_i(U^+)$.

Proof. It is clear that O_i is a $\mathbf{Q}(v)$-subalgebra of U^+. It is easy to check that the generators of O_i' are contained in O_i. It follows that $O_i'\subset O_i$.

By using the method in the proof of [L1, 1.8] we see that $O_i''\subset O_i'$. As the same way of the proof of $R_i\subset R_i''$ in [L3, 2.4] (notations in loc. cit) we get $O_i\subset O_i''$. The lemma is proved.

LEMMA 3.3. *We fix* $i\in[1,n]$. *Let* $P_i=\{\xi\in U^+\mid F_i\xi-\xi F_i\in K_iU^+\}$. *Let* P_i' *be the* $\mathbf{Q}(v)$*-subalgebra of* U^+ *generated by the elements* $T_i'(E_j)$, $T_i'T_j'(E_i)$, $T_i'T_j'T_i'(E_j)$, $T_i'T_j'T_i'T_j'(E_i)$ *for* j *such that* $a_{ij}a_{ji}=3$, *the elements* $T_i'(E_j)$, $T_i'T_j'(E_i)$ *for* j *such that* $a_{ij}a_{ji}=2$, *the elements* $T_i'(E_j)$ *for* j *such that* $a_{ij}a_{ji}=1$, *and by* E_j *for* $j\neq i$. *Choose a reduced expression* $s_{k_1}s_{k_2}s_{k_3}\cdots s_{k_{\nu-1}}s_{k_\nu}$ *of* w_0 *be such that* $k_1=i$. *Let* P_i'' *be the* $\mathbf{Q}(v)$*-subspace of* U^+ *spanned by the elements* G^c *(defined in* 3.1) *for various* $c=(c_1,\ldots,c_\nu)\in\mathbf{N}^\nu$ *such that* $c_1=0$. *We have* $P_i=P_i'=P_i''=U^+\cap T_i'(U^+)$.

The proof is similar.

3.4. For $\lambda\in\mathbf{N}R^+$, we denote U_λ the set of all elements $\xi\in U$ such that $K_i\xi K_i^{-1}=v^{d_i\langle\alpha_i^\vee,\lambda\rangle}\xi$. Let $U_\lambda^+=U^+\cap U_\lambda$.

LEMMA 3.5. *Let* $Q_i=O_i\cap P_i=\{\xi\in U^+\mid F_i\xi=\xi F_i\}$. *We have* $s_i(\lambda)\geq\lambda$ *if* $Q_i\cap U_\lambda^+\neq\{0\}$.

Proof. Let U_A be the $A=\mathbf{Q}[v]$-subalgebra of U generated by all E_j, F_j, K_j, K_j^{-1}. Regard $\mathbf{Q}$ as a $\mathbf{Q}[v]$-algebra by specializing v to 1. Thus we can get the $\mathbf{Q}$-algebra

$$U_1=U_A\otimes_A\mathbf{Q}/<K_1-1,K_2-1,\ldots,K_n-1\rangle,$$

which is just the universal enveloping algebra of the simple Lie algebra corresponding to the Cartan matrix (a_{ij}). Let f_i, U_1^+, $U_{1,\lambda}^+$, be the images of F_i, U^+, U_λ^+, respectively. According to the commutation relations between root vectors in U_1 and PBW Theorem one can check easily that the subalgebra $Q_{1,i}=\{x\in U^+\mid f_ix=xf_i\}$ is generated by $e_\alpha(\alpha\in R^+)$ such that $\alpha-\alpha_i\notin R$, where e_α is a root vector in U_1^+ of root α. Note that $\alpha-\alpha_i\notin R$ implies that $s_i(\alpha)\geq\alpha$, we see that $Q_{1,i}\cap U_{1,\lambda}^+\neq\{0\}$ implies that $s_i(\lambda)\geq\lambda$. Our assertion follows from this and that $Q_{1,i}\cap U_{1,\lambda}^+\neq\{0\}$ if $Q_i\cap U_\lambda^+\neq\{0\}$. The lemma is proved.

3.6. REMARK. By 3.2 and 3.3 we know that $Q_i=O_i\cap P_i=U^+\cap T_i(U^+)\cap T_i'(U^+)$. It is likely that Q_i is the $\mathbf{Q}(v)$-subalgebra of U^+ generated by the elements $T_kT_i(E_j)$ for j,k with $a_{ij}a_{ji}>0$, $a_{ik}a_{ki}=1$, and by E_j for $j\neq i$.

4. Root vectors

4.1. Keep the notation in section 1 and section 3. In this section we describe the set of all root vectors of a given root. Theorem 4.4 is the main result.

Let Π denote also the set of simple roots in R. Given a positive root α in R^+. Let Y_α be the set of all root vectors of root α. Recall that we have defined the set Π_α in Proposition 2.7 (iii). According to Proposition 2.9 (ii), for each $\beta \in \Pi_\alpha$, there exists a unique element $w \in W$ such that $w(\beta) = \alpha$ and $\ell(w) = h'(\alpha)$. We shall denote the element by $w_{\alpha,\beta}$ or by $w_{\alpha,k}$ when $\beta = \alpha_k$. Suppose $m = h'(\alpha)$, we fix a reduced expression $s_{j_1}s_{j_2}\ldots s_{j_m}$ of $w_{\alpha,\beta}$, $\beta \in \Pi_\alpha$. For any simple root γ we set $E_\gamma := E_i$ when $\gamma = \alpha_i$. Define $Y'_\alpha = \{T_{\alpha,\beta,a}(E_\beta) \mid a \in I_\alpha\}$, where $T_{\alpha,\beta,a} = T_{j_1}^{a_1}T_{j_2}^{a_2}\ldots T_{j_m}^{a_m}$, $a = (a_1, a_2, \ldots, a_m) \in I_\alpha := \{1, -1\}^m$. When $h'(\alpha) = 0$, we set $I_\alpha = \{e\}$ and $T_{\alpha,\beta,e} = \mathrm{id}_U$, where e is the neutral element of W.

LEMMA 4.2. *Keep the notations in* 4.1.

(i) *The set Y'_α is independent of the choice of the reduced expression and the choice of β, so only depends on α.*

(ii) *The elements $T_{\alpha,\beta,a}(E_\beta)$, $a \in I_\alpha$ are linearly independent over* $\mathbf{Q}(v)$. *In particular, the set Y'_α contains $2^{h'(\alpha)}$ elements.*

Proof. (i) Using Proposition 2.7 (iv) and induction on $h'(\alpha)$ we see that Y'_α is independent of the choice of the reduced expression. According to the proof of Proposition 2.7 (iii) and 1.3 (d) we know that Y'_α does not depend on the choice of β.

(ii) If each $j \in [1, n]$ appears in the sequence $j_1, j_2, \ldots, j_m, j_{m+1}$ ($\alpha_{j_{m+1}} := \beta$) at most two times, then we can choose the reduced expression such that $j_1, j_2, \ldots, j_p$ is a subsequence (disregard order) of $j_{p+1}, j_{p+2}, \ldots, j_m, j_{m+1}$ for some p and $j_{p+1}, j_{p+2}, \ldots, j_m, j_{m+1}$ are pairwise different. Thus for any $a \in I_\alpha$, $T_{j_1}^{a_1}T_{j_2}^{a_2}\ldots T_{j_q}^{a_q}(F_{j_{q+1}}) \in U^- = \Omega(U^+)$ for any $q \le p-1$, and $T_{j_{p+1}}^{a_{p+1}}T_{j_{p+2}}^{a_{p+2}}\ldots T_{j_m}^{a_m}(E_\beta) \in U^+$, since both $j_1, j_2, \ldots, j_p$ and $j_{p+1}, j_{p+1}, \ldots, j_m$ are pairwise different. Combine these and using induction on m we see that in the expression

$$T_{j_1}T_{j_2}^{a_2}\ldots T_{j_m}^{a_m}(E_\beta) = \sum_{\substack{c',c \in \mathbf{N}^\nu \\ r \in \mathbf{Z}^n}} \rho_{c',r,c}F^{c'}K^rE^c, \qquad \rho_{c',r,c} \in \mathbf{Q}(v),$$

$$\left(\text{resp. } T_{j_1}^{-1}T_{j_2}^{a_2}\ldots T_{j_m}^{a_m}(E_\beta) = \sum_{\substack{c',c \in \mathbf{N}^\nu \\ r \in \mathbf{Z}^n}} \rho'_{c',r,c}H^{c'}K^rG^c, \qquad \rho'_{c',r,c} \in \mathbf{Q}(v),\right)$$

if $\rho_{c',r,c} \neq 0$ (resp. $\rho'_{c',r,c} \neq 0$), then $E^c \in O_{j_1}$ (resp. $G^c \in P_{j_1}$), where $F^{c'}, E^c, H^{c'}$, G^c, K^r are defined as in 3.1, we choose the reduced expression of w_0 such that $k_1 = j_1$. According to Proposition 2.9 (iii) we see that

$$s_{j_1}s_{j_2}\dots s_{j_r}(\alpha_{j_{r+1}}) \geq s_{j_2}\dots s_{j_r}(\alpha_{j_{r+1}}) \qquad \text{for any } 1 \leq r \leq m. \tag{$*$}$$

Therefore if $\rho_{c',r,c} \neq 0$ (resp. $\rho'_{c',r,c} \neq 0$), then $E^c \in U_\lambda^+$ (resp. $G^c \in U_\lambda^+$) for some $\lambda \in \mathbf{N}R^+$ such that $s_{j_1}(\lambda) < \lambda$. Using Lemma 3.5 we see that if

$$\sum_{a \in I_\alpha} \rho_a T_{\alpha,\beta,a}(E_\beta) = 0, \qquad \rho_a \in \mathbf{Q}(v),$$

then

$$\sum_{\substack{a \in I_\alpha \\ a_1 = 1}} \rho_a T_{\alpha,\beta,a}(E_\beta) = 0, \qquad \sum_{\substack{a \in I_\alpha \\ a_1 = -1}} \rho_a T_{\alpha,\beta,a}(E_\beta) = 0.$$

Using induction we know that $\rho_a = 0$ for all $a \in I_\alpha$. Thus we have proved (ii) for type A_n, B_n, C_n, D_n, G_2.

In general we argue as follows.

Let

$$T_{j_1}^{a_1}T_{j_2}^{a_2}\dots T_{j_m}^{a_m}(E_\beta) = \xi_a + \xi'_a,$$

where

$$\xi_a = \sum_{\substack{c',c \in \mathbf{N}^\nu \\ r \in \mathbf{Z}^n \\ E^c \in O_{j_1}}} \rho_{c',r,c} F^{c'}K^rE^c, \qquad \xi'_a = \sum_{\substack{c',c \in \mathbf{N}^\nu \\ r \in \mathbf{Z}^n \\ E^c \notin O_{j_1}}} \rho'_{c',r,c} F^{c'}K^rE^c, \qquad \text{if } a_1 = 1,$$

$$\xi_a = \sum_{\substack{c',c \in \mathbf{N}^\nu \\ r \in \mathbf{Z}^n \\ G^c \in P_{j_1}}} \rho_{c',r,c} H^{c'}K^rG^c, \qquad \xi'_a = \sum_{\substack{c',c \in \mathbf{N}^\nu \\ r \in \mathbf{Z}^n \\ G^c \notin P_{j_1}}} \rho'_{c',r,c} H^{c'}K^rG^c, \qquad \text{if } a_1 = -1,$$

$\rho_{c',r,c} \in \mathbf{Q}(v)$, $\rho'_{c',r,c} \in \mathbf{Q}(v)$.

Note that

($**$) The image of $T_{j_1}^{a_1}T_{j_2}^{a_2}\dots T_{j_r}^{a_r}(F_{j_{r+1}})$ $(1 \leq r \leq m)$ in U_1^- (see the proof of Lemma 3.5) is not zero,

and α_{j_1}, $s_{j_1}s_{j_2}\dots s_{j_r}(\alpha_{j_{r+1}})$, $1 \leq r \leq m$ are pairwise different. Using induction on m and the fact ($*$) it is not difficult to check that if $\rho_{c',r,c} \neq 0$, $E^c \in O_{j_1} \cap U_\lambda$ (resp. $\rho'_{c',r,c} \neq 0$, $G^c \in P_{j_1} \cap U_\lambda$), then $s_{j_1}(\lambda) < \lambda$, and that the set $\{\xi_a \mid a_1 = 1\}$ (resp. $\{\xi_a \mid a_1 = -1\}$) is $\mathbf{Q}(v)$-linearly independent. By these and Lemma 3.5 we see that (ii) is true.

4.3 REMARK. By (*) and (**) in the proof of Lemma 4.2 we know that if $T_{j_r}^{a_r}T_{j_{r+1}}^{a_{r+1}}\dots T_{j_m}^{a_m}(E_\beta)\notin U^+$ for some $r\le m$, then $T_{\alpha,\beta,a}(E_\beta)\notin U^+$.

Set $Y=\bigcup_{\alpha\in R^+}Y_\alpha$, $Y'=\bigcup_{\alpha\in R^+}Y'_\alpha$.

THEOREM 4.4. *Keep the notations in* 4.1. *Let* $\alpha\in R^+$, *then*

(i) *The set* Y_α *is stable under the anti-automorphism* Ψ (*see* 1.3 (a) *for the definition*), *i.e.* $\Psi(Y_\alpha)=Y_\alpha$. *In particular,* $\Psi(Y)=Y$.

(ii) *We have* $Y_\alpha\subset Y'_\alpha\cap U^+$. *In particular, all root vectors in* U *are linearly independent over* $\mathbf{Q}(v)$.

(iii) *Recall that we have defined the set* $\tilde{\mathscr{H}}$ *in* 2.10. *The map* $\Theta:(w,\beta)^\sim\to T_w(E_\beta)$ *defines a bijection between* $\tilde{\mathscr{H}}$ *and* Y. *Moreover* $\Theta(\tilde{\mathscr{H}}_\alpha)=Y_\alpha$, *where* $\tilde{\mathscr{H}}_\alpha=\{(w,\beta)^\sim\in\tilde{\mathscr{H}}\mid w(\beta)=\alpha\}$.

(iv) *Let* $(w,\beta)^\sim\in\tilde{\mathscr{H}}$, *then* $\Theta((w,\beta)_*^\sim)=\Psi\cdot\Theta((w,\beta)^\sim)$.

Proof. Let δ be a simple root and $x\in W$ such that $E:=T_x(E_\delta)$ is an element in Y_α.

(i) Choose $y\in W$ be such that $y^{-1}x=y^{-1}\cdot x$ and $\varepsilon:=y^{-1}x(\delta)$ is a simple root, according to 1.3 (a–b) we get $\Psi(E)=T_y(E_\varepsilon)\in Y_\alpha$.

(ii) When $h'(\alpha)=0$, the assertion is obvious. Since $h'(s_{j_1}(\alpha))<h'(\alpha)$, we shall use induction on $h'(\alpha)$. By induction hypothesis we see that there exist $a_2,\dots,a_m\in\{1,-1\}$, such that $T_z(E_\delta)=T_{j_2}^{a_2}\dots T_{j_m}^{a_m}(E_\beta)$, where $z=s_{j_1}x$. Therefore $T_x(E_\delta)=T_{j_1}T_{j_2}^{a_2}\dots T_{j_m}^{a_m}(E_\beta)$, if $\ell(x)=\ell(s_jx)+1$; and $T_x(E_\delta)=T_{j_1}^{-1}T_{j_2}^{a_2}\dots T_{j_m}^{a_m}(E_\beta)$, if $\ell(x)=\ell(s_jx)-1$. Hence $E\in Y'_\alpha\cap U^+$.

(iii) By 1.3 (b) we know that Θ is well defined and is surjective. We use induction on $h'(\alpha)$ to prove that Θ is injective. Assume that $\Theta((w,\beta)^\sim)=\Theta((u,\gamma)^\sim)$. Let $w'=s_{j_1}w$, $u'=s_{j_1}u$. Using (i), (ii), 1.3(a) and Proposition 2.12 (ii) we may assume that $w'<w$, $u'\le u$. By induction hypothesis we get $(w',\beta)^\sim=(u',\gamma)^\sim$, using Proposition 2.12 (i) we have in addition $(w,\beta)^\sim=(u,\gamma)^\sim$.

(iv) It follows from the proof of (i).

The theorem is proved.

REMARK. (i) It is likely that $Y=Y'\cap U^+$.

(ii) For any $v_0\in\mathbf{C}^*$, we regard $\mathbf{Q}(v_0)$ as an $A=\mathbf{Q}[v]$-algebra by specializing v to v_0. Let $U_{v_0}=U_A\otimes_A\mathbf{Q}(v_0)$. If $v^{2d}\ne1$ for any $1\le d\le\max\{d_i\}$, the same arguments show that Lemma 4.2, 4.3 and Theorem 4.4 are true for U_{v_0}. If $v_0^2=1$, then for each $\alpha\in R$, there is a unique (up to ±1) root vector of root α.

COROLLARY 4.5. *Notations are as in* 4.1. *Let* $E = T_{\alpha,\beta,a}(E_\beta) \in Y'_\alpha$, $a = (a_1, a_2, \ldots, a_m)$, $m = h'(\alpha)$, *then*

(i) *The element* E *is a root vector if and only if* $\Psi(E)$ *is a root vector. When* $a_1 = 1$, *then* E *is a root vector if and only if* $T_{j_2}^{a_2} \ldots T_{j_m}^{a_m}(E_\beta) = T_u(E_\delta) \in Y$ *for some* $u \in W$, $\delta \in \Pi$ *and* $s_{j_1}u \geq u$.

(ii) *For any* $1 \leq i \leq m$, $T_{j_i}^{a_i} T_{j_{i+1}}^{a_{i+1}} \ldots T_{j_m}^{a_m}(E_\beta)$ *is a root vector if* E *is a root vector.*

(iii) *If* $T_{j_p}^{a_p} T_{j_{p+1}}^{a_{p+1}} \ldots T_{j_m}^{a_m}(E_\beta)$ *is not a root vector for some* $1 \leq p \leq m$, *then* E *is not a root vector, i.e.* $E \notin Y_\alpha$.

Proof.

(i) The first assertion follows from Theorem 4.4 (i). The second follows from the proof of Theorem 4.4 (ii).

(ii) Suppose that $E = T_x(E_\delta)$ for some $x \in W$ and some simple root δ. As in the proof of Theorem 4.4 (ii) we see $T_{x'}(E_\delta) = T_{j_i}^{a_i} T_{j_{i+1}}^{a_{i+1}} \ldots T_{j_m}^{a_m}(E_\beta)$, where $x' = s_{j_{i-1}} s_{j_{i-2}} \ldots s_{j_1} w$.

(iii) It follows from (ii).

For any $E \in Y$, we shall denote the shortest elements in $\Theta^{-1}(E)$, $\Theta^{-1}(\Psi(E))$ by (w_E, α_{k_E}), $(w_E^*, \alpha_{k_E^*})$ respectively.

COROLLARY 4.6. *Let* α, j_1 *be as in* 4.1 *and let* $E \in Y_\alpha$.

(i) *We have* $s_{j_1} w_E \leq w_E$ *if and only if* $s_{j_1} w_E^* \geq w_E^*$.

(ii) *Let* W_α *is the subgroup of* W *generated by these simple reflections* s_m *such that* $\alpha_m \leq \alpha$. *Then* $w_E, w_E^* \in W_\alpha$ *and* $\alpha_{k_E}, \alpha_{k_E^*} \in \Pi_\alpha$.

(iii) *We have* $w_E^{-1} w_E^* = w_E^{-1} \cdot w_E^*$ *and* $w_E^{-1} w_E^*(\alpha_{k_E^*}) = a_{k_E}$.

Proof. (i) Let $a \in I_\alpha$ be such that $E = T_{\alpha,\beta,a}(E_\beta)$ (notations as in 4.1). By Theorem 4.4 (ii) and its proof we see that $s_{j_1} w_E \leq w_E$ if and only if $a_1 = 1$. Since $\Psi(E) = T_{j_1}^{-a_1} \ldots T_{j_m}^{-a_m}(E_\beta)$, we know that our assertion is true.

(ii) From the proof of Proposition 2.12 (ii) we see that $w_E \in W_\alpha$ if and only if $w_E^* \in W_\alpha$. Thus we may assume that $a_1 = 1$ to prove (ii). In this case, according to Corollary 4.5 (i), Theorem 4.4 (iii) and Proposition 2.12 (i), it is obvious that we have $w_E = s_{j_1} w_{E'}$, where $E' = T_{j_2}^{a_2} \ldots T_{j_m}^{a_m}(E_\beta)$. Thus we can use induction on $h'(\alpha)$ to prove the result since $h'(s_{j_1}(\alpha)) = h'(\alpha) - 1$.

(iii) It follows from the proof of Proposition 2.12 (ii).

By means of Ψ we can describe the antipode $S(E)$ for a root vector $E \in Y_\alpha$.

THEOREM 4.7. *Suppose* $\alpha = m_1\alpha_1 + m_2\alpha_2 + \cdots + m_n\alpha_n \in R^+$. *For any root vector* E *in* Y_α, *we have* $S(E) = \rho_\alpha K_\alpha^{-1} \Psi(E)$, *where*

$$\rho_\alpha = (-1)^{m_1+m_2+\cdots+m_n}\prod_{k=1}^{n} v^{m_k(m_k-1)d_k}\prod_{k=1}^{n-1} v^{m_k d_k(m_{k+1}a_{k,k+1}+\cdots+m_n a_{k,n})},$$

$$K_\alpha = K_1^{m_1}K_2^{m_2}\dots K_n^{m_n}.$$

Note that $\Psi(E)$ is also a root vector in Y_α.

Proof. It follows from $K_i^{-1}E_iK_j^{-1}E_j = v^{d_i a_{ij}}K_i^{-1}K_j^{-1}E_iE_j = v^{d_j a_{ji}}K_i^{-1}K_j^{-1}E_iE_j$ and the definitions of S, Ψ (see 1.1 and 1.3 (a)).

PROPOSITION 4.8. *We have $\# Y_\alpha \le 2^{h'(\alpha)}$. The equality holds if and only if $j_1, j_2, \dots, j_m, j_{m+1}$ ($\alpha_{j_{m+1}} := \beta$) (notations as 4.1) are pairwise different.*

Proof. The first part is obvious.

Thanks to Corollary 4.5 (i) and Corollary 4.6 (ii) we see the "if" part of the second assertion is true.

Assume that $j_k = j_{k'}$ for some different k, k'. Using Corollary 4.5 (iii) we may suppose that α, R is one of the following cases: $\alpha_1+\alpha_2+\alpha_3+\alpha_4, D_4$; $\alpha_1+2\alpha_2+2\alpha_3, B_3$; $\alpha_1+2\alpha_2+\alpha_3, C_3$; $3\alpha_1+2\alpha_2, G_2$; $2\alpha_1+\alpha_2, G_2$. (Here we number the simple roots in Π as usual). Then it is easy to check that the following elements are not in U^+ by using 1.3 (h–k): $T_2^{-1}T_1T_3T_4(E_2), D_4$; $T_2T_1^{-1}T_3^{-1}(E_2), B_3$; $T_2T_3^{-1}(E_2), C_3$; $T_2T_1^{-1}(E_2), G_2$; $T_1T_2^{-1}(E_1), G_2$. In particular, they are not root vectors. The proposition is proved.

4.9 REMARK. Let $\alpha = m_1\alpha_1 + m_2\alpha_2 + \cdots + m_n\alpha_n \in R^+$. Using PBW Theorem and Proposition 4.8 we see that U_α^+ is spanned by Y_α if all $m_k \le 1$. It seems that U_α^+ is not spanned by Y_α if $m_k \ge 2$ for some $k \in [1, n]$.

5. An example, type A_n

5.1. It is easy to say a little more for type A_n. In this section we shall assume that R is of type A_n, number its simple roots as usual and fix $\alpha := \alpha_i + \alpha_{i+1} + \cdots + \alpha_j (i \le j)$. We choose all d_k to be 1. We have

(i) $h'(\alpha) = j - 1$.
(ii) $\Pi_\alpha = \{\alpha_i, \alpha_{i+1}, \dots, \alpha_j\}$.
(iii) $w_{\alpha,k} = s_js_{j-1}\cdots s_{k+1}s_is_{i+1}\cdots s_{k-1}$, $i \le k \le j$.
(iv) $W_\alpha = \langle s_i, s_{i+1}, \dots, s_j\rangle$.

(v) We have $\# Y_\alpha = \# Y'_\alpha = 2^{j-i}$. So $\# Y = 2^{n+1} - n - 2$.

(vi) Let $E = T_j^{a_j} T_{j-1}^{a_{j-1}} \cdots T_{i+1}^{a_{i+1}}(E_i)$, $(a_j, \dots, a_{i+1}) \in I_\alpha$, then we have

$$\text{(a)} \quad E = \begin{cases} v^{-1}E_iE' - E'E_i, & \text{if } a_{i+1} = 1, \\ v^{-1}E'E_i - E_iE', & \text{if } a_{i+1} = -1. \end{cases}$$

$$\text{(b)} \quad E = \begin{cases} v^{-1}E''E_j - E_jE'', & \text{if } a_j = 1, \\ v^{-1}E_jE'' - E''E_j, & \text{if } a_j = -1, \end{cases}$$

where $E' = T_j^{a_j} T_{j-1}^{a_{j-1}} \cdots T_{i+2}^{a_{i+2}}(E_{i+1})$, $E'' = T_{j-1}^{a_{j-1}} \cdots T_{i+1}^{a_{i+1}}(E_i)$. Moreover, $E_jE = v^{\pm a_j}EE_j$, $E_iE = v^{\mp a_i}EE_i$.

Proof. (i–v) is obvious by results in sections 2 and 4. Now we prove (vi).

The assertion (a) is obvious. Note that $E = T_i^{-a_i+1} T_{i+1}^{-a_{i+1}+2} \cdots T_{j-1}^{-a_j}(E_j)$, we get (b). The remain part of (vi) can be easily deduced from the definition relations of U.

Let O_{ij} be the set of monomials in $E_i, E_{i+1}, \dots, E_j$ such that in any of which E_k $(i \le k \le j)$ appears exactly once. It is obvious that $O_{ij} = \{E_jE, EE_j \mid E \in O_{i,j-1}\}$ (we define $O_{i,j-1}$ similarly), so there are at most 2^{j-i} elements in O_{ij}. But each element $E \in Y_\alpha$ is a $\mathbf{Q}(v)$-linear combination of elements in O_{ij}, thus (v) implies that O_{ij} has exactly 2^{j-i} elements which are linearly independent over $\mathbf{Q}(v)$ (one also can get this from PBW Theorem).

Using (vi) and induction on $j - i$ it is easy to see that the determinate of the transformation matrix from the set Y_α to the set O_{ij} is $\pm(v^{-2} - 1)^{(j-i)2^{j-i-1}}$.

We give some properties for (w_E, α_{k_E}), $E \in Y_\alpha$. We need the following lemma.

LEMMA 5.2. *Given $(w, \alpha_k) \in \mathscr{H}$ and let $t_q t_{q-1} \cdots t_2 t_1$ be a reduced expression of w. If*

$$t_p t_{p-1} t_{p-2} \cdots t_1(\alpha_k) < t_{p-1} t_{p-2} \cdots t_1(\alpha_k) \ge t_{p-2} \cdots t_1(\alpha_k) \ge \cdots \ge t_1(\alpha_k) \ge \alpha_k$$

for some $1 < p \le q$, then (w, α_k) is shortable.

Proof. The element (w, α_k) is obvious shortable when there exists some simple reflection s in $\mathscr{R}(w) = \{s_i \mid ws_i \le w,\ i \in [1, n]\}$ such that $s(\alpha_k) = \alpha_k$. Suppose that there exists no simple reflection s in $\mathscr{R}(w)$ such that $s(\alpha_k) = \alpha_k$, then $\#\mathscr{R}(w) = 1$ or 2. When $\mathscr{R}(w) = 1$, it is easy to see that $w = u \cdot s_k s_{k-1}$ or $w = u \cdot s_k s_{k+1}$ for some $u \in W$, so (w, α_k) is shortable. When $\#\mathscr{R}(w) = 2$, we have $\mathscr{R}(w) = \{s_{k-1}, s_{k+1}\}$, and $w = w_1 s_k \cdot s_{m_1} s_{m_1-1} \cdots s_{k+2} s_{k+1} s_{n_1} s_{n_1-1} \cdots s_{k-2} s_{k-1}$ for some $m_1 > k$, $n_1 < k$, where $w_1 s_k$ is the shortest element in the coset wW'_k, W'_k is the subgroup of W generated by those s_i such that $i \neq k$. Our assumption on $\mathcal{R}(w)$ implies that $w_1 = w_2 s_k \cdot s_{m_2} s_{m_2-1} \cdots s_{k+2} s_{k+1} s_{n_2} s_{n_2-1} \cdots s_{k-2} s_{k-1}$ or $w_2 s_k \cdot s_{m_2} s_{m_2-1} \cdots s_{k+2} s_{k+1}$ or $w_2 s_k \cdot s_{n_2} s_{n_2-1} \cdots s_{k-2} s_{k-1}$ for some $m_2 > k$, $n_2 < k$, where $w_2 s_k$ is the shortest element in the coset $w_1 W'_k$. If $m_2 \geq m_1$ or $n_2 \leq n_1$, we have $w = u \cdot s_k s_{k-1}$ or $w = u \cdot s_k s_{k+1}$ for some $u \in W$, so the assertion is true. If $m_2 < m_1$ and $n_2 > n_1$, we continue this process, finally we see that $w = u \cdot s_k s_{k-1}$ or $w = u \cdot s_k s_{k+1}$ for some $u \in W$, which is what we need.

REMARK. In general Lemma 5.2 is not true. For type D_4, let $w = s_2 s_1 s_3 s_4 s_2 s_1 s_3 s_4$, then (w, α_2) is the shortest element in $(w, \alpha_2)^{\sim}$, but $w(\alpha_2) < s_2 w(\alpha_2)$, so Lemma 5.2 is false for type D_4. Here we number the simple roots in R as usual.

PROPOSITION 5.3. *Let* $E = T_j^{a_j} T_{j-1}^{a_{j-1}} \ldots T_{i+1}^{a_{i+1}}(E_i) \in Y_\alpha$, $(a_j, a_{j-1}, \ldots, a_{i+1}) \in I_\alpha$. *Then*

(i) *We have* $w_E = s_i w_{E'}$ *if* $a_{i+1} = -1$, *and* $w_E = s_j w_{E''}$ *if* $a_j = 1$, *where*

$$E' = T_j^{a_j} T_{j-1}^{a_{j-1}} \cdots T_{i+2}^{a_{i+2}}(E_{i+1}), \qquad E'' = T_{j-1}^{a_{j-1}} \cdots T_{i+1}^{a_{i+1}}(E_i).$$

(ii) *We have* $w_E = s_k s_{k+1} \cdots s_j w_G$ *if* $a_j = a_{j-1} = \cdots = a_{k+1} = -1$, $a_k = 1$, $j > k > i$, *where* $G = T_{j-1}^{-1} \cdots T_k^{-1} T_{k-1}^{a_{k-1}} \cdots T_{i+1}^{a_{i+1}}(E_i)$.

(iii) *We have* $w_E = u_E \cdot w_{\alpha,k_E}$ *for some* $u_E \in W_{\alpha - \alpha_i - \alpha_j}$ (*if* $\alpha - \alpha_i - \alpha_j \notin R^+$ *we set* $W_{\alpha - \alpha_i - \alpha_j} = \{e\}$). *Moreover* $w_E = w_{\alpha,k_E}$ *when* $k_E = i$ *or* j.

(iv) *We have* $\#\{E \in Y_\alpha \mid k_E = k\} = C_{j-i}^{k-i}$. *Note that* C_{j-i}^{k-i} *is also the number of different reduced expressions of* w_{α,k_E}.

(v) *Set* $Y_{\alpha,k} = \{E \in Y_\alpha \mid k_E = k\}$ $(i \leq k \leq j)$, *then* $\Psi(Y_{\alpha,k}) = Y_{\alpha,j-k+i}$.

Proof. (i) Note that we also have $E = T_i^{-a_{i+1}} T_{i+1}^{-a_{i+2}} \ldots T_{j-1}^{-a_j}(E_j)$, we see that (i) was already proved in the argument of Corollary 4.6 (ii).

(ii) Let $w = s_k s_{k+1} \cdots s_j w_G$ and let $w_E = s_h s_{h+1} \cdots s_j w_1, i < h < j$. Then $T_w(E_k) = E$ for some $k \in [i, j-1]$ (in fact $k = k_G$). Since $w, w_E \in W_\alpha$, by Proposition 2.12 (i) we can find some $x \in W_\alpha$ such that $w = w_E \cdot x$. But $w(\alpha_k) = \alpha$, we necessarily have $x \in W_{\alpha - \alpha_j}$. This forces that $k = h$. We then have $T_{w_1}(E_{k_E}) = T_{w_G}(E_k)$. Therefore $w_1 = w_G$ since w_E is the shortest element in $\Theta^{-1}(E)$. The assertion (ii) is proved.

(iii) If $k_E = i$ or j, by Lemma 5.2 we see that $w_E = w_{\alpha,k_E}$. If $k_{E \neq j}$, by the proof of (ii) we see that $w_E = s_h s_{h+1} \cdots s_j w_G$, $k_E = k_G$ for some $h \in [i+1, j]$, $G \in Y_{s_j(\alpha)}$. Using induction hypothesis we know that $w_G = u_G \cdot w_{s_j(\alpha),k_G}$ for some $u_G \in W_{s_j(\alpha) - \alpha_i - \alpha_{j-1}}$. So we have $s_j u_G = u_G s_j$. Note that $s_j w_{s_j(\alpha),k_G} = w_{\alpha,k_E}$, we see (iii) is true in this case.

From the proof of (ii) it is easy to see that $k_E = k$ if and only if $\#\{m \in [i+1, j] \mid a_m = -1\} = k - i$. Thus we get (v), and (iv) follows from 5.1 (v). The proposition is proved.

5.4. We shall give a clear formula for the coproduct of a root vector. We need some preparation.

Let α be as in 5.1. For any $\beta \in \mathbf{N}R^+$, let $c(\beta)$ be the number of connected components of β. When $\beta \le \alpha$, $c(\beta)$ is just the minimal number of roots in R^+ whose sum is β.

Let $E = T_j^{a_j} T_{j-1}^{a_{j-1}} \cdots T_{i+1}^{a_{i+1}}(E_i) = T_i^{-a_{i+1}} T_{i+1}^{-a_{i+2}} \cdots T_{j-1}^{-a_j}(E_j)$ be a root vector in Y_α. Let $\beta \in \mathbf{N}R^+$ be such that $\beta \le \alpha$. If $\beta = 0$ we set $E_\beta = 1$, $K_\beta = 1$, if $\beta = \alpha_k + \alpha_{k+1} + \cdots + \alpha_l$ $(i \le k \le l \le j)$ we set $E_\beta = T_l^{a_l} T_{l-1}^{a_{l-1}} \cdots T_{k+1}^{a_{k+1}} E_k$, $K_\beta = K_l K_{l-1} \cdots K_{k+1} K_k$, if $\beta_1, \ldots, \beta_{c(\beta)}$ are connected components of β and $\beta = \beta_1 + \cdots + \beta_{c(\beta)}$, we set $E_\beta = E_{\beta_1} \ldots E_{\beta_{c(\beta)}}$, $K_\beta = K_{\beta_1} \ldots K_{\beta_{c(\beta)}}$. The elements E_β, K_β are well defined since for different connected components β_h, β_m we have $E_{\beta_h} E_{\beta_m} = E_{\beta_m} E_{\beta_h}$, $K_{\beta_h} K_{\beta_m} = K_{\beta_m} K_{\beta_h}$.

We define X_E inductively as follows: If $j - i \le 2$, we set

$$X_E = \{\gamma \in \mathbf{N}R^+ \mid \gamma \le \alpha, w_E^{-1}(\gamma) \ge 0\}.$$

Assume that $X_{E'}$ is well defined for $E' = T_j^{a_j} T_{j-1}^{a_{j-1}} \ldots T_{i+2}^{a_{i+2}}(E_{i+1}) \in Y_{\alpha'}$, $\alpha' = \alpha - \alpha_i$, when $a_{i+1} = 1$, we set

$$X_E = \{\gamma + \alpha_i, \gamma' \mid \gamma, \gamma' \in X_{E'}, \alpha' - \gamma' \ge \alpha_{i+1}\};$$

when $a_{i+1} = -1$, we set

$$X_E = \{\gamma + \alpha_i, \gamma' \mid \gamma, \gamma' \in X_{E'}, \gamma \ge \alpha_{i+1}\}.$$

Now we can state our second main result.

THEOREM 5.5.

(i) *Keep the notations in* 5.4, *then*

$$\Delta(E) = \sum_{\gamma \in X_E} (v^{-1} - v)^{c(\alpha - \gamma) + c(\gamma) - 1} K_\gamma E_{\alpha - \gamma} \otimes E_\gamma.$$

(*When* $a_j = \cdots = a_{i+1} = 1$, *this formula appears in* [*R*].)

(ii) $S(E) = (-1)^{i-j+1} v^{i-j} K_\alpha^{-1} \Psi(E)$.

Proof. When $j = i$, it follows from the definition of the coproduct. Now assume that $j > i$. Let $E' = T_j^{a_j} T_{j-1}^{a_{j-1}} \dots T_{i+2}^{a_{i+2}}(E_{i+1}) \in Y_{\alpha'}$, $\alpha' = \alpha - \alpha_i$. We use induction on $j - i$.

If $a_{i+1} = 1$, then (see 5.1 (vi)) $E = v^{-1}E_iE' - E'E_i$. By induction hypothesis we get

$$(1)\quad \Delta(E) = v^{-1}(E_i \otimes 1 + K_i \otimes E_i)\left(\sum_{\substack{\gamma' \in X_{E'} \\ \beta' = \alpha' - \gamma'}} (v^{-1} - v)^{c(\beta') + c(\gamma') - 1} K_{\gamma'} E_{\beta'} \otimes E_{\gamma'}\right)$$
$$- \left(\sum_{\substack{\gamma' \in X_{E'} \\ \beta' = \alpha' - \gamma'}} (v^{-1} - v)^{c(\beta') + c(\gamma') - 1} K_{\gamma'} E_{\beta'} \otimes E_{\gamma'}\right)(E_i \otimes 1 + K_i \otimes E_i)$$

If $\gamma' \geq \alpha_{i+1}$, then we have

$$(2)\quad E_iK_{\gamma'} = vK_{\gamma'}E_i,\ E_{\beta'}E_i = E_iE_{\beta'}.$$

$$v^{-1}E_iE_{\gamma'} - E_{\gamma'}E_i = E_{\gamma' + \alpha_i},\ E_{\beta'}K_i = K_iE_{\beta'},\ c(\gamma' + \alpha_i) = c(\gamma').$$

If $\beta' \geq \alpha_{i+1}$, then we have

$$(3)\quad v^{-1}E_iE_{\beta'} - E_{\beta'}E_i = E_{\beta' + \alpha_i},\ K_{\gamma'}E_i = E_iK_{\gamma'},\ c(\beta' + \alpha_i) = c(\beta').$$

$$E_iE_{\gamma'} = E_{\gamma'}E_i = E_{\gamma' + \alpha_i},\ E_{\beta'}K_i = vK_iE_{\beta'},\ c(\gamma' + \alpha_i) = c(\gamma') + 1.$$

If $a_{i+1} = -1$, then (see 5.1 (vi)) $E = v^{-1}E'E_i - E_iE'$. By induction hypothesis we get

$$(4)\quad \Delta(E) = v^{-1}\left(\sum_{\substack{\gamma' \in X_{E'} \\ \beta' = \alpha' - \gamma'}} (v^{-1} - v)^{c(\beta') + c(\gamma') - 1} K_{\gamma'} E_{\beta'} \otimes E_{\gamma'}\right)(E_i \otimes 1 + K_i \otimes E_i)$$
$$- (E_i \otimes 1 + K_i \otimes E_i)\left(\sum_{\substack{\gamma' \in X_{E'} \\ \beta' = \alpha' - \gamma'}} (v^{-1} - v)^{c(\beta') + c(\gamma') - 1} K_{\gamma'} E_{\beta'} \otimes E_{\gamma'}\right).$$

If $\gamma' \geq \alpha_{i+1}$, then we have

$$(5)\quad E_iK_{\gamma'} = vK_{\gamma'}E_i,\ E_{\beta'}E_i = E_iE_{\beta'} = E_{\beta' + \alpha_i},\ c(\beta' + \alpha_i) = c(\beta') + 1.$$

$$v^{-1}E_{\gamma'}E_i - E_iE_{\gamma'} = E_{\gamma' + \alpha_i},\ E_{\beta'}K_i = K_iE_{\beta'},\ c(\gamma' + \alpha_i) = c(\gamma').$$

If $\beta' \geq \alpha_{i+1}$, then we have

(6) $v^{-1}E_{\beta'}E_i - E_iE_{\beta'} = E_{\beta'+\alpha_i}$, $K_{\gamma'}E_i = E_iK_{\gamma'}$, $c(\beta'+\alpha_i) = c(\beta')$.

$E_iE_{\gamma'} = E_{\gamma'}E_i$, $E_{\beta'}K_i = vK_iE_{\beta'}$.

Combine (1–6) and the definition of X_E we see (i) is true. (ii). It follows from Theorem 4.7.
The theorem is proved.

REMARK. For other types it is not difficult to get the formula $\Delta(E)$ for $E \in Y_\alpha$ when the $j_1, j_2, \ldots, j_m, j_{m+1}$ are pairwise different (see Proposition 4.8 for notations).

5.6. We shall write E_{ij} for the root vector $T_jT_{j-1}\cdots T_{i+1}(E_i)$. In particular we have $E_{ii} = E_i$. The set $\{E_{ij} \mid 1 \leq i \leq j \leq n\}$ first appears in [J] and corresponds to the reduced expression $s_ns_{n-1}s_{n-2}s_{n-1}s_n \cdots s_1s_2 \cdots s_{n-2}s_{n-1}s_n$ of the longest element of W (see [L2]). In this subsection we list some formulas concerned with E_{ij}, $F_{ij} = \Omega(E_{ij})$, $K_{ij} = T_jT_{j-1}\cdots T_{i+1}(K_i)$, one can prove them by direct computations or see [L1, R] for some of them.

The indices i, j, k, l always indicate numbers in $[1, n]$, and M, N always indicate non-negative positives, we also assume that $i \leq j$ and $k \leq l$.

$$E_{ij}E_{kl} = \begin{cases} E_{kl}E_{ij}, & \text{if } j < k-1 \text{ or } k < i \leq j < l, \\ vE_{kl}E_{ij}, & \text{if } k < i < j = l, \\ v^{-1}E_{kl}E_{ij}, & \text{if } i = k \leq j < l \text{ or } i < k \leq j = l, \\ vE_{il} + vE_{kl}E_{ij}, & \text{if } j = k-1, \\ E_{kl}E_{ij} + (v^{-1}-v)E_{il}E_{kj}, & \text{if } i < k \leq j < l. \end{cases} \tag{d0}$$

we set $E_{ij}^{(N)} = E_{ij}^N/[N]!$, $F_{ij}^{(N)} = F_{ij}^N/[N]!$, where $[N]! = \Pi_{i=1}^N (v^i - v^{-i})/(v - v^{-1})$ if $n \geq 1$, $[0]! = 1$. Let c be an integer, we set

$$\begin{bmatrix} K_{ij}, c \\ N \end{bmatrix} = \prod_{r=1}^{N} \frac{K_{ij}v^{c-r+1} - K_{ij}^{-1}v^{-c+r-1}}{v^r - v^{-r}}.$$

$$E_{ij}^{(M)}E_{kl}^{(N)} = E_{kl}^{(N)}E_{ij}^{(M)} \quad \text{if } j < k-1 \text{ or } k < i \leq j < l. \tag{d1}$$

$$E_{ij}^{(M)}E_{kl}^{(N)} = v^{MN}E_{kl}^{(N)}E_{ij}^{(M)} \quad \text{if } k < i < j = l. \tag{d2}$$

$$E_{ij}^{(M)}E_{kl}^{(N)} = v^{-MN}E_{kl}^{(N)}E_{ij}^{(M)} \qquad \text{if } i = k \le j < l, \text{ or } i < k \le j = l. \tag{d3}$$

$$E_{ij}^{(M)}E_{kl}^{(N)} = \sum_{\substack{p>0, q\ge 0\\ p+q=N\\ q+r=M}} v^{rp+q}E_{kl}^{(p)}E_{il}^{(q)}E_{ij}^{(r)} \qquad \text{if } j = k-1. \tag{d4}$$

$$E_{ij}^{(M)}E_{kl}^{(N)} = \sum_{0\le t\le M,N} v^{-t(t-1)/2}(v^{-1}-v)^t[t]!E_{kj}^{(t)}E_{kl}^{(N-t)}E_{ij}^{(M-t)}E_{il}^{(t)} \tag{d5}$$

if $i < k \le j < l$.

$$E_{ij}F_{kl} = \begin{cases} F_{kl}E_{ij}, & \text{if } j < k \text{ or } k < i \le j < l,\\ F_{kl}E_{ij} + v^{-1}K_{k,j}^{-1}E_{i,k-1}, & \text{if } i < k \le j = l,\\ F_{kl}E_{ij} - F_{j+1,l}K_{ij}^{-1}, & \text{if } i = k \le j < l,\\ F_{kl}E_{ij} + \begin{bmatrix} K_{ij}, 0\\ 1\end{bmatrix}, & \text{if } i = k, j = l,\\ F_{kl}E_{ij} + v^{-1}(v-v^{-1})F_{j+1,l}K_{k,j}^{-1}E_{i,k-1}, & \text{if } i < k \le j < l.\end{cases} \tag{e0}$$

$$E_{ij}^{(M)}F_{kl}^{(N)} = F_{kl}^{(N)}E_{ij}^{(M)} \qquad \text{if } j < k \text{ or } k < i \le j < l. \tag{e1}$$

$$E_{ij}^{(M)}F_{kl}^{(N)} = \sum_{0\le t\le M,N} v^{t(N-t-1)}F_{kj}^{(N-t)}K_{kj}^{-t}E_{ij}^{(M-t)}E_{i,k-1}^{(t)} \qquad \text{if } i < k \le j = l. \tag{e2}$$

$$E_{ij}^{(M)}F_{kl}^{(N)} = \sum_{0\le t\le M,N} (-1)^t v^{t(M-t)}F_{j+1,l}^{(t)}F_{kl}^{(N-t)}K_{ij}^{-t}E_{ij}^{(M-t)} \qquad \text{if } i = k \le j < l. \tag{e3}$$

$$E_{ij}^{(M)}F_{ij}^{(N)} = \sum_{0\le t\le M,N} F_{ij}^{(N-t)}\begin{bmatrix} K_{ij}, 2t-M-N\\ t\end{bmatrix}E_{ij}^{(M-t)} \tag{e4}$$

$$E_{ij}^{(M)}F_{kl}^{(N)} = \sum_{0\le t\le M,N} v^{-(t(2N+t-1))/2}(v-v^{-1})^t[t]!F_{kl}^{(N-t)}F_{j+1,l}^{(t)} \cdot K_{kj}^{-t}E_{ij}^{(M-t)}E_{i,k-1}^{(t)} \qquad \text{if } i < k \le j < l. \tag{e5}$$

We have $X_{E_{ij}} = \{0, \alpha_{ii}, \alpha_{i,i+1}, \ldots, \alpha_{ij}\}$ (see 5.4 for notations), so we get

$$\Delta(E_{ij}) = E_{ij}\otimes 1 + K_{ij}\otimes E_{ij} + (v^{-1}-v)\sum_{i\le k<j} K_{ik}E_{k+1,j}\otimes E_{ik}. \tag{f0}$$

$$\Delta(E_{ij}^{(M)}) = \sum_{\substack{m_0, m_i, m_{i+1} \cdots m_j \geq 0 \\ m_0 + m_i + m_{i+1} + \cdots + m_j = M}} \xi_{\mathbf{m}} K_{\mathbf{m}} E_{\mathbf{m}} \otimes E'_{\mathbf{m}}, \tag{f1}$$

where $\mathbf{m} = (m_0, m_i, m_{i+1}, \ldots, m_j)$, $K_{\mathbf{m}} = K_{ii}^{m_i} K_{i,i+1}^{m_{i+1}} \ldots K_{ij}^{m_j}$,

$$\xi_{\mathbf{m}} = v^{-m_0(M-m_0)} \prod_{r=i}^{j-1} (v^{-1} - v)^{m_r} [m_r]! v^{m_r(m_r-1)/2},$$

$$E_{\mathbf{m}} = E_{j,j}^{(m_{j-1})} E_{j-1,j}^{(m_{j-2})} \cdots E_{i+1,j}^{(m_i)} E_{i,j}^{(m_0)}, \qquad E'_{\mathbf{m}} = E_{ii}^{(m_i)} E_{i,i+1}^{(m_{i+1})} \cdots E_{ij}^{(m_j)}.$$

$$S(E_{ij}) = (-1)^{-i-j+1} v^{i-j} K_{ij} \Psi(E_{ij}). \tag{g0}$$

$$S(E_{ij}^{(M)}) = (-1)^{M(i-j+1)} v^{M(i-j)+M(M-1)} K_{ij}^M \Psi(E_{ij}^{(M)}). \tag{g1}$$

Note that $\Psi(E_{ij}) = T_i T_{i+1} \cdots T_{j-1}(E_j)$ is also a root vector.
Apply Ω one can get more formulas.

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