On the Gauss curvature of maximal surfaces in the 3-dimensional Lorentz-Minkowski space.

Autor(en): **Estudillo, F. J.M. / Romero, Alfonso**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **69 (1994)**

PDF erstellt am: **15.08.2024**

Persistenter Link: https://doi.org/10.5169/seals-52244

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.
Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.
Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Ein Dienst der *ETH-Bibliothek*
ETH Zürich, Rämistrasse 101, 8092 Zürich, Schweiz, www.library.ethz.ch

http://www.e-periodica.ch

Comment. Math. Helvetici 69 (1994) 1–4 0010-2571/94/010001-04$1.50 + 0.20/0
© 1994 Birkhäuser Verlag, Basel

On the Gauss curvature of maximal surfaces in the 3-dimensional Lorentz–Minkowski space

FRANCISCO J. M. ESTUDILLO AND ALFONSO ROMERO*

Several authors have dealt with maximal surfaces in the Lorentz–Minkowski space $\mathbb{L}^3$, [1], [2], [6], [8], from diverse points of view. The most remarkable result on this family of surfaces can be enounced as follows, [1], [6],

(C) *Space-like planes are the only complete maximal surfaces in* $\mathbb{L}^3$.

The same conclusion is reached if the assumption "complete" is replaced by "closed", [2]. In particular, this gives an affirmative answer to the Bernstein problem for maximal surfaces of $\mathbb{L}^3$, [1]. Consequently, the global geometry of maximal surfaces was completed by these results. If we remove the regularity condition, we can then consider generalized maximal surfaces. A systematic study of their branch points, including an extension of Theorem (C) above, is given in [3]. The main purpose of this paper is to obtain the following universal inequality of the Gauss curvature at any point p, $K(p)$, of a maximal surface M with boundary in $\mathbb{L}^3$,

$$K(p) \leq \frac{4}{d(p, \partial M)^2}, \qquad \text{for any } p \in M, \tag{0.1}$$

where d is the distance on M.

Remember that $K(p) \geq 0$ and therefore (0.1) clearly implies Theorem (C). Our main idea is to use Schwarz' Lemma to control curvature, since the Gauss map of a maximal surface can be viewed as a holomorphic function with values in the unit disk. Observe also that no assumption on the normals to M in $\mathbb{L}^3$ is made in order to obtain (0.1). On the other hand, under various conditions on the Gauss map, analogous inequalities to (0.1) were obtained for minimal surfaces in the 3-dimensional euclidean space, [4], [7], [9].

---

* Partially supported by DGICYT Grant PB 90-0014-C03-02.

1. Preliminaries

We consider the Lorentz–Minkowski space $\mathbb{L}^3$ with its usual Lorentzian metric $dx_1^2 + dx_2^2 - dx_3^2$. Let M be an orientable Riemannian 2-manifold, with metric ds^2, which is isometrically immersed with zero mean curvature in $\mathbb{L}^3$. As usual, we call M a maximal surface in $\mathbb{L}^3$. At any point of M we have local isothermal coordinates (u, v), (see [5], pp. 34–35). In a natural way we then induce a conformal structure on M. If we put $\phi_k = (\partial x_k/\partial u) - i(\partial x_k/\partial v)$, $k = 1, 2, 3$, then the holomorphic functions ϕ_k, $k = 1, 2, 3$, satisfy $\phi_1^2 + \phi_2^2 - \phi_3^2 = 0$ and $|\phi_1|^2 + |\phi_2|^2 - |\phi_3|^2 > 0$ everywhere. If $\phi_2 \neq i\phi_1$ then we have a globally defined holomorphic 1-form ω on M and a meromorphic function g on M, constructed locally as $\omega = (\phi_2 - i\phi_1)\, dz$ and $g = \phi_3/(\phi_2 - i\phi_1)$. The poles of g with order m coincide with the zeroes of ω with order $2m$. It is known, [3], [6], [8], that $ds^2 = (1/4)(1 - |g|^2)^2|\omega|^2$. The Gauss map N is valued in the two-sheet unit hyperboloid (i.e. the two-sheet hyperbolic plane) in $\mathbb{L}^3$ and then we get either $|g| < 1$ or $|g| > 1$ everywhere. The map g represents, after stereographic projection, the Gauss map N. Finally we observe that if $\omega = f\, dz$ locally, the Gauss curvature K of M is locally obtained as

$$K = [4|g'|/(|f|(|g|^2 - 1)^2)]^2, \tag{1.1}$$

therefore $K \geq 0$ everywhere and K has only isolated zeroes whereas $K \neq 0$.

2. Main result and consequences

In order to get (0.1) we first give the following result, inspired in [9], Theorem 1, and [10].

THEOREM 1. *Let M be a maximal surface in the Lorentz–Minkowski space $\mathbb{L}^3$. Let p be a point of M and U be an open neighborhood of p having the property that for some positive real number β, the normal at each point of U makes a hyperbolic angle of less than β with the normal at p. Then the Gauss curvature at p, $K(p)$, satisfies*

$$K(p) \leq \left(\frac{4}{\delta^2}\right)\left(\tanh\frac{\beta}{2}\right)^2, \tag{2.1}$$

where δ is a positive real number such that the distance along M from p to the boundary of U is at least δ.

Proof. We may assume that the open neighborhood U is 1-connected, otherwise we shall change it by its universal covering. We consider the (local) Enneper–Weierstrass representation $(\tilde{f}, \tilde{g})$ on U. Assume $\tilde{g}(p) = 0$ by using perhaps a rigid motion on $\mathbb{L}^3$. Thus we have $|\tilde{g}| < 1$ everywhere on U and, in particular, $\tilde{g}$ has no pole on U. Therefore, $\tilde{f}$ has no zero on U. This provides us with the following flat Riemannian metric $ds_1^2 = (1/4)|\tilde{f}\,dz|^2$ on U. Let $D(0, r)$ be the greatest disc around the origin in the tangent plane to M at p, on which the exponential map relative to ds_1^2, $\exp_p$, can be defined as a local isometry. Consider now the Enneper–Weierstrass representation (f, g) with respect to the conformal parameter $w \in D(0, r)$. It is clear that $f(w)\,dw = \tilde{f}(z)\,dz$ and $|dw|^2 = ds_1^2$. Therefore, we get $|f(w)| = 2$ at any $w \in D(0, r)$. On the other hand, from $|\tilde{g}| < 1$ we have $|g| < 1$. Let w_0 be a point on the boundary of $D(0, r)$ such that $\exp_p w_0$ lies on the boundary U. The curve $\gamma(t) = \exp_p(tw_0)$, $t \in [0, 1)$, is divergent in U. If δ represents a positive number less or equal to the distance, with respect to ds^2, from p to the boundary of U then we get

$$\delta \leq \int_\gamma ds = \int_\gamma (1 - |g(w)|^2)|dw| \leq \int_\gamma |dw| = r. \tag{2.2}$$

Now note that $N_p = (0, 0, -1)$ from our assumption $\tilde{g}(p) = 0$ above. It is easy to see that the radius R of the image by g of $D(0, r)$ is given by $\tanh(\beta/2)$. Schwarz' Lemma for the holomorphic function $G : D(0, 1) \to D(0, 1)$, $G(\eta) = (1/R) \cdot g(r\eta)$, $\eta \in D(0, 1)$, and (1.1), provide us with

$$K(p) \leq 4(R/r)^2. \tag{2.3}$$

Finally, from (2.3), using (2.2) and taking into account the value for R obtained above, we complete the proof of Theorem 1.

Clearly the inequality (0.1) follows from (2.1) above.

Remark. (1) In the proof of Theorem 1 we have found the following slightly stronger inequality $K(p) \leq (4/r(p)^2)$, where $r(p)$ is the infimum of the lengths of divergent curves starting from the point p, with respect to the flat metric ds_1^2. It is straightforward to show that a metric homothetical to ds_1^2 bounds from above to the induced metric on U by the usual Euclidean one of $\mathbb{R}^3$. Thus, we can modify last inequality to reprove that a closed maximal surface in $\mathbb{L}^3$ must be totally geodesic.

(2) A similar argument as in Theorem 1 permits us to state that if p is a point of a maximal surface M and V is an open neighborhood of p having the property that the normal at any point of V makes a hyperbolic angle of at least $\beta \geq 0$ with

some fixed timelike vector, then $K(p) \leq (4/\delta^2)((1+\cosh\alpha)^2/(1+\cosh\beta)^2)$ where $\alpha \geq \beta$ is the hyperbolic angle of the normal at p with the fixed timelike vector, and $\delta > 0$ is less than or equal to the distance from p to the boundary of V, (compare with [9], Theorem 2).

Acknowledgements

We would like to express our thanks to A. Ros and J. Pérez Muñoz for useful discussions related with this paper. We also thank the referee for his valuable suggestions.

REFERENCES

[1] E. Calabi, *Examples of Bernstein problems for some non-linear equations*, Proc. Symp. Pure Math. *15* (1970), 223–230.
[2] S. Y. Cheng, S. T. Yau, *Maximal space-like hypersurfaces in the Lorentz–Minkowski space*, Ann. of Math. *104* (1976), 407–419.
[3] F. J. M. Estudillo, A. Romero, *Generalized maximal surfaces in the Lorentz–Minkowski space in* $\mathbb{L}^3$, Math. Proc. Camb. Phil. Soc. (1992), 111–515.
[4] H. Fujimoto, *On the number of exceptional values of the Gauss map of minimal surfaces*, J. Math. Soc. Japan *40* (1988), 237–249.
[5] J. L. Kazdan, *Some applications of partial differential equations to problems in geometry*, Surveys in Geom. Ser. Tokyo Univ. 1983.
[6] O. Kobayashi, *Maximal surfaces in the 3-dimensional Minkowski space* $\mathbb{L}^3$, Tokyo J. Math. *6* (1983), 297–309.
[7] H. B. Lawson, Jr., *Lectures on Minimal Surfaces*, Publish or Perish, 1980.
[8] L. V. McNertney, *One-parameter families of surfaces with constant mean curvature in Lorentz 3-space*, Doctoral thesis, Brown Univ., 1980.
[9] R. Osserman, *On the Gauss curvature of minimal surfaces*, Trans. Amer. Math. Soc. *12* (1960), 115–128.
[10] R. Osserman, *Minimal surfaces in the large*, Comm. Math. Helvetici *35* (1960), 65–76.

Dep. de Matemáticas
F. de Ciencias Económicas
y Empresariales
Universidad de Córdoba
14004-Córdoba, Spain

and

Dept. Geometría Topología
F. Ciencias
Universidad de Granada
18071-Granada, Spain

Received November 26, 1991