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On the Gauss curvature of maximal surfaces in the 3-dimensional Lorentz–Minkowski space

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Several authors have dealt with maximal surfaces in the Lorentz–Minkowski space $\mathbb{L}^3$, [1], [2], [6], [8], from diverse points of view. The most remarkable result on this family of surfaces can be enounced as follows, [1], [6],

(C) *Space-like planes are the only complete maximal surfaces in* $\mathbb{L}^3$.

The same conclusion is reached if the assumption "complete" is replaced by "closed", [2]. In particular, this gives an affirmative answer to the Bernstein problem for maximal surfaces of $\mathbb{L}^3$, [1]. Consequently, the global geometry of maximal surfaces was completed by these results. If we remove the regularity condition, we can then consider generalized maximal surfaces. A systematic study of their branch points, including an extension of Theorem (C) above, is given in [3]. The main purpose of this paper is to obtain the following universal inequality of the Gauss curvature at any point p, $K(p)$, of a maximal surface M with boundary in $\mathbb{L}^3$,

$$K(p) \le \frac{4}{d(p, \partial M)^2}, \qquad \text{for any } p \in M, \tag{0.1}$$

where d is the distance on M.

Remember that $K(p) \ge 0$ and therefore (0.1) clearly implies Theorem (C). Our main idea is to use Schwarz' Lemma to control curvature, since the Gauss map of a maximal surface can be viewed as a holomorphic function with values in the unit disk. Observe also that no assumption on the normals to M in $\mathbb{L}^3$ is made in order to obtain (0.1). On the other hand, under various conditions on the Gauss map, analogous inequalities to (0.1) were obtained for minimal surfaces in the 3-dimensional euclidean space, [4], [7], [9].

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1. Preliminaries

We consider the Lorentz–Minkowski space $\mathbb{L}^3$ with its usual Lorentzian metric $dx_1^2 + dx_2^2 - dx_3^2$. Let M be an orientable Riemannian 2-manifold, with metric ds^2, which is isometrically immersed with zero mean curvature in $\mathbb{L}^3$. As usual, we call M a maximal surface in $\mathbb{L}^3$. At any point of M we have local isothermal coordinates (u, v), (see [5], pp. 34–35). In a natural way we then induce a conformal structure on M. If we put $\phi_k = (\partial x_k/\partial u) - i(\partial x_k/\partial v)$, $k = 1, 2, 3$, then the holomorphic functions ϕ_k, $k = 1, 2, 3$, satisfy $\phi_1^2 + \phi_2^2 - \phi_3^2 = 0$ and $|\phi_1|^2 + |\phi_2|^2 - |\phi_3|^2 > 0$ everywhere. If $\phi_2 \neq i\phi_1$ then we have a globally defined holomorphic 1-form ω on M and a meromorphic function g on M, constructed locally as $\omega = (\phi_2 - i\phi_1)\,dz$ and $g = \phi_3/(\phi_2 - i\phi_1)$. The poles of g with order m coincide with the zeroes of ω with order $2m$. It is known, [3], [6], [8], that $ds^2 = (1/4)(1 - |g|^2)^2|\omega|^2$. The Gauss map N is valued in the two-sheet unit hyperboloid (i.e. the two-sheet hyperbolic plane) in $\mathbb{L}^3$ and then we get either $|g| < 1$ or $|g| > 1$ everywhere. The map g represents, after stereographic projection, the Gauss map N. Finally we observe that if $\omega = f\,dz$ locally, the Gauss curvature K of M is locally obtained as

$$K = [4|g'|/(|f|(|g|^2 - 1)^2)]^2, \tag{1.1}$$

therefore $K \geq 0$ everywhere and K has only isolated zeroes whereas $K \neq 0$.

2. Main result and consequences

In order to get (0.1) we first give the following result, inspired in [9], Theorem 1, and [10].

THEOREM 1. *Let M be a maximal surface in the Lorentz–Minkowski space $\mathbb{L}^3$. Let p be a point of M and U be an open neighborhood of p having the property that for some positive real number β, the normal at each point of U makes a hyperbolic angle of less than β with the normal at p. Then the Gauss curvature at p, $K(p)$, satisfies*

$$K(p) \leq \left(\frac{4}{\delta^2}\right)\left(\tanh\frac{\beta}{2}\right)^2, \tag{2.1}$$

where δ is a positive real number such that the distance along M from p to the boundary of U is at least δ.

Proof. We may assume that the open neighborhood U is 1-connected, otherwise we shall change it by its universal covering. We consider the (local) Enneper–Weierstrass representation $(\tilde{f}, \tilde{g})$ on U. Assume $\tilde{g}(p) = 0$ by using perhaps a rigid motion on $\mathbb{L}^3$. Thus we have $|\tilde{g}| < 1$ everywhere on U and, in particular, $\tilde{g}$ has no pole on U. Therefore, $\tilde{f}$ has no zero on U. This provides us with the following flat Riemannian metric $ds_1^2 = (1/4)|\tilde{f}\,dz|^2$ on U. Let $D(0, r)$ be the greatest disc around the origin in the tangent plane to M at p, on which the exponential map relative to ds_1^2, $\exp_p$, can be defined as a local isometry. Consider now the Enneper–Weierstrass representation (f, g) with respect to the conformal parameter $w \in D(0, r)$. It is clear that $f(w)\,dw = \tilde{f}(z)\,dz$ and $|dw|^2 = ds_1^2$. Therefore, we get $|f(w)| = 2$ at any $w \in D(0, r)$. On the other hand, from $|\tilde{g}| < 1$ we have $|g| < 1$. Let w_0 be a point on the boundary of $D(0, r)$ such that $\exp_p w_0$ lies on the boundary U. The curve $\gamma(t) = \exp_p(tw_0)$, $t \in [0, 1)$, is divergent in U. If δ represents a positive number less or equal to the distance, with respect to ds^2, from p to the boundary of U then we get

$$\delta \leq \int_\gamma ds = \int_\gamma (1 - |g(w)|^2)|dw| \leq \int_\gamma |dw| = r. \tag{2.2}$$

Now note that $N_p = (0, 0, -1)$ from our assumption $\tilde{g}(p) = 0$ above. It is easy to see that the radius R of the image by g of $D(0, r)$ is given by $\tanh(\beta/2)$. Schwarz' Lemma for the holomorphic function $G : D(0, 1) \to D(0, 1)$, $G(\eta) = (1/R) \cdot g(r\eta)$, $\eta \in D(0, 1)$, and (1.1), provide us with

$$K(p) \leq 4(R/r)^2. \tag{2.3}$$

Finally, from (2.3), using (2.2) and taking into account the value for R obtained above, we complete the proof of Theorem 1.

Clearly the inequality (0.1) follows from (2.1) above.

Remark. (1) In the proof of Theorem 1 we have found the following slightly stronger inequality $K(p) \leq (4/r(p)^2)$, where $r(p)$ is the infimum of the lengths of divergent curves starting from the point p, with respect to the flat metric ds_1^2. It is straightforward to show that a metric homothetical to ds_1^2 bounds from above to the induced metric on U by the usual Euclidean one of $\mathbb{R}^3$. Thus, we can modify last inequality to reprove that a closed maximal surface in $\mathbb{L}^3$ must be totally geodesic.

(2) A similar argument as in Theorem 1 permits us to state that if p is a point of a maximal surface M and V is an open neighborhood of p having the property that the normal at any point of V makes a hyperbolic angle of at least $\beta \geq 0$ with

some fixed timelike vector, then $K(p) \leq (4/\delta^2)((1+\cosh\alpha)^2/(1+\cosh\beta)^2)$ where $\alpha \geq \beta$ is the hyperbolic angle of the normal at p with the fixed timelike vector, and $\delta > 0$ is less than or equal to the distance from p to the boundary of V, (compare with [9], Theorem 2).

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