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Graphs of actions on R-trees

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Abstract. Let G be a finitely generated group acting on an $\mathbf{R}$-tree T. First assume that the action is free, and minimal (there is no proper invariant subtree), or more generally that it satisfies a certain finiteness condition. Then it may be described as a *graph of transitive actions*: the action may be recovered from a finite graph, together with additional data; in particular, every vertex v carries an action (G_v, T_v) whose orbits are dense. For the action (G, T), it follows for instance that the closure of any orbit is a discrete union of closed subtrees: it cannot meet a segment in a Cantor set.

Now let ℓ be the length function for an arbitrary action of G. For $\varepsilon > 0$ small enough, the subgroup $G(\varepsilon) \subset G$ generated by elements g with $\ell(g) \leq \varepsilon$ is independent of ε, and $G/G(\varepsilon)$ is free. Several interpretations are given for the rank of $G/G(\varepsilon)$.

Introduction and statement of results

Let G be a finitely generated group acting on an $\mathbf{R}$-tree T. Assume that the action is minimal (there is no proper invariant subtree), and simplicial. Then the quotient space T/G is a finite graph. By Bass–Serre theory [Se], one may recover the action (G, T) from this graph, together with some additional data: groups attached to vertices and edges of T/G, and monomorphisms between them.

Now consider an arbitrary minimal action of G. The first problem one encounters when trying to generalize Bass–Serre theory is that in general T/G is not a graph, indeed it may even fail to be Hausdorff. To get a Hausdorff space, we note that T/G carries a natural pseudometric, and we define $\widehat{T/G}$ as the associated metric space.

Let $\pi : T \to \widehat{T/G}$ be the projection. Let $B \subset T$ be the set of branch points (recall that x is a *branch point* if $T \setminus \{x\}$ has more than two components). If G acts freely, work by Jiang [Ji] and Rips (see [GLP]) implies that the action of G on B has only finitely many orbits. In particular, $\pi(B)$ is a finite set (if B is dense in T, it follows that every orbit of (G, T) is dense).

In general, we say that an action is a *J-action* if $\pi(B)$ is finite. For instance, very small or geometric actions of free groups are J-actions [GL].

THEOREM 1. *Let G be a finitely generated group acting minimally on an $\mathbf{R}$-tree T. If the action is free, or more generally if it is a J-action, then $\widehat{T/G}$ is a finite graph.*

Edges of $\widehat{T/G}$ correspond to orbits of the action of G on $\pi_0(T\backslash\bar{B})$. The key fact is that this action has finitely many orbits (geometrically, this means that $\widehat{T/G}$ contains no "Hawaiian earring"). For an arbitrary action, we shall see that $\widehat{T/G}$ has the homotopy type of a finite graph.

In order to recover a J-action (G, T) from $\widehat{T/G}$, we need additional information. Given $v \in \pi(B)$ (a vertex of the graph), the preimage $\pi^{-1}(v)$ is a disjoint union of closed subtrees which are all congruent under the action of G. Let T_v be one of them, and $G_v \subset G$ its stabilizer. The action of G_v on T_v has dense orbits, we say that it is *transitive*. [In dynamical systems, an action is called transitive if some orbit is dense, minimal if every orbit is dense. For an action by isometries, the two are equivalent. We use the word transitive, since minimal has a different meaning in the theory.]

The knowledge of these transitive actions (G_v, T_v) is the main piece of information that is needed to reconstruct the original action. Details will be given in section I.3. As a corollary, we show that, for a J-action, the closure of any orbit Gx is a discrete union of closed subtrees. More precisely:

THEOREM 2. *Let G be a finitely generated group acting on an* **R**-*tree. Assume the action is free (or is a J-action). Given* $x \in T$, *there exists* $\delta > 0$ *such that* $\overline{Gx} \cap J$ *is connected for every segment J of length less than* δ.

Now we apply these results to an arbitrary action of G, using the following fact [Le, Theorem 2]: there is a (canonical) normal subgroup $H_0 \subset G$ such that $\widehat{T/H_0}$ is an **R**-tree and the action of G/H_0 on $\widehat{T/H_0}$ is free. Note however that this action need not be minimal, even if G acts minimally.

Given a length function ℓ on G, let $m(\ell)$ be the rank of the free abelian group $M(\ell) = \{\tau \in \mathrm{Hom}(G, \mathbf{Z}) \mid \exists a > 0 \text{ such that } |\tau| \le a\ell\}$. For $\varepsilon > 0$, let $G(\varepsilon)$ be the subgroup of G generated by $\ell^{-1}([0, \varepsilon])$.

THEOREM 3. *Let G be a finitely generated group acting on an* **R**-*tree T, with length function* ℓ.

(1) *There exists* $\varepsilon_0 > 0$ *such that* $G(\varepsilon)$ *is independent of* ε *for* $\varepsilon < \varepsilon_0$. *Denote it by* G_0.
(2) *The quotient* G/G_0 *is free of rank* $m(\ell)$.
(3) *The space* $\widehat{T/G}$ *has the homotopy type of a wedge of* $m(\ell)$ *circles. It has a universal covering: the* **R**-*tree* $\widehat{T/G_0}$.

COROLLARY. *If the action is free, there is a decomposition* $G \simeq G/G_0 * H_1 * \cdots * H_p$, *where each* $H_i \subset G$ *acts transitively on its minimal invariant subtree (compare [MS, Corollary 3.5]).*

COROLLARY. *The following conditions are equivalent*:

(1) $\widehat{T/G}$ *is an* **R**-*tree*.

(2) $\widehat{T/G}$ *is simply connected*.

(3) *If* $\ell \geq a|\tau|$, *with* $a > 0$ *and* $\tau : G \to \mathbf{Z}$ *a homomorphism, then* $\tau = 0$.

(4) *G is generated by* $\ell^{-1}([0, \varepsilon])$ *for every* $\varepsilon > 0$.

(5) *Given* $g \in G$ *and* $\varepsilon > 0$, *there exist* g_1 *and* g_2 *such that* $g = g_1 g_2$ *and* $\max(\ell(g_1), \ell(g_2)) < \frac{2}{3}\ell(g) + \varepsilon$.

The implication $1 \Rightarrow 5$ follows from an argument of [Le]. For free actions, we have:

THEOREM 4. *Let G be a finitely generated group acting freely and minimally on an* **R**-*tree T. The graph* $\widehat{T/G}$ *is homeomorphic to a segment* (*or a point*) *if and only if, given* $g \in G$ *and* $\varepsilon > 0$, *there exist* g_1 *and* g_2 *such that* $g = g_1 g_2$ *and* $\max(\ell(g_1), \ell(g_2)) < \frac{1}{2}\ell(g) + \varepsilon$.

This theorem cannot be generalized to non-free actions (see III.2).

I. Minimal J-actions

In this section we study minimal actions with $\pi(B)$ finite. As mentioned above, this includes minimal free actions: if G acts freely, it is a free product of free abelian groups and surface groups by Rips' theorem (see [GLP]); for free minimal actions of such groups, Jiang has proved that B/G is finite [Ji].

We rule out the trivial case $T = \mathbf{R}$.

I.1. The space of orbits carries a pseudodistance: denoting by d the distance on T, set $d(Gx, Gy) = \inf_{g,g' \in G} d(gx, g'y)$. Let $\widehat{T/G}$ be the associated metric space, obtained by identifying points at distance 0 from each other. Note that, for any $u \in \widehat{T/G}$, the set $F_u = \pi^{-1}(u)$ is closed, G-invariant, and the action of G on F_u is transitive (recall that $\pi : T \to \widehat{T/G}$ is the natural projection).

Minimality of the action implies that $\widehat{T/G}$ is compact. In fact, it has finite length in the following sense. Fix x in T, and choose a finite generating system $\{g_i\}$ for G. Then the images of the segments $[x, g_i x]$ cover $\widehat{T/G}$.

Note that $\pi(\bar{B}) = \pi(B)$ is finite. Define *flat pieces* of T as components of $T \setminus \bar{B}$. Using minimality, we see that flat pieces are isometric to open segments (a, b), with a, b in $\bar{B}$ (but not necessarily in B).

If no element of G interchanges a and b, the restriction of π to (a, b) is injective. The image of $[a, b]$ in $\widehat{T/G}$ is then a segment between $\pi(a)$ and $\pi(b)$ if $\pi(a) \neq \pi(b)$, or a circle containing $\pi(a)$ if $\pi(a)=\pi(b)$. If some element of G interchanges a and b, let m be the midpoint of $[a, b]$. Then π sends $[a, m]$ and $[m, b]$ injectively onto a segment joining $\pi(a)=\pi(b)$ and $\pi(m)$.

Theorem 1 now follows from the following fact:

LEMMA. *The action of G on $\pi_0(T\backslash\bar{B})$ has finitely many orbits.*

Proof. First consider flat pieces (a, b) such that $\pi(a)\neq\pi(b)$. Since $\pi(\bar{B})$ is finite, their length is bounded away from 0. They fall into finitely many orbits because $\widehat{T/G}$ has finite length (see above).

It is thus enough to fix $v\in\pi(\bar{B})$, and to show that flat pieces with $\pi(a)=\pi(b)=v$ form finitely many orbits. Let (a_i, b_i) be a finite family of such pieces, with midpoints m_i. Since the orbit of m_i is discrete, it defines a homomorphism c_i from G to $\mathbf{Z}/2\mathbf{Z}$: choose any $x\notin Gm_i$ and define $c_i(g)$ as the number of times the segment $[x, gx]$ meets Gm_i (counted mod 2).

Let $\delta>0$ be smaller than the length of (a_i, b_i) for all i. Since G acts transitively on $F_v=p^{-1}(v)$, there exists $g_i\in G$ such that $d(g_i a_i, b_i)<\delta$. If the pieces (a_i, b_i) belong to distinct orbits, we have $c_i(g_j)=\delta_{ij}$ and the c_i's are linearly independent in $\operatorname{Hom}(G, \mathbf{Z}/2\mathbf{Z})$. The required finiteness follows. □

In the case of a free action, the set of vertices of the graph $\Gamma=\widehat{T/G}$ is $\pi(B)$, while edges are orbits of the action of G on $\pi_0(T\backslash\bar{B})$. If n is the minimal number of generators of G, the number of vertices is lesser than or equal to $2n-2$ by [Ji]. Assertion 3 of Theorem 3 will lead to the upper bound $3n-3$ for the number of edges.

I.2. Using the lemma, we choose $\varepsilon_0>0$ such that every flat piece has length $>\varepsilon_0$. Given $v\in\pi(B)$, the set F_v is a union of closed subtrees which are at least ε_0 apart from each other. Since G acts transitively on F_v, all components of F_v are congruent and the stabilizer G_v of a component $T_v\subset F_v$ acts transitively on T_v.

We can now prove Theorem 2 for a minimal J-action. If $x\notin\bar{B}$, its orbit is discrete and the result is clear. If $x\in\bar{B}$, let $v=\pi(x)$. The closure of Gx is F_v, and we can take $\delta=\varepsilon_0$. Theorem 2 for non-minimal actions will be proved in II.2.

I.3. Our next goal is to define *graphs of actions* (compare [CL], [Sk2]). Let T' be the **R**-tree one gets from T by collapsing every component of $\bar{B}$ to a point. It may be given a simplicial structure, for which G acts simplicially and without inversions. There are two types of vertices: components of $\bar{B}$, and midpoints of flat pieces (a, b) such that some $g\in G$ interchanges a and b. Edges are flat pieces, or halves of flat pieces. Obviously T'/G is isometric to $\widehat{T/G}$.

By Bass–Serre theory [Se], the action of G on T' corresponds to a graph of groups: groups G_v (resp. G_e) are attached to vertices (resp. edges) of $\Gamma = T'/G$, and every oriented edge $\bar{e}$ gives a monomorphism $i_{\bar{e}} : G_e \to G_v$. If the action of G on T is free, then the set of vertices of T'/G is $\pi(\bar{B})$, and all edge groups are trivial.

To recover T from T', we need to know the **R**-trees T_v, and the way flat pieces are attached to them. This leads to the following definition.

DEFINITION. A *graph of actions* $\mathscr{G}$ consists of:

(1) a metric graph Γ with vertex groups G_v, edge groups G_e, and monomorphisms $i_{\bar{e}}$.
(2) for every vertex v, an action of G_v on an **R**-tree T_v.
(3) for every oriented edge $\bar{e}$, a point of T_v fixed under the action of $i_{\bar{e}}(G_e)$.

We define $\pi_1(\mathscr{G})$ as the fundamental group of the underlying graph of groups. We say that $\mathscr{G}$ is *finite* if Γ is finite, that $\mathscr{G}$ is a graph of *transitive* actions (resp. of *free* actions) if every action (G_v, T_v) is transitive (resp. free). Note that in a graph of free actions every edge group is trivial.

A graph of actions $\mathscr{G}$ leads to an action of $\pi_1(\mathscr{G})$ on an **R**-tree $T(\mathscr{G})$. We have proved:

THEOREM 5. *Every minimal J-action* (*resp. every minimal free action*) *may be represented as a finite graph of transitive actions* (*resp. of transitive free actions*). □

I.4. We now prove Theorem 3 for minimal J-actions. Consider the action of G on the simplicial tree T'. Say that $g \in G$ is *elliptic* if it acts with a fixed point.

LEMMA. *For $\varepsilon < \varepsilon_0$, the group $G(\varepsilon) \subset G$ is the subgroup generated by the elliptic elements of the action (G, T').*

Proof. Assume g is elliptic. Recall that vertices of T' are components of $\bar{B}$ or midpoints of flat pieces. If g fixes a component T_v of $\bar{B}$, it is a product of elements g_i with $\ell(g_i) < \varepsilon$ since the stabilizer G_v acts transitively on T_v. If g fixes a midpoint, then $\ell(g) = 0$.

Conversely, assume $\ell(g) < \varepsilon_0$. If g acts with a fixed point on T, it also has a fixed point on T'. If g has no fixed point, its translation axis cannot meet any flat piece. It is contained in some component of $\bar{B}$, so that g acts with a fixed point on T'. □

Let G_0 be $G(\varepsilon)$ for $\varepsilon < \varepsilon_0$. Since the action of G on T' is simplicial and G_0 is the subgroup generated by the elliptic elements, the quotient G/G_0 is isomorphic to

$\pi_1(T'/G)$ and T'/G_0 is the universal covering of T'/G. We have already mentioned that T'/G is isometric to $\widehat{T/G}$. Also note that T'/G_0 is isometric to $\widehat{T/G_0}$.

To complete the proof of Theorem 3, it remains to check that $G/G_0 \simeq \pi_1(\widehat{T/G})$ has rank $m(\ell)$. Any homomorphism $\tau : G \to \mathbf{Z}$ belonging to $M(\ell)$ vanishes on G_0, so that the rank of G/G_0 is greater than or equal to $m(\ell)$.

To prove the opposite inequality, consider the length functions $\ell' : G \to \mathbf{R}$ and $\ell_0 : G/G_0 \to \mathbf{R}$ associated to the actions (G, T') and $(G/G_0, T'/G_0)$ respectively. We note that $\ell' \leq \ell$ and $\ell_0(gG_0) = \inf_{g_0 \in G_0} \ell'(gg_0)$, so that $m(\ell) \geq m(\ell') \geq m(\ell_0)$. Since the action of G/G_0 on T'/G_0 is free and simplicial, the rank of G/G_0 equals $m(\ell_0)$, and the result follows.

Note the following consequence: *if* (G, T) *is a minimal J-action with* $m(\ell) > 0$, *then branch points are not dense in* T.

I.5. Finally, we prove the first corollary of Theorem 3. We may assume that the action is minimal. We know from I.3 that G is the fundamental group of a graph of groups T'/G whose edge groups are trivial, so that G is a free product whose factors are $\pi_1(T'/G) \simeq G/G_0$ and the vertex groups. A vertex group G_v acts transitively on T_v, hence also on its minimal invariant subtree.

II. Arbitrary actions

In this section we consider an arbitrary action of a finitely generated group G on an **R**-tree T.

II.1. Let $T_m \subset T$ be the set of points fixed by every $g \in G$ if there are any, the minimal G-invariant subtree otherwise. Since $\widehat{T_m/G}$ is compact (see I.1), we have $\widehat{\overline{T_m}/G} = \widehat{T_m/G}$. Given $x \in T$, there is a unique $p(x) \in \overline{T_m}$ such that $d(x, p(x)) = d(x, \overline{T_m})$.

For $t \in [0, 1]$, define $p(x, t)$ by following the segment $[x, p(x)]$ with constant speed. We get a G-equivariant strong deformation retraction from T to $\overline{T_m}$, so that $\widehat{T/G}$ has the same homotopy type as $\widehat{T_m/G}$.

If the action is free, we know by I.4 that $\widehat{T_m/G_0}$ is a (simplicial) tree. It follows that $\widehat{T/G_0}$ is also an **R**-tree (using for instance Theorem 1 of [Le]). Since the projection $\widehat{T_m/G_0} \to \widehat{T_m/G}$ is a covering map, the projection $\widehat{T/G_0} \to \widehat{T/G}$ is one also. We have thus proved Theorem 3 for non-minimal free actions.

II.2. Now we prove Theorem 2. By I.2, it holds for the action of G on $\overline{T_m}$, so that δ exists if $x \in \overline{T_m}$. Now consider $x \notin \overline{T_m}$. If the action is free, the orbit of x is discrete since $d(x, gx) > 2d(x, \overline{T_m})$ for $g \neq \mathrm{id}$. If the action is a J-action, the segment $[x, p(x)]$ contains only finitely many branch points (since it projects injectively into $\widehat{T/G}$), and Gx is discrete in this case also.

II.3. To deal with completely arbitrary actions, we need the following two facts from [Le]:

THEOREM 6. *Let G be a countable group acting on an* **R***-tree with length function ℓ.*

(1) *There exists a normal subgroup $H_0 \subset G$ such that $\widehat{T/H_0}$ is an* **R***-tree and the natural action of G/H_0 on $\widehat{T/H_0}$ is free.*

(2) *Suppose $\widehat{T/G}$ is an* **R***-tree. Given $\varepsilon > 0$ and $g \in G$, there exist g_1, g_2 such that $g = g_1 g_2$ and* $\max(\ell(g_1), \ell(g_2)) < \frac{2}{3}\ell(h) + \varepsilon$. *In particular, G is generated by $\ell^{-1}([0, \varepsilon])$ for every $\varepsilon > 0$.*

We write $\bar{G} = G/H_0$ and $\bar{T} = \widehat{T/H_0}$. Let $\rho : G \to \bar{G}$ be the quotient map. The length function $\bar{\ell}$ of the action $(\bar{G}, \bar{T})$ is given by $\bar{\ell}(gH_0) = \inf_{h \in H_0} \bar{\ell}(gh)$. By I.4, there exists $\varepsilon_0 > 0$ such that the subgroup of $\bar{G}$ generated by $\bar{\ell}^{-1}([0, \varepsilon])$ is a fixed group $\bar{G}_0$ for $\varepsilon \in (0, \varepsilon_0)$. Let $G_0 = \rho^{-1}(\bar{G}_0)$.

LEMMA. *In the above situation*:

(1) $m(\bar{\ell}) = m(\ell)$.

(2) $G(\varepsilon) = G_0$ *for* $0 < \varepsilon < \varepsilon_0$.

Proof. Since H_0 is generated by its elements of length $<\varepsilon$ for every $\varepsilon > 0$, any homomorphism $\tau : G \to \mathbf{Z}$ in $M(\ell)$ vanishes on H_0 and thus factors through a homomorphism $\bar{\tau} : \bar{G} \to \mathbf{Z}$ belonging to $M(\bar{\ell})$. This defines an isomorphism between $M(\ell)$ and $M(\bar{\ell})$.

Clearly $\rho(g) \in \bar{G}_0$ if $\ell(g) < \varepsilon_0$, so that $G(\varepsilon) \subset G_0$. Conversely, assume $\rho(g) \in \bar{G}_0$. Write $\rho(g) = \bar{g}_1 \cdots \bar{g}_p$ with $\bar{\ell}(\bar{g}_i) < \varepsilon$, and choose $g_i \in \rho^{-1}(\bar{g}_i)$ with $\ell(g_i) < \varepsilon$. Then $g = hg_1 \cdots g_p$ with $h \in H_0$. Since H_0 is contained in $G(\varepsilon)$ by assertion 2 of Theorem 6, we get $g \in G(\varepsilon)$. □

We have proved assertion 1 of Theorem 3. Assertion 2 is clear since $G/G_0 \simeq \bar{G}/\bar{G}_0$ is free of rank $m(\ell) = m(\bar{\ell})$. To prove assertion 3, note that $\widehat{T/G}$ is equal to $\widehat{\bar{T}/\bar{G}}$. By II.1, it has the homotopy type of a wedge of $m(\ell)$ circles. Its universal covering is the **R**-tree $\widehat{\bar{T}/\bar{G}_0}$ which is equal to $\widehat{T/G_0}$. The second corollary of Theorem 3 is proved by using assertion 2 of Theorem 6.

Note that G_0 *is the largest subgroup $H \subset G$ such that $\widehat{T/H}$ is an* **R***-tree.*

III. Proof of Theorem 4

III.1. Let G be a countable group acting on an **R**-tree T with length function ℓ. Let $c \in [0, 1]$. We say that *ℓ satisfies condition* $(*_c)$ if, given $g \in G$ and $\varepsilon > 0$, there exist $g_1, g_2 \in G$ such that $g = g_1 g_2$ and:

$$\begin{cases} \ell(g_1) + \ell(g_2) < \ell(g) + \varepsilon \\ \max(\ell(g_1), \ell(g_2)) < (1-c)\ell(g) + \varepsilon \end{cases}$$

The following facts are proved in [Le] (Theorem 1 and Remark III.1):

THEOREM 7

(1) *If ℓ satisfies $(*_c)$ for some $c > 0$, then $\widehat{T/G}$ is an* **R**-*tree.*

(2) *If $\widehat{T/G}$ is an* **R**-*tree and $c \in [0, 1/3]$, then ℓ satisfies $(*_c)$.*

(3) *If $\widehat{T/G}$ is homeomorphic to a subinterval of* **R** *and $c \in [0, 1/2]$, then ℓ satisfies $(*_c)$.*

We now prove the following result, which strengthens Theorem 4:

THEOREM 8. *Consider a minimal free action of a finitely generated group G. If ℓ satisfies $(*_c)$ for some $c > 1/3$, then $\widehat{T/G}$ is homeomorphic to a (possibly degenerate) segment (so that ℓ satisfies $(*_c)$ if $c \in [0, 1/2]$).*

III.2. Before proving this theorem, we consider the following example. Let Γ be the simplicial tree having 3 vertices a_1, a_2, a_3 of valence 1, and 1 vertex v of valence 3. Make it into a graph of groups by setting $G_{a_i} = \mathbf{Z}/2\mathbf{Z}$ (edge groups and G_v are trivial), and consider the associated action of $G = \mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}$.

Vary the metric on Γ by choosing the 3 numbers $d_i = d(v, a_i)$ independently. This gives topologically conjugate actions of G whose length functions have the following form: if w is a cyclically reduced word in the generators g_1, g_2, g_3, then $\ell(w) = 2\sum_{i=1}^{3} m_i d_i$, where m_i is the number of occurrences of g_i in w (with an exception: $\ell(w) = 0$ if $\sum m_i \le 1$).

If $d_1 = d_2 = d_3$, taking $g = g_1 g_2 g_3$ shows that ℓ satisfies $(*_c)$ if and only if $c \le 1/3$. If $d_3 = 2d_1 = 2d_2$, one can show that ℓ satisfies $(*_c)$ for $c \le 1/2$. This means that Theorem 4 does not extend to non-free actions.

III.3. To prove Theorem 8, assume that ℓ satisfies $(*_c)$ with $c > 1/3$. By I.1 and assertion 1 of Theorem 7, we know that $\widehat{T/G}$ is a finite tree whose set of vertices is

$\pi(\bar{B})$. By I.2, the preimage $\pi^{-1}(v)$ of a vertex v consists of congruent subtrees, and the stabilizer G_v of such a subtree T_v acts transitively on T_v.

Say that a vertex v of $\widehat{T/G}$ is trivial if the vertex group G_v is trivial. Minimality of (G, T) implies that a vertex of valence 1 cannot be trivial, while freeness implies that T_v is not a point if v is nontrivial.

Assume $\widehat{T/G}$ is not homeomorphic to a segment. Then we can find a vertex v of valence ≥ 3 and three nontrivial vertices a_1, a_2, a_3 such that the open segments (va_i) are disjoint. Set $d_i = d(v, a_i)$, and choose $\delta_i > 0$ such that $\lambda = 2d_i + \delta_i$ is independent of $i = 1, 2, 3$. Finally, choose $\varepsilon > 0$ small with respect to the d_i's and $c - 1/3$.

In this proof, we shall say that two points (or two numbers) are close if their distance does not exceed a fixed multiple of ε (which we do not bother to specify each time). Similarly, we write $p \gtrsim q$ if $p - q$ is greater than a fixed negative multiple of ε.

Definition of $\widehat{T/G}$ implies the following lifting property: given two points a, b at distance d in $\widehat{T/G}$, and a point $\tilde{a} \in \pi^{-1}(a)$, there exists $\tilde{b} \in \pi^{-1}(b)$ such that $d(\tilde{a}, \tilde{b})$ is close to d.

Fix $\tilde{v} \in \pi^{-1}(v)$. Choose $\tilde{a}_1 \in \pi^{-1}(a_1)$ with $d(\tilde{v}, \tilde{a}_1)$ close to d_1. Let T_1 be the component of $\pi^{-1}(a_1)$ that contains $\tilde{a}_1$, and G_1 the stabilizer of T_1. Choose $h_1 \in G_1$ such that $d(\tilde{a}_1, h_1 \tilde{a}_1)$ is close to δ_1 (this is where we use freeness of the action, cf. III.2). Define $\tilde{v}_1 = h_1 \tilde{v}$, and note that $d(\tilde{v}, \tilde{v}_1)$ is close to λ. Perform this operation twice more, passing from $\tilde{v}_1$ to $\tilde{v}_2 = h_2 \tilde{v}_1$ via some $\tilde{a}_2 \in \pi^{-1}(a_2)$, and from $\tilde{v}_2$ to $\tilde{v}_3 = h_3 \tilde{v}_2$ via some $\tilde{a}_3 \in \pi^{-1}(a_3)$.

Let $g = h_1 h_2 h_3$ be the element of G that takes $\tilde{v}$ to $\tilde{v}_3$, and A_g its translation axis. Since $\pi([\tilde{v}, \tilde{a}_1])$ and $\pi([h_3 \tilde{a}_3, \tilde{v}_3])$ are almost disjoint, the translation length $\ell(g)$ is close to $d(\tilde{v}, \tilde{v}_3) \approx 3\lambda$. Also note that, if $h(\tilde{v})$ and $h'(\tilde{v})$ are both close to A_g (with $h, h' \in G$), then $(1/\lambda) d(h(\tilde{v}), h'(\tilde{v}))$ is close to some integer k; if $k \geq 2$, then $(1/\lambda)\ell(h'h^{-1}) \gtrsim 2$.

Write $g = g_1 g_2$ with $\ell(g_1) + \ell(g_2) < \ell(g) + \varepsilon$. We shall show that $\ell(g_1)$ or $\ell(g_2)$ is $\gtrsim 2\lambda$. We may assume that A_{g_2} meets A_g, since otherwise we get $\ell(g_1) > \ell(g_1 g_2) + \ell(g_2^{-1}) > \ell(g)$.

First suppose that $g_2 \tilde{v}$ is close to A_g. Then at least one of the numbers $d(\tilde{v}, g_2 \tilde{v})$ or $d(g_2 \tilde{v}, g\tilde{v})$ is $\gtrsim 2\lambda$, and we get $\ell(g_i) \gtrsim 2\lambda$ for $i = 1$ or 2.

If $g_2 \tilde{v}$ is not close to A_g, there exists $x \in [\tilde{v}, g\tilde{v}] \cap A_{g_2}$ such that the open segment $(x, g_2 x)$ is disjoint from A_g. Let y be the point of $[x, gx]$ whose distance to gx is $\ell(g_2)$. It has to be close to $g_1 x$, since $\ell(g_1) < \ell(g) - \ell(g_2) + \varepsilon = d(x, y) + \varepsilon$. If x is close to $G\tilde{v}$, we argue as in the first case. If not, one of the segments $[x, y]$ or $[y, gx]$ is disjoint from $G\tilde{v}$, so that the other one contains two points of $G\tilde{v}$ whose distance is $\gtrsim 2\lambda$. It follows that $\ell(g_1)$ or $\ell(g_2)$ is $\gtrsim 2\lambda$.

IV. Questions and remarks

IV.1. How general are J-actions? For instance, do J-actions include all small minimal actions of finitely presented groups with proper direction-to-vertex stabilizers (see [BF, p. 140])? We can prove it for actions of free groups.

IV.2. Is there a generalization of Theorems 1 and 2 to arbitrary minimal actions?

IV.3. Inspired by terminology from dynamical systems and foliation theory, define a *minimal set* to be a nonempty, closed, G-invariant subset of T which is minimal for these three properties. Minimal sets are precisely preimages $\pi^{-1}(x)$.

Theorem 2 implies that a minimal set of a J-action is a discrete union of subtrees. What can be said in general (assuming G is finitely generated)? For instance, can a minimal set be *exceptional* (i.e. intersect a segment in a Cantor set), or be a closed non-discrete orbit (compare IV.5)?

IV.4. Consider a transitive free action of a finitely generated group G. Can G be written as a free product in such a way that each factor acts on its minimal invariant subtree with the property that every orbit meets every nondegenerate segment? Since free actions of surface groups are classified [Sk1], the interesting case is when G is free. This may be related (via Theorem III.7 of [Le]) to the following geometric question.

Consider a nonsingular, codimension one, measured foliation $\mathcal{F}$ on an open manifold M. Can $\mathcal{F}$ have exceptional leaves if $\pi_1 M$ is finitely generated?

IV.5. Theorem 3 becomes false if G is only assumed to be countable. The following simple example was pointed out by G. Meigniez. Let G be the free group on countably many generators $\{g_n\}$. Let T be the Cayley graph of G, with a metric giving length $1/n$ to the edges corresponding to g_n. The natural action of G on T is free and minimal.

The sequence $G(1/n)$ is strictly decreasing. The space $\widehat{T/G}$ is a Hawaiian earring. In particular, it does not have the homotopy type of a wedge of circles. It has no universal covering, and its fundamental group is uncountable. Also note that the orbit consisting of the vertices of T is closed but not discrete.

For any action, there is a natural homomorphism $\varphi : G \to \pi_1(\widehat{T/G})$: choose a basepoint $x \in T$, and send $g \in G$ to the projection of the segment $[x, gx]$. If G is finitely generated, Theorem 3 says that φ is an epimorphism whose kernel is G_0. In the above example, φ is not onto.

Theorem 7, however, is valid for infinitely generated groups.

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