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The geometric invariants of direct products of virtually free groups

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1. Introduction

1.1. *Summary.* The purpose of this paper is to compute the homological and the homotopical geometric invariants of [Bi-Re] and [Re 88] for direct products $G = G_1 \times G_2 \times \cdots \times G_l$ of finitely generated virtually free groups. As an application we determine the finiteness properties "type FP_m" and "type F_m" for all subgroups of G above the commutator subgroup G'.

1.2. Recall that a group (or a monoid) G is said to be of type FP_m, where $m \in \mathbf{N}_0$, if the trivial G-module $\mathbf{Z}$ admits a projective $\mathbf{Z}G$-resolution, which is finitely generated in all dimensions $\leq m$ [Bi 76/81]. Moreover, a group G is of type F_m if an Eilenberg–McLane complex $K(G, 1)$ for G with finite m-skeleton exists [Wa]. Type F_m always implies type FP_m, but it's not known whether the converse is true. More details can be found in [Bi 76/81], [Br], [Rat].

The homological invariants $\Sigma^m(G; \mathbf{Z})$ and the homotopical invariants $\Sigma^m(G)$ referred to above are conical subsets of the real vector space $V(G) := \mathrm{Hom}\,(G; \mathbf{R})$. They can be defined in terms of FP_m-properties of certain submonoids of G in the homological case and in terms of connectivity properties of pieces of universal coverings of certain $K(G, 1)$-complexes in the homotopical case. We will give the definitions in Section 2; for a survey the reader is referred to [Bi 93], [Bi-Str].

1.3. *The result.* Let $G = G_1 \times G_2 \times \cdots \times G_l$ be the direct product of l finitely generated virtually free groups. We denote by $\mathscr{L}$ the lattice of all subsets of

$$\mathscr{I} := \{j \in \{1, \ldots, l\} \mid G_j/G_j' \text{ infinite and } G_j \text{ virtually (free of rank } \geq 2)\}$$

and if $\sigma \in \mathscr{L}$ we write $|\sigma|$ for its cardinality. For $\sigma \in \mathscr{L}$ we consider the subgroup $H_\sigma \leq G$ generated by the union of all G_i, $i \in \sigma$. If ω is the complement of σ in $\mathscr{I}$, then G is the direct product $H_\sigma \times H_\omega \times H$, where H is the subgroup of G generated by all G_i with $i \notin \mathscr{I}$. Now, the canonical projection $\pi_\sigma : G \twoheadrightarrow H_\sigma$ induces an injective $\mathbf{R}$-linear map $\pi_\sigma^* : V(H_\sigma) \rightarrowtail V(G)$, and we can state our main result.

THEOREM. *Let $G = G_1 \times \cdots \times G_l$ be the direct product of l finitely generated virtually free groups. Then the homological and the homotopical geometric invariants of G coincide and their complements in $V(G)$ are given by the formula*

$$\Sigma^m(G; \mathbf{Z})^c = \Sigma^m(G)^c = \left(\bigcup_{\sigma \in \mathscr{L}, |\sigma| \le m} \pi_\sigma^* V(H_\sigma) \right) - \{0\}. \tag{$*$}$$

Note that $\Sigma^m(G; \mathbf{Z})^c = \Sigma^m(G)^c$ are equal to $\pi_\mathscr{I}^* V(H_\mathscr{I}) - \{0\}$ if $m \ge |\mathscr{I}|$. Moreover, the theorem says, in other words, that *a non-zero homomorphism $\chi : G \to \mathbf{R}$ is in $\Sigma^m(G; \mathbf{Z})^c = \Sigma^m(G)^c$ if and only if its kernel contains $H_\omega \times H$ for some $\omega \in \mathscr{L}$ with $|\omega| \ge |\mathscr{I}| - m$.*

The three inclusions which are necessary to prove the theorem will be established in Paragraph 2.3, Proposition 3.7 and Proposition 4.3.

1.4. *Remarks.* 1) Sometimes it might be convenient to replace $\mathscr{I}$ by the set of all j such that G_j is virtually (free of rank ≥ 2). This yields the same result because groups with finite Abelianization do not admit any non-zero homomorphism into the reals.

2) The homological part of the theorem is essentially contained in the author's diploma thesis [Mei 90]. However, all proofs given here are new.

1.5. The problem of how to compute the invariants of a direct product in terms of the invariants of the factors is still open. It is conceivable that the answer is given by the

CONJECTURE. *If $G = G_1 \times G_2$ is of type F_m then*

$$\Sigma^m(G_1 \times G_2)^c = \bigcup_{p+q=m} (\pi_1^* \Sigma^p(G_1)^c + \pi_2^* \Sigma^q(G_2)^c),$$

where $\pi_i^* : V(G_i) \rightarrowtail V(G)$ is induced by the projection $\pi_i : G \twoheadrightarrow G_i$ and $+$ denotes the complex-sum in the real vector space $V(G)$.

The conjecture is true for $m = 1$ [Bi-Neu-Str] (also see [Bi-Str]) and $m = 2$ [Geh]; the inclusion $\subseteq$ holds for arbitrary m [Geh]. Gehrke's method also gives a formula for $\Sigma^m(G)^c$ if G is the direct product of l groups $G_1, G_2, \ldots, G_l$ of type F_m with the property that $\Sigma^1(G_i) = \Sigma^m(G_i)$ for all $1 \le i \le l$. For example, f.g. virtually free groups, 1-relator groups, polycyclic groups or fundamental groups of compact 3-manifolds are of that type for all m. In this case $\Sigma^m(G)^c$ is the union of all subsets $\pi_{i_1}^* \Sigma^1(G_{i_1})^c + \cdots + \pi_{i_k}^* \Sigma^1(G_{i_k})^c$ of $V(G)$ with $1 \le i_1 < \cdots < i_k \le l$ and $k \le m$. Our theorem follows from Gehrke's result, but his proof is much longer and needs totally different techniques.

1.6. *Normal subgroups with Abelian quotient.* Let N be a normal subgroup of $G = G_1 \times \cdots \times G_l$ with Abelian quotient G/N. We define the *depth* $\vartheta(N)$ *of* N by

$$\vartheta(N) := \min\{d \in \mathbf{N}_0 \mid NHH_\omega \text{ has finite index in } G \text{ for every } \omega \in \mathscr{L} \text{ with } |\omega| = d\}.$$

Note that $0 \leq \vartheta(N) \leq |\mathscr{I}|$, that $\vartheta(N) = 0$ if and only if G/NH is finite, that $\vartheta(N)$ is equal to $1 + \#\{j \in \mathscr{I} \mid |G_j : G_j \cap N| < \infty\}$ if G/N has torsion free rank 1 and G/NH is infinite and that $\vartheta(G') = |\mathscr{I}|$. We say that a group is of type F_∞ if it is of type F_m for all m and note that G has this property. Now, the finiteness properties of N can be read off from the depth $\vartheta(N)$.

COROLLARY. *Let N be a normal subgroup of the direct product $G = G_1 \times \cdots \times G_l$ of l finitely generated virtually free groups and assume that G/N is Abelian. If $\vartheta(N) = 0$ then N is of type F_∞, and if $\vartheta(N) > 0$ then N is of type F_m and not of type FP_{m+1}, where $m = |\mathscr{I}| - \vartheta(N)$.*

Proof. The linear subspace of $V(G)$ consisting of all homomorphisms $\chi : G \to \mathbf{R}$ which vanish on N will be denoted by $V(G; N)$. Then we use the following result of R. Bieri and B. Renz ([Bi-Re], [Re 88]; see also [Bi 93] or [Bi-Str]): N is of type FP_m (resp. F_m) if and only if $V(G; N) \subseteq \Sigma^m(G; \mathbf{Z})$ (resp. $V(G; N) \subseteq \Sigma^m(G)$).

Now, by formula $(*)$ a non-zero homomorphism $\chi \in V(G)$ is an element of $\Sigma^m := \Sigma^m(G; \mathbf{Z}) = \Sigma^m(G)$ if and only if its kernel does not contain any $H_\omega \times H$ with $|\omega| \geq |\mathscr{I}| - m$. Next, we observe that the existence of a non-zero homomorphism $\chi : G \to \mathbf{R}$ whose kernel contains N and $H_\omega \times H$ for some $\omega \in \mathscr{L}$ is equivalent with the assertion that the Abelian group G/NHH_ω be infinite. From this we infer that $V(G; N) \subseteq \Sigma^m$ if and only if NHH_ω has finite index in G for all $\omega \in \mathscr{L}$ with $|\omega| \geq |\mathscr{I}| - m$.

Now, $\vartheta(N) = 0$ implies $V(G; N) \subseteq \Sigma^m$ for all $m \in \mathbf{N}_0$, so N is of type F_∞ by the result quoted above. If we assume $\vartheta(N) > 0$, it follows that $V(G; N) \subseteq \Sigma^m$ if and only if $\vartheta(N) \leq |\mathscr{I}| - m$. In other words, N is of type FP_m if and only if N is of type F_m if and only if $m \leq |\mathscr{I}| - \vartheta(N)$. □

1.7. A concrete example is given as follows. Let $D_m := \langle x_1, y_1 \mid - \rangle \times \cdots \times \langle x_m, y_m \mid - \rangle$, define a D_m-action on F, the free group on generators $\{a_k \mid k \in \mathbf{Z}\}$, by $x_i \cdot a_k := a_{k+1} =: y_i \cdot a_k$ and put $A_m := F \rtimes D_m$. If G is the direct product of $m + 1$ free groups of rank 2 consider the homomorphism $\chi : G \twoheadrightarrow \mathbf{Z}$ which sends each basis element of each free factor of G onto 1. Then A_m is isomorphic to the kernel

N of χ and the depth of N is $\vartheta(N) = 1$. Hence A_m is of type F_m and not of type FP_{m+1} by our corollary.

The groups A_m were introduced in [Bi 76] to establish the existence of groups of type FP_m which are not of type FP_{m+1} for $m \in \mathbb{N}$, where the case $m = 2$ is due to J. R. Stallings [Sta].

1.8. Recently, S. M. Gersten proved that each of the groups A_m, $m \geq 2$, satisfies a fifth degree polynomial isoperimetric inequality [Ger]. On the other hand these groups are neither combable nor asynchronously automatic (see [ECHLPT]) since groups with one of these properties are of type F_∞ ([Al], [ECHLPT], [Ger]). No examples of groups with sub-exponential isoperimetric function which are not combable were known before.

Now, one can use the corollary above to characterize all combable normal subgroups N with Abelian quotient of a direct product G of finitely many free groups of finite rank ≥ 2. Using [Al], [ECHLPT], [Ger] and our result that N is of type F_∞ if and only if N has finite index in G, one can conclude: N is combable (automatic, asynchronously automatic, biautomatic) if and only if N has finite index in G.

1.9. There is a slight overlap with work of G. Baumslag and J. E. Roseblade [Bau-Ro]. One of their main theorems states that every finitely presented subgroup S of a direct product of two free groups is a finite extension of a direct product of two free groups (of finite rank). If S contains the derived subgroup G', then we recover their result from our corollary. In fact, if G is a direct product of l free groups of finite rank ≥ 2, then every normal subgroup N of type FP_l with $G' \leq N$ has finite index in G. In particular, N is a finite extension of a direct product of l free groups (of finite rank). Hence we have enough examples to ask:

QUESTION. *Let G be the direct product of l free groups of finite rank ≥ 2. Is every subgroup of type* FP_l *in G a finite extension of a direct product of l free groups* (*of finite rank*)?

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2. The geometric invariants

2.1. *The homological invariants.* Let G be a group and $\chi : G \to \mathbf{R}$ a homomorphism. Then we consider the submonoid $G_\chi := \{g \in G \mid \chi(g) \geq 0\}$ of G and put for $m \in \mathbf{N}_0$

$$\Sigma^m(G; \mathbf{Z}) := \{\chi \in V(G) \mid G_\chi \textit{ is of type } \mathrm{FP}_m\} \subseteq V(G).$$

The complement of $\Sigma^m(G; \mathbf{Z})$ in $V(G)$ will be denoted by $\Sigma^m(G; \mathbf{Z})^c$. It follows from [Bi-Re] that $\Sigma^m(G; \mathbf{Z}) \neq \emptyset$ if and only if $0 \in \Sigma^m(G; \mathbf{Z})$ if and only if G is of type FP_m.

2.2. *The homotopical invariants.* Let G be a group of type F_m and X the universal cover complex of a $K(G, 1)$-complex with finite m-skeleton. If $\chi \in V(G)$, then G acts via χ on $\mathbf{R}$ and any continuous G-equivariant map $h = h_\chi : X \to \mathbf{R}$ shall be called a *height function* (*with respect to* χ). For a real number r we denote by $X_h^{[r,\infty)}$ the maximal subcomplex of X contained in $h^{-1}([r, \infty))$. $X_h^{[r,\infty)}$ is called *essentially k-connected in X* for some $k \geq -1$, if there is a $d \geq 0$ with the property that the map $\pi_i(X_h^{[r,\infty)}) \to \pi_i(X_h^{[r-d,\infty)})$ induced by inclusion is trivial for all $i \leq k$. Then we define

$$\Sigma^m(G) := \{\chi \in V(G) \mid X_h^{[0,\infty)} \textit{ is essentially } (m-1)\textit{-connected in } X\} \subseteq V(G)$$

and $\Sigma^m(G)^c := V(G) - \Sigma^m(G)$. This definition does not depend on the choice of X and h [Bi-Str], and we always have $0 \in \Sigma^m(G)$.

2.3. It is an open problem as to whether the two invariants coincide if both are defined. However, $\Sigma^0(G) = \Sigma^0(G; \mathbf{Z}) = V(G)$ for all groups, $\Sigma^1(G) = \Sigma^1(G; \mathbf{Z})$ for all finitely generated groups and by a result of Renz (see [Bi 93] or [Bi-Str]) $\Sigma^m(G) = \Sigma^2(G) \cap \Sigma^m(G; \mathbf{Z})$ holds for every group G of type F_m if $m \geq 2$. This proves the first inclusion, $\Sigma^m(G; \mathbf{Z})^c \subseteq \Sigma^m(G)^c$, of our theorem.

3. The homotopical part of the theorem

The aim of this section is to prove that $\Sigma^m(G)^c$ is contained in the right hand side of formula $(*)$. However, we start with two easy results on arbitrary groups. Recall that the subspace of $V(G)$ consisting of all homomorphisms which vanish on a subgroup $S \leq G$ is denoted by $V(G; S)$.

3.1. LEMMA. *Let $Z = Z(G)$ be the centre of a group G of type* F_m. *Then $\Sigma^m(G)$ contains the complement of the subspace $V(G; Z)$.*

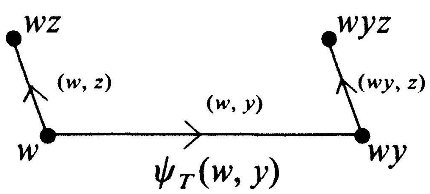
Proof. Exactly as in the homological case ([Bi-Re], Lemma 5.2) using the homotopical version of the Σ^m-criterion ([Bi 93], Theorem A; [Bi-Str]). □

3.2. LEMMA. *Let G be a group of type* F_m *and let $S \leq G$ be a subgroup of finite index. If $\chi : G \to \mathbf{R}$ is a homomorphism, then $\chi \in \Sigma^m(G)$ if and only if $\chi|_S \in \Sigma^m(S)$.*

Proof. Let X be the universal cover of a $K(G, 1)$-complex with finite m-skeleton and let $h : X \to \mathbf{R}$ be a height function with respect to $\chi : G \to \mathbf{R}$. Then X is the universal cover of a $K(S, 1)$ with finite m-skeleton and h is also a height function with respect to $\chi|_S : S \to \mathbf{R}$. Now the claim is obvious by the definition of $\Sigma^m(-)$. □

3.3. *A construction.* We now turn to free groups F of finite rank. Let $\mathscr{Y} \subseteq F$ be a finite set of free generators and consider the Cayley graph $T := \Gamma(F; \mathscr{Y})$ of F with respect to $\mathscr{Y}$. This is a combinatorial F-tree with set of vertices V the elements of F, with set of oriented edges E the pairs $e = (w, y) \in F \times \mathscr{Y}$, the origin of e given by w and the terminus given by wy (cf. [Serre]). By the inverse edge e^- we mean e with the opposite orientation and by $P(T)$ we denote the set of all edge paths of T.

Now, let $\chi : F \to \mathbf{R}$ be a non-zero homomorphism. Without loss of generality we may assume that there is an element $z \in \mathscr{Y}$ with $\chi(z) > 0$. Then we define F-maps $\psi_T : V \to V$ and $\psi_T : E \to P(T)$ by putting $\psi_T(w) := wz$ for $w \in V$, $\psi_T(w, z) := (wz, z)$ and $\psi_T(w, y) := (w, z)^-(w, y)(wy, z)$ for $(w, y) \in E$ with $y \neq z$. Moreover, we define a combinatorial height function $h_T : V \to \mathbf{R}$ by $h_T(w) := \chi(w)$ for $w \in V$.



The geometric realisation X of T is a contractible 1-dimensional CW-complex, on which F acts freely by permuting the cells, i.e. X is the universal cover of a finite 1-dimensional $K(F, 1)$. By linear extension of h_T we equip X with a height function $h : X \to \mathbf{R}$ with respect to χ. Now, by a suitable realisation of ψ_T we obtain for every $\varepsilon > 0$ a continuous cellular F-equivariant map $\psi : X \to X$ with $h(\psi(x)) \geq h(x) - \varepsilon$ for all $x \in X$ and $h(\psi(x^0)) = h(x^0) + \chi(z)$ for all 0-cells $x^0 \in X^0$. □

3.4. Let $G = F_1 \times \cdots \times F_l$ be the direct product of l free groups of finite rank. Then $\mathscr{J} = \{j \mid \operatorname{rk} F_j \geq 2\}$ and the subgroup H generated by all F_i with $i \notin \mathscr{J}$ is equal to the centre $Z = Z(G)$ of G. Let $\chi : G \to \mathbf{R}$ be a non-zero homomorphism and recall that $\mathscr{L}$ is the lattice of all subsets of $\mathscr{I}$. Then the crucial step is the following:

3.5. PROPOSITION. *Suppose there is an element $\sigma \in \mathscr{L}$ with the properties that $|\sigma| > m$ and that $\chi(F_i) \neq \{0\}$ for all $i \in \sigma$. Then $\chi \in \Sigma^m(G)$.*

Proof. Put $\chi_i := \chi|_{F_i}$ for $i = 1, \ldots, l$ and choose the universal covering X_i of a finite 1-dimensional $K(F_i, 1)$-complex together with the height function $h_i : X_i \to \mathbf{R}$ as in 3.3. Then $X := X_1 \times \cdots \times X_l$ is the universal cover of a finite l-dimensional $K(G, 1)$-complex and $h : X \to \mathbf{R}$ defined by $h := h_1 p_1 + \cdots + h_l p_l$ is a height function with respect to χ if p_i is the projection $X \twoheadrightarrow X_i$. Now, by 3.3 again there is a $\delta > 0$ and there are continuous cellular F_i-equivariant maps $\psi_i : X_i \to X_i$ for all $i \in \sigma$ with the property that $h_i(\psi_i(x_i)) \geq h_i(x_i) - \delta/l$ for all $x_i \in X_i$ and $h_i(\psi_i(x_i^0)) \geq h_i(x_i^0) + \delta$ for all 0-cells $x_i^0 \in X_i^0$ (recall that the definition of ψ_i depends on a *non-zero* homomorphism χ_i whereas the definition of X_i and h_i does not).

Next, we put $\varphi : X \to X$ to be the product map $\varphi := \prod_{i=1}^{l} \varphi_i$, where $\varphi_i := \psi_i$ if $i \in \sigma$ and $\varphi_i := \mathrm{Id}_{X_i}$ otherwise. Then φ is a continuous cellular G-equivariant map with $h(\varphi(x)) \geq h(x) + \delta/l$ for all $x \in X^m$. To see this let $x = (x_1, \ldots, x_l) \in X^m$ and note that the number of x_k with $x_k \notin X_k^0$ is at most $m < |\sigma| \leq l$. Hence there is at least one $i \in \sigma$ such that $x_i \in X_i^0$. Consequently $h(\varphi(x)) \geq h(x) + \delta - m \cdot \delta/l \geq h(x) + \delta/l$.

Using the homotopical version of the Σ^m-criterion ([Bi 93], Theorem A; [Bi-Str]) we see that $\chi \in \Sigma^m(G)$. □

3.6. *Remarks.* 1) Note that the height functions h_i and h used above are valuations in the sense of [Re 87] (Remark on p. 468) and [Re 88].

2) One can prove that the following assertion is valid for arbitrary groups G_1 and G_2 of type F_m, where $m = m_1 + m_2 + 1$ with $m_i \in \mathbf{N}_0$. If $\chi_i \in \Sigma^{m_i}(G_i) - \{0\}$, then $\chi_1 \times \chi_2 \in \Sigma^m(G_1 \times G_2)$ (see [Geh]). A similar result holds for the homological invariants.

Now we are ready to prove the homotopical part of our theorem.

3.7. PROPOSITION. *Let $G = G_1 \times \cdots \times G_l$ be the direct product of l finitely generated virtually free groups. Then*

$$V(G) - \left(\bigcup_{\sigma \in \mathscr{L}, |\sigma| \leq m} \pi_\sigma^* V(H_\sigma) \right) \subseteq \Sigma^m(G).$$

Proof. Let $\chi : G \to \mathbf{R}$ be a homomorphism in the left hand side. Then either (i) χ does not vanish on the subgroup $H \leq G$ generated by all G_i with $i \notin \mathscr{I}$, where $\mathscr{I}$

is the set of all j with G_j/G_j' infinite and G_j virtually (free of rank ≥ 2), or (ii) there exists a $\sigma \in \mathscr{L}$, the lattice of all subsets of $\mathscr{I}$, with $|\sigma| > m$ and $\chi(G_i) \neq \{0\}$ for all $i \in \sigma$.

Next, we consider a subgroup $S = F_1 \times \cdots \times F_l$ of finite index in G with $F_i \leq G_i$ free of finite rank. By Lemma 3.2 we have $\chi \in \Sigma^m(G)$ if and only if $\chi|_S \in \Sigma^m(S)$. Now, in case (i) χ does not vanish on the subgroup of G generated by all virtually (infinite cyclic) factors G_i. Hence $\chi|_S$ is non-trivial on the centre $Z(S)$ of S so the result follows from Lemma 3.1, and case (ii) is obviously covered by Proposition 3.5. □

4. The homological part of the theorem

In this section we prove the remaining inclusion of formula (*). As in Section 3 we begin with a result on the Σ's of arbitrary groups.

4.1. PROPOSITION. *Suppose that* $N \rightarrowtail G \overset{\pi}{\twoheadrightarrow} Q$ *is a short exact sequence of groups of type* FP_m *and let* $\psi : Q \to \mathbf{R}$ *be a homomorphism. Then* $\psi \in \Sigma^m(Q; \mathbf{Z})$ *if and only if* $\psi \circ \pi \in \Sigma^m(G; \mathbf{Z})$.

Proof. We may assume that $m \geq 1$ and we put $\chi := \psi \circ \pi$, so that N is contained in the kernel of χ. The obvious ring homomorphism $\pi_* : \mathbf{Z}G_\chi \twoheadrightarrow \mathbf{Z}Q_\psi$ induces spectral sequences

$$\mathrm{Tor}_p^{\mathbf{Z}Q_\psi}(\mathrm{Tor}_q^{\mathbf{Z}G_\chi}(\prod \mathbf{Z}G_\chi; \mathbf{Z}Q_\psi); \mathbf{Z}) \underset{p}{\Rightarrow} \mathrm{Tor}_{p+q}^{\mathbf{Z}G_\chi}(\prod \mathbf{Z}G_\chi; \mathbf{Z})$$

for arbitrary direct products $\Pi\, \mathbf{Z}G_\chi$ of copies of $\mathbf{Z}G_\chi$ ([Rot], Theorem 11.62).

Since $\mathbf{Z}G_\chi$ is a free $\mathbf{Z}N$-module and $\mathbf{Z}G_\chi \otimes_{\mathbf{Z}N} \mathbf{Z} \cong \mathbf{Z}Q_\psi$ as G_χ-modules with the obvious actions, a change-of-ring isomorphism ([Rot], Theorem 11.64) yields $\mathrm{Tor}_q^{\mathbf{Z}G_\chi}(\Pi\, \mathbf{Z}G_\chi; \mathbf{Z}Q_\psi) \cong \mathrm{Tor}_q^{\mathbf{Z}N}(\Pi\, \mathbf{Z}G_\chi; \mathbf{Z})$. Now, N is of type FP_m, hence $\mathrm{Tor}_q^{\mathbf{Z}N}(-; \mathbf{Z})$ commutes with direct products for $q < m$ ([Bi 76/81], Theorem 1.3), and we obtain $\mathrm{Tor}_q^{\mathbf{Z}N}(\Pi\, \mathbf{Z}G_\chi; \mathbf{Z}) = 0$ if $1 \leq q < m$ and $\cong \Pi\, (\mathbf{Z}Q_\psi)$ if $q = 0$.

We find that the above spectral sequence has enough collapsing to yield isomorphisms $\mathrm{Tor}_n^{\mathbf{Z}Q_\psi}(\Pi\, \mathbf{Z}Q_\psi; \mathbf{Z}) \cong \mathrm{Tor}_n^{\mathbf{Z}G_\chi}(\Pi\, \mathbf{Z}G_\chi; \mathbf{Z})$ for $n < m$ and arbitrary direct products Π. Another appeal to Theorem 1.3 of [Bi 76/81] now gives the result by the definition of $\Sigma^m(-; \mathbf{Z})$. □

4.2. *Remarks.* 1) A similar result holds for the homotopical geometric invariants [Mei 93].

2) If N satisfies the weaker condition that the Abelian groups $H_i(N; \mathbf{Z})$ are finitely generated for $1 \le i \le m-1$, and G is of type FP_m, then $\psi \circ \pi \in \Sigma^m(G; \mathbf{Z})$ implies $\psi \in \Sigma^m(Q; \mathbf{Z})$.

Now everything is present to complete the proof of our theorem.

4.3. PROPOSITION. *Let $G = G_1 \times \cdots \times G_l$ be the direct product of l finitely generated virtually free groups. Then*

$$\left(\bigcup_{\sigma \in \mathscr{L}. |\sigma| \le m} \pi_\sigma^* V(H_\sigma) \right) - \{0\} \subseteq \Sigma^m(G; \mathbf{Z})^c.$$

Proof. Let $m > 0$ and let $\chi : G \to \mathbf{R}$ be a non-zero homomorphism with $\chi \in \pi_\sigma^* V(H_\sigma)$ for some $\sigma \in \mathscr{L}$ with $|\sigma| \le m$. Then there is a non-zero $\chi_\sigma \in V(H_\sigma)$ such that $\chi = \chi_\sigma \circ \pi_\sigma$.

Let ω be the complement of σ in $\mathscr{I}$. Then $G \cong H_\sigma \times H_\omega \times H$ and Proposition 4.1 asserts that $\chi \in \Sigma^m(G; \mathbf{Z})^c$ if and only if $\chi_\sigma \in \Sigma^m(H_\sigma; \mathbf{Z})^c$ since $H_\omega \times H$ is of type F_∞. Now, H_σ has a subgroup $S = F_1 \times \cdots \times F_{|\sigma|}$ of finite index which is a direct product of $|\sigma|$ free groups of finite rank ≥ 2. By the analogue of Lemma 3.1, the homological finite index result [Bi-Str], we find that $\chi_\sigma \in \Sigma^m(H_\sigma; \mathbf{Z})^c$ if and only if $\chi_\sigma|_S \in \Sigma^m(S; \mathbf{Z})^c$. In view of the inequality $|\sigma| \le m$ the result follows once we have established the next lemma. □

4.4. LEMMA. *Let $S = F_1 \times \cdots \times F_s$ be the direct product of s free groups of finite rank ≥ 2. Then $\Sigma^s(S; \mathbf{Z}) = V(S) - \{0\}$.*

Proof. For each $i = 1, \ldots, s$ there is a free F_i-resolution $\mathbf{E}_i \twoheadrightarrow \mathbf{Z}$ of the form $0 \to (\mathbf{Z}F_i)^{r_i} \to \mathbf{Z}F_i \to \mathbf{Z} \to 0$, where $r_i \ge 2$ is the rank of F_i. Putting $\mathbf{E} := \mathbf{E}_1 \otimes_{\mathbf{Z}} \cdots \otimes_{\mathbf{Z}} \mathbf{E}_s$ yields a free S-resolution $\mathbf{E} \twoheadrightarrow \mathbf{Z}$ with $E_n \cong (\mathbf{Z}S)^{k_n}$ and $k_n = 0$ if $n > s$. Moreover, $\mathbf{E}$ has the additional property that $k_{s+1} - k_s + k_{s-1} - \cdots \pm k_0 = -(r_1 - 1)(r_2 - 1) \cdots (r_s - 1) < 0$ as is easily seen by induction on $s \in \mathbf{N}$. Now, a result on the partial Euler characteristics [Bi-Str] asserts that $\Sigma^s(S; \mathbf{Z}) - \{0\} = \emptyset$. □

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