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Stability problems in a theorem of F. Schur

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Abstract. Schur's theorem states that an isotropic Riemannian manifold of dimension greater than two has constant curvature. It is natural to guess that compact almost isotropic Riemannian manifolds of dimension greater than two are close to spaces of almost constant curvature. We take the curvature anisotropy as the discrepancy of the sectional curvatures at a point. The main result of this paper is that Riemannian manifolds in Cheeger's class $\mathfrak{R}(n, d, V, \Lambda)$ with L_1-small integral anisotropy have L_p-small change of the sectional curvature over the manifold. We also estimate the deviation of the metric tensor from that of constant curvature in the W_p^2-norm, and prove that compact almost isotropic spaces inherit the differential structure of a space form. These stability results are based on the generalization of Schur' theorem to metric spaces.

1. Introduction

1.1. *Stability results*

A classical theorem of F. Schur asserts that a Riemannian space of dimension greater than two that is isotropic at all its points is a space of constant curvature.

Let $\langle M, g\rangle$ be an n-dimensional C^∞-Riemannian manifold. Denote by $K_\sigma(P)$ the sectional curvature at the point P in the direction of the plane element $\sigma \subset M_P$. In what follows $S(P)$ is the scalar curvature at P. The function ε_g,

$$\varepsilon_g(P) = \sup_{\sigma \subset M_P} \left\{\left|K_\sigma(P) - \frac{S(P)}{n(n-1)}\right|\right\},$$

is said to be the *curvature anisotropy* of $\langle M, g\rangle$ at the point P. Assume that $\text{Vol}(M) \neq \infty$ and the average scalar curvature of $\langle M, g\rangle$, i.e.,

$$S_a = \frac{1}{\text{Vol}_g(M)} \int_M S(P)dVol_g(P),$$

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is finite. Then the function

$$\omega_g(P) = |S(P) - S_a|$$

is called the *curvature oscillation.*

Thus, Schur's Theorem says that Riemannian manifolds of dimension greater than 2 with zero curvature anisotropy have zero curvature oscillation. It is natural to conjecture that a Riemannian manifold with small anisotropy has small oscillation of the curvature (*metric stability*). This conjecture was stated by Yu. G. Reshetnyak in 1969.

Another aspect of the problem of stability in Schur's Theorem is *topological stability*: *Does the smallness of the curvature anisotropy of a Riemannian manifold imply that the manifold is diffeomorphic to a space form?*

Note that the diffeomorphism problem in the Sphere Theorem can be viewed as the problem of topological stability in Schur's theorem in the class of positively curved manifolds. Consequently, well-known works by R. S. Hamilton (1982) [14], E. Ruh (1982) [26] and G. Huisken (1985) [15] solve the problem of topological stability for positively curved manifolds. In the recent work by R. Ye (1989) [30], topological stability was also proved for negatively curved manifolds. In this work the small of the curvature anisotropy in the uniform norm was replaced by small of anisotropy in the L_2-norm.

We present here the following results.

We use the L_1-norm to measure anisotropy, namely, a compact Riemannian manifold $\langle M, g\rangle$ has *δ-small integral anisotropy* if

$$|\varepsilon_g(P)|_{L_1(M)} \leq \delta.$$

Let $\mathfrak{R}(n, V, k)$ be the class of compact C^∞-Riemannian manifolds $\langle M, g\rangle$ with

$$\dim M = n, \mathrm{Vol}(M) \geq V > 0, \quad \text{and} \quad \mathrm{diam}^2(M)|K_g|_{C^0(M)} \leq \kappa.$$

THEOREM A. *Let $n \geq 3$. Then there is a positive constant $\delta(n, V, \kappa)$ such that any Riemannian manifold in class $\mathfrak{R}(n, V, \kappa)$ with δ-small integral anisotropy and $0 \leq \delta \leq \delta(n, V, \kappa)$ is diffeomorphic to a space form.*

The example by M. Gromov and W. Thurston [13] shows that at least for $n \geq 4$ the restriction on volume or diameter cannot be removed.

Note that our method is based on different approach. In contrast to [30] the sign of the curvature in Theorem A may be arbitrary and we use a weaker norm to measure anisotropy. However, the constant $\delta(n, V, k)$ is not estimated explicitly.

In 1980 I. Gribkov [10] showed that for an arbitrary small anisotropy of the curvature of an open manifold the oscillation can be arbitrary large (see also [11]). A compact example with this curvature behaviour was constructed by R. J. Currier [7].

Our principal result is Theorem B that solves the problem of metric stability. First we recall the definition of Cheeger's class $\mathfrak{R}(n, d, V, \Lambda)$, which consists of compact n-dimensional C^∞-Riemannian manifolds $\langle M, g\rangle$ for which the following conditions hold:

$$\dim M = n, \qquad \operatorname{diam}(M) \le d, \qquad \operatorname{Vol}(M) \ge V > 0, \qquad |K_g| \le \Lambda.$$

We claim that the L_p-norm of the curvature oscillation converges to 0 as anisotropy goes to 0, i.e., the curvature of an almost isotropic space is almost constant in the L_p-class:

THEOREM B. *Let $n \ge 3$ and $1 \le p < +\infty$. Then given $v > 0$ there is a positive constant $\delta(n, d, V, \Lambda, p, v)$ such that the curvature oscillation of any Riemannian manifold $\langle M, g\rangle$ in Cheeger's class $\mathfrak{R}(n, d, V, \Lambda)$ with δ-small integral anisotropy and $0 \le \delta < \delta(n, d, V, \Lambda, p, v)$ satisfies*

$$|\omega_g(P)|_{L_p(M)} \le v.$$

Note that an *a priori* estimate for the deviation of the metric tensor of an almost isotropic Riemannian manifold from that of a space of constant curvature was known only in the class of multidimensional conformally flat metrics ([28]); the anisotropy was measured in the L_p-norm for $p > 1$ and the deviation was measured in the L_∞-norm.

Theorem B yields the result that the metric tensor of the Riemannian manifold with small integral anisotropy is close to the metric tensor of constant curvature in the W_p^2 and consequently in the $C^{1,\alpha}$-norm for $p \in [1, +\infty)$ and $\alpha \in (0, 1)$. The W_p^2-norm is defined w.r.t. harmonic coordinates, see Subsec. 4.3.

We say that the metric tensor g is *v-close to the metric tensor of a space of constant curvature in the W_p^2-norm* is there exists a metric tensor g^c of constant curvature c on M such that

$$|g - g^c|_{W_p^2(M)} \le v.$$

THEOREM C. *Let $n \ge 3$ and $1 \le p < +\infty$. Then given $v > 0$ there is a positive constant $\tilde{\delta}(n, d, V, \Lambda, p, v)$ such that the metric tensor of any Riemannian manifold*

$\langle M, g\rangle$ in Cheeger's class $\mathfrak{R}(n, d, V, \Lambda)$ with δ-small integral anisotropy and $0 \le \delta < \bar{\delta}(n, d, V, \Lambda, p, \nu)$ is ν-close to the metric tensor of a space of constant curvature in the W_p^2-norm.

Note that similar results can be obtained for almost Einstein manifolds. The author plans to present these results elsewhere.

1.2. *Generalization of Schur's theorem to metric spaces*

It is not difficult to define the idea of isotropicity in terms of distances only, see Subsec. 2.4. In 1982 A. D. Aleksandrov conjectured that Schur's theorem can be proved for metric spaces, i.e., it is not necessary to assume that the space is a smooth Riemannian manifold or even a topological manifold. The following theorem, which is the foundation of our stability results, gives the affirmative answer on Aleksandrov's conjecture.

GENERALIZED SCHUR'S THEOREM. *Suppose that (M, ρ) is a (locally compact) geodesically complete isotropic metric space (with intrinsic metric) of Urysohn-Menger dimension* ([16]) *greater than two. Then (M, ρ) is isometric to a Riemannian manifold of constant curvature.*

Note that our result is new even for C^2-smooth Riemannian manifolds; the Bianchi identities that are the basis of the classical proof require at least three derivatives of the metric tensor.

As a direct corollary we answer an old question from Distance Geometry (see [20] and [18]).

THEOREM D. *Let (M, ρ) be a locally compact geodesically complete metric space with intrinsic metric and of Urysohn-Menger dimension greater than two. Assume that the Wald curvature $K_w(P)$ (see Subsec. 2.5) exists at each point $P \in M$. Then (M, ρ) is isometric to a Riemannian manifold of constant curvature.*

We prove generalized Bianchi identities in order to bring into consideration the derivatives of the metric tensor of order not greater than 2. In addition, the proof of the generalized Schur's theorem is based on the approximation of spaces of bounded curvature by Riemannian manifolds with controlled bounds of sectional curvatures and hard technical result, Theorem 2.1 in [25] (see Subsec. 2.1).

The Generalized Schur's theorem was announced in [23].

1.3. *Outline of the proof of stability results*

The basic idea of our method is to prove the stability results as corollaries of the Generalized Schur's Theorem.

To prove Theorems A, B we make use of the standard method of application of Cheeger's finiteness and Gromov's compactness theorems, see Subsec. 2.1. Namely, if Theorem A does not hold, there exists a sequence $\{\langle M_m, g_m \rangle\}_{m=1,2,\dots}$ in class $\mathfrak{R}(n, V, \kappa)$ such that $|\varepsilon_{g_m}|_{L_1(M_m)} \to 0$ as $m \to \infty$ but for sufficiently large m the manifold M_m carries no metric of constant curvature. Without loss of generality (changing the metrics by a scalar factor if necessary) we may assume that the manifolds M_m, $m = 1, 2, \dots$ are in class $\mathfrak{R}(n, 1, V, \kappa)$. Gromov's theorem is used to yield the limit space M_∞, which is an Aleksandrov's space of bounded curvature. Assuming that the curvature anisotropy of the limit space is zero we immediately arrive at a contradiction, since by the Generalized Schur's Theorem the limit space is isometric to a space form and by Cheeger's finiteness theorem it is diffeomorphic to M_m, for sufficiently large m. Therefore, to prove Theorem A we need a statement on convergence of the second derivatives of the metric tensors. Below we give an example of a sequence of Riemannian manifolds in Cheeger's class for which the sequence of curvature tensors of each subsequence has no limit even almost everywhere. Instead we prove the *weak convergence* of the curvature tensors. This is enough to prove that the limit space has zero anisotropy.

Similar method is applied to prove Theorem B. Note that to apply Gromov's compactness theorem we need to improve the weak convergence of the curvature tensors of Riemannian manifolds $\langle M_m, g_m \rangle$ to L_p-convergence. We apply the ideas from the work by E. Ruh [26] to prove that the L_1-convergence to zero of the curvature anisotropies implies the L_p-convergence of the curvature tensors.

2. Generalized Schur's Theorem

2.1. *Spaces of bounded curvature*

For metric spaces, the concepts of geodesic segment (or shortest geodesic), (upper) angle between geodesics, and triangle made up of geodesic segments make sense ([1]).

The area $s(T)$ of a triangle T in a metric space is defined to be equal to the area of a Euclidean triangle of the same edge lengths. The excess $\delta(T)$ of the triangle T is understood to be the sum of the angles at the vertices T minus π.

Define the upper and lower curvatures $\bar{K}(T)$ and $\underline{K}(T)$ as follows. If $s(T) \neq 0$, then

$$\bar{K}(T) = \underline{K}(T) = \delta(T)/s(T).$$

For a degenerate triangles (i.e., $s(T) = 0$), set

$$\bar{K}(T) = \begin{cases} +\infty & \text{if } \delta(T) > 0 \\ -\infty & \text{if } \delta(T) \le 0, \end{cases} \qquad \underline{K}(T) = \begin{cases} +\infty & \text{if } \delta(T) \ge 0 \\ -\infty & \text{if } \delta(T) < 0. \end{cases}$$

The *upper* and *lower curvatures* of a locally compact metric space M with intrinsic metric ρ at a point $P \in M$ are

$$\bar{K}_M(P) = \limsup\{\bar{K}(T)\}, \qquad \underline{K}_M(P) = \liminf\{\underline{K}(T)\}; \qquad T \to P.$$

A locally compact, geodesically complete metric space M with intrinsic metric ρ is called a *space of bounded curvature* if for each point $P \in M$ the upper and lower curvatures at P satisfy the inequalities: $\bar{K}_M(P) < +\infty$ and $\underline{K}_M(P) > -\infty$.

SMOOTHNESS THEOREM ([21], [22]). *In a space of bounded curvature (M, ρ) it is possible to introduce the structure of a Riemannian manifold with the help of local harmonic coordinates, which form an atlas $\mathfrak{I}$ of smoothness $C^{3,\alpha}$, and the metric tensor in the harmonic coordinates belongs to least to $W_p^2 \cap C^{1,\alpha}$ for each $p \in [1, +\infty)$ and $\alpha \in (0, 1)$.*

We recall that a system of coordinates $x : U \subset M \to R^n$ in a Riemannian manifold $\langle M, g\rangle$ is said to be *harmonic* if $\Delta_g x = 0$. We denote by W_p^2 Sobolev's class of functions having second generalized derivatives summable to the power p.

The most important applications of spaces of bounded curvature are connected with Cheeger's finiteness [6] and Gromov's compactness [12] theorems. By Cheeger's finiteness theorem there are only finitely many diffeomorphism types of manifolds in class $\mathfrak{R}(n, d, V, \Lambda)$. Gromov's compactness theorem states that Cheeger's class is relatively compact w.r.t. Lipschitz distance, that is, from any sequence $\{\langle M_k, g_k\rangle\}$ in $\mathfrak{R}(n, d, V, \Lambda)$ one can extract a Lipschitz convergent subsequence. Limit spaces in Gromov's theorem turn out to be spaces of bounded curvature. Moreover, the completion $\bar{\mathfrak{R}}(n, d, V, \Lambda)$ of Cheeger's class consists of compact n-dimensional Aleksandrov's spaces of bounded curvature with the same restrictions on dimension, diameter, volume and (upper and lower) curvatures as in Gromov's compactness theorem and is compact space w.r.t. Lipschitz distance [24], [25].

For further details concerning spaces of bounded curvature, see surveys [2] and [3].

2.2. *Formal curvature tensor*

The Smoothness Theorem enables us to define the formal curvature tensor almost everywhere (a.e.) in M. Namely, let $x : U \subset M \to G \subset R^n$ ($x^i = x^i(X)$) be a harmonic system of coordinates in a neighborhood U of a point P in a space of bounded curvature M. We let $\{g_{ij}\}_{i,j=1,2,\dots,n,}$ denote the components of the metric tensor w.r.t. a harmonic system of coordinates $(x^1, \dots, x^n)$. We get formal equivalents of several standard notions of Riemannian geometry. The metric tensor $\{g_{ij}\}$ defines the Christoffel symbols $\{\Gamma^p_{ij}\}$, which determine the covariant derivative operator ∇.

By the Smoothness Theorem one can introduce a.e. in G the *formal Riemannian curvature tensor* $\{R^j_{irq}\}$ and define the *formal sectional curvature* K_σ in the direction of the plane element σ given by a nonzero bivector. Then $R_{ijrq} = g_{ik}R^k_{jrq}$; $R^{ij}_{rq} = g^{ik}R^j_{krq}$. the (*formal*) Ricci curvature R_{ij} and the scalar curvature S are the contractions $R_{ij} = R^k_{ijk}$, $S = g^{ij}R_{ij}$.

By the Smoothness Theorem

$$g_{ij} \in W^2_p(G) \cap C^{1,\alpha}(G); \qquad \Gamma^h_{ij} \in W^1_p(G) \cap C^{0,\alpha}(G); \qquad i, j, h = 1, 2, \dots, n$$

for any $p \in [1, +\infty)$, $\alpha \in (0, 1)$. The Smoothness Theorem also yields that components of the curvature tensor are summable to an arbitrary large finite power p. In [25] (Theorem 2.1) we proved that a.e. the formal sectional curvature can be computed by means of limits of ratios $\delta(T)/s(T)$. In particular this implies that $R^k_{ilj} \in L_\infty(G)$, $i, j, l, k = 1, \dots, n$.

We denote by $\Lambda^p_{m,n}(G)$ the set of differential forms of degree p with values in the space of (m, n)-tensor fields. The operator ∇ produces Cartan's absolute exterior differential D, which assigns to a differentiable form in $\Lambda^p_{m,n}(G)$ a form in $\Lambda^{p+1}_{m,n}(G)$. The absolute exterior differential of a smooth form $\Phi = \{\Phi^{ij}\}$ in $\Lambda^2_{2,0}(G)$ is given by

$$D\Phi^{ij} = d\Phi^{ij} + \Phi^{kj} \wedge \omega^i_k + \Phi^{ik} \wedge \omega^j_k, \tag{1}$$

where $\omega^k_i = \Gamma^k_{ij}\, dx^j$.

We define the (*formal*) *curvature form* $\Omega = \{\Omega^j_i\}_{i,j=1,2,\dots,n}$ in $\Lambda^2_{1,1}(G)$ by

$$\Omega^j_i = d\omega^j_i - \omega^k_i \wedge \omega^j_k = \frac{1}{2} R^j_{irq}\, dx^r \wedge dx^q, \qquad \Omega^{ij} = g^{ik}\Omega^j_k = \frac{1}{2} R^{ij}_{rq}\, dx^r \wedge dx^q.$$

2.3. *Generalized Bianchi identities*

To motivate our generalization of the classical Bianchi identities we first assume that the components of the metric tensor $\{g_{ij}\}$ belong to class C^∞. For smooth Riemannian manifolds the Bianchi identities are

$$D\Omega^{ij}=0, \qquad i,j=1,2,\ldots,n$$

(see [5], Sec. 191).

Let $p\in[1,+\infty)$. We denote by $L_p\Lambda^2_{2,0}(G)$ the set of forms $\Phi=\{\Phi^{ij}\}$ in $\Lambda^2_{2,0}(G)$ for which $\Phi^{ij}_{rq}\in L_p(G)$, $i,j,r,q=1,2,\ldots,n$ $(\Phi^{ij}=\Phi^{ij}_{rq}\,dx^r\wedge dx^q)$. We denote by $\Lambda^m_0(G)$ the set of C^∞-smooth scalar differential forms χ of degree m, which are compactly supported in the domain G.

Any form $\Phi\in L_p\Lambda^2_{2,0}(G)$ defines a tensor current in G, namely

$$(\Phi,\chi)=\{(\Phi^{ij},\chi)\}; \qquad (\Phi^{ij},\chi)=\int_G \Phi^{ij}\bigwedge\chi,\ \chi\in\Lambda^{n-2}_0(G).$$

We define the absolute exterior differential $(D\Phi,\chi)=\{(D\Phi^{ij},\chi)\}$ as

$$(D\Phi^{ij},\chi)=\int_G\left[-\Phi^{ij}\bigwedge d\chi+\left(\Phi^{kj}\bigwedge\omega^i_k+\Phi^{ik}\bigwedge\omega^j_k\right)\bigwedge\chi\right]; \qquad \chi\in\Lambda^{n-3}_0(G).$$

Integration by parts and Eq. (1) show that in the case of C^∞-Riemannian manifolds the Bianchi identities can be rewritten as

$$(D\Omega,\chi)=0, \qquad \text{for every } \chi\in\Lambda^{n-3}_0(G).$$

Now we turn to the case of a space of bounded curvature. We will need the construction of *Sobolev's averaging operator* ([8]). Recall that the C^∞-function $\varphi: R^n\to R_+$ is the *averaging kernel* if the support of φ is contained in the unit ball $B(0,1)\subset R^n$ and $\int_{R_n}\varphi(u)\,du=1$. Consider a domain $G'\supset G$ and for the sake of simplicity assume that G is a bounded domain in R^n. Then for sufficiently small ε the operator $\mathbf{A}_\varepsilon: L_p(G')\to C^\infty(G)$,

$$f^\varepsilon(x)=\frac{1}{\varepsilon^n}\int_G\varphi\left(\frac{x-u}{\varepsilon}\right)f(u)\,du, \quad x\in G; \qquad f^\varepsilon=\mathbf{A}_\varepsilon(f), \quad f\in L_p(G')$$

is well defined.

LEMMA 2.1 (Generalized Bianchi identities). *Eq.* (2) *holds in a space of bounded curvature.*

Proof. Let $x : U \subset M \to G \subset R^n$ be a harmonic system of coordinates in a space of bounded curvature (M, ρ). Consider $\chi \in \Lambda_0^{n-3}(G)$ and a domain G_0 such that $\operatorname{supp} \chi \subset G_0 \subset G$. Sobolev's averaging operator $\mathbf{A}_\varepsilon : L_p(G_0) \to C^\infty(G_0)$ is used to construct the *averaged metric tensor* $\{g_{ij}^\varepsilon\}_{i,j=1,2,\dots,n}$ in $G_0(g_{ij}^\varepsilon = \mathbf{A}_\varepsilon(g_{ij}))$. The Smoothness Theorem and elementary properties of the operator $\mathbf{A}_\varepsilon$ (see [8]) yield

$$|g_{ij}^\varepsilon - g_{ij}|_{C^{1,\alpha}(G_0)}, \quad |g_{ij}^\varepsilon - g_{ij}|_{W_p^2(G_0)} \to 0 \quad \text{as } \varepsilon \to 0+ (\alpha \in (0, 1), p \in [1, +\infty)).$$

Then the differential forms $\omega_i^{\varepsilon j}$ and $\Omega_i^{\varepsilon j}$, $i, j = 1, 2, \dots, n$, computed by metric tensor $\{g_{ij}^\varepsilon\}$ satisfy the following estimates

$$|\omega_i^{\varepsilon j} - \omega_i^j|_{C^{0,\alpha}(G_0)}, \quad |\Omega_i^{\varepsilon j} - \Omega_i^j|_{L_p(G_0)} \to 0 \qquad \text{as } \varepsilon \to 0+. \tag{3}$$

Since $\langle G_0, g_{ij}^\varepsilon \rangle$ is C^∞,

$$(D^\varepsilon \Omega^\varepsilon, \chi) = 0, \qquad \text{for every } \chi \in \Lambda_0^{n-3}(G_0).$$

By (3) the limit of the above equation as $\varepsilon \to 0$ is Eq. (2). □

2.4. *Isotropic metric spaces*

Recall that a neighborhood of a point in a metric space is *linear* if it is isometric to a straight line.

We say that *isotropic curvature* $K(P)$ exists at the point P of a locally compact metric space M with intrinsic metric ρ if no neighborhood of P is linear and $\bar{K}_M(P) = \underline{K}_M(P) \neq \infty$. The isotropic curvature at a point of a Riemannian manifold exists if and only if the sectional curvature at the point does not depend on plane elements.

A locally compact metric space M with intrinsic metric ρ is said to be *isotropic* if the isotropic curvature exists at each point of M.

Note that existence of the isotropic curvature at each point implies continuity of the function $K(P)$. We will also use the obvious fact that a geodesically complete isotropic space is a space of bounded curvature.

2.5. *Wald's curvature*

In Distance Geometry (see [4]), a large role is played by Wald's curvature [29], [20]. Wald's definition is equivalent to the following:

A quadruple of points in a metric space has *embedding curvature* equal to k if it is isometric to some quadruple of points on the simply connected surface of constant curvature k. A triple of points is called *linear* if it is isometric to a triple in the straight line. Let (M, ρ) be a locally compact metric space with intrinsic metric ρ in which no neighborhood is linear. Then (M, ρ) has *Wald's curvature* $K_W(p)$ at P if for each $\varepsilon > 0$ there is a $\sigma > 0$ such that each quadrupole Q of points that contains a linear triple of points and is in the ball of radius σ about P has embedding curvature $k(Q)$ admitting the estimate

$$|K_W(P) - k(Q)| \leq \varepsilon.$$

(Wald's curvature can also be defined equivalently in terms of embedding of quadruples in 3-dimensional spaces of constant curvature.) After this definition was introduced, it was developed in a two-dimensional theory. It was conjectured to apply to more general spaces than those of constant curvature in dimensions greater than two [18]. In fact, as follows from Theorem D, constant curvature spaces are the only higher-dimensional spaces possessing Wald's curvature.

Note that by Theorem 3.1 in [18]

$$K(P) = K_W(P),$$

whenever one of these curvatures exists. Thus, the class of isotropic spaces coincides with the class of spaces possessing Wald's curvature.

2.6. *Proof of the Generalized Schur's theorem*

Observe that (M, ρ) is a space of bounded curvature. By isotropicity and Theorem and 2.1 in [25],

$$\Omega^{ij} = -K(x)\, dx^i \wedge dx^j, \qquad i, j = 1, 2, \ldots, n, \qquad i \neq j.$$

Lemma 2.1 yields that for $i, j = 1, 2, \ldots, n$, $i \neq j$, and each $\chi \in \Lambda_0^{n-3}(G)$

$$\int_G K(x)\left[-dx^i \wedge dx^j \wedge d\chi + \left(dx^k \wedge dx^j \wedge \omega_k^i + dx^i \wedge dx^k \wedge \omega_k^j\right) \wedge \chi\right] = 0. \quad (4)$$

We specify the form χ as follows

$$\chi = f(x)\varsigma^{ijk};$$

$$\varsigma^{ijk} = dx^1 \wedge dx^2 \wedge \cdots \wedge d\hat{x}^i \wedge \cdots \wedge d\hat{x}^j \wedge \cdots \wedge d\hat{x}^k \wedge \cdots \wedge dx^n,$$

where $k \in \{1, 2, \ldots, n\}/\{i, j\}$; $f \in \Lambda_0^0(G)$, the notation $\hat{x}^i$ etc. means that x^i is missed.

Observe that

$$\Gamma^i_{kq}\, dx^k \wedge dx^j \wedge dx^q \wedge \varsigma^{ijk} = (\Gamma^i_{kq} - \Gamma^i_{qk})\, dx^k \wedge dx^j \wedge dx^q \wedge \varsigma^{ijk} = 0.$$

Then Eq. (4) for the specified choice of χ becomes

$$\int_G K(x) \frac{\partial f}{\partial x^k}(x)\, dx = 0; \qquad k = 1, 2, \ldots, n; \qquad f \in \Lambda_0^0(G).$$

The latter equation means that the generalized derivatives of the distribution $f \to \int_G K(x) f(x)\, dx$ vanishes and therefore $K(x)$ equals a constant c a.e. As mentioned above the function $K(x)$ is continuous and consequently equals c everywhere in G. Then by [1] (M, ρ) is isometric to a Riemannian manifold of constant curvature. □

Remark 2.1. Let (M, ρ) be a space of bounded curvature with isotropic formal sectional curvature. Making use of results in [27] it is not difficult to show that the formal sectional curvature can be extended to a continuous function and we conclude that (M, ρ) is isometric to a space of constant curvature.

3. Weak convergence of curvatures in class $\mathfrak{R}(n, d, V, \Lambda)$

3.1. *Example*

Consider the sequence of C^∞-Riemannian metrics

$$ds_k^2 = \exp(\lambda_k(x, y))(dx^2 + dy^2), \qquad k = 1, 2, \ldots,$$

$$\lambda_k(x, y) = \frac{1}{k^2}(\cos \pi k x + \cos \pi k y), \qquad x, y \in (-1, 1).$$

Observe that the sequence $\{ds_k^2\}_{k=1,2,\ldots}$ converges to $ds_0^2 = dx^2 + dy^2$ in the $C^{1,\alpha}$-norm for every $\alpha \in (0, 1)$. A direct computation shows that

$$K(ds_k^2) = \frac{\pi^2}{2}\exp\left(-\frac{1}{k^2}(\cos \pi kx + \cos \pi ky)\right)(\cos \pi kx + \cos \pi ky).$$

Since $\lambda_k(x, y)$, $k = 1, 2, \ldots$, are periodic functions of variables x and y, the above metrics can be realized on a torus T^2. Clearly the sequence $\{|K(ds_k^2)|\}$ is uniformly bounded. The sequence of Riemannian manifolds $\{\langle T^2, ds_k^2\rangle\}_{k=1,2,\ldots}$ is $C^{1,\alpha}$-convergent to $\langle T^2, ds_0^2\rangle$ and consequently the manifolds $\langle T^2, ds_k^2\rangle$ are in the class $\mathfrak{R}(2, 2, \frac{1}{2}, 2\pi^2)$ for sufficiently large k. At the same time the sequence $\{K(ds_k^2)\}_{k=1,2,\ldots}$ converges to zero in the weak sense and there is no subsequence converging to zero almost everywhere (in particular, in the L_p-norm).

3.2. *The $C^{1,\alpha}$-convergence of metrics in Cheeger's class*

We will need the following specific results on harmonic coordinates and convergence of metrics in the class $\mathfrak{R}(n, d, V, \Lambda)$.

(I) [17]. Let $\langle M, g\rangle \in \mathfrak{R}(n, d, V, \Lambda)$ Then there is a subatlas $\mathfrak{J}'$ in the atlas $\mathfrak{J}$ of harmonic coordinates in M such that any harmonic system of coordinates $x : U \subset M \to R^n$ in $\mathfrak{J}'$ is defined on the ball U of the radius $r = r(n, d, V, \Lambda) > 0$ and for each $\alpha \in (0, 1)$, the contravariant and covariant components of the metric tensor g w.r.t. arbitrary system of coordinates x in $\mathfrak{F}'$ satisfy the following *a priori* estimate:

$$|g^{ij}|_{C^{1,\alpha}}, |g_{ij}|_{C^{1,\alpha}} \leq C(n, d, V, \Lambda, \alpha), \qquad i, j = 1, 2, \ldots, n.$$

(II) [9]. Let $\{\langle M_k, g_k\rangle\}_{k=1,2,\ldots}$ be a sequence in $\mathfrak{R}(n, d, V, \Lambda)$, which is Lipschitz convergent to $\langle M, g\rangle$, where M is a C^∞-smooth differentiable manifold. Then there is a subsequence $\{\langle M_{k_m}, g_{k_m}\rangle\}_{m=1,2,\ldots}$ and for any $m = 1, 2, \ldots$, there exists a diffeomorphism $i_m : M \to M_{k_m}$ such that

(i) for every pair of systems of coordinates $x : U \subset M \to R^n$ in $\mathfrak{J}'(M)$ and $y : V \subset M \to R^n$ in $\mathfrak{J}'(M_{k_m})$ $(V \subset \imath_m(U))$ the coordinate functions $y^q = \imath_m^q(x^1, x^2, \ldots, x^n)$ satisfy the following estimate for any $\alpha \in (0, 1)$:

$$|\imath_m^q(x^1, x^2, \ldots, x^n)|_{C^{2,\alpha}} \leq \tilde{C}(n, d, V, \Lambda, \alpha) \tag{6}$$

(estimate (6) holds also for the inverse map $\imath_m^{-1}$);

(ii) given any point $P \in M$, there is an open set $U \subset M$ containing P and a local coordinate chart $x : U \subset M \to G \subset R^n$ in $\mathfrak{J}'(M)$ such that

$$\lim_{m \to \infty} |\tilde{g}_{ij}^{k_m} - g_{ij}|_{C^{1,\alpha}(G)} = 0, \tag{7}$$

where $\tilde{g}_{k_m} = \imath^* g_{k_m}$ and $\{\tilde{g}_{ij}^{k_m}\}_{i,j=1,2,\ldots,n}$ are components of the metric tensor $\tilde{g}_{k_m}$ w.r.t. the system of coordinates $x : U \to G$.

In what follows we will refer to the sequence $\langle M_{k_m}, g_{k_m} \rangle$ as *$C^{1,\alpha}$-convergent*.

Remark 3.1. It is possible to improve estimate (6) to a $C^{3,\alpha}$-bound. The proof of (6) in [9] was based on the following estimate. Let $x_l : V_l \subset M \to R^n$, $l = 1, 2, \ldots, N$, be harmonic coordinates in M. Assume that $V_\mu \cap V_\nu \neq \emptyset$. Then the functions $y^q = y^q_{\mu,\nu}(x^1, x^2, \ldots, x^n)$, μ, $\nu = 1, 2, \ldots, N$ ($y_{\mu,\nu} = x_\nu \circ x_\mu^{-1}$) satisfy the elliptic equation

$$g^{hl} \frac{\partial^2 y^q}{\partial x^h \, \partial x^l} - g^{rt} \Gamma^l_{rt} \frac{\partial y^q}{\partial x^l} = 0, \qquad q = 1, 2, \ldots, n. \tag{8}$$

On account of (I), (8) and usual Schauder-estimates argument, $|y^q|_{C^{2,\alpha}}$ is bounded by a constant depending only on n, d, V, Λ and α.

It is well-known that a system of coordinates $x_l : V_l \subset M \to R^n$ is harmonic if

$$g^{rt} \Gamma^l_{rt} = 0$$

(see Eq. (1) in [27]). Then Eq. (8) takes the form

$$g^{tl} \frac{\partial^2 y^q}{\partial x^t \, \partial x^l} = 0, \qquad q = 1, 2, \ldots, n.$$

By (I) and Schauder estimates we conclude that the $C^{3,\alpha}$-norm of the functions $y^q(x^1, x^2, \ldots, x^n)$ is bounded by a constant depending only on n, d, V, Λ and α. Then by following arguments in [9] (see p. 133) we improve the bound (6) to the $C^{3,\alpha}$-norm.

3.3. *Weak convergence of curvatures*

Let $\langle M, g_k \rangle_{k=1,2,\ldots}$ be a sequence in the class $\mathfrak{R}(n, d, V, \Lambda)$. Assume that

$$\lim_{k \to \infty} g_k = g.$$

Consider a domain $U \subset M$. A pair of smooth vector fields X, Y on U is called admissible if

$$v = v(X, Y) = \inf_{P \in U} \{(|X|^2|Y|^2) - \langle X, Y\rangle^2)|_P\} > 0.$$

w.r.t. the scalar product given by g.

In what follows $\sigma|_P$ means the plane element defined by the bivector $X \wedge Y|_P$.

The sequence $\{K(g_k)\}_{k=1,2,\ldots}$ is said to be *weakly convergent* to $K(g)$ (notation: $K(g_k) \rightharpoonup K(g)$) if for any point $P \in M$ there is a neighborhood $U \subset M$ of the point P and a coordinate chart $x : U \to G \subset R^n$ in $\mathfrak{J}$ with the following property:

For every admissible pair of vector fields X, Y on U and for every smooth compactly supported function $f \in \Lambda_0^0(G)$

$$\lim_{k \to \infty} \int_G [K_\sigma(g_k) - K_\sigma(g)]|_x f(x)\, dx = 0.$$

LEMMA 3.1. *Let $\{\langle M, g_k\rangle\}_{k=1,2,\ldots}$ be a sequence in $\mathfrak{R}(n, d, V, \Lambda)$ that is $C^{1,\alpha}$-convergent to a compact space of bounded curvature $\langle M, g\rangle$. Then*

$$K(g_k) \rightharpoonup K(g).$$

Proof. Consider the coordinate chart $x : U \to G \subset R^n$ defined in (ii). Let R_k be the curvature tensor of the Riemannian manifold $\langle M, g_k\rangle$. Then

$$K_\sigma(g_k) = \langle R_k(X, Y)Y, X\rangle v_k^{-1},$$

$$v_k(x) = [g_k(X, X)g_k(Y, Y) - g_k(X, Y)^2]|_x, \qquad x \in G.$$

Let $v(x) = [g(X, X)g(Y, Y) - g(X, Y)^2]|_x$. Observe that $v_k(x) \geq v/2 > 0$ for sufficiently large k and consequently $\lim_{k\to\infty} v_k^{-1}(x) = v(x)^{-1}$. We conclude that

$$\lim_{k \to \infty} |\langle R_k(X, Y)Y, X\rangle v_k^{-1} - \langle R_k(X, Y)Y, X\rangle v^{-1}|_{C^0(U)} = 0.$$

So, to prove the lemma we need to establish that

$$\lim_{k \to \infty} \int_U (\langle R_k(X, Y)Y, X\rangle - \langle R(X, Y)Y, X\rangle)|_x \frac{f(x)\, dx}{v(x)} = 0.$$

It is enough to show that

$$\lim_{k\to\infty}\int_G (R^{(k)}_{rjpq} - R_{rjpq})\phi(x)\,dx = 0$$

for every smooth function ϕ compactly supported in G.

Integration by parts yields

$$\int_G (R^{(k)}_{rjpq} - R_{rjpq})\phi(x)\,dx = \int_G \left[-(\Gamma^{(k)}_{jpq} - \Gamma_{jpq})\frac{\partial\phi}{\partial x^r} + (\Gamma^{(k)}_{rpq} - \Gamma_{rpq})\frac{\partial\phi}{\partial x^j} \right.$$
$$\left. + (\Gamma^{(k)l}_{jp} - \Gamma^{l}_{jp}\Gamma_{lrq})\phi - (\Gamma^{(k)l}_{rp}\Gamma^{(k)}_{jlq} - \Gamma^{l}_{rp}\Gamma_{jlq})\phi \right] dx.$$

The $C^{1,\alpha}$-convergence ensures that the limit of the right hand-size equals zero. This completes the proof of the lemma. □

The weak convergence of the scalar curvature is defined similarly.

COROLLARY 3.1. *$S(g_k) \rightharpoonup S(g)$ as $k \to \infty$.*

3.4. *Proof of Theorem A*

By Subsec. 1.3., Theorem A is a direct corollary of the following lemma.

LEMMA 3.2. *Let $\{\langle M, g_k\rangle\}_{k=1,2,\dots}$ be a sequence in $\mathfrak{R}(n, d, V, \Lambda)$ that is $C^{1,\alpha}$-convergent to $\langle M, g\rangle$. Assume that the curvature anisotropies ε_{g_k} satisfy*

$$\lim_{k\to\infty}\int_M \varepsilon_{g_k}(P)\phi(P)\,d\,\mathrm{Vol}_{g_k}(P) = 0 \tag{9}$$

for every smooth compactly supported function ϕ on M. Then $\langle M, g\rangle$ is a space of constant curvature.

Note that L_1-convergence, i.e.

$$\lim_{k\to\infty}\int_M \varepsilon_{g_k}(P)\,d\,\mathrm{Vol}_{g_k}(P) = 0,$$

implies (9).

Proof. Let $x : U \subset M \to G \subset R^n$ be a coordinate chart in $\mathfrak{J}'$. Assume that $\operatorname{supp} \phi \subset U$. Consider a pair of admissible vector fields X and Y and the plane element σ defined by the bivector $X \wedge Y$. We integrate the inequality

$$-\frac{1}{\eta^n}\varphi\left(\frac{x-u}{\eta}\right)\varepsilon_{g_k}(u)|g_k(u)| \le \frac{1}{\eta^n}\left[K_\sigma^{(k)}(u) - \frac{S^{(k)}(u)}{n(n-1)}\right]\varphi\left(\frac{x-u}{\eta}\right)|g_k(u)|$$
$$\le \frac{1}{\eta^n}\varphi\left(\frac{x-u}{\eta}\right)\varepsilon_{g_k}(u)|g_k(u)|, \qquad \eta > 0,$$

over G, where $|g_k| = \sqrt{\det(g_{ij}^k)}$. Then

$$-\frac{1}{\eta^n}\int_G \varphi\left(\frac{x-u}{\eta}\right)\varepsilon_{g_k}(u)|g_k(u)|\,du \le \mathbf{A}_\eta\left(\left(K_\sigma^{(k)}(u) - \frac{S^{(k)}(u)}{n(n-1)}\right)|g_k(u)|\right)$$
$$\le \frac{1}{\eta^n}\int_G \varphi\left(\frac{x-u}{\eta}\right)\varepsilon_{g_k}(u)|g_k(u)|\,du.$$

Observe that the function

$$\psi(u) = \varphi\left(\frac{x-u}{\eta}\right)$$

is smooth and compactly supported in G for any $x \in G$ and $\eta > 0$. Hypothesis (9) ensures that for any $x \in G$ and $\eta > 0$ the limits of both sides of inequalities (10) equal zero. On the other hand, Lemma 3.1 and Corollary 3.1 yield that

$$0 = \lim_{k\to\infty}\left[\mathbf{A}_\eta\left(\left(K_\sigma^{(k)} - \frac{S^{(k)}}{n(n-1)}\right)|g_k|\right)\right]\Big|_x = \mathbf{A}_\eta(K_\sigma|g|)|_x - \mathbf{A}_\eta\left(\frac{S}{n(n-1)}|g|\right)\Big|_x.$$

Consider the limit of the right-hand side of the above equation w.r.t. η and observe that

$$K_\sigma(x)|g(x)| = \frac{S(x)}{n(n-1)}|g(x)|$$

for almost all $x \in G$. Since the function $|g(x)|$ is positive, we conclude that

$$\varepsilon_g(x) = 0 \text{ a.e.}$$

The Generalized Schur's Theorem together with Remark 2.1 ensure that the limit manifold $\langle M, g\rangle$ is a space of constant curvature. □

4. Metric stability

4.1. L_p-bounds connected with scalar curvature

Let $\langle M, g\rangle$ be a Riemannian manifold in $\mathfrak{R}(n, d, V, \Lambda)$, $r > 0$ and $x : U \subset M \to B(6r) \subset R^n$ be a harmonic system of coordinates in $\mathfrak{J}'$ satisfying (5).

Let $\bar{S}(x) = \mathbf{A}_r(S)|_x$, i.e.

$$\bar{S}(x) = \frac{1}{r^n} \int_{B(5r)} \varphi\left(\frac{x-u}{r}\right) S(u)\, du, \qquad x \in B(r).$$

Consider a C^∞-smooth cut-off function $\theta : R^n \to R$ such that

(i) $\operatorname{supp} \theta \subset B(2r)$.
(ii) $\theta(z) = 1$ for $z \in B(r)$.

Hereafter $n \geq 3$. We specify the function $Q(x)$ as follows.

$$Q(x) = \frac{1}{c_n} \int_{B(3r)} \frac{\theta(|x-y|)}{|x-y|^{n-2}} [\bar{S}(y) - S(y)]\, dy, \qquad x \in B(r),$$

where $c_n = -n(n-2)k_n$ and k_n equals the volume of the unit ball in R^n.

We will need the following modifications of Lemmas 1, 2 in [26], which one can easily get from [26].

LEMMA 4.1. *Let* $1 < p < +\infty$. *Then*

$$\max_{i,j=1,2,\ldots,n} \left| \frac{\partial^2 Q}{\partial x^i\, \partial x^j} \right|_{L_p(B(r))} \leq C(n, d, V, \Lambda, p, r) |\varepsilon_g|_{L_p(M)}.$$

Denote by Δ the Laplace operator $\partial^2/\partial x^{1^2} + \partial^2/\partial x^{2^2} + \cdots + \partial^2/\partial x^{n^2}$.

LEMMA 4.2. *Let* $1 \leq p \leq +\infty$. *Then*

$$|\Delta Q - (\bar{S} - S)|_{L_p(B(r))} \leq C(n, d, V, \Lambda, p, r) |\varepsilon_g|_{L_p(M)}.$$

COROLLARY 4.1. $|\bar{S} - S|_{L_p(M)} \leq C'(n, d, V, \Lambda, p, r) |\varepsilon_g|_{L_p(M)}$.

Note that in contrast to [26] we will apply Corollary 4.1 for fixed r.

4.2. *Proof of Theorem B*

Let $\langle M, g\rangle$ be a C^1-Riemannian manifold and $1 \le p \le +\infty$. Recall that the set of measurable functions $\psi : M \to R$ for which

$$|\psi|_{L_p(M)} = \left[\int_M |\psi(P)|^p \, d\,\mathrm{Vol}_g(P)\right]^{1/p} < +\infty,$$

forms the class $L_p(M)$. For $p = +\infty$ we require that

$$|\psi|_{L_\infty(M)} = \mathrm{essup}\{|\psi(P)|\} < +\infty.$$

We argue by contradiction. Let $1 \le p < +\infty$. Assume that there is $v > o$ and there exists a sequence of Riemannian manifolds $\{\langle M_k, g_k\rangle\}_{k=1,2,\dots}$ in the class $\mathfrak{R}(n, d, V, \Lambda)$ such that,

(i) $\lim_{k\to\infty} |\varepsilon_{g_k}|_{L_1(M_k)} = 0$.
(ii) $|S^{(k)}(P) - S_a^{(k)}|_{L_p(M)} \ge v$, $k = 1, 2, \dots$

Because of Cheeger's theorem we may assume that $M_k = M$ for $k = 1, 2, \dots$. Then Theorem B immediately follows from:

LEMMA 4.3. *Let* $1 \le p < +\infty$, $n \ge 3$ *and* $\{\langle M, g_k\rangle\}_{k=1,2,\dots}$ *be a sequence in the class* $\mathfrak{R}(n, d, V, \Lambda)$ *for which (i) is satisfied. Then there exists a subsequence* $\{\langle M, g_{k_m}\rangle\}_{m=1,2,\dots}$ *such that*

$$\lim_{m\to\infty} |S^{(k_m)}(P) - S_a^{(k_m)}|_{L_p(M)} = 0.$$

Proof. We begin with the following remark. Let $\{f_k : M \to R\}_{k=1,2,\dots}$ be a sequence of uniformly bounded measurable functions on M, which is L_1-convergent to the function f, namely,

$$|f_k - f|_{L_1(M)} = \int_M |f_k(x) - f(x)| \, d\,\mathrm{Vol}_{g_k}(x) \to 0, \qquad k \to \infty;$$

$$|f_k|_{L_\infty} \le c < \infty, \qquad k = 1, 2, \dots.$$

On account of bound (5) the L_1-norm defined w.r.t. g_k is equivalent to the L_1-norm w.r.t. a fixed metric g_{k_0}. Then L_1-convergence ensures that there is a subsequence $\{f_{k_m}\}$ such that

$$\lim_{m\to\infty} f_{k_m}(x) = f(x) \text{ a.e. in } M$$

Since M is compact and the sequence $\{f_{k_m}\}_{m=1,2,\ldots}$ is uniformly bounded, Lebesgue's theorem is applied to yield that for each $1 \leq p < +\infty$

$$\lim_{m\to\infty} |f_{k_m} - f|_{L_p(M)} = 0.$$

Thus, without loss of generality we may assume that the L_1-norm in (i) can be replaced by the L_p-norm for $p \in (1, +\infty)$. This remark will allow us to apply the results of Subsec. 4.1.

Consider $\bar{S}^{(k)}(x) = \mathbf{A}_r(S^{(k)})|_x$, $k = 1, 2, \ldots$. Then

$$\mathbf{A}_r(S^{(k)})|_x = \frac{1}{r^n}\int_{B(5r)} \varphi\left(\frac{x-u}{r}\right) S^{(k)}(u)\, du$$
$$= \frac{1}{r^n}\int_{R^n} \varphi\left(\frac{x-u}{r}\right) S^{(k)}(u)\, du, \qquad x \in B(r),$$

where we put $S(u) = 0$ outside the ball $B(5r)$.

Observe that

$$\bar{S}_a^{(k)} = \mathbf{A}_r(S_a^{(k)}) = S_a^{(k)}.$$

Gromov's compactness theorem, (II), Corollary 3.1 and Lemma 3.2 ensure that there is a subsequence $\{\langle M, g_{k_m}\rangle\}_{m=1,2,\ldots}$ with the following properties:

(a) The sequence $\{\langle M, g_{k_m}\rangle\}_{m=1,2,\ldots}$ $C^{1,\alpha}$-converges to a space of constant curvature $\langle M, g\rangle$.

(b) $S^{(k_m)}(x) \to S_a$.

Since the sequence $\{S_a^{(k_m)}\}_{m=1,2,\ldots}$ is uniformly bounded, without loss of generality we may assume that

(c) $\lim_{m\to\infty} S_a^{(k_m)} = S_a$.

Note that for each x in the ball $B(r)$ the function $\psi(u) = \varphi\ ((x-u)/r)$ is C^∞-smooth and compactly supported in R^n. Because of (b) we conclude that the operator, $\mathbf{A}_r$ transforms the weakly convergent sequence $\{S^{(k_m)}(x)\}_{m=1,2,\ldots}$ into the point-wise convergent sequence $\{\bar{S}^{(k_m)}(x)\}_{m=1,2,\ldots}$, namely,

$$\lim_{m\to\infty} \bar{S}^{(k_m)}(x) = \bar{S}_a = \mathbf{A}_r(S_a) = S_a, \qquad x \in B(r).$$

It is well-known that

$$|\mathbf{A}_r(S^{(k_m)})|_{L_p} \le |S^{(k_m)}|_{L_p}, \qquad 1 \le p \le +\infty.$$

Thus, we conclude that the sequence $\{|\bar{S}^{(k_m)}|_{L_\infty}\}_{m=1,2,\ldots}$ is uniformly bounded. Then Lebesgue's theorem yields that

$$\lim_{s\to\infty} |\bar{S}^{(k_m)}(x) - S_a|_{L_p} = 0, \qquad 1 \le p < \infty.$$

By the triangle inequality

$$|S^{(k_m)}(x) - S_a^{(k_m)}|_{L_p} \le |S^{(k_m)}(x) - \bar{S}^{(k_m)}(x)|_{L_p} + |\bar{S}^{(k_m)}(x) - S_a|_{L_p} + |\bar{S}_a^{(k_m)} - S_a|_{L_p}.$$

Let $1 < p < +\infty$. By (i) and Corollary 4.1 we obtain that

$$\lim_{s\to\infty} |S^{(k_m)}(x) - \bar{S}^{(k_m)}(x)|_{L_p} = 0,$$

and we prove (11) for $p > 1$. We apply the remark at the beginning of the proof to find a subsequence of the sequence $\{\langle M, g_{k_m}\rangle\}_{m=1,2,\ldots}$, for which (11) can be improved to $p = 1$. This completes the proof of the lemma. □

4.3. *Proof of Theorem C*

Let $\langle M, g\rangle \in \mathfrak{R}(n, d, V, \Lambda)$. Consider a metric tensor $\tilde{g}$ on M. Then

$$|\tilde{g}|_{W_p^2(M)} = \sup \max_{i,j=1,2,\ldots,n} \{|\tilde{g}_{ij}|W_p^2(G)\}, \qquad 1 \le p < +\infty,$$

where sup is taken over the set $\mathfrak{I}'(\langle M, g\rangle)$ of harmonic systems of coordinates

$$x : U \subset M \to G \subset R^n, \qquad U = B(r(n, d, V, \Lambda)/2) \tag{12}$$

and $\{\tilde{g}_{ij}\}$ are components of the metric tensor $\tilde{g}$ in the coordinate chart (12).

LEMMA 4.4. *Let* $1 \le p < +\infty$. *There is a constant* $C(n, d, V, \Lambda, p) > 0$ *such that, for every* $\langle M, g\rangle$ *in* $\mathfrak{R}(n, d, V, \Lambda)$

$$|g|_{W_p^2(M)} \le C(n, d, V, \Lambda, p).$$

Proof. It is well-known that components of the metric tensor in harmonic coordinates satisfy the following elliptic equation

$$g^{tl}\frac{\partial^2 g_{ij}}{\partial x^t\,\partial x^l} = 2R_{ij} + I(g), \qquad i,j = 1,2,\ldots,n, \tag{13}$$

where $I(g)$ depends only on g_{ij} and their first derivatives, see Eq. (7) in [27]. Consider a C^∞-cut-off function $\varsigma(P)$ that is compactly supported in the ball $B(r(n, d, V, \Lambda))$ and equals 1 on U. Components of the tensor $\varsigma(P)g(P)$ vanish on the boundary of the ball $B(r(n, d, V, \Lambda))$. Meanwhile, components of ςg satisfy an elliptic equation similar to (13) with the right-hand side depending in addition on derivatives of ς. By (5), Theorem 15.1 in [19] yields the W_p^2-bound of the metric tensor g. □

Now we turn to the proof of Theorem C. Again, we argue by contradiction. Let $1 \le p < +\infty$. Assume that there is $\nu > o$ and a sequence of Riemannian manifolds $\{\langle M_k, g_k\rangle\}_{k=1,2,\ldots}$ in the class $\mathfrak{R}(n, d, V, \Lambda)$ such that (i) holds and

(ii)′ There is no constant curvature metric g_k^c on M_k, $k = 1, 2, \ldots$ such that the following inequality does hold

$$|g_k - g_k^c|_{W_p^2(M_k)} \le \nu.$$

Then Theorem C immediately follows from:

LEMMA 4.5. *Let $1 \le p < +\infty$, $n \ge 3$ and $\{\langle M_k, g_k\rangle\}$ be a sequence in the class $\mathfrak{R}(n, d, V, \Lambda)$ for which (i) is satisfied. Then there is a subsequence $\{\langle M_{k_m}, g_{k_m}\rangle\}$ such that for any $m = 1, 2, \ldots$, there exists a metric of constant curvature $g_{k_m}^c$ on M with the following property*:

$$\lim_{m\to\infty} |g_{k_m} - g_{k_m}^c|_{W_p^2(M_{k_m})} = 0. \tag{14}$$

Proof. Let $1 \le p < +\infty$. Gromov's compactness theorem, (II) and Theorem B ensure that there is a subsequence $\{\langle M_{k_m}, g_{k_m}\rangle\}_{m=1,2,\ldots}$ with the following properties:

(a) The sequence $\{\langle M_{k_m}, g_{k_m}\rangle\}_{m=1,2,\ldots}$ $C^{1,\alpha}$-converges to a space of constant curvature $\langle M, g^c\rangle$.

(b) The sequence of the scalar curvatures $\{S^{(k_m)}(\tilde{g}_{k_m})\}$ is L_p-convergent to the constant c.

Here we keep the notation of Subsec. 3.2.

Consider a coordinate chart (12) in $\mathfrak{J}'(\langle M, g^c\rangle)$. We claim that

$$\lim_{m\to\infty} |\tilde{g}_{ij}^{(k_m)} - g_{ij}^c|_{W_p^2(G)} = 0. \tag{15}$$

As in the proof of Theorem B we may assume that p is sufficiently large. Let $\langle M, g\rangle \in \mathfrak{R}(n, d, V, \Lambda)$ and l_g be the following differential operator

$$l_g(u) = g^{tl}\frac{\partial^2 u}{\partial x^t\, \partial x^l}, \qquad u \in C^2(G).$$

Let $\varsigma(P)$ be the cut-off function that was defined in Lemma 4.4. We also introduce the notation: $G' = x(B(r, n, d, V, \Lambda)/2)$ so that $\varsigma(x) = 1$ for $x \in G'$ and $\operatorname{supp}\varsigma \subset G$. We let $u(x)$ be $\varsigma(x)v(x)$; $v(x) = g_{ij}(x)$, $x \in G$. By (13)

$$l_g(u) = 2\varsigma R_{ij} + \varsigma I(g) + l_g(\varsigma)v + l_g(\varsigma, v), \tag{16}$$

where

$$l_g(\varsigma, v) = g^{tl}\frac{\partial\varsigma}{\partial x^t}\frac{\partial v}{\partial x^l}.$$

We introduce the following notation:

$$l_{\tilde{g}_{k_m}} = l_m, \qquad l_{g^c} = l,$$

$$v_m(x) = \tilde{g}_{ij}^{(k_m)}, \qquad u_m(x) = \varsigma(x)v_s(x),$$

$$v_0(x) = g_{ij}^c, \qquad u_0(x) = \varsigma(x)v_0(x), \qquad x \in G, \qquad m = 1, 2, \ldots.$$

Let $h_m(x) = u_m(x) - u_0(x)$. Then

$$l(h_m) = [l_m(u_m) - l(u)] - (l_m - l)(u_m). \tag{17}$$

On account of (a) and Lemma 4.4,

$$\lim_{s\to\infty} |(l_m - l)(u_m)|_{L_p(M)} = 0.$$

On the other hand, due to Eq. (16),

$$l_m(u_m) - l(u) = 2\varsigma[R_{ij}(\tilde{g}_{k_m}) - R_{ij}(g^c)] + \varsigma(I(\tilde{g}_{k_m}) - I(g^c)]$$
$$+ [l_m(\varsigma)v_m - l(\varsigma)v] + [l_m(\varsigma, v_m) - l(\varsigma, v)].$$

Because of (a) the limits of the L_p-norms

$$|\varsigma[I(\tilde{g}_{k_m}) - I(g^c)]|_{L_p(G)}, \qquad |l_m(\varsigma)v_m - l(\varsigma)v|_{L_p(G)}, \qquad |l_m(\varsigma, v_m) - l(\varsigma, v)|_{L_p(G)}$$

are zero.

As in the proof of Lemma 4.3 we may assume that the convergence in (i) can be improved to the L_p-convergence for arbitrary $p > 1$. This together with (b) ensures that

$$\lim_{m\to\infty} |R_{ij}(\tilde{g}_{k_m}) - R_{ij}(g^c)|_{L_p(G)} = 0.$$

Finally we conclude that the sequence $\{|l(h_m)|_{L_p(G)}\}$ converges to zero as $m \to \infty$ and consequently the right-hand side of Eq. (17) converges to zero in the L_p-norm. In other words, the functions $h_m(x, y)$, $m = 1, 2, \ldots$ satisfy the following equation:

$$l(h_m) = \Phi_m, \qquad \text{where } \lim_{m\to\infty} |\Phi_m|_{L_p(G)} = 0 \quad \text{and} \quad h_m(x) = 0, \qquad x \in \partial G.$$

In addition the property of the $C^{1,\alpha}$-convergence ensures that

$$\lim_{m\to\infty} |h_m|_{C(G)} = 0.$$

Then Eq. (15) follows from *a priori bound* (11.8) of Ch. III in [19].

Observe that due to the improved bound (6) (see Remark 3.1), (14) follows from (15) if we put

$$g^c_{k_m} = \varsigma_{k_m}^{-1}{}^*g^c.$$

This completes the proof of Lemma 4.5. □

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