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Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **70 (1995)**

PDF erstellt am: **17.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-53004

Comment. Math. Helvetici 70 (1995) 383–402 0010-2571/95/030383-20$1.50 + 0.20/0

Graded Poisson structures on the algebra of differential forms

J. V. Beltrán and J. Monterde

Abstract. We study the graded Poisson structures defined on $\Omega(M)$, the graded algebra of differential forms on a smooth manifold M, such that the exterior derivative is a Poisson derivation. We show that they are the odd Poisson structures previously studied by Koszul, that arise from Poisson structures on M. Analogously, we characterize all the graded symplectic forms on $\Omega(M)$ for which the exterior derivative is a Hamiltonian graded vector field. Finally, we determine the topological obstructions to the possibility of obtaining all odd symplectic forms with this property as the image by the pullback of an automorphism of $\Omega(M)$ of a graded symplectic form of degree 1 with respect to which the exterior derivative is a Hamiltonian graded vector field.

Introduction

There are many examples in differential geometry of constructions, definitions, objects, etc., on the graded commutative algebra of exterior differential forms, $\Omega(M)$, on a smooth manifold M, which depend on its grading. One of these objects is the Poisson bracket on differential forms once a Poisson bracket on functions is given [Kz].

The purpose of this paper is to characterize such Poisson brackets by application of the techniques of graded manifold theory to the particular graded manifold defined by the sheaf of differential forms, i.e., using graded symplectic forms on this graded manifold and their corresponding graded Poisson brackets (cf. [Mo], [Ro]).

It has been shown in [Kz] that a Poisson bracket $\{\ ,\ \}$ on M defines a graded Poisson bracket $[\![\ ,\ ]\!]$ on $\Omega(M)$. We shall call it Koszul-Schouton bracket. Its values on smooth functions and on exact 1-forms are given by $[\![f, g]\!] = 0$, $[\![df, g]\!] = \{f, g\}$ and $[\![df, dg]\!] = d\{f, g\}$, respectively, and it is then extended to all differential forms by the graded Leibniz rule. Two properties follow immediately for such a Poisson bracket: First, the exterior derivative is a Poisson derivation for it; that is,

$$d[\![\alpha, \beta]\!] = [\![d\alpha, \beta]\!] + (-1)^{|\alpha|-1}[\![\alpha, d\beta]\!],$$

1991 *Mathematics Subject Classification.* Primary: 58F05 58A50 Secondary: 58A10 53C15.

for any $\alpha, \beta \in \Omega(M)$. Graded Poisson brackets with this property will be called *differential.* The second property is that with respect to the $\mathbb{Z}$-grading, it is of $\mathbb{Z}$-degree -1, i.e., for homogeneous differential forms α and β, the $\mathbb{Z}$-degree of $[\![\alpha, \beta]\!]$ is $|\alpha|+|\beta|-1$.

Our first aim is to characterize the graded Poisson brackets that satisfy these two properties. We show in section 1 that among all graded Poisson brackets on $\Omega(M)$ of $\mathbb{Z}$-degree -1, only the Koszul-Schouten brackets are differential.

The notion of graded Poisson bracket may also be formulated for the $\mathbb{Z}_2$-grading of $\Omega(M)$. The corresponding homogeneous brackets in this case are called even or odd. We have found a negative answer for the existence of nondegenerate differential even Poisson brackets (cf. Prop. 6.1 below).

In section 2 we characterize all graded Poisson brackets by means of graded derivations on $\Omega(M)$. In particular, we give an expression for the Koszul-Schouten bracket in these terms.

Our second aim is to study the analogous problem for graded symplectic forms. So, in section 3 we review, following [Ro] and [Mo], the graded techniques needed to characterize the graded symplectic forms by means of tensor fields. It is shown that any graded symplectic form on $\Omega(M)$ of $\mathbb{Z}$-degree 1 is uniquely determined by a linear isomorphism $\mathbf{L}: T^*M \to TM$. Then, in section 4 we show that in the nondegenerate case, the Koszul-Schouten bracket comes from a graded symplectic exact form. That is, the graded symplectic forms on $\Omega(M)$ of $\mathbb{Z}$-degree 1 that produce differential graded Poisson brackets are determined by the isomorphism $\mathbf{P}$ defined by an invertible Poisson bivector, i.e., a symplectic structure on M. On the other hand it is shown in section 5 that any odd symplectic form is the image of a graded symplectic form on $\Omega(M)$ of $\mathbb{Z}$-degree 1 by the pullback of an automorphism of the algebra $\Omega(M)$.

These two results lead naturally to the question of whether or not an odd symplectic form that produces a differential graded Poisson bracket is the image of a differential graded symplectic form of $\mathbb{Z}$-degree 1 by the pullback of an automorphism of the algebra $\Omega(M)$ that commutes with the exterior derivative. By application of the results obtained in section 2, we obtain an affirmative answer to this question if the Betti numbers $b_{2k}, k \geq 2$ of M, vanish them all (Theorem 5.2). In particular, this cannot be the case in compact symplectic manifolds of dimension greater than 2 (see e.g. [Po]).

1. Graded Poisson structures on $\Omega(M)$

Let M be a differentiable manifold of dimension n. Let $\Omega(M) = \oplus_{k=0}^{n} \Omega^k(M)$ be the algebra of differential forms on M, and let $\Omega(M; TM) = \oplus_{k=0}^{n} \Omega^k(M; TM)$ be the graded left $\Omega(M)$-module of vector-valued differential forms. We adopt the convention that if v is an element of a graded module and the notation $|v|$ is used we are tacitly assuming that v is homogeneous of degree $|v|$. $\Omega(M; TM)$ can also be viewed as a graded right $\Omega(M)$-module with the multiplication $S \wedge \alpha = (-1)^{|S||\alpha|}\alpha \wedge S$, for $\alpha \in \Omega(M)$ and $S \in \Omega(M; TM)$. Let Der $\Omega(M)$ be the graded right $\Omega(M)$-module of all derivations on $\Omega(M)$. Der $\Omega(M)$ is a graded Lie algebra with the usual graded commutator. Unless otherwise stated, *linear* will mean $\mathbb{R}$-linear.

DEFINITION 1.1. A graded Poisson structure of $\mathbb{Z}$-degree $k, k \in \mathbb{Z}$ (resp. $\mathbb{Z}_2$-degree $k, k \in \mathbb{Z}_2$) on $\Omega(M)$ is a map $[\![\ ,\ ]\!] : \Omega(M) \times \Omega(M) \to \Omega(M)$ such that

(1) $[\![\ ,\ ]\!]$ is $\mathbb{R}$-bilinear.

(2) $|[\![\alpha, \beta]\!]| = |\alpha| + |\beta| + k$ (resp. mod. 2).

(3) $[\![\alpha, \beta]\!] = -(-1)^{(|\alpha|+k)(|\beta|+k)}[\![\beta, \alpha]\!]$.

(4) $[\![\alpha, \beta \wedge \gamma]\!] = [\![\alpha, \beta]\!] \wedge \gamma + (-1)^{(|\alpha|+k)|\beta|}\beta \wedge [\![\alpha, \gamma]\!]$.

(5) $[\![\alpha, [\![\beta, \gamma]\!]]\!] = [\![[\![\alpha, \beta]\!], \gamma]\!] + (-1)^{(|\alpha|+k)(|\beta|+k))}[\![\beta, [\![\alpha, \gamma]\!]]\!]$.

The natural grading of the algebra $\Omega(M)$ is the $\mathbb{Z}$-grading, but sometimes we will also refer to the $\mathbb{Z}_2$-grading. In this case, homogeneous elements, or any other homogeneous structures, will be called even, if $k = 0$, or odd if $k = 1$.

Note that a graded Poisson bracket is completely determined by its value on differentiable functions and on their differentials. This is an immediate consequence of the Leibniz rule (4).

A Lie algebra structure on 1-forms can be defined by means of a linear isomorphism $\mathbf{L} : T^*M \to TM$. A special case is when the linear isomorphism is the inverse of the morphism defined by a symplectic form on M ([A-M], pages 191–194).

Moreover, by transfering the Schouten-Nijenhuis bracket on multivectors to differential forms via the homomorphism extension $\Lambda T^*M \to \Lambda TM$ of a linear isomorphism $\mathbf{L} : T^*M \to TM$ (which will also be denoted by L) we obtain a graded Poisson structure on the full algebra $\Omega(M)$, whose restriction to smooth functions vanishes identically. We thus have the following:

DEFINITION 1.2. Let $\mathbf{L} : T^*M \to TM$ be a linear isomorphism. We define the Poisson bracket of degree -1 on $\Omega(M)$, $[\![\ ,\ ]\!]_{\mathbf{L}}$, as

$$[\![\alpha, \beta]\!]_{\mathbf{L}} = -\mathbf{L}^{-1}[\mathbf{L}(\alpha), \mathbf{L}(\beta)]_{SN},$$

for $\alpha, \beta \in \Omega(M)$, where $[\ ,\ ]_{SN}$ denotes the Schouten-Nijenhuis bracket of multivector fields.

The minus sign appears in order for this definition to coincide with that of [A-M].

Note that the antisymmetry, the Leibniz rule and the Jacobi identity for this graded Poisson bracket are obtained from the same properties of the Schouten-Nijenhuis bracket of multivector fields on M.

An odd Poisson bracket on $\Omega(M)$ was defined by Koszul using differential operators [Kz] as follows (see also [Kr1], [Kr2]): Let P be a Poisson bivector (not necessarily nondegenerate). We denote by $\mathscr{L}_P$ the second order differential operator acting on differential forms defined as $\mathscr{L}_P = i_P \circ d - d \circ i_p$, where i_P is the total insertion operator with respect to the bivector P.

DEFINITION 1.3. The bracket $[\![\ ,\ ]\!]_{KS(P)}$ defined by

$$[\![\alpha, \beta]\!]_{KS(P)} = (-1)^{\alpha}(\mathscr{L}_P(\alpha \wedge \beta) - (\mathscr{L}_P\alpha) \wedge \beta - (-1)^{\alpha}\alpha \wedge (\mathscr{L}_P\beta)),$$

for $\alpha, \beta \in \Omega(M)$, defines an odd Poisson structure on $\Omega(M)$, which is called the Koszul-Schouten bracket associated to P.

The Koszul-Schouten bracket is characterized by its value on smooth functions and on exact 1-forms as follows,

$$\begin{aligned} &[\![f, g]\!]_{KS(P)} = 0, \qquad [\![f, dg]\!]_{KS(P)} = P(df, dg) = \{f, g\}_P, \\ &[\![df, g]\!]_{KS(P)} = P(df, dg) = \{f, g\}_P, \ [\![df, dg]\!]_{KS(P)} = d(P(df, dg)) = d\{f, g\}_P, \end{aligned} \qquad (*)$$

for all $f, g \in C^\infty(M)$.

Remark. This bracket has been used recently in the Batalin-Vilkovisky quantization and in BRST theories [Ne, Sc].

Remark. If P is a nondegenerate Poisson bivector and $\mathbf{P} : T^*M \to TM$ is the associated linear isomorphism, then the brackets $[\![\ ,\ ]\!]_{\mathbf{P}}$ and $[\![\ ,\ ]\!]_{KS(P)}$ coincide. This can be proved easily by checking that their values on functions and on their differentials agree. Use has to be made of the fact that $\mathbf{P}\, d(P(df, dg)) = [\mathbf{P}\, df, \mathbf{P}\, dg]$.

The main property of the bracket defined by Koszul is that the exterior derivative is a Poisson derivation of degree 1 in the graded Poisson algebra $(\Omega(M), [\![\ ,\ ]\!])$. This is easily shown using the identity $\mathscr{L}_P \circ d = -d \circ \mathscr{L}_P$ (see [Kz]). So we cast this property in the following.

DEFINITION 1.4. A graded Poisson structure $[\![\ ,\ ]\!]$ on $\Omega(M)$ of degree $k \in \mathbb{Z}_2$ (resp. $\mathbb{Z}$) is differential if the exterior derivative is a Poisson derivation of degree 1 in the graded Poisson algebra ($\Omega(M)$, $[\![\ ,\ ]\!]$); that is, if for any $\alpha, \beta \in \Omega(M)$ we have

$$d[\![\alpha, \beta]\!] = [\![d\alpha, \beta]\!] + (-1)^{|\alpha|+k}[\![\alpha, d\beta]\!].$$

What we want to show now is that this simple condition characterizes completely the Koszul-Schouten brackets.

THEOREM 1.5. *A graded Poisson bracket of degree* -1 *on* $\Omega(M)$ *is differential if and only if it is the Koszul-Schouten bracket associated to a Poisson structure on* M.

Proof. Let $[\![\ ,\ ]\!]$ be a differential graded Poisson bracket of degree -1. We define the map $\{\ ,\ \} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$, $(f, g) \mapsto \{f, g\} = [\![f, dg]\!]$, and we show that $\{\ ,\ \}$ is a Poisson bracket;

(1) $\{f, g\} = -\{g, f\}$ since $[\![f, g]\!] = 0$, and $[\![\ ,\ ]\!]$ is differential, then $0 = d[\![f, g]\!] = [\![df, g]\!] - [\![f, dg]\!] = -[\![g, df]\!] - [\![f, dg]\!]$.

(2) the Leibniz rule $\{f, gh\} = [\![f, d(gh)]\!] = [\![f, h\, dg]\!] + [\![f, g\, dh]\!] = h[\![f, dg]\!] + g[\![f, dh]\!] = \{f, g\}h + g\{f, h\}$.

(3) the Jacobi identity $\{f, \{g, h\}\} = [\![f, d[\![g, dh]\!]]\!] = [\![f, [\![dg, dh]\!]]\!] = [\![[\![f, dg]\!], dh]\!] + [\![dg, [\![f, dh]\!]]\!] = \{\{f, g\}, h\} + \{g, \{f, h\}\}$.

So, $\{\ ,\ \}$ is a Poisson bracket on $C^\infty(M)$. Let P be its associated Poisson bivector. The Koszul-Schouten bracket is characterized by the relations (*). Thus, we finally have to show that a differential graded Poisson bracket $[\![\ ,\ ]\!]$ of degree -1 satisfies the relations (*). The only one of those which is not yet obvious from the definition of $\{\ ,\ \}$ is the last one. But,

$$[\![df, dg]\!] = d[\![f, dg]\!] = d\{f, g\} = d(P(df, dg)). \qquad \square$$

2. Hamiltonian graded vector fields

Let $[\![\ ,\ ]\!]$ be a Poisson bracket on $\Omega(M)$ of degree $k \in \mathbb{Z}_2$ (resp. $\mathbb{Z}$), and let α be a differential form on M. Condition (4) of definition 1.1 implies that the operator $D_\alpha = [\![\alpha,\ ]\!]$ is a derivation of $\Omega(M)$ of degree $|\alpha| + k$.

DEFINITION 2.1. The derivation $D_\alpha = [\![\alpha,\ ]\!]$ will be called the Hamiltonian graded vector field associated to α.

It is well known [F-N] that any derivation of $\Omega(M)$ can be uniquely written as a sum of an algebraic derivation and a derivation that commutes with the exterior derivative. Let us recall this decomposition:

If $K = \alpha \oplus X \in \Omega^{|\alpha|}(M; TM)$, the algebraic derivation $i_K \in \mathrm{Der}_{|\alpha|-1}\, \Omega(M)$, is defined by $i_K(\beta) = \alpha \wedge i_X(\beta)$, where i_X is the usual insertion operator. For arbitrary $K \in \Omega(M; TM)$, i_K is defined by linear extension. The graded commutator $[i_K, d]$ is a derivation that commutes with the exterior derivative. It will be denoted by $\mathscr{L}_K$. If K is homogeneous of degree k then $\mathscr{L}_K$ is of degree k.

A fundamental result of [F-N] is the following

THEOREM 2.2. *Any derivation* $D \in \mathrm{Der}\, \Omega(M)$ *can be uniquely written as* $\mathscr{L}_Q + i_L$, *where* $Q, L \in \Omega(M; TM)$.

We apply this result to the derivation D_α, to conclude that there exist vector-valued forms $Q_\alpha \in \Omega^{|\alpha|+k}(M; TM)$ and $L_\alpha \in \Omega^{|\alpha|+k+1}(M; TM)$, such that, $D_\alpha = \mathscr{L}_{Q_\alpha} + i_{L_\alpha}$. It is now possible to define two derivations from $\Omega(M)$ into $\Omega(M; TM)$ related to Q_α and L_α:

PROPOSITION 2.3. *Let* $[\![\ ,\ ]\!]$ *be a Poisson bracket of degree* $k \in \mathbb{Z}_2$ *(resp.* $\mathbb{Z}$*), and*

(1) *let* $K : \Omega(M) \to \Omega(M; TM)$ *be the map defined by* $K(\alpha) = (-1)^{|\alpha|k} Q_\alpha$. *Then,* K *is a derivation of degree* k.
(2) *let* $H : \Omega(M) \to \Omega(M; TM)$ *be the map defined by* $H(\alpha) = (-1)^{|\alpha|(k+1)} L_\alpha + K(d\alpha)$. *Then,* H *is a derivation of degree* $k+1$.

Proof. In the sequel, K_α and H_α will be used instead of $K(\alpha)$ and $H(\alpha)$. Condition (4) of Definition 1.1 can be written, with the help of condition (3), as

$$[\![\alpha \wedge \beta, \gamma]\!] = (-1)^{|\beta|(|\gamma|+k)} [\![\alpha, \gamma]\!] \wedge \beta + \alpha \wedge [\![\beta, \gamma]\!].$$

Writing this formula in terms of derivations and having in mind that for $S \in \Omega(M; TM)$ and $\alpha \in \Omega(M)$

$$\mathscr{L}_{\alpha \wedge S} = \alpha \wedge \mathscr{L}_S + (-1)^{|\alpha|+|S|} i_{(d\alpha \wedge S)},$$

we get the following expressions

$$\begin{aligned} Q_{\alpha \wedge \beta} &= (-1)^{|\beta|k} Q_\alpha \wedge \beta + \alpha \wedge Q_\beta, \\ L_{\alpha \wedge \beta} &= (-1)^{|\beta|(k+1)} L_\alpha \wedge \beta + \alpha \wedge L_\beta - (-1)^{|\beta|(k+1)} Q_\alpha \wedge d\beta \\ &\quad - (-1)^{|\alpha|+|\beta|+k} d\alpha \wedge Q_\beta. \end{aligned}$$

The first formula says that K is a derivation of degree k. Using this fact the second formula can be rewritten as

$$(-1)^{(|\alpha|+|\beta|)(k+1)}L_{\alpha\wedge\beta}+K(d(\alpha\wedge\beta))=((-1)^{|\alpha|(k+1)}L_\alpha+K(d\alpha))\wedge\beta$$
$$+(-1)^{|\alpha|(k+1)}\alpha\wedge((-1)^{|\beta|(k+1)}L_\beta+K(d\beta)).$$

But this is precisely the fact that H is a derivation of degree $k+1$. □

In particular, it is now evident that we can write the graded Poisson bracket of degree k in terms of derivations as follows

$$[\![\alpha,\beta]\!]=(-1)^{|\alpha|k}(\mathscr{L}_{K_\alpha}+(-1)^{|\alpha|}i_{H_\alpha-K_{d\alpha}})\beta.$$

The following proposition characterizes the differential brackets in terms of the derivations K and H.

PROPOSITION 2.4. *A graded Poisson bracket of degree $k\in\mathbb{Z}_2$ (resp. $\mathbb{Z}$) is differential if and only if the associated derivation H vanishes.*

Proof. An easy computation shows that

$$d[\![\alpha,\beta]\!]-[\![d\alpha,\beta]\!]-(-1)^{|\alpha|+k}[\![\alpha,d\beta]\!]=(-1)^{(|\alpha|+1)k}(-\mathscr{L}_{H_\alpha}+(-1)^{|\alpha|}i_{H_{d\alpha}})\beta.$$

Then, d is a Poisson derivation if and only if the derivation $-\mathscr{L}_{H_\alpha}+(-1)^{|\alpha|}i_{H_{d\alpha}}$ is identically zero, i.e., if and only if $H\equiv 0$. □

COROLLARY 2.5. *If d is a Poisson derivation for $[\![\ ,\ ]\!]$, then $L_\alpha=-(-1)^{|\alpha|+k}Q_{d\alpha}$ and*

$$[\![\alpha,\beta]\!]=(-1)^{|\alpha||k|}(\mathscr{L}_{K_\alpha}-(-1)^{|\alpha|}i_{K_{d\alpha}})\beta.$$

Let us determine the derivation K that defines a Koszul-Schouten bracket. First, let $\mathbf{P}$ be the linear map associated to the bivector P, and let us denote by ρ the extension of this linear map to a derivation from $\Omega(M)$ into $\Omega(M;TM)$ of degree -1, i.e., $\rho(f)=0$ for $f\in C^\infty(M)$, and

$$\rho(\alpha_1\wedge\cdots\wedge\alpha_k)=\sum_{i=1}^{k}(-1)^{i-1}\alpha_1\wedge\cdots\wedge\hat{\alpha}_i\wedge\cdots\wedge\alpha_k\otimes\rho(\alpha_i),\quad \alpha_i\in\Omega^1(M).$$

PROPOSITION 2.6. *The Koszul-Schouten bracket is given by*

$$[\![\alpha,\beta]\!]_{KS(P)}=(-1)^{|\alpha|}(\mathscr{L}_{\rho(\alpha)}-(-1)^{|\alpha|}i_{\rho(d\alpha)})\beta,$$

for α and β in $\Omega(M)$.

Proof. The Koszul-Schouten bracket is a differential Poisson bracket of degree -1. Thus by corollary 2.5, we have to show that the derivation K is equal to ρ. Since any derivation from $\Omega(M)$ into $\Omega(M; TM)$ is determined by its value on 0-forms and on exact 1-forms it suffices to determine K_f and K_{df}, for $f \in C^\infty(M)$. An application of the definition of the Koszul-Schouten bracket yields, for all $g \in C^\infty(M)$

$$[\![f, g]\!]_{KS(P)} = 0 = \mathscr{L}_{K_f} g, \qquad [\![df, g]\!]_{KS(P)} = P(df, dg) = -\mathscr{L}_{K_{df}} g,$$

so, we deduce that $K_f = 0$ and $K_{df} = \rho(df)$. Whence, $K = \rho$. □

Remark. Let $\bar{\omega}_P$ be the symplectic form associated to a nondegenerate Poisson bivector P and $[\![\ ,\ ]\!]_{KS(P)}$ its Koszul-Schouten bracket. Then, the Hamiltonian graded vector field associated to $-\bar{\omega}_P$ is the exterior derivative (see [K-M], formula 6.12),

$$D_{-\bar{\omega}_P} = [\![-\bar{\omega}_P, \ ]\!]_{KS(P)} = \mathscr{L}_{\rho(-\bar{\omega}_P)} = \mathscr{L}_{Id} = d.$$

3. Graded symplectic forms on $\Omega(M)$

The remaining part of the paper will be concerned with nondegenerate graded Poisson structures, i.e., graded symplectic forms. This section is devoted to a review of the graded symplectic geometry that will be needed to state our main results in sections 4 and 5 below. From now on, we will be interested mainly in the $\mathbb{Z}_2$-grading.

Graded differential forms. Recall that the elements of the graded $\Omega(M)$-module of derivations, $\mathrm{Der}\,\Omega(M)$, can be regarded as graded vector fields on the graded manifold with structure sheaf $\Omega(M)$. By analogy, a graded differential form is an $\Omega(M)$-multilinear alternating graded homomorphism from the module of graded vector fields into $\Omega(M)$. (We shall refer to [Ko] for definitions.)

Being a graded homomorphism of graded modules, a graded differential form has a degree. Thus, we can define a $\mathbb{Z} \times \mathbb{Z}$-bigrading on the module of graded differential forms. We will say that a graded differential form, λ, has bidegree $(p, k) \in \mathbb{Z} \times \mathbb{Z}$, if

$$\lambda : \mathrm{Der}\,\Omega(M) \times \overset{p}{\ldots} \times \mathrm{Der}\,\Omega(M) \to \Omega(M)$$

is such that, for all $D_1, \ldots, D_p \in \mathrm{Der}\,\Omega(M)$,

$$|\langle D_1, \dots, D_p; \lambda \rangle| = \sum_{i=1}^{p} |D_i| + k.$$

Using this bigrading, any graded p-differential form λ can be decomposed as a sum $\lambda = \lambda_{(0)} + \cdots + \lambda_{(n)}$, where $\lambda_{(i)}$ is a graded form of bidegree (p, i).

We shall denote by d^G the graded exterior differential. (See [Ko] for details.) In particular, for a graded 0-form $\alpha \in \Omega(M)$, $\langle D; d^G\alpha \rangle = D(\alpha)$, and for a graded 1-form, λ, on $\Omega(M)$ we have

$$\langle D_1, D_2; (d^G\lambda) \rangle = D_1(\langle D_2; \lambda \rangle) - (-1)^{|D_1||D_2|} D_2(\langle D_1; \lambda \rangle) - \langle [D_1, D_2]; \lambda \rangle.$$

The graded exterior differential is an operator of bidegree $(1, 0)$.

A fundamental result is the following corollary to Kostant's Theorem 4.7 [Ko]. A very simple proof of this fact has been obtained by Tuynman [Tu].

COROLLARY 3.1. *Every d^G-closed graded form of bidegree (p, k) with $k > 0$ is exact.*

Other familiar operators on ordinary manifolds also have counterparts on graded manifolds. If $D \in \operatorname{Der} \Omega(M)$, then:

(1) The insertion operator, $\iota(D)$, is defined by

$$\langle D_1, \dots, D_{p-1}; \iota(D)\lambda \rangle = (-1)^{p-1} \langle D_1, \dots, D_{p-1}, D; \lambda \rangle.$$

Note that $\iota(D)$ is an operator of bidegree $(-1, |D|)$.

(2) The Lie operator, $\mathscr{L}_D^G$, is defined by

$$\mathscr{L}_D^G = \iota(D) \circ d^G + d^G \circ \iota(D).$$

Note that $\mathscr{L}_D^G$ is an operator of bidegree $(0, |D|)$.

Graded symplectic forms. Let ω be a graded symplectic form of degree $k \in \mathbb{Z}_2$ (resp. $\mathbb{Z}$) on $\Omega(M)$. By definition, this is a d^G-closed, nondegenerate graded 2-form.

We observe that

(1) A nondegenerate graded Poisson bracket $[\![\ ,\ ]\!]_\omega$ is defined by

$$[\![\alpha, \beta]\!]_\omega = D_\alpha^\omega(\beta),$$

for $\alpha, \beta \in \Omega(M)$, and where, D_α^ω is the unique graded vector field such that $\iota(D_\alpha^\omega)\omega = d^G\alpha$.

(2) Conversely, a nondegenerate Poisson bracket of degree k on $\Omega(M)$, defines a graded symplectic form of degree k by means of the formula

$$[\![\alpha, \beta]\!] = -\langle D_\alpha, D_\beta; \omega\rangle,$$

where, D_α (resp. D_β) is the Hamiltonian graded vector field associated to α (resp. β) (Definition 2.1). In this case, we have $D^\omega_\beta = D_\beta$ for all $\beta \in \Omega(M)$. In fact,

$$-\langle D_\alpha, D_\beta; \omega\rangle = [\![\alpha, \beta]\!] = D_\alpha(\beta) = \langle D_\alpha; d^G\beta\rangle = -\langle D_\alpha, D^\omega_\beta; \omega\rangle.$$

Therefore $\langle D_\alpha, (D_\beta - D^\omega_\beta); \omega\rangle = 0$ for all $\alpha \in \Omega(M)$, and since ω is nondegenerate, $D_\beta = D^\omega_\beta$.

Note that if ω is a graded symplectic form of bidegree (2, 1), then, D_α is a derivation of $\mathbb{Z}$-degree $|\alpha| - 1$ and the associated graded Poisson bracket is of $\mathbb{Z}$-degree -1.

DEFINITION 3.2. A graded vector field D is locally Hamiltonian if the graded 1-form $\imath(D)\omega$ is d^G-closed.

Remark. As a consequence of corollary 3.1, every locally Hamiltonian graded vector field of positive degree is a Hamiltonian graded vector field.

Graded symplectic forms of $\mathbb{Z}$-degree 1. Next, we will characterize the graded symplectic forms of $\mathbb{Z}$-degree 1. This characterization is a particular case of the one obtained in [Mo] for graded manifolds in general. We shall include the proof for the graded manifold of differential forms because it does not make use of linear connections as opposed to the original approach in [Mo].

Note that $\operatorname{Der}\Omega(M)$ is a graded locally free $\Omega(M)$-module. This is a consequence of the Frölicher-Nijenhuis theorem 2.2. So, any graded form is uniquely determined by its action on derivations $\{i_X, \mathscr{L}_X\}$, where i_X and $\mathscr{L}_X$ are the insertion operator and the Lie derivative with respect to a vector field X on M, respectively.

PROPOSITION 3.3. *There is a one-to-one correspondence between graded symplectic forms, ω, of $\mathbb{Z}$-degree* 1 *and linear isomorphisms* $\mathbf{L} : T^*M \to TM$.

Proof. First, given a linear isomorphism $\mathbf{L} : T^*M \to TM$, let us denote by $\lambda_{\mathbf{L}}$ the graded form of bidegree (1, 1) defined by:

$$\langle i_X; \lambda_{\mathbf{L}}\rangle = 0, \qquad \langle \mathscr{L}_X; \lambda_{\mathbf{L}}\rangle = \mathbf{L}^{-1}(X).$$

It is easy to prove that the graded differential form $\omega_{\mathbf{L}} = d^G\lambda_{\mathbf{L}}$ is symplectic.

Conversely, if ω is a graded symplectic form of $\mathbb{Z}$-degree 1, corollary 3.1 implies that $\omega = d^G\lambda$, for some graded form λ of bidegree $(1, 1)$. The value of λ on the derivations i_X, where X is a vector field on M, determines a differential 1-form, α, on $M : \langle i_X; \lambda \rangle = \alpha(X)$. Now, consider the graded 1-form $\bar{\lambda} = \lambda - d^G\alpha$. Clearly $\langle i_X; \bar{\lambda} \rangle = 0$ and we can characterize this graded 1-form by its value on the derivations $\mathscr{L}_X$ namely, $\langle \mathscr{L}_X; \bar{\lambda} \rangle = \mathbf{A}(X) \in \Omega^1(M)$, where $\mathbf{A}$ is a $C^\infty(M)$-linear mapping from TM to T^*M. Note that $\mathbf{A}$ is a tensor field because $\langle i_X; \bar{\lambda} \rangle = 0$. As ω is nondegenerate, $\mathbf{A}$ is a linear isomorphism. Let $\mathbf{L} = \mathbf{A}^{-1}$. Then $\bar{\lambda} = \lambda_{\mathbf{L}}$, and $\omega = \omega_{\mathbf{L}}$. □

COROLLARY 3.4. *Any graded symplectic form of $\mathbb{Z}$-degree* 1 *can be written as* $\omega_{\mathbf{L}} = \mathrm{d}^G\lambda_{\mathbf{L}}$, where $\lambda_{\mathbf{L}}$ *is the graded form of bidegree* $(1, 1)$ *defined by the linear isomorphism* $\mathbf{L}$ *associated to the graded symplectic form by the previous proposition.*

Let $\mathbf{L} : T^*M \to TM$ be a linear isomorphism. Let us determine the graded Poisson bracket $[\![\ ,\ ]\!]_{\omega_{\mathbf{L}}}$ associated to $\omega_{\mathbf{L}} = d^G\lambda_{\mathbf{L}}$. To do this we determine the Hamiltonian graded vector fields $D_f^{\mathbf{L}}$ and $D_\alpha^{\mathbf{L}}$, for $f \in C^\infty(M)$ and $\alpha \in \Omega^1(M)$.

The derivation $D_f^{\mathbf{L}}$, defined by $\imath(D_f^{\mathbf{L}})\omega_{\mathbf{L}} = d^Gf$, is of degree -1. Thus, there exists a unique vector field Y on M such that $D_f^{\mathbf{L}} = i_Y$. The vector field Y is determined by the value of $\imath(D_f^{\mathbf{L}})\omega_{\mathbf{L}}$ on graded vector fields of the type $\mathscr{L}_X$ for a vector field X on M,

$$\langle \mathscr{L}_X; \imath(D_f^{\mathbf{L}})\omega_{\mathbf{L}} \rangle = -\langle \mathscr{L}_X, D_f^{\mathbf{L}}; d^G\lambda_{\mathbf{L}} \rangle = i_Y(\mathbf{L}^{-1}(X)) = ((\mathbf{L}^{-1})^*(Y))(X),$$

where the asterisk denotes the dual map. On the other hand, $\langle \mathscr{L}_X; d^Gf \rangle = df(X)$. Therefore $Y = \mathbf{L}^*(df)$.

The derivation $D_\alpha^{\mathbf{L}}$, defined by $\imath(D_\alpha^{\mathbf{L}})\omega_{\mathbf{L}} = d^G\alpha$, is of degree 0. Thus, $D_\alpha^{\mathbf{L}} = \mathscr{L}_Z + i_K$ where Z is a vector field on M and $K \in \Omega^1(M; TM)$. Then

$$\langle i_X; \imath(D_\alpha^{\mathbf{L}})\omega_{\mathbf{L}} \rangle = -\langle i_X, \mathscr{L}_Z; \omega_{\mathbf{L}} \rangle = -i_X(\mathbf{L}^{-1}Z).$$

On the other hand, $\langle i_X; d^G\alpha \rangle = i_X(\alpha)$, and then $Z = -\mathbf{L}(\alpha)$. In order to determine K, we have

$$\begin{aligned} \langle \mathscr{L}_X; \imath(D_\alpha^{\mathbf{L}})\omega_{\mathbf{L}} \rangle &= +\langle \mathscr{L}_X, \mathscr{L}_{\mathbf{L}(\alpha)}; \omega_{\mathbf{L}} \rangle - \langle \mathscr{L}_X, i_K; \omega_{\mathbf{L}} \rangle \\ &= \mathscr{L}_X\alpha - \mathscr{L}_{\mathbf{L}(\alpha)}(\mathbf{L}^{-1}X) - \mathbf{L}^{-1}[X, \mathbf{L}(\alpha)] + i_K(\mathbf{L}^{-1}X) \\ &= \mathscr{L}_X\alpha - ((\mathscr{L}_{\mathbf{L}(\alpha)}\mathbf{L}^{-1}) \circ \mathbf{L})(\mathbf{L}^{-1}X) + i_K(\mathbf{L}^{-1}X). \end{aligned}$$

But $\langle \mathscr{L}_X; d^G\alpha \rangle = \mathscr{L}_X\alpha$. So, $K = (\mathscr{L}_{\mathbf{L}(\alpha)}\mathbf{L}^{-1}) \circ \mathbf{L} \in \Omega^1(M; TM)$. Then the graded Poisson bracket associated to $\omega_{\mathbf{L}} = d^G\lambda_{\mathbf{L}}$ satisfies

$$[\![f, g]\!]_{\omega_{\mathbf{L}}} = 0, \qquad [\![\alpha, g]\!]_{\omega_{\mathbf{L}}} = -dg(\mathbf{L}(\alpha)),$$

$$[\![f, \alpha]\!]_{\omega_{\mathbf{L}}} = \alpha(\mathbf{L}^*(df)), \qquad [\![\alpha, \beta]\!]_{\omega_{\mathbf{L}}} = -\mathbf{L}^{-1}[\mathbf{L}\alpha, \mathbf{L}\beta],$$

for $f, g \in C^\infty(M)$ and $\alpha, \beta \in \Omega^1(M)$. Comparing those expressions with formulae (*) in section 1, we have $[\![\ ,\ ]\!]_{\omega_{\mathbf{L}}} = [\![\ ,\ ]\!]_{\mathbf{L}}$.

4. Differential graded symplectic forms

In this section we are going to translate the condition that a nondegenerate graded Poisson bracket must satisfy in order to be differential into a condition on its associated graded symplectic form. From now on we assume that the dimension of M is $2n$.

DEFINITION 4.1. A graded symplectic form on $\Omega(M)$ *is differential if the exterior derivative is a Hamiltonian graded vector field on* $\Omega(M)$ *with respect to this graded symplectic form.*

LEMMA 4.2. *A graded symplectic form,* ω, *is differential if and only if* $\mathcal{L}_d^G \omega = 0$.

Proof. If ω is a differential graded symplectic form, then $\mathcal{L}_d^G\omega = d^G\iota(d)\omega = 0$. Conversely, if $\mathcal{L}_d^G\omega = d^G\iota(d)\omega = 0$, then d is a locally Hamiltonian graded vector field of degree 1. Therefore by the remark following definition 3.2, d is globally Hamiltonian. □

PROPOSITION 4.3. *A graded symplectic form of degree* $k \in \mathbb{Z}_2$ (resp. $\mathbb{Z}$) *is differential if and only if the associated graded Poisson bracket is differential.*

Proof. Let ω be a graded symplectic form of degree $k \in \mathbb{Z}_2$ (resp. $\mathbb{Z}$) and let $[\![\ ,\ ]\!]_\omega$ be its associated graded Poisson bracket. This graded Poisson bracket is differential if and only if

$$-d\langle D_\alpha, D_\beta; \omega\rangle + \langle D_{d\alpha}, D_\beta; \omega\rangle + (-1)^{|\alpha|+k}\langle D_\alpha, D_{d\beta}; \omega\rangle = 0$$

or equivalently,

$$d(D_\alpha(\beta)) - D_{d\alpha}(\beta) - (-1)^{|\alpha|+k}D_\alpha(d\beta) = 0,$$

for any $\alpha, \beta \in \Omega(M)$.

Therefore, the graded Poisson bracket is differential if and only if $[d, D_\alpha] = D_{d\alpha}$. Now, by the definition of the Lie operator,

$$(-1)^{|\alpha|+|\beta|}\langle D_\alpha, D_\beta; \mathscr{L}_d^G\omega\rangle = d\langle D_\alpha, D_\beta; \omega\rangle - \langle [d, D_\alpha], D_\beta; \omega\rangle \\ -(-1)^{|\alpha|+k}\langle D_\alpha, [d, D_\beta]; \omega\rangle.$$

If the graded Poisson bracket is differential, then

$$(-1)^{|\alpha|+|\beta|}\langle D_\alpha, D_\beta; \mathscr{L}_d^G\omega\rangle = d\langle D_\alpha, D_\beta; \omega\rangle - \langle D_{d\alpha}, D_\beta; \omega\rangle \\ -(-1)^{|\alpha|+k}\langle D_\alpha, D_{d\beta}; \omega\rangle = 0.$$

That is, d is a Hamiltonian graded vector field for ω.

Conversely, let us suppose that d is a Hamiltonian graded vector field for ω. The graded commutator of two Hamiltonian graded vector fields D_α and D_β is again a Hamiltonian graded vector field, i.e.,

$$[D_\alpha, D_\beta] = D_{[\![\alpha,\beta]\!]} = D_{D_\alpha\beta}.$$

Applying this to d, we have $[d, D_\alpha] = D_{d\alpha}$; in other words, the associated graded Poisson bracket is differential. □

The graded symplectic forms of $\mathbb{Z}$-degree 1 are those that produce graded Poisson brackets of degree -1. Thus, according to Theorem 1.5, we have the following

COROLLARY 4.4. *A symplectic form of $\mathbb{Z}$-degree* 1, $\omega_L = d^G\lambda_L$, *is differential if and only if the nondegenerate tensor field L is a Poisson bivector.*

The proof follows from the previous proposition, Theorem 1.5 and the expression of $[\![\ ,\ ]\!]_{\omega_L}$ computed in section 3.

A consequence of this corollary is that the Koszul-Schouten bracket defined by a nondegenerate bivector field P comes from an odd symplectic exact form $\omega_P = d^G\lambda_P$, where $\mathbf{P}: T^*M \to TM$ is the linear isomorphism associated to the bivector P.

5. Odd symplectic forms under the action of the automorphism group of $\Omega(M)$

The group of automorphisms, *Aut* $\Omega(M)$, of the algebra of differential forms acts by pullback on the space of graded symplectic forms. In this section we first study the space of odd symplectic forms under the action of the subgroup $Aut_0 \Omega(M) \subset Aut\, \Omega(M)$ consisting of those automorphisms that induce the identity on M.

THEOREM 5.1. *Any odd graded symplectic form is of the form* $\psi^*(\omega_{\mathbf{L}})$, *where* $\omega_{\mathbf{L}}$ *is the graded symplectic form of* $\mathbb{Z}$*-degree* 1 *defined by a linear isomorphism* $\mathbf{L} : T^*M \to TM$, *and* ψ *is an automorphism of* $\Omega(M)$ *that induces the identity on* M.

The result follows by an argument similar to that of [Ro] (Theorem 4) applied now to the odd case. We develop it here because we shall refer to it later in the proof of Theorem 5.2.

Proof. Any odd graded symplectic form, ω, can be written as a sum $\omega = \omega_{(1)} + \omega_{(3)} + \dots$, where $\omega_{(1)}$ is of bidegree (2, 1), $\omega_{(3)}$ is of bidegree (2, 3), etc. Now it is easy to see that $\omega_{(1)}$ is again a graded symplectic form; so, by corollary 3.4 there exists a linear isomorphism $\mathbf{L} : T^*M \to TM$ such that $\omega_{(1)} = \omega_{\mathbf{L}}$.

From $d^G\omega = 0$ we have that $d^G\omega_{(3)} = 0$, thus by Corollary 3.1, $\omega_{(3)} = d^G\mu_{(3)}$ with $\mu_{(3)}$ of bidegree (1, 3). Since $\omega_{\mathbf{L}}$ is nondegenerate, then there is a unique graded vector field $Y_{(2)}$ defined by

$$\imath(Y_{(2)})\omega_{\mathbf{L}} = -\mu_{(3)}.$$

Note that $Y_{(2)}$ is a nilpotent derivation and therefore its formal exponential exp (D), is finite and defines an automorphism of $\Omega(M)$. Let us consider the graded symplectic form

$$(\exp(Y_{(2)}))^*(\omega_{\mathbf{L}} + \omega_{(3)} + \cdots),$$

where $(\exp(Y_{(2)}))^*$ is the pull-back defined by the automorphism $\exp(Y_{(2)})$.

An easy computation shows that the part of bidegree (2, 3) of $(\exp(Y_{(2)}))^*(\omega_{\mathbf{L}} + \omega_{(3)} + \cdots)$ is

$$\mathscr{L}^G_{Y_{(2)}}\omega_{\mathbf{L}} + \omega_{(3)} = d^G\imath(Y_{(2)})\omega_{\mathbf{L}} + \omega_{(3)} = -d^G\mu_{(3)} + \omega_{(3)} = 0.$$

So, we have

$$(\exp(Y_{(2)}))^*\omega = \omega_{\mathbf{L}} + \tilde{\omega}_{(5)} + \cdots,$$

where $\tilde{\omega}_{(2k-1)}$ is a graded form of bidegree $(2, 2k-1)$. By applying iteratively the same argument we obtain at the end, an automorphism ψ of $\Omega(M)$, which is the composition of automorphisms of the type $\exp(Y_{(2k)})$, such that $\psi^*\omega$ is exactly $\omega_{\mathbf{L}}$. Just note that with every iteration we raise the $\mathbb{Z}$-degree of the remainder terms, so we finally exceed the dimension of M. □

Now we study the space of differential odd symplectic forms under the action of the subgroup, $Aut_0^d\,\Omega(M) \subset Aut_0\,\Omega(M)$ consisting of automorphisms taken from $Aut_0\,\Omega(M)$ that commute with the exterior derivative.

In section 4 we have characterized those graded symplectic forms of $\mathbb{Z}$-degree 1 that produce differential graded Poisson brackets, i.e., the differential graded symplectic forms of $\mathbb{Z}$-degree 1. They are of the form $\omega_{\mathbf{P}}$, where $\mathbf{P}$ is the isomorphism defined by a nondegenerate Poisson bivector P. On the other hand Theorem 5.1 gives us a characterization of all odd graded symplectic forms. Therefore, it is natural to ask if any differential odd symplectic form is the image of a differential symplectic form of $\mathbb{Z}$-degree 1 by the pullback of an algebra automorphism of $\Omega(M)$ that commutes with the exterior derivative. The rest of this section is devoted to ascertaining the validity of this conjecture.

Our question is prompted by the following two facts: First, if $\omega_{(1)}$ is a differential graded symplectic form of $\mathbb{Z}$-degree 1 and ψ is an automorphism of $\Omega(M)$ that commutes with the exterior derivative, then, $\psi^*(\omega_{(1)})$ is a differential odd symplectic form. This assertion follows from $\imath(d) \circ \psi^* = \psi^* \circ \imath(d)$ and $d^G \circ \psi^* = \psi^* \circ d^G$. And second, as we have said before, any odd symplectic form is the image of a graded symplectic form of $\mathbb{Z}$-degree 1 by the pullback of an automorphism of the algebra $\Omega(M)$, and the graded symplectic form of $\mathbb{Z}$-degree 1 is of the form $\omega_{\mathbf{L}}$ where $\mathbf{L} : T^*M \to TM$ is an isomorphism. This automorphism is not unique; it depends on the choice of a representative of the cohomology class of closed graded differential forms of $\mathbb{Z}$-degree greater than zero. (See for example the choice of $\mu_{(3)}$ in the proof of Theorem 5.1.) So, in some cases it may be possible to choose this representative in such a way that the resulting automorphism commutes with the exterior derivative.

The answer to the question is settled by the following

THEOREM 5.2. *The following two conditions are equivalent:*

(i) *The Betti numbers of M, b_{2k}, vanish for $k = 2, \dots, n$.*

(ii) *Any differential odd symplectic form is the image of a differential graded symplectic form of $\mathbb{Z}$-degree 1 by the pullback of an automorphism of $\Omega(M)$ that induces the identity on M and that commutes with the exterior derivative.*

Proof. (i) ⇒ (ii). We will prove that it is possible to choose suitable representatives of the corresponding graded cohomology classes such that the resulting automorphism of $\Omega(M)$ commutes with the exterior derivative.

Let us recall that the resulting automorphism in Theorem 5.1 is the composition of the exponentials of graded vector fields of different degrees, i.e.,

$$\psi = \exp(Y_{(2n)}) \circ \cdots \circ \exp(Y_{(2)}),$$

where the iteratively defined even vector fields $Y_{(2k)}$ can be written as (see Theorem 2.2)

$$Y_{(2k)} = \mathscr{L}_{Q_{(2k)}} + i_{L_{(2k+1)}}, \quad \text{where} \quad \begin{aligned} &Q_{(2k)} \in \Omega^{2k}(M; TM), \\ &L_{(2k+1)} \in \Omega^{2k+1}(M; TM). \end{aligned}$$

Such automorphisms commute with the exterior derivative if and only if the graded vector fields commute, i.e., if and only if they have no algebraic part. So, the choice of representatives will be made in such a way as to delete the algebraic part of the graded vector fields.

We are going to do this for the first step of the iterative process. For the rest, similar arguments can be applied.

Let us suppose that ω is a differential odd symplectic form that can be written as $\omega = \omega_{\mathbf{P}} + \omega_{(2k-1)} + \ldots$, where $\mathbf{P}: T^*M \to TM$ is a linear isomorphism that comes from a nondegenerate Poisson bivector P, k is a natural number greater than 1, and the dots denote terms of $\mathbb{Z}$-degree greater than $2k-1$. Recall that the graded vector field $Y_{(2k-2)}$ is defined by

$$\imath(Y_{(2k-2)})\omega_{\mathbf{P}} = -\mu_{(2k-1)},$$

where $\mu_{(2k-1)}$ is a graded differential form of bidegree $(1, 2k-1)$ such that $d^G\mu_{(2k-1)} = \omega_{(2k-1)}$. By definition, d is a Hamiltonian graded vector field for ω, so

$$\imath(d)\omega = \imath(d)\omega_{\mathbf{P}} + \imath(d)\omega_{(2k-1)} + \cdots = -d^G\bar{\omega}_P + d^G\bar{\omega}_{(2k-1)} + \ldots,$$

where $\bar{\omega}_P$ is the symplectic form on M associated to P, and $\bar{\omega}_{(2k-1)}$ is a differential form on M of degree $(2k-1)$. Let us show that the commutator $[d, Y_{(2k-2)}]$ is a Hamiltonian graded vector field of $\mathbb{Z}$-degree $(2k-1)$ for $\omega_{\mathbf{P}}$. We have

$$\begin{aligned} \imath([d, Y_{(2k-2)}])\omega_{\mathbf{P}} &= [\mathscr{L}_d^G, \imath(Y_{(2k-2)})]\omega_{\mathbf{P}} = \mathscr{L}_d^G(\imath(Y_{(2k-2)})\omega_{\mathbf{P}}) - \imath(Y_{(2k-2)})(\mathscr{L}_d^G\omega_{\mathbf{P}}) \\ &= -\mathscr{L}_d^G\mu_{(2k-1)} = -\imath(d)(d^G\mu_{(2k-1)}) - d^G\imath(d)\mu_{(2k-1)} \\ &= -d^G(\bar{\omega}_{(2k-1)} + \imath(d)\mu_{(2k-1)}). \end{aligned}$$

Thus, $[d, Y_{(2k-2)}]$ is the Hamiltonian graded vector field associated to $(-\bar{\omega}_{(2k-1)} - \imath(d)\mu_{(2k-1)})$.

On the other hand, the graded symplectic form ω_P produces a nondegenerate Koszul-Schouten bracket; thus, applying proposition 2.6 to $[d, Y_{(2k-2)}]$, there exists $\gamma \in \Omega^{2k}(M)$ such that

$$[d, Y_{(2k-2)}] = D_\gamma = [\![\gamma, \,]\!]_{\omega_P} = \mathscr{L}_{\rho(\gamma)} - i_{\rho(d\gamma)}.$$

Since the derivation $[d, Y_{(2k-2)}]$ commutes with the exterior derivative, it has no algebraic part and this implies that $d\gamma = 0$. The uniqueness of the decomposition of derivations of $\Omega(M)$ (see Theorem 2.2) implies that $Y_{(2k-2)} = \mathscr{L}_{Q_{2k-2}} - i_{\rho(\gamma)}$.

Using $b_{2k} = 0$, we obtain $\gamma = d\theta$, with $\theta \in \Omega^{2k-1}(M)$. Note that $-(\mu_{(2k-1)} + d^G\theta)$ is a graded form of bidegree $(1, 2k-1)$ such that $d^G(-\mu_{(2k-1)} - d^G\theta) = \omega_{(2k-1)}$.

Now the derivation

$$Z_{(2k-2)} = Y_{(2k-2)} - D_\theta = \mathscr{L}_{Q_{(2k-2)}} - i_{\rho(\gamma)} - (-\mathscr{L}_{\rho(\theta)} - i_{\rho(d\theta)}) = \mathscr{L}_{Q_{(2k-2)}+\rho(\theta)},$$

where $D_\theta = [\![\theta, \,]\!]_{\omega_P}$, satisfies $\imath(Z_{(2k-2)})\omega_P = \omega_{(2k-1)}$. So, we have obtained a graded vector field that commutes with the exterior derivative.

Using this fact, $\exp(Z_{(2k-2)})^*(\omega)$ is a differential graded symplectic form that can be written as $\omega_P + \omega_{(2k+1)} + \dots$. This iterative process produces a symplectic form which, at each step, yields ω_P plus a term of $\mathbb{Z}$-degree higher than the previous one. It obviously terminates when the $\mathbb{Z}$-degree exceeds the dimension of the manifold M.

(ii) $\Rightarrow$ (i). Given $k \in \{2, \dots, n\}$, let α be a differential form on M of degree $2k$, such that $d\alpha = 0$. We shall show that α is an exact form.

Let $\lambda_{(2k-1)}$ be the graded form of bidegree $(1, 2k-1)$ defined as follows,

$$\langle i_X; \lambda_{(2k-1)} \rangle = 0, \qquad \langle \mathscr{L}_X; \lambda_{(2k-1)} \rangle = -i_X\alpha.$$

Now consider the graded symplectic form $\omega = \omega_P + d^G\lambda_{(2k-1)}$.

The Hamiltonian graded vector field for ω associated to the differential form on M $(-\bar{\omega}_P + (2k-1)\alpha)$ is the exterior derivative: that is, $\imath(d)\omega = d^G(-\bar{\omega}_P + (2k-1)\alpha)$.

By hypothesis, every differential graded symplectic form, ω, is of the type $\omega = \psi^*(\omega_P)$ where ψ is an automorphism that commutes with the exterior derivative. So, we obtain

$$d^G(-\bar{\omega}_P + (2k-1)\alpha) = \imath(d)\omega = \imath(d)\psi^*\omega_P = \psi^*\imath(d)\omega_P = d^G(\psi(-\bar{\omega}_P)),$$

that is, $\psi(-\bar{\omega}_P) = -\bar{\omega}_P + (2k-1)\alpha$. Note that $\psi(-\bar{\omega}_P) \in \Omega(M)$ is equal to a sum of two terms of degree 2 and $2k$ respectively.

At this point, let us recall a result from [U] that guarantees that any automorphism that commutes with the exterior derivative can be written as,

$$\psi = \exp(\mathscr{L}_{K_{(2n)}}) \circ \cdots \circ \exp(\mathscr{L}_{K_{(2)}}) \circ \phi^*,$$

where ϕ is a diffeomorphism of M and $K_{(2r)} \in \Omega^{2r}(M; TM)$, for $r = 1, \ldots, n$.

We apply this fact to our case and equate in $\psi(-\bar{\omega}_P) = -\bar{\omega}_P + (2k-1)\alpha$ the terms of the same degrees. The only terms that do not vanish are those of degree 2 and $2k$, i.e.,

$$(\psi(-\bar{\omega}_P))_{(2)} = \phi(-\bar{\omega}_P) = -\bar{\omega}_P,$$

$$(\psi(-\bar{\omega}_P))_{(2k)} = (\mathscr{L}_{K_{(2k-2)}})(-\bar{\omega}_P) = -di_{K_{(2k-2)}}\bar{\omega}_P = (2k-1)\alpha.$$

Therefore, $\alpha = -1/(2k-1)\, di_{K_{(2k-2)}}\bar{\omega}_P$ is exact. □

Let us suppose that M is a compact symplectic manifold with dimension greater than 2. It is well known, see e.g. [Po] Theorem 8.8, that M has non zero even Betti numbers. In particular the Betti number b_4 does not vanish. Therefore, condition (ii) in Theorem 5.2 is not satisfied. This means that there exist differential odd symplectic forms on M which are not in the $Aut_0^d\,\Omega(M)$-orbit of a differential graded symplectic form of $\mathbb{Z}$-degree 1.

6. Nondegenerate even Poisson structures

In the previous sections we have studied the class of nondegenerate odd Poisson structures that are differential. The situation is different for nondegenerate even Poisson structures. The same condition, *d is a Poisson derivation*, does not distinguish a class of even Poisson brackets.

PROPOSITION 6.1. *There are no nondegenerate differential even Poisson brackets.*

Proof. We are going to prove that there are no differential even symplectic forms. An even symplectic form can be written as $\omega = \omega_{(0)} + \omega_{(2)} + \ldots$. In particular, it defines a symplectic form on the manifold M, $\tilde{\omega}$, determined by the following expression

$$\tilde{\omega}(X, Y) := \langle \mathscr{L}_X, \mathscr{L}_Y; \omega_{(0)} \rangle = \pi_{(0)}(\langle \mathscr{L}_X, \mathscr{L}_Y; \omega \rangle),$$

where $\pi_{(0)}(\alpha) \in \Omega^0(M)$ denotes the degree 0 component of the differential form $\alpha \in \Omega(M)$, and X and Y are vector fields on M.

If d is a locally Hamiltonian graded vector field for ω, then $\imath(d)\omega = \imath(d)\omega_{(0)} + \imath(d)\omega_{(2)} + \dots$ is a d^G-closed graded 1-form of odd degree, but it is easy to check that $\imath(d)\omega_{(0)}$ is not d^G-closed. Indeed, if X, Y are two vector fields such that $\tilde{\omega}(X, Y) \neq 0$, then

$$\begin{aligned}
&\langle \mathscr{L}_X, i_Y; d^G(\imath(d)\omega_{(0)})\rangle \\
&= \mathscr{L}_X(\langle i_Y; \imath(d)\omega_{(0)}\rangle) - i_Y(\langle \mathscr{L}_X; \imath(d)\omega_{(0)}\rangle) - \langle [\mathscr{L}_X, i_Y]; \imath(d)\omega_{(0)}\rangle \\
&= i_Y(\langle \mathscr{L}_X, d; \omega_{(0)}\rangle) = \tilde{\omega}(X, Y) \neq 0. \qquad \square
\end{aligned}$$

Acknowledgements

We would like to thank the referee for his careful reading of the manuscript, and his most valuable suggestions, and helpful comments which clarified and improved the quality of our exposition.

The first author was partially supported by Fundació Caixa Castelló and the second author was partially supported by the Spanish DGICYT grant *#PB91-0324.*

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Received April 11, 1994; February 13, 1995