# Isospectral deformations on Riemannian manifolds which are diffeomorphic to compact Heisenberg manifolds.

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# Isospectral deformations on Riemannian manifolds which are diffeomorphic to compact Heisenberg manifolds

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*Abstract.* It is known that if $H_m$ is the classical $(2m + 1)$-dimensional Heisenberg group, $\Gamma$ a cocompact discrete subgroup of $H_m$ and $g$ a left invariant metric, then $(\Gamma\backslash H_m, g)$ is infinitesimally spectrally rigid within the family of left invariant metrics. The purpose of this paper is to show that for every $m \geq 2$ and for a certain choice of $\Gamma$ and $g$, there is a deformation $(\Gamma\backslash H_m, g_\alpha)$ with $g = g_1$ such that for every $\alpha \neq 1$, $(\Gamma\backslash H_m, g_\alpha)$ *does* admit a nontrivial isospectral deformation. For $\alpha \neq 1$ the metrics $g_\alpha$ will not be $H_m$-left invariant, and the $(\Gamma\backslash H_m, g_\alpha)$ will not be nilmanifolds, but still solvmanifolds.

## Introduction

Let $H_m$ be the classical $(2m + 1)$-dimensional Heisenberg group, $\Gamma$ a cocompact discrete subgroup of it, and $g^t$ a continuous family of left invariant metrics. The $g^t$ descend to metrics $g^t$ on $\Gamma\backslash H_m$ which we call left invariant again. It was shown by Ouyang and Pesce [OP] in 1991 that if all the manifolds occurring during the deformations $(\Gamma\backslash H_m, g^t)$ are isospectral, i.e., have the same Laplace spectra, then this deformation must be trivial. In other words, $(\Gamma\backslash H_m, g)$ is infinitesimally spectrally rigid within the family of left invariant metrics.

What Ouyang and Pesce actually showed was that the *only* way of constructing isospectral deformation on 2-step nilmanifolds is the so-called method of almost inner automorphisms which was developed by Gordon, Wilson, and DeTurck ([GW2], [DG], [Go]) and which can be applied in higher-step nilpotent and solvable Lie groups, too. But the only almost inner automorphisms of $H_m$ are inner and hence give rise to trivial deformations.

In the present paper we will use the method of almost inner automorphisms to show:

For every $m \geq 2$ there is a cocompact discrete subgroup $\Gamma$ of $H_m$ and a continuous 2-parameter family of metrics $g_\alpha^t$ $(0 < \alpha \leq 1, t \in \mathbb{R})$ on $H_m$ such that for every fixed $\alpha$, the deformation $(\Gamma\backslash H_m, g_\alpha^t)$ with varying $t$ is isospectral; for $\alpha = 1$ the $g_1^t$ are $H_m$-left invariant, and this deformation is trivial. But for every $0 < \alpha < 1$, the corresponding isospectral deformation is nontrivial. For fixed $t$ and varying $\alpha$ the deformation $(\Gamma\backslash H_m, g_\alpha^t)$ is *not* isospectral. The $g_\alpha^t$ with $\alpha < 1$ are not $H_m$-left invariant, but left invariant with respect to a different group structure.

The key of the construction is introducing this different group structure on the manifold $H_m$. It will not nilpotent but still solvable (however, not exponentially solvable). The new group will be called $G_m$, and it will turn out that for a certain choice of left invariant metrics $g$ resp. $h$ on $G_m$ resp. $H_m$ the manifolds $(G_m, g)$ and $(H_m, h)$ are isometric; for a certain pair of cocompact discrete subgroups $\Gamma$ resp. $\tilde{\Gamma}$, the isometry will even descend to the compact quotients. While $H_m$ does not admit any almost inner, non-inner automorphisms, $G_m$ admits a 1-parameter family $\Phi_t$ of such automorphisms. For every $G_m$-left invariant metric $g$ the family $(\Gamma\backslash G_m, \Phi_t^* g)$ is isospectral. If we start with $g = g_1$ (the metric which is isometric to the $H_m$-left invariant metric $h$) then $\Phi_t^* g$ is a trivial deformation. But if we first deform $g_1$ into a slightly different metric $g_\alpha$ (which is no longer $H_m$-left invariant) then $(\Gamma\backslash G_m, \Phi_t^* g_\alpha)$ with fixed $\alpha$ and varying $t$ will turn out to be a nontrivial isospectral deformation.

We will show the nontriviality of these deformations by geometrical observations.

There are certain isometry invariants which can certainly *not* be used for this because isospectral manifolds share them with each other. For example, isospectral manifolds always have the same dimension, volume, and total scalar curvature (see e.g. [BGM]). Isospectral manifolds which are constructed using the method of almost inner automorphisms are always locally isometric (note that $(\Gamma\backslash G, \Phi^* g)$ is isometric to $(\Phi(\Gamma)\backslash G, g)$), and moreover, as was shown by Carolyn Gordon in [Go], they always have the same length spectrum, even the same marked length spectrum.

DeTurck, Gordon, Gluck, and Webb ([DGGW1–DGGW6]) developed the following approach for detecting the change in geometry during an isospectral deformation $(\Gamma\backslash G, g^t)$: Try to find two 1-dimensional integral homology classes whose mass minimizing (with respect to $g^t$) classical cycles foliate two submanifolds $M_1(t)$, $M_2(t)$ of $\Gamma\backslash G$ for which $d(t) := \mathrm{dist}_{g^t}(M_1(t), M_2(t))$ is continuous and nonconstant. This implies the nontriviality of the deformation because the set of all integral homology classes is countable. (The mentioned authors generalized this approach to the case of more than two such submanifolds with nonconstant "multidistance", see e.g. [DGGW5]; another related approach developed by these authors is looking for a higher-dimensional integral homology class whose mass-minimizing classical cycles have nonconstant mass, see e.g. [DGGW1], or looking for a codimension one integral homology class for which the geometry inside the fibers changes during the deformation, see e.g. [DGGW4, 6].)

The method of calibrations is used there extensively to find the mass minimizing cycles in the homology classes under consideration. Details and many examples (always with nilpotent $G$) can be found in the mentioned papers.

Although this beautiful argument works in many examples, it has certain disadvantages:

Firstly, a certain isospectral deformation on a 5-dimensional solvmanifold $\Gamma\backslash G$ (where $G$ is not nilpotent, but exponentially solvable) which was – just as the above family of deformations $(\Gamma\backslash G_m, g_\alpha^t)$ – constructed in the author's thesis [Sch] (see also the Appendix below) shows that the homology of $\Gamma\backslash G$ can happen to be too poor for this kind of argument; in the mentioned example, one can find two free homotopy classes whose $g^t$-shortest closed geodesics are homologous to zero (and therefore in particular not detectable by calibrations), but foliate again two submanifolds $M_1(t)$, $M_2(t)$ with nonconstant $g^t$-distance as desired, while such a phenomenon cannot be obtained in that example when homology classes are used instead of homotopy classes.

Secondly, the above method does not automatically supply an isometry invariant for the deformation because even for different $d(t)$ and $d(t')$, there could be an isometry between $g^t$ and $g^{t'}$ which does not keep the two chosen homology (or free homotopy) classes invariant. Therefore, we will use a related approach which is, however, conceptually different:

For each of our isospectral deformations $(\Gamma\backslash G_m, g_\alpha^t)$ with fixed $0 < \alpha < 1$, we will construct two submanifolds $M_1(t)$, $M_2(t)$ of $\Gamma\backslash G_m$, depending on the metric at time $t$, by purely geometrical steps – without making choices as in the above approach. The varying distance $d(t)$ between $M_1(t)$ and $M_2(t)$ will then indeed be an isometry invariant, i.e., $d(t) \neq d(t')$ will imply that $(\Gamma\backslash G_m, g_\alpha^t)$ is not isometric to $(\Gamma\backslash G_m, g_\alpha^{t'})$. Thereby we will be able to determine the precise parameter of the deformations.

One of the steps that will be used in the construction is determining the Riemannian submersion whose image is the set of orbits of the flow of all Killing vectorfields in the manifold $(\Gamma\backslash G_m, g_\alpha^t)$. After doing this, we will split up the quotient manifold obtained in this way into an orthogonal product where one of the factors is determined geometrically as the integral manifold of the distribution consisting of the 0-eigenspaces for the Ricci tensor; this factor will be a flat torus. Next, we determine all closed geodesics of a certain length tangent to this torus. $M_1(t)$ will then be defined as the union of those horizontal lifts of these geodesics which stay closed with the same length in $(\Gamma\backslash G_m, g_\alpha^t)$. $M_2(t)$ will be defined in a similar way by purely geometrical steps, namely as the preimage in $(\Gamma\backslash G_m, g_\alpha^t)$ of the globally shortest closed geodesics in the Riemannian manifold which is obtained by first dividing the Riemannian manifold $(\Gamma\backslash G_m, g_\alpha^t)$ by the flow of its Killing fields and then splitting off the mentioned flat torus.

The shortest geodesics just mentioned *are* detectable by calibrations, but the calibrations will be given by 1-forms which are *not* left invariant. This contrasts with the situation in [DGGW1–DGGW6] where the calibrations were always given by left invariant forms. –

The isospectral deformations constructed in this paper are interesting not only because they contrast with the infinitesimal spectral rigidity of the Heisenberg manifolds but also because they are the first examples of isospectral deformations of solvmanifolds for which the underlying Lie group is not exponentially solvable, i.e., the exponential map from the Lie algebra to the Lie group is not bijective.

Even in the exponentially solvable case, there are very few examples until now. A certain 9-dimensional example with an exponentially solvable, non-nilpotent Lie group was given in [GW2], but that example can be reduced to a nilpotent example; in fact, all the algebraic and geometrical aspects that are important for the isospectral deformation are related there only to the nilradical of the underlying group. In order to fill this gap in the field of solvable examples a little more, we will at the end of this paper very briefly discuss the 5-dimensional exponentially solvable example which we already mentioned above. –

The paper is organized as follows:

In §1 we shortly recall the basic definitions, the method of almost inner automorphisms and the rigidity theorem of Ouyang and Pesce.

In §2 we formulate the main theorem and give the explicit construction of the 2-parameter family $(\Gamma\backslash G_m, g_\alpha^t)$.

§3 contains the geometrical arguments described above which will show the nontriviality of the deformations for fixed $\alpha < 1$ and varying $t$.

The Appendix shortly describes the above-mentioned 5-dimensional exponentially solvable example, in particular the aspects in which it is similar to resp. different from the deformations $(\Gamma\backslash G_m, g_\alpha^t)$.

## §1 Preliminaries

Let $(M, g)$ be a Riemannian manifold, and let $\Delta_g$ be the Laplacian acting on functions by

$$(\Delta_g f)(p) := -\sum_{i=1}^{n} \frac{d^2}{dt^2}\bigg|_{t=0} f(c_i(t)) \qquad \text{for } p \in M,$$

where the $c_i$ are geodesics starting in $p$ such that $\{\dot{c}_1(0), \ldots, \dot{c}_n(0)\}$ is an orthonormal basis for $T_pM$. The discrete sequence

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$$

of the eigenvalues of $\Delta_g$ is called the *spectrum* of $(M, g)$. Two Riemannian manifolds are said to be *isospectral* if their spectra are equal.

An importnat instrument for the construction of isospectral manifolds are the almost inner automorphisms which are defined as follows: Let $G$ be a Lie group and

$\Gamma$ a subgroup of $G$. Then

$$\text{AIA}(G;\Gamma) := \{\Phi \in \text{Aut}(G) \mid \forall \gamma \in \Gamma\ \exists a \in G : \Phi(\gamma) = a\gamma a^{-1}\}$$

is called the group of *$\Gamma$-almost inner automorphisms* of $G$.

1.1 THEOREM (DeTurck and Gordon [DG]). *Let $G$ be a Lie group, $\Gamma$ a cocompact discrete subgroup of $G$, and $g$ a left invariant metric on $G$. Let $\Phi \in \text{AIA}(G;\Gamma)$ have the property*

$$(\text{Det})\quad \begin{array}{l}\textit{Whenever } \gamma \in \Gamma \textit{ and } a \in G \textit{ satisfy } \Phi(\gamma) = a\gamma a^{-1}\\ \textit{then } \det((I_a^{-1} \circ \Phi)_{*e \mid T_eG_\gamma}) = 1,\end{array}$$

*where $I_a$ denotes conjugation by $a$ and $G_\gamma$ is the centralizer of $\gamma$ in $G$. Then $(\Gamma\backslash G, g)$ and $(\Gamma\backslash G, \Phi^*g)$ are isospectral.*

Here, $g$ and $\Phi^*g$ denote the induced metrics, too. By abuse of notation, we will call metrics on quotients of the form $\Gamma\backslash G$ left invariant if they are induced by left invariant metrics on $G$.

Theorem 1.1 is a special case of [DG, Theorem 1.16]; it is not stated separately there but can be extracted from the proof of [DG, Theorem 2.7].

The isospectral manifolds in Theorem 1.1 are not only isospectral for the Laplacian on functions but also on $p$-forms (see [DG]).

Condition (Det) is automatically satisfied if $G$ is nilpotent, because in this case $\Phi_{*e}$ is always unipotent if $\Phi$ is almost inner.

Using Theorem 1.1 (or slightly modified formulations), Gordon, Wilson, DeTurck et al. constructed many examples of isospectral deformations, i.e., *continuous* isospectral families. Good references are [GW2], [Go], [DG], [DGGW1]–[DGGW6].

In the converse direction, there is the following classification theorem for continuous isospectral families in the 2-step nilpotent case:

1.2 THEOREM (Ouyang and Pesce [OP]). *Let $G$ be a simply connected 2-step nilpotent Lie group, $\Gamma$ a cocompact discrete subgroup of $G$ and $g^t$ a continuous family of left invariant metrics on $G$ such that all $(\Gamma\backslash G, g^t)$ are isospectral. Then there exists a continuous family $\Phi_t \in \text{AIA}(G;\Gamma)$ with $\Phi_0 = \text{id}$ and $g^t = \Phi_t^*(g^0)$.*

This means that Theorem 1.1 is the *only* way of constructing isospectral deformations of the form $(\Gamma\backslash G, g^t)$ with all $g^t$ left invariant and $G$ 2-step nilpotent and simply connected.

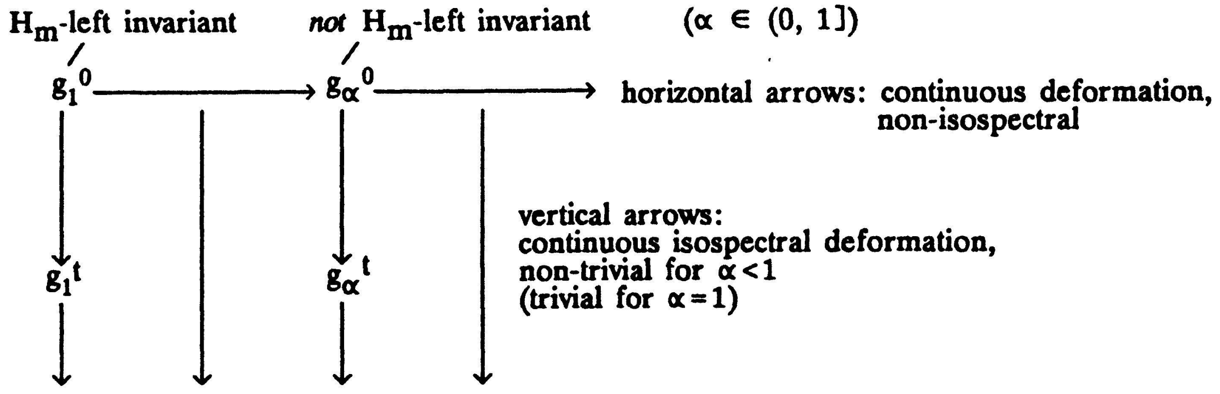
Denote by $\mathfrak{h}_m$ the classical $(2m+1)$-dimensional 2-step nilpotent *Heisenberg algebra* spanned by $\{X_1, X_1, \ldots, X_m, Y_m, Z\}$ whose nontrivial Lie brackets are given by $[X_i, Y_i] = Z \; \forall i = 1, \ldots, m$. Let $H_m$ be the corresponding simply connected Lie group. For any cocompact discrete subgroup of $H_m$, all $\Phi \in \mathrm{AIA}(G; \Gamma)$ are inner automorphisms. Since inner automorphisms give rise to trivial deformations, Theorem 1.2 in particular implies:

1.3 COROLLARY ([OP]). *For every cocompact discrete subgroup $\Gamma$ of $H_m$ and every left invariant metric g on $H_m$, $(\Gamma \backslash H_m, g)$ is infinitesimally spectrally rigid within the family of left invariant metrics.*

## §2 Construction of isospectral deformations on manifolds which are arbitrarily close to Riemannian Heisenberg manifolds

In contrast to Corollary 1.3 we will show:

2.1 THEOREM. *For every $m \geq 2$ there is a cocompact discrete subgroup $\Gamma$ of $H_m$ and a continuous 2-parameter family of metrics $g_\alpha^t$ $(0 < \alpha \leq 1, t \in \mathbb{R})$* on $\Gamma \backslash H_m$ *with the following properties: The $g_1^t$ are $H_m$-left invariant and pairwise isometric. For every fixed $0 < \alpha < 1$, the deformation $g_\alpha^t$ is isospectral and nontrivial. The $g_\alpha^t$ with $0 < \alpha < 1$ are not $H_m$-left invariant, but left invariant with respect to a certain solvable group structure on the manifold $H_m$.*




2.2 DEFINITION. Let $\mathfrak{g}_m$ be the solvable Lie algebra spanned by $\{X_1, Y_1, \ldots, X_m, Y_m, Z\}$ whose nontrivial Lie brackets are given by $[X_1, X_2] = Y_2$, $[X_1, Y_2] = -X_2$, $[X_i, Y_i] = Z \; \forall i = 1, \ldots, m$.

Let $G_m$ be the simply connected Lie group with Lie algebra $\mathfrak{g}_m$. Let $H_m$ be the Heisenberg group defined in §1. Define ideals $\mathfrak{g}'_m$ resp. $\mathfrak{h}'_m$ as span$\{Y_1, X_2, Y_2, \ldots, Z\}$ in $\mathfrak{g}_m$ resp. $\mathfrak{h}_m$. Let $G'_m := \exp \mathfrak{g}'_m$, $H'_m := \exp \mathfrak{h}'_m$, and denote $\exp(y_1 Y_1 +$

$x_2X_2+y_2Y_2+\cdots+x_mX_m+y_mY_m+zZ)$ by $(y_1, x_2, y_2, \ldots, x_m, y_m, z)$ in $G'_m$ resp. $H'_m$. Let the map $F: G_m \to H_m$ be defined by

$$F\colon G_m \ni (y_1, x_2, y_2, \ldots, z)\cdot \exp x_1X_1 \mapsto (y_1, x_2, y_2, \ldots, z)\cdot \exp x_1X_1 \in H_m.$$

Define left invariant metrics $g$ resp. $h$ on $G_m$ resp. $H_m$ by letting $\{X_1, Y_1, \ldots, X_m, Y_m, Z\}$ be an orthonormal basis of $\mathfrak{g}_m$ resp. $\mathfrak{h}_m$.

$G_m$ is a solvable but not exponentially solvable Lie group because ad $X_1$ has purely imaginary eigenvalues. $G_m$ and $H_m$ can be viewed as fiber bundles over $\mathbb{R} = \exp \mathbb{R}X_1$ where the fiber is the nilpotent normal subgroup $\exp(\mathrm{span}\{Y_1, X_2, Y_2, \ldots, Z\})$. The map $F$ identifies the fiber over $x_1 \in \mathbb{R}$ in $G_m$ with the corresponding fiber in $H_m$, but this in a way which depends on $x_1$.

2.3 LEMMA. *$F$ is an isometry between $(G_m, g)$ and $(H_m, h)$.*

*Proof.* Note that

$$I_{\exp x_1X_1}\left(y_1, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, x_3, \ldots, z\right) = \left(y_1, D^{x_1}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, x_3, \ldots, z + x_1y_1\right)$$

in $G_m$, but

$$I_{\exp x_1X_1}\left(y_1, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, x_3, \ldots, z\right) = \left(y_1, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, x_3, \ldots, z + x_1y_1\right)$$

in $H_m$ (where $D^{x_1}$ denotes rotation by the angle $x_1$ in $\mathbb{R}^2$). Let $p = (y_1, x_2, y_2, \ldots, z)\cdot \exp x_1X_1 \in G_m$. Then we have

$F_{*p}(X_1) = X_1|_{F(p)}$ immediately from the definition of $F$,

$$\begin{aligned} F_{*p}(Y_1) &= \frac{d}{dt}\Big|_{t=0} F((y_1, \ldots, z)\cdot \exp x_1X_1 \cdot \exp tY_1) \\ &= \frac{d}{dt}\Big|_{t=0} (y_1 + t, x_2, \ldots, y_m, z + tx_1)\cdot \exp x_1X_1 \\ &= \frac{d}{dt}\Big|_{t=0} (y_1, \ldots, z)\cdot \exp x_1X_1 \cdot \exp tY_1 = Y_1|_{F(p)}, \end{aligned}$$

$$F_{*p}(X_2) = \frac{d}{dt}\Big|_{t=0} F((y_1, \ldots, z) \cdot \exp x_1 X_1 \cdot \exp tX_2)$$
$$= \frac{d}{dt}\Big|_{t=0} F((y_1, \ldots, z) \cdot \exp t(\cos(x_1)X_2 + \sin(x_1)Y_2) \cdot \exp x_1 X_1)$$
$$= \frac{d}{dt}\Big|_{t=0} (y_1, \ldots, z) \cdot \exp x_1 X_1 \cdot \exp t(\cos(x_1)X_2 + \sin(x_1)Y_2)$$
$$= \cos(x_1)X_2 \,|_{F(p)} + \sin(x_1)Y_2|_{F(p)},$$
$$F_{*p}(Y_2) = -\sin(x_1)X_2|_{F(p)} + \cos(x_1)Y_2|_{F(p)}$$

analogously, and for $W \in \{X_3, Y_3, \ldots, Z\}$:

$$F_{*p}(W) = W|_{F(p)}$$

since $W$ commutes with $X_1$ in both $\mathfrak{g}_m$ and $\mathfrak{h}_m$. Thus $F$ maps an orthogonal basis of $T_pG_m$ to an orthonormal basis of $T_{F(p)}H_m$. □

Next we construct cocompact discrete subgroups of $G_m$ and $H_m$ such that the isometry $F$ even descends to the compact quotients:

2.4 DEFINITION. Define $\Gamma'$ resp. $\tilde{\Gamma}'$ as $\{(b_1/2\pi, a_2, b_2, \ldots, a_m, b_m, c/2) \mid a_i, b_i, c \in \mathbb{Z}\}$ in $G'_m$ resp. $H'_m$, and $\Gamma$ resp. $\tilde{\Gamma}$ as the subgroup generated by the union of $\{\exp(2k\pi X_1) \mid k \in \mathbb{Z}\}$ with $\Gamma'$ resp. $\tilde{\Gamma}'$ in $G_m$ resp. $H_m$.

2.5 LEMMA.
(i) *$\Gamma$ and $\tilde{\Gamma}$ are discrete and cocompact in* $G_m$ resp. $H_m$.
(ii) $F(\Gamma) = \tilde{\Gamma}$.
(iii) $F(\gamma \cdot p) = F(\gamma) \cdot F(p)\ \forall \gamma \in \Gamma\ \forall p \in G_m$.

*Proof.* (i) $\Gamma'$ is a subgroup of the 2-step nilpotent subgroup $G'_m$ of $G_m$ (by the Campbell-Baker-Hausdorff formula for 2-step nilpotent groups: $\exp(U) \cdot \exp(V) = \exp(U + V + \frac{1}{2}[U, V])$). $\Gamma'$ is discrete by definition. $G'_m$ is normal in $G_m$ and hence $\Gamma$ is the union of all sets $(\Gamma \cap G'_m) \cdot \exp(2k\pi X_1)$, $k \in \mathbb{Z}$. Thus we have to show that $\Gamma \cap G'_m$ is discrete. But

$$\Gamma \cap G'_m = \bigcup_{k \in \mathbb{Z}} I_{\exp(2k\pi X_1)}(\Gamma') = \Gamma' \qquad \text{because of}$$

$$I_{\exp(2k\pi X_1)}\left(b_1/2\pi, \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}, a_3, \ldots, c/2\right) = \left(b_1/2\pi, D^{2k\pi}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}, a_3, \ldots, c/2 + b_1\right) \in \Gamma'$$

(note that $D^{2k\pi} = \mathrm{id}$).

These arguments are true in $H_m$ analogously, except that $D^{2k\pi}$ does not even come in here. One can easily check that the domains

$$\bigcup_{0\le x_1<2\pi} \{(y_1, x_2, y_2, \dots, z) \mid 0 \le y_1 < 1/2\pi, 0 \le x_i, y_i < 1, \\ 0 \le z < 1/2\} \cdot \exp x_1 X_1$$

are fundamental domains for $\Gamma$ resp. $\tilde{\Gamma}$ which are therefore cocompact.

(ii) By the proof of (i), $\Gamma$ and $\tilde{\Gamma}$ consist precisely of the elements $(b_1/2\pi, a_2, b_2, \dots, a_m, b_m, c/2) \cdot \exp(2k\pi X_1)$ with $a_i$, $b_i$, $c$, $k \in \mathbb{Z}$, in $G_m$ resp. $H_m$. Thus $F(\Gamma) = \tilde{\Gamma}$ can be read off from the definition of $F$.

(iii) Let $\gamma = (b_1/2\pi, a_2, b_2, \dots, c/2) \cdot \exp(2k\pi X_1) \in \Gamma, p' = (y_1, x_2, y_2, \dots, z) \in G'_m$, $p = p' \cdot \exp x_1 X_1 \in G_m$. Then

$$\begin{aligned} F(\gamma \cdot p) &= F((b_1/2\pi, a_2, b_2, \dots, c/2) \cdot I_{\exp(2k\pi X_1)}(p') \cdot \exp(x_1 + 2k\pi)X_1) \\ &= F((b_1/2\pi, a_2, b_2, \dots, c/2) \cdot (y_1, x_2, y_2, \dots, z + 2k\pi y_1) \cdot \\ &\quad \cdot \exp(x_1 + 2k\pi)X_1) \\ &\quad \text{(note again } D^{2k\pi} = \mathrm{id}) \\ &= (b_1/2\pi, a_2, b_2, \dots, c/2) \cdot (y_1, x_2, y_2, \dots, z + 2k\pi y_1) \cdot \exp(x_1 + 2k\pi)X_1 \\ &\quad \text{(because the multiplication laws in } G'_m \text{ and } H'_m \text{ are the same)} \\ &= (b_1/2\pi, a_2, b_2, \dots, c/2) \cdot \exp(2k\pi X_1) \cdot (y_1, x_2, y_2, \dots, z) \cdot \exp x_1 X_1 \\ &= F(\gamma) \cdot F(p) \end{aligned}$$

□

2.6 COROLLARY. *$\Gamma$ is isomorphic to $\tilde{\Gamma}$ and thus 2-step nilpotent, although $\Gamma$ is cocompact in the solvable and non-nilpotent group $G_m$. $F$ induces an isometry, which we again call $F$, between the compact Riemannian manifolds $(\Gamma \backslash G_m, g)$ and $(\tilde{\Gamma} \backslash H_m, h)$.*

Thus we have realized the same Riemannian manifold as a nilmanifold $(\tilde{\Gamma} \backslash H_m, h)$ as well as a solvmanifold $(\Gamma \backslash G_m, g)$. The motivation for doing this is the fact that $G_m$ admits nontrivial (i.e., non-inner) almost inner automorphisms, which $H_m$ does not.

2.7 DEFINITION. Define an automorphism $\Phi_t$ of $G_m$ by letting $(\Phi_t)_{*e}$: $Y_1 \mapsto Y_1 + tZ$, $W \mapsto W$ for $W \in \{X_1, X_2, Y_2, \dots, X_m, Y_m, Z\}$.

2.8 LEMMA.

(i) *For every* $t \in \mathbb{R}$ *we have* $\Phi_t \in \mathrm{AIA}(G_m; \Gamma)$; *more strongly, even* $\Phi_t \in \mathrm{AIA}(G_m; G_m)$.

(ii) *For every* $t \in \mathbb{R}$, $\Phi_t$ *satifies condition* (Det) *from Theorem* 1.1 (*even for all* $p \in G_m$ *instead of all* $\gamma \in \Gamma$).

(iii) *For* $t \notin 2\pi\mathbb{Z}$, $\Phi_t$ *is not an inner automorphism of* $G_m$.

*Proof.* (i) and (ii):
Let $p = (y_1, x_2, y_2, \ldots, z) \cdot \exp x_1 X_1 \in G_m$.

*Case* 1. $x_2 = y_2 = 0$. Then $\Phi_t(p) = I_a(p)$ with $a := \exp tX_1$, and $T_e G_p$ is a certain subspace of $\mathfrak{g}_m$ which is either contained in span $\{X_1, Y_1, X_3, Y_3, \ldots, Z\}$ (if $x_1 \notin 2\pi\mathbb{Z}$) and fix under $(I_a^{-1} \circ \Phi_t)_{*e}$, or contains span $\{X_2, Y_2\}$ (if $x_1 \in 2\pi\mathbb{Z}$) on which $(I_a^{-1} \circ \Phi_t)_{*e}$ acts as a rotation while it fixes a complement of span $\{X_2, Y_2\}$ in $T_e G_p$. In either case, $(I_a^{-1} \circ \Phi_t)_{*e}$ has determinant 1 on $T_e G_p$.

*Case* 2. $(x_2 \neq 0 \vee y_2 \neq 0) \wedge x_1 = 0$. If $x_2 \neq 0$ and $x_1 = 0$, then $\Phi_t(p) = I_a(p)$ with $a := \exp(-ty_1/x_2 \cdot Y_2)$, and $T_e G_p$ is a certain subspace of $\mathfrak{g}'_m$ (!) on which $(I_a^{-1} \circ \Phi_t)_{*e}$ acts as the restriction of the unipotent endomorphism of $\mathfrak{g}'_m$ defined by $Y_1 \mapsto Y_1 + tZ$, $X_2 \mapsto X_2 - ty_1/x_2 \cdot Z$, $W \mapsto W$ for $W \in \{Y_2, \ldots, Z\}$. Thus $(I_a^{-1} \circ \Phi_t)_{*e}$ has determinant 1 on $T_e g_p$. The case $y_2 \neq 0$ and $x_1 = 0$ is similar.

*Case* 3. $(x_2 \neq 0 \vee y_2 \neq 0) \wedge x_1 \neq 0$. Then $\Phi_t(p) = I_a(p)$ with $a := \exp(-ty_1/x_1 \cdot Y_1)$, and $T_e G_p$ is a certain subspace of $\mathfrak{g}'_m$ on which $\mathrm{Ad}_a$ acts as the identity and $(\Phi_t)_{*e}$ as the restriction of the unipotent endomorphism of $\mathfrak{g}'_m$ defined by $Y_1 \mapsto Y_1 + tZ$, $W \mapsto W$ for $W \in \{X_2, Y_2, \ldots, Z\}$. Again, $(I_a^{-1} \circ \Phi_t)_{*e}$ has determinant 1 on $T_e G_p$.

(iii) Any inner automorphism $\Psi$ of $G_m$ satifies: If $\Psi_*$ fixes span $\{X_2, Y_2\}$ then $\Psi_*(Y_1) \in Y_1 + 2\pi\mathbb{Z} \cdot Z$. The behaviour of $\Phi_t$ for $t \notin 2\pi\mathbb{Z}$ violates this rule. □

2.9 COROLLARY. *If* $g^0$ *is any left invariant metric on* $G_m$ *then by Theorem* 1.1 *and Lemma* 2.8(i)&(ii), $(\Gamma \backslash G_m, g^t)$ *with* $g^t = \Phi_t^* g^0$ *is an isospectral deformation.*

2.10 DEFINITION. For $0 < \alpha \leq 1$ define left invariant metrics $g_\alpha$ on $G_m$ by letting $\{X_1, Y_1, X_2, \alpha Y_2, X_3, Y_3, \ldots, Z\}$ be an orthonormal basis of $\mathfrak{g}_m$, and define $g_\alpha^t := \Phi_t^* g_\alpha$.

*Remarks.* For fixed $\alpha \in (0, 1]$ and varying $t$, $(\Gamma \backslash G_m, g_\alpha^t)$ is an isospectral deformation by Corollary 2.9. If $0 < \alpha < 1$ then this deformation is nontrivial, as we will show in §3.

$g_1^0$ is just our initial metric $g$ for which $(\Gamma\backslash G_m, g)$ is isometric to $(\tilde{\Gamma}\backslash H_m, h)$ with the $H_m$-left invariant metric $h$ (Corollary 2.6). The deformation $g_1^t$ is trivial since $(I_{\exp(-tX_1)} \circ \Phi_t)_{*e}$ acts as a rotation on span $\{X_2, Y_2\}$ which is an isometry for the metric $g_1^0$ (but not for $g_\alpha^0$), and thus $I_{\exp(-tX_1)}$ and by left invariance also $R_{\exp tX_1}$ (which descends to the compact quotients) is an isometry from $g_1^0$ to $\Phi_t^* g_1^0 = g_1^t$.

The deformation $g_\alpha^0$ with varying $\alpha$ is (unfortunately!) not isospectral because the volume of $(\Gamma\backslash G_m, g_\alpha^0)$ is not constant; even after fixing the volume by rescaling the metrics, the resulting manifolds have non-constant total scalar curvature which is impossible for an isospectral deformation (see e.g. [BGM]).

## §3 The changing geometry during the isospectral deformations constructed in §2

For each fixed $0 < \alpha < 1$, we will now construct two purely geometrically defined submanifolds $M_1(t)$, $M_2(t)$ of $\Gamma\backslash G_m$, depending only on the metric $g_\alpha^t$, whose $g_\alpha^t$-distance to each other is nonconstant in $t$, which will prove the nontriviality of the isospectral deformation $(\Gamma\backslash G_m, g_\alpha^t)$ for varying $t$. For the construction of these submanifolds we will make use of the Killing fields on $(\Gamma\backslash G_m, g_\alpha^t)$. In order to determine them, we first need the following two lemmas.

3.1 LEMMA. *Let* $0 < \alpha < 1$ *and* $t \in \mathbb{R}$. *Then* $I(G_m, g_\alpha^t) = K_{g_\alpha^t}(G_m) \ltimes L_{G_m}$, *where* $I(G_m, g_\alpha^t)$ *denotes the full isometry group of* $(G_m, g_\alpha^t)$, $K_{g_\alpha^t}(G_m) := \{\Psi \in \mathrm{Aut}(G_m) \mid \Psi^*(g_\alpha^t) = g_\alpha^t\}$, *and* $L_{G_m}$ *is the group of left translations by elements of* $G_m$.

For exponentially solvable Lie groups $G$ with only real roots and a left invariant metric $g$, the analog of this statement is always true, as shown by Gordon and Wilson in [GW1, Theorem 4.3]. Since $G_m$ is not exponentially solvable one has to analyse the steps of the proof of that theorem in order to see that if $0 < \alpha < 1$, the statement is true nevertheless for $(G_m, g_\alpha^t)$. It is easy to observe that this is not the case for $\alpha = 1$; one can find $g_1^t$-isometries which fix $e$ but which are not automorphisms of $G_m$.

*Proof of Lemma* 3.1. Let $G$ be any simply connected unimodular solvable Lie group with Lie algebra $\mathfrak{g}$ and a left invariant metric $g$. Let $I_0(G, g)$ be the identity component of the full isometry group $I(G, g)$. By [GW1, Theorem 4.2], there is exactly one subgroup $S$ of $I_0(G, g)$ which is in "standard position"; moreover, this $S$ is then normal in $I(G, g)$ by [GW1, Corollary 1.12]. Now Proposition 3.3, Definition 3.4 ("standard modification algorithm") and Theorem 3.5 from [GW1] together imply that if

(*) $\operatorname{tr}(\varphi \circ \operatorname{ad} X) = 0$ for all $X \in \mathfrak{g}$ and all $g$-skew-symmetric derivations $\varphi$ of $\mathfrak{g}$

then $S$ just equals $L_G$, and hence $L_G$ is normal in $I(G, g)$. But this implies that for any $\Psi \in I(G, g)$ with $\Psi(e) = e$ we have $\Psi \circ L_x \circ \Psi^{-1} = L_{\Psi(x)}$ for all $x \in G$, which means that $\Psi$ is an automorphism; hence $I(G, g) = K_g(G) \ltimes L_G$.

It remains to show that if $0 < \alpha < 1$, property $(*)$ is satisfied for $\mathfrak{g}_m$ and $g'_\alpha$.

Let $\varphi$ be a $g'_\alpha$-skew-symmetric derivation of $\mathfrak{g}_m$. As a derivation, $\varphi$ keeps the nilradical $\mathfrak{g}'_m$ invariant, and by skew-symmetry, $\varphi(X_1) = 0$. Hence $\varphi$ commutes with $\operatorname{ad} X_1$ and keeps the eigenspaces of $\operatorname{ad} X_1$ invariant. In particular, $\varphi$ keeps $\operatorname{span}\{X_2, Y_2\}$ invariant. By skew-symmetry there is a $\lambda \in \mathbb{R}$ with $\varphi(X_2) = \lambda(\alpha Y_2)$, $\varphi(\alpha Y_2) = -\lambda X_2$. Because of $\alpha \neq 1$ and since $\varphi$ commutes with $\operatorname{ad} X_1 \colon X_2 \mapsto Y_2$, $Y_2 \mapsto -X_2$, we must have $\lambda = 0$. Moreover, $\varphi(Z) = 0$ because $\varphi$ must keep the center $\operatorname{span}\{Z\}$ of $\mathfrak{g}_m$ invariant. Thus $\varphi$ annihilates $[\mathfrak{g}_m, \mathfrak{g}_m] = \operatorname{span}\{X_2, Y_2, Z\}$ which implies $\varphi \circ \operatorname{ad} X = 0$ for all $X \in \mathfrak{g}_m$, in particular $(*)$. □

3.2 LEMMA. *If* $\Phi \in \operatorname{Aut}(G_m)$ *and* $\Phi|_\Gamma = \mathrm{id}$, *then* $\Phi = \mathrm{id}$.

For exponentially solvable Lie groups with a cocompact discrete subgroup $\Gamma$, the analog of this statement is obvious. But since $G_m$ is not exponentially solvable, explicit calculations have to be carried out here.

*Proof of Lemma* 3.2. For the nilradical $G'_m$ (see Definition 2.2) and $\Gamma' = \Gamma \cap G_m$ (see Definition 2.4 and the proof of Lemma 2.5), $\Phi|_{\Gamma'} = \mathrm{id}$ implies $\Phi|_{G'_m} = \mathrm{id}$, $\Phi_{*e}|_{\mathfrak{g}'_m} = \mathrm{id}$. It remains to show that $\Phi_{*e}(X_1) = X_1$. Let $\gamma_0 := \exp(2\pi X_1)$. Then $\Phi(\gamma_0) = \gamma_0$ implies $\Phi_{*e}(2\pi X_1) \in \exp^{-1}(\gamma_0)$. The difficulty is that $\exp^{-1}(\gamma_0)$ does not consist of only one point. Obviously, $\exp^{-1}(\gamma_0)$ is contained in $2\pi X_1 + [\mathfrak{g}_m, \mathfrak{g}_m] = 2\pi X_1 + \operatorname{span}\{X_2, Y_2, Z\}$. Explicit calculation of $\exp \colon \mathfrak{g}_m \to G_m$ shows (see [Sch], Lemma II.1.8):

$$\begin{aligned}
&\exp(\xi X_1 + xX_2 + yY_2 + zZ) \\
&\quad = \left(0, \xi^{-1} \cdot D^{90^\circ}(\mathrm{id}\text{-}D^\xi) \cdot \begin{pmatrix} x \\ y \end{pmatrix}, 0, \ldots, 0, z + (2\xi^2)^{-1}(\xi - \sin\xi)(x^2 + y^2)\right) \\
&\qquad \cdot \exp \xi X_1
\end{aligned}$$

for all $\xi, x, y, z \in \mathbb{R}$ (with the notation from Definition 2.2), where $D^\xi$ denotes rotation by the angle $\xi$. Thus

$$\exp^{-1}(\gamma_0) = \{2\pi X_1 + xX_2 + yY_2 - ((x^2 + y^2)/4\pi)Z \mid x, y \in \mathbb{R}\}.$$

Let $x$, $y$ be such that $\Phi_{*e}(X_1) = 2\pi X_1 + xX_2 + yY_2 - ((x^2 + y^2)/4\pi)Z$. Then by $\Phi_{*e}|_{\mathfrak{g}'_m} = \mathrm{id}$, $(\Phi_{*e})^2(2\pi X_1) = 2\pi X_1 + 2xX_2 + 2yY_2 - ((x^2 + y^2)/2\pi)Z$. But this has again to be an element of $\exp^{-1}(\gamma_0)$, hence $(x^2+y^2)/2\pi = (4x^2+4y^2)/4\pi$ and therefore $x = y = 0$. Thus indeed we have $\Phi_{*e}(X_1) = X_1$. □

3.3 PROPOSITION. *Let* $0 < \alpha < 1$, $t \in \mathbb{R}$, *and* $V$ *be a vectorfield on* $(\Gamma\backslash G_m, g_\alpha^t)$. *Then* $V$ *is a Killing field if and only if* $V$ *is a central vectorfield.*

By central vectorfields we denote vectorfields which are induced from $G_m$ by some scalar multiple of the vectorfield $p \mapsto Z|_p$.

*Proof.* The "if" part is obvious because $p \mapsto Z|_p$ is a left *and* right invariant vectorfield on $G_m$; therefore it is a Killing field on $(G_m, g_\alpha^t)$ which descends to $\Gamma\backslash G_m$.

Now let $V$ be a Killing field on $(\Gamma\backslash G_m, g_\alpha^t)$ and $\Psi^s$ its flow, consisting of isometries. Since $G_m$ is simply connected, there are lifts $\bar{\Psi}^s : G_m \to G_m$ which can be chosen such that $u_s := \bar{\Psi}^s(e)$ is a smooth curve with $u_0 = e$. Define $F_s := L_{u_s}^{-1} \circ \bar{\Psi}^s$. Then $F_s$ is an isometry with $F_s(e) = e$. Hence by Lemma 3.1, $F_s \in \mathrm{Aut}(G_m)$. Consequently, the map $R_{u_s}^{-1} \circ \bar{\Psi}^s = I_{u_s} \circ F_s$ (where $R_{u_s}$ denotes right translation by $u_s$), too, is an automorphism of $G_m$, which moreover fixes $\Gamma$ because $\bar{\Psi}^s(\Gamma) = \Gamma \cdot u_s$ (remember that $\bar{\Psi}^s$ descends to $\Gamma\backslash G_m$). By Lemma 3.2, $\bar{\Psi}^s = R_{u_s}$ for all $s$. Thus $R_{u_s}$ and also $I_{u_s}$ are isometries of $(G_m, g_\alpha^t)$. Let $U := \dot{u}_0 \in \mathfrak{g}_m$. Then $\mathrm{ad}\, U$ must be $g_\alpha^t$-skew-symmetric which is the case if and only if $U \in \mathrm{span}\{Z\}$. So $V$ is induced on $\Gamma\backslash G_m$ by the central left invariant vectorfield $U$ on $G_m$. □

3.4 DEFINITION. Consider the projection $\pi : G_m \to G_m/\exp(\mathrm{span}\{Z\}) =: \tilde{G}_m$. Let $\tilde{\Gamma} := \pi(\Gamma)$. Note that $\tilde{\Gamma}$ is discrete and cocompact in $\tilde{G}_m$ because so is $\Gamma \cap \exp(\mathrm{span}\{Z\})$ in $\exp(\mathrm{span}\{Z\})$. Denote the projection $\Gamma\backslash G_m \to \tilde{\Gamma}\backslash\tilde{G}_m$ by $\pi$ again. Let $\tilde{\mathfrak{g}}_m$ be the Lie algebra of $\tilde{G}_m$; we identify $\tilde{\mathfrak{g}}_m$ with $\mathrm{span}\{X_1, Y_1, \ldots, X_m, Y_m\}$ with the nontrivial Lie brackets $[X_1, X_2] = Y_2$, $[X_1, Y_2] = -X_2$. For $0 < \alpha \le 1$ let $\tilde{g}_\alpha$ be the unique left invariant metric on $\tilde{G}_m$ with respect to which $\pi$ becomes a Riemannian submersion, i.e., the metric with orthonormal basis $\{X_1, Y_1, X_2, \alpha Y_2, X_3, Y_3, \ldots, X_m, Y_m\}$.

*Remark.* Let $0 < \alpha < 1$, $t \in \mathbb{R}$. Then by Proposition 3.3, the Riemannian manifold $(\tilde{\Gamma}\backslash\tilde{G}_m, \tilde{g}_\alpha)$ is obtained from $(\Gamma\backslash G_m, g_\alpha^t)$ in a purely geometrical way, namely by dividing $(\Gamma\backslash G_m, g_\alpha^t)$ by the flow of its Killing fields.

We now want to repeat this step, namely to find a geometrical characterization of the central directions $\mathrm{span}\{Y_1, X_3, Y_3, \ldots, X_m, Y_m\} =: \mathfrak{z}(\tilde{\mathfrak{g}}_m)$ in $(\tilde{\Gamma}\backslash\tilde{G}_m, \tilde{g}_\alpha)$. As was shown by the author in [Sch, Theorem II.2.8], the central vectorfields are *not* the only Killing fields on $(\tilde{\Gamma}\backslash\tilde{G}_m, \tilde{g}_\alpha)$ because the *right* invariant vectorfields corresponding to vectors in $\mathrm{span}\{X_2, Y_2\}$, too, happen to descend to vectorfields on $\tilde{\Gamma}\backslash\tilde{G}_m$. Nevertheless it can be shown (cf. [Sch, Theorem II.2.9]) that the central vectorfields are just the *parallel* Killing fields on $(\tilde{\Gamma}\backslash\tilde{G}_m, \tilde{g}_\alpha)$.

A slightly shorter way which we will use here is noting that $\mathfrak{z}(\tilde{\mathfrak{g}}_m)$ is just the 0-eigenspace for the Ricci tensor associated to $\tilde{g}_\alpha$:

3.5 LEMMA. *Let* $0 < \alpha < 1$. *Then the orthonormal basis* $\{X_1, Y_1, X_2, \alpha Y_2, \ldots, X_m, Y_m\}$ *of* $\tilde{\mathfrak{g}}_m$ *consists of eigenvectors for the Ricci tensor associated to* $\tilde{g}_\alpha$, *and the eigenvalues are*

$$\operatorname{ric}(X_1) = 1 - (\alpha^2 + \alpha^{-2})/2 < 0, \qquad \operatorname{ric}(X_2) = (\alpha^2 - \alpha^{-2})/2 < 0,$$

$$\operatorname{ric}(\alpha Y_2) = (\alpha^{-2} - \alpha^2)/2 > 0, \qquad \operatorname{ric}(W) = 0 \quad \textit{for all } W \in \mathfrak{z}(\tilde{\mathfrak{g}}_m).$$

*Proof.* The statements for $\mathfrak{z}(\tilde{\mathfrak{g}}_m)$ follow from the fact that $\tilde{\mathfrak{g}}_m$ is the direct orthogonal product of span $\{X_1, X_2\ Y_2\}$ and $\mathfrak{z}(\tilde{\mathfrak{g}}_m)$. Let $\tilde{Y}_2 := \alpha Y_2$. Using the well-known Koszul formula for the Levi-Cività connection $\nabla$, one gets

$$\nabla_{X_1} X_2 = ((\alpha + \alpha^{-1})/2)\tilde{Y}_2, \qquad \nabla_{X_2} X_1 = ((\alpha - \alpha^{-1})/2)\tilde{Y}_2,$$

$$\nabla_{X_1} \tilde{Y}_2 = -((\alpha + \alpha^{-1})/2)X_2, \qquad \nabla_{\tilde{Y}_2} X_1 = ((\alpha - \alpha^{-1})/2)X_2,$$

$$\nabla_{X_2} \tilde{Y}_2 = ((\alpha^{-1} - \alpha)/2)X_1 = \nabla_{\tilde{Y}_2} X_2, \qquad \nabla_{X_1} X_1 = \nabla_{X_2} X_2 = \nabla_{\tilde{Y}_2} \tilde{Y}_2 = 0.$$

Thereby the mixed Ricci terms are easily seen to vanish which shows that the basis vectors are eigenvectors for the Ricci tensor. The nontrivial sectional curvatures are

$$K(X_1, X_2) = (\alpha + \alpha^{-1})^2/4 - \alpha^{-2}, \qquad K(X_1, \tilde{Y}_2) = (\alpha + \alpha^{-1})^2/4 - \alpha^2,$$

$$K(X_2, \tilde{Y}_2) = (\alpha - \alpha^{-1})^2/4.$$

The statement now follows by straightforward calculation. □

3.6 DEFINITION.

(i) Let $\bar{\mathfrak{g}} := \operatorname{span}\{X_1, X_2, Y_2\}$ with nontrivial Lie brackets $[X_1, X_2] = Y_2$, $[X_1, Y_2] = -X_2$. Let $\bar{G}$ be the simply connected Lie group with Lie algebra $\bar{\mathfrak{g}}$. Let $\bar{g}_\alpha$ be the left invariant metric with respect to which $\{X_1, X_2, \alpha Y_2\}$ is an orthonormal basis. Define $\bar{\Gamma} := \{\exp(2\pi X_1 + a_2 X_2 + b_2 Y_2) \mid a_2, b_2 \in \mathbb{Z}\}$. Then obviously $(\tilde{\Gamma}\backslash\tilde{G}_m, \tilde{g}_\alpha)$ is the Riemannian product of $(\bar{\Gamma}\backslash\bar{G}, \bar{g}_\alpha)$ and the orthogonal flat torus $T := \mathbb{R}^{2m-3}/((\mathbb{Z}/2\pi) \times \mathbb{Z} \times \cdots \times \mathbb{Z})$, endowed with the euclidean standard metric. Denote the projection of $(\tilde{\Gamma}\backslash\tilde{G}_m, \tilde{g}_\alpha)$ onto the factor $(\bar{\Gamma}\backslash\bar{G}, \bar{g}_\alpha)$ by $\tilde{\pi}$.

(ii) For $0<\alpha<1$, consider those closed geodesics on $(\tilde{\Gamma}\backslash\tilde{G}_m, \tilde{g}_\alpha)$ which are tangent to the $T$-factor and have length $1/2\pi$. Now consider those horizontal lifts of these curves to $(\Gamma\backslash G_m, g_\alpha^t)$ which are again closed with length $1/2\pi$. Define $M_1(t) \subseteq \Gamma\backslash G_m$ as the union of all the curves obtained in this way.

(iii) For $0<\alpha<1$, let $\bar{M} \subseteq \bar{\Gamma}\backslash\bar{G}$ be the union of all globally shortest geodesics in $(\bar{\Gamma}\backslash\bar{G}, \bar{g}_\alpha)$. Define $M_2(t) := (\tilde{\pi}\circ\pi)^{-1}(\bar{M}) \subseteq \Gamma\backslash G_m$.

(iv) For $x \in \mathbb{R}$, define a submanifold $M_x \subseteq \Gamma\backslash G_m$ as the preimage under the projection $G_m \to \Gamma\backslash G_m$ of $G'_m \cdot \exp xX_1$ (notation from Definition 2.2). Note that $M_x = M_{x'}$ is equivalent to $x = x' \bmod 2\pi$. Analogously, let $\bar{G}' := \exp(\operatorname{span}\{X_2, Y_2\})$ and define $\bar{M}_x$ as the preimage under the projection $\bar{G} \to \bar{\Gamma}\backslash\bar{G}$ of $\bar{G}' \cdot \exp xX_1$. Note that $M_x = (\tilde{\pi}\circ\pi)^{-1}(\bar{M}_x)$.

*Remark.* The isometric orthogonal splitting $(\tilde{\Gamma}\backslash\tilde{G}_m, \tilde{g}_\alpha) = (\bar{\Gamma}\backslash\bar{G}, \bar{g}_\alpha) \times T$ is defined geometrically for $0<\alpha<1$ because in each point of $\tilde{\Gamma}\backslash\tilde{G}_m$ the directions tangent to the $T$-factor span the 0-eigenspace for the Ricci tensor by Lemma 3.5. Hence also $M_1(t)$ and $M_2(t)$ (whatever they will turn out to be) are defined by purely geometrical steps.

Note moreover that $\bar{G}$ is isomorphic to the group $E(2)$ of motions of the euclidean plane, that $\bar{\Gamma}$ is abelian, and that $(\bar{\Gamma}\backslash\bar{G}, \bar{g}_\alpha)$ is a torus (see 3.8(i)) which is non-flat except for $\alpha = 1$.

3.7 LEMMA. $M_1(t) = M_t \cup M_{t+\pi} \subseteq \Gamma\backslash G_m$.

*Proof.* Because of $1/2\pi < 1$, the only closed geodesics of $(\tilde{\Gamma}\backslash\tilde{G}_m, \tilde{g}_\alpha)$ tangent to the $T$-factor and of length $1/2\pi$ are the integral curves $p \cdot \exp sY_1$ of the left invariant vectorfield $Y_1$ (up to parametrization). Since $Y_1 - tZ \perp Z$ with respect to $g_\alpha^t$, the horizontal lifts to $(\Gamma\backslash G_m, g_\alpha^t)$ of such curves are of the form $p \cdot \exp s(Y_1 - tZ)$, $p \in \Gamma\backslash G_m$. For this curve to close up again after time $1/2\pi$, we must have $q \cdot \exp((Y_1 - tZ)/2\pi) \in \Gamma \cdot q$ where $q \in G_m$ is a representative of $p \in \Gamma\backslash G_m$, equivalently: $I_q(\exp((Y_1 - tZ)/2\pi)) \in \Gamma$. Let $q = q' \cdot \exp xX_1$ with $q' \in G'_m$, $x \in \mathbb{R}$. Since $Y_1$ is central in $\mathfrak{g}'_m$, the condition now becomes: $\exp((Y_1 + (x-t)Z)/2\pi) \in \Gamma$, which by definition of $\Gamma$ is equivalent to $(x-t)/2\pi \in \mathbb{Z}/2$, i.e., $x \in t + \pi\mathbb{Z}$. But this just means that $p \in M_t \cup M_{t+\pi}$. □

In order to determine the shortest closed geodesics in $(\bar{\Gamma}\backslash\bar{G}, \bar{g}_\alpha)$ and to describe $M_2(t)$, we first need some preparations.

3.8 DEFINITION AND REMARKS.

(i) Denote $\exp(x_2X_2 + y_2Y_2) \in \bar{G}'$ by $(x_2, y_2)$. Define a diffeomorphism $\bar{F}: \bar{G} \to \mathbb{R}^3$ by

$$\bar{F}((x_2, y_2) \cdot \exp x_1X_1) := (x_1, x_2, y_2) \in \mathbb{R}^3.$$

$\bar{F}$ induces a diffeomorphism (which we again call $\bar{F}$) from $\bar{\Gamma}\backslash\bar{G}$ to the torus $\Lambda\backslash\mathbb{R}^3$ with $\Lambda := \{(2\pi a_1, a_2, b_2) \mid a_1, a_2, b_2 \in \mathbb{Z}\}$. This can be shown analogously to the proof of Lemma 2.5. Moreover, just as in the proof of Lemma 2.3 one gets for $p = p' \cdot \exp x_1 X_1$ with $p' \in \bar{G}'$, $x_1 \in \mathbb{R}$:

$$\bar{F}_{*p}(X_1) = X_1|_{\bar{F}(p)},$$

$$\bar{F}_{*p}(X_2) = \cos(x_1)X_2|_{\bar{F}(p)} + \sin(x_1)Y_2|_{\bar{F}(p)},$$

$$\bar{F}_{*p}(Y_2) = -\sin(x_1)X_2|_{\bar{F}(p)} + \cos(x_1)Y_2|_{\bar{F}(p)},$$

where on the right side, $X_1$, $X_2$, $Y_2$ denote the constant standard basis vectorfields on $\mathbb{R}^3$.

(ii) Let $dx_1$, $dx_2$, and $dy_2$ be the standard 1-forms on $\mathbb{R}^3$ dual to $X_1$, $X_2$, $Y_2$. For $s \in \mathbb{R}$ define a 1-form $\beta^s$ on $\bar{\Gamma}\backslash\bar{G}$ by letting $\beta^s := \bar{F}^*(\cos(s)\,dx_2 + \sin(s)\,dy_2)$. By (i) we know for $p' \in \bar{G}'$, $x_1 \in \mathbb{R}$, $p = p' \cdot \exp x_1 X_1 \in \bar{G}$ that $\beta^s(X_1) = 0$, $\beta^s(X_2) = \cos(s - x_1)$, $\beta^s(Y_2) = \sin(s - x_1)$. Note that the pullback of $\beta^s$ to $\bar{G}$ is not a left invariant 1-form.

In the following we will call a vectorfield on $\bar{\Gamma}\backslash\bar{G}$ left invariant if it is induced from $\bar{G}$ by a left invariant vectorfield, and we will denote two such vectorfields by the same name.

3.9 LEMMA.

(i) *$\beta^s$ is closed.*

(ii) *Let $0 < \alpha < 1$ and $W \in T_p(\bar{\Gamma}\backslash\bar{G})$ with $\|W\|_{\bar{g}_\alpha} = 1$. Then $\beta^s(W) \leq 1$, and equality holds if and only if either $W = X_2|_p \wedge p \in \bar{M}_s$ or $W = -X_2|_p \wedge p \in \bar{M}_{s+\pi}$.*

*Proof.*

(i) This is clear because $\beta^s$ was defined as the pullback of a closed form.

(ii) Let $W = uX_1 + vX_2 + w \cdot \alpha Y_2$, $u^2 + v^2 + w^2 = 1$, $p \in \bar{G}$, $p = p' \cdot \exp x_1 X_1$ with $p' \in \bar{G}'$, $x_1 \in \mathbb{R}$. Then $\beta^s(W) = v \cdot \cos(s - x_1) + \alpha w \cdot \sin(s - x_1) \leq 1$, and because of $\alpha < 1$, equality holds if and only if $u = w = 0$ and either $v = 1 \wedge x_1 = s \bmod 2\pi$ or $v = -1 \wedge x_1 = (s + \pi) \bmod 2\pi$. □

Lemma 3.9 just says that $\beta^s$ is a closed form of comass 1, hence a *calibration*. See [HL] or [DGGW5, DGGW6] for the theory of calibrations and the notions of mass and comass. We will now use the calibrations $\beta^s$ in order to determine the shortest closed geodesics in $(\bar{\Gamma}\backslash\bar{G}, \bar{g}_\alpha)$.

3.10 PROPOSITION. *Let* $0 < \alpha < 1$.

(i) *Let* $k, n \in \mathbb{Z}$ and $\gamma := \exp(kX_2 + nY_2) \in \bar{\Gamma}$. Since $\bar{\Gamma}$ *is abelian*, $\gamma$ *precisely corresponds to a free homotopy class in* $\bar{\Gamma}\backslash\bar{G}$. *The shortest closed geodesics in this class have length* $L := (k^2 + n^2)^{1/2}$ *and foliate* $\bar{M}_s \cup \bar{M}_{s+\pi} \subseteq \bar{\Gamma}\backslash\bar{G}$ *where* $s$ *is such that* $k = L \cdot \cos(s)$, $n = L \cdot \sin(s)$. *Those of these geodesics which lie in* $\bar{M}_s$ *are integral curves of* $X_2$, *those in* $\bar{M}_{s+\pi}$ *are integral curves of* $-X_2$.

(ii) *The shortest closed geodesics of* $(\bar{\Gamma}\backslash\bar{G}, \bar{g}_\alpha)$ *belong (up to inversion of their directions) to the free homotopy classes corresponding to* $\exp X_2$, $\exp Y_2 \in \bar{\Gamma}$ *and foliate the submanifold* $\bar{M} = \bar{M}_0 \cup \bar{M}_{\pi/2} \cup \bar{M}_\pi \cup \bar{M}_{3\pi/2} \subseteq \bar{\Gamma}\backslash\bar{G}$. *Here* $\bar{M}_0 \cup \bar{M}_\pi$ *is foliated by geodesics which belong to* $\exp X_2$ *and* $\bar{M}_{\pi/2} \cup \bar{M}_{3\pi/2}$ *by those goedesics which belong to* $\exp Y_2$. *All of these geodesics are integral curves of the left invariant vectorfields* $X_2$ *or* $-X_2$.

(iii) $M_2(t) = M_0 \cup M_{\pi/2} \cup M_\pi \cup M_{3\pi/2} \subseteq \Gamma\backslash G_m$.

Note that statement (ii) does not hold for $\alpha = 1$ because $(\bar{\Gamma}\backslash\bar{G}, \bar{g}_1)$ is a flat torus in which the shortest geodesics consequently lie everywhere.

*Proof of Proposition* 3.10.

(i) Let $p \in \bar{M}_s$ and $q$ be a representative of $p$ in $\bar{G}$, $q = q' \cdot \exp sX_1$, $q' \in \bar{G}'$. Then we have (with the notation from Definition 3.8(i)): $q \cdot \exp LX_2 = q \cdot (L, 0) = I_q(L, 0) \cdot q = (L \cdot \cos(s), L \cdot \sin(s)) \cdot q = (k, n) \cdot q = \gamma \cdot q$. Thus any $X_2$-integral curve in $\bar{M}_s$ closes up with length $L$, and this closed curve belongs to the free homotopy class associated to $\gamma$. The analogous statement for the $(-X_2)$-integral curves in $\bar{M}_{s+\pi}$ can be shown similarly. The two families of closed curves just described are calibrated by the 1-form $\beta^s$.

Now let $c$ be any closed curve in the same free homotopy class (and thereby a forteriori a cycle in the same homology class) as the above curves, with a length of at most $L$. Then $c$, too, has to be calibrated by $\beta^s$, which by Lemma 3.9(ii) implies that $c$ lies in one of the two families already described. (For details of the calibration argument see [Sch, proof of Proposition II.3.8] or the literature cited above.)

(ii) With the notations from (i), for $\gamma = \exp X_2$ we have $L = 1$, $s = 0$, $\bar{M}_s \cup \bar{M}_{s+\pi} = \bar{M}_0 \cup \bar{M}_\pi$, and for $\gamma = \exp Y_2 : L = 1$, $s = \pi/2$, $\bar{M}_s \cup \bar{M}_{s+\pi} = \bar{M}_{\pi/2} \cup \bar{M}_{3\pi/2}$. Thus the statement follows immediately from (i), provided that those closed geodesics in $(\bar{\Gamma}\backslash\bar{G}, \bar{g}_\alpha)$ which belong to a free homotopy class of the form $\gamma' \cdot \exp(2\pi m X_1)$ with $\gamma' \in \exp(\text{span}_\mathbb{Z}\{X_2, Y_2\})$ and $m \in \mathbb{Z} - \{0\}$ always have a length bigger than 1. In fact, they always have a length bigger or equal to $2\pi$. This follows by considering the Riemannian submersion $(\bar{\Gamma}\backslash\bar{G}, \bar{g}_\alpha) \to S^1_{2\pi}$ which is induced by the projection $\bar{G} \to \bar{G}/\exp(\text{span}\{X_2, Y_2\})$, and using the fact that Riemannian submersions do not increase the length.

(iii) This is clear by (ii) and the definition of $M_2(t)$. □

*Remarks*

(i) Even if $2\pi$ were not bigger than 1 by chance (which we used in the proof of (ii)), in other words: even if the norm of $X_1$ were chosen arbitrarily small, the submanifold $\bar{M} = \bar{M}_0 \cup \bar{M}_{\pi/2} \cup \bar{M}_\pi \cup \bar{M}_{3\pi/2}$ would still stay geometrically distinguished. For example, one can show that it is the union of the shortest closed geodesics within the family of those geodesics which are tangent to Killing fields on $(\bar{\Gamma}\backslash\bar{G}, \bar{g}_\alpha)$ (if $0 < \alpha < 1$), independently of the norm of $X_1$ (see [Sch], Remark II.3.22).

(ii) Note that the calibrations $\beta^s$ which we used to determine the shortest closed geodesics in $(\bar{\Gamma}\backslash\bar{G}, \bar{g}_\alpha)$ are not left invariant 1-forms as it is always the case for the calibrations considered in [DGGW1–DGGW6]. This is related to the fact that $\bar{\Gamma}\backslash\bar{G}$ is diffeomorphic to a 3-dimensional torus (see 3.8(i)), and its cohomology is *not* isomorphic to the cohomology of $\bar{G}$-left invariant differential forms. Such a thing can never happen in the case of nilmanifolds (see [No]).

3.11 THEOREM. *Let* $0 < \alpha < 1$.

(i) *The nonconstant function* $d(t) := \operatorname{dist}_{g_\alpha^t}(M_1(t), M_2(t)) = \operatorname{dist}(t, (\pi/2)\mathbb{Z})$, *is an isometry invariant for the family* $g_\alpha^t$, *i.e., if* $\operatorname{dist}(t, (\pi/2)\mathbb{Z}) \neq \operatorname{dist}(t', (\pi/2)\mathbb{Z})$ *then* $(\Gamma\backslash G_m, g_\alpha^t)$ *is not isometric to* $(\Gamma\backslash G_m, g_\alpha^{t'})$.

(ii) *The converse holds, too: If* $d(t) = d(t')$ *then* $(\Gamma\backslash G_m, g_\alpha^t)$ *and* $(\Gamma\backslash G_m, g_\alpha^{t'})$ *are isometric. Thus* $\operatorname{dist}(t, (\pi/2)\mathbb{Z})$ *is the precise parameter of the deformation* $(\Gamma\backslash G_m, g_\alpha^t)$ *with fixed* $0 < \alpha < 1$ *and varying* $t$.

*Proof.*

(i) $d(t) \leq \operatorname{dist}(t, (\pi/2)\mathbb{Z})$ is obvious, because there are curves of this length which join $M_1(t)$ to $M_2(t)$, namely of the form $p \cdot \exp sX_1$. The "$\geq$" statement follows by considering the Riemannian submersion onto the $X_1$-direction (compare the proof of 3.10(ii)).

(ii) Let $\Psi \in \operatorname{Aut}(G_m)$ be defined by $\Psi_*: X_1 \mapsto -X_1$, $Y_1 \mapsto Y_1$, $Z \mapsto -Z$, $X_i \mapsto X_i$, $Y_i \mapsto -Y_i$ $(i = 2, \ldots, m)$. We have $\Psi_*(\Gamma) = \Gamma$ and $\Psi_*(Y_1 - tZ) = Y_1 - (-tZ)$, hence $\Psi$ induces an isometry from $(\Gamma\backslash G_m, g_\alpha^t)$ to $(\Gamma\backslash G_m, g_\alpha^{-t})$. It remains to find an isometry from $(\Gamma\backslash G_m, g_\alpha^t)$ to $(\Gamma\backslash G_m, g_\alpha^{t-\pi/2})$. Let $\Theta \in \operatorname{Aut}(G_m)$ be defined by $\Theta_*: X_1 \mapsto -X_1$, $Y_1 \mapsto Y_1$, $X_2 \mapsto Y_2$, $Y_2 \mapsto X_2$, $Z \mapsto -Z$, $X_i \mapsto X_i$, $Y_i \mapsto -Y_i$ $(i = 3, \ldots, m)$. Obviously $\Theta(\Gamma) = \Gamma$. Moreover, we have $\operatorname{Ad}_{\exp(\pi/2)X_1} \circ \Theta_*: X_1 \mapsto -X_1$, $Y_1 \mapsto Y_1 + (\pi/2)Z$, $X_2 \mapsto -X_2$, $Y_2 \mapsto Y_2$, $Z \mapsto -Z$, $X_i \mapsto X_i$, $Y_i \mapsto -Y_i$ $(i = 3, \ldots, m)$. In particular, $Y_1 - tZ$ is mapped to $Y_1 - (t - \pi/2)Z$, and span $\{X_2\}$, span $\{Y_2\}$ are invariant again (which they were not under $\Theta_*$). Thus the map $R^{-1}_{\exp(\pi/2)X_1} \circ \Theta$ which descends to $\Gamma\backslash G_m$ is an isometry from $g_\alpha^t$ to $g_\alpha^{t-\pi/2}$. □

## Appendix

We conclude this paper by some comments on another isospectral deformation on a solvmanifold which we constructed in [Sch], too. The underlying Lie group $G$ is exponentially solvable here. Except this example and our above examples $(\Gamma\backslash G_m, g_\alpha^t)$ there are nearly no explicit examples known until now in the solvable case, just one example in [GW2] which in some sense actually belongs to the nilpotent case (see the Introduction). Moreover, in the following example one cannot use the method of calibrations in the geometrical discussion as one does in the nilpotent case (see [DGGW1–DGGW6]) and also in our above examples $(\Gamma\backslash G_m, g_\alpha^t)$. The reason is that here the "interesting" closed geodesics will be homologous to (although not homotopic to) zero.

Consider the 5-dimensional Lie group $G$ with Lie algebra $\mathfrak{g}$ spanned by $\{X_1, Y_1, X_2, Y_2, Z\}$ with the nontrivial Lie brackets $[X_1, X_2] = X_2$, $[X_1, Y_2] = -Y_2$, $[X_i, Y_i] = Z$ for $i = 1, 2$. Instead of having purely imaginary eigenvalues as in our above $\mathfrak{g}_m$, $\operatorname{ad} X_1$ now has only real eigenvalues. Consequently $G$ is exponentially solvable. There is a cocompact discrete subgroup $\Gamma \subseteq G$ generated by $\{\exp t_0X_1, \exp(1/t_0)Y_1, \exp Q, \exp U \exp(1/2)Z\}$ with $t_0 = \ln(c^2)$ where $c$ is the positive solution of $x^{-1} = x - 1$, $Q = \{X_2 - Y_2)/\lambda$, $U = (c^{-1}X_2 + cY_2)/\lambda$ with $\lambda = (c + c^{-1})^{1/2}$. Let $g$ be the left invariant metric with orthonormal basis $\{X_1, Y_1, X_2, Y_2, Z\}$, $\Phi_t \in \operatorname{Aut}(G)$ with $\Phi_{t*} : Y_1 \mapsto Y_1 + tZ$, $W \mapsto W$ for the other basis vectors $W$, and let $g^t := \Phi_t^* g$. It is shown in [Sch] that $(\Gamma\backslash G, g^t)$ is a nontrivial isospectral deformation which can be discussed in a very similar way as the isospectral deformations $(\Gamma\backslash G_m, g_\alpha^t)$ constructed above, in spite of the fact that the topology here is quite different because $\Gamma$ is not nilpotent (as our above $\Gamma$ happened to be), but only solvable.

Some steps in the discussion are even simpler here; the analog of Lemma 3.1 always holds for exponentially solvable $G$ ([GW1, Theorem 4.3]), and the analog of Lemma 3.2 is clear because $\exp : \mathfrak{g} \to G$ is bijective now. The Killing fields on $(\Gamma\backslash G, g^t)$ are just the central vectorfields again; analogously to Definition 3.4, define $(\tilde{\Gamma}\backslash\tilde{G}, \tilde{g})$ as the manifold obtained by dividing by the $Z$-direction. The central direction $Y_1$ of $(\tilde{\Gamma}\backslash\tilde{G}, \tilde{g})$ can either be split off by analogus curvature arguments as in 3.5, 3.6, or by noting that here, the central vectorfields are once more just the Killing fields on $(\tilde{\Gamma}\backslash\tilde{G}, \tilde{g})$ (as it was not the case on $(\tilde{\Gamma}\backslash\tilde{G}, \tilde{g})$ (as it was not the case on $(\tilde{\Gamma}\backslash\tilde{G}_m, \tilde{g}_\alpha)$, see the remark preceding Lemma 3.5). We get a 3-dimensional solvmanifold $(\bar{\Gamma}\backslash\bar{G}, \bar{g})$ with $\bar{\mathfrak{g}} = \operatorname{span}\{X_1, X_2, Y_2\}$, $[X_1, X_2] = X_2$, $[X_1, Y_2] = -Y_2$, where $\bar{g}$ has orthonormal basis $\{X_1, X_2, Y_2\}$. $\bar{G}$ is isomorphic to the group $E(1, 1)$ of motions of $(\mathbb{R}^2, dx^2 - dy^2)$. Submanifolds $M_1(t)$, $M_2(t) \subseteq \Gamma\backslash G$ are defined in complete analogy to Definition 3.6. The fact $M_1(t) = M_t \cup M_{t+t_0/2}$ follows analogously to 3.7.

However, a major difference to the above discussion in 3.8–3.10 occurs when one attempts to determine the shortest closed geodesics in $(\bar{\Gamma}\backslash\bar{G}, \bar{g})$. These turn out to be again tangent to the distribution given by span $\{X_2, Y_2\}$, in fact, to foliate $\bar{M}_0 \cup \bar{M}_{t_0/2}$, but to be homologous to zero. More precisely, they belong to certain free homotopy classes $[\gamma]_{\bar{\Gamma}}$ with $\gamma \in [\bar{\Gamma}, \bar{\Gamma}]$. (Note that our "old" $\bar{\Gamma}$ from §3 happened to be abelian due to the fact that $I_{\exp(2\pi X_1)}$ was the identity.) So these shortest closed geodesics can*not* be detected by calibrations; instead, one has to study explicitly the geodesic equations $(\bar{G}, \bar{g})$ and to derive an estimate for the length of closed geodesics in $(\bar{\Gamma}\backslash\bar{G}, \bar{g})$ which are not integral curves of a left invariant vectorfield (see [Sch], II.3.12–II.3.15). Thereby one can show that $M_2(t) = M_0 \cup M_{t_0/2}$. One gets $\mathrm{dist}_{g^t}(M_1(t), M_2(t)) = \mathrm{dist}(t, (t_0/2)\mathbb{Z})$ as the precise parameter of the deformation $(\Gamma\backslash G, g^t)$.

We finally remark that $(\Gamma\backslash G, g^t)$ can be regarded as a kind of companion to the 5-dimensional example $(\Gamma\backslash G_2, g^t_\alpha)$ which in some aspects behaves very similarly, but differently in others. One can also construct $(2m+1)$-dimensional analogs of $(\Gamma\backslash G, g^t)$ which are related to $(\Gamma\backslash G_m, g^t_\alpha)$ in the same way as $(\Gamma\backslash G, g)$ is to $(\Gamma\backslash G_2, g^t_\alpha)$.

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