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Constructing taut foliations

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0. Introduction

In ([5]–[13]), David Gabai uses taut foliations to prove an impressive collection of results. In this paper, we use a refinement of Gabai's techniques to construct taut foliations which permit a pleasingly simple description. For certain knots k, the foliations constructed allow one to conclude that k has Property P: no nontrivial Dehn surgery on k yields a simply-connected manifold. In particular, it follows that a large class of alternating knots satisfies Property P. More precisely, we obtain the following.

THEOREM 0.1. *Let k be a knot in S^3 with alternating projection in which there is nontrivial nesting among the Seifert cycles. Then any manifold obtained from S^3 by nontrivial Dehn surgery along k contains a taut foliation. Hence, nontrivial Dehn surgery along k yields a manifold with infinite fundamental group.*

THEOREM 0.2. *Let k be a nontrivial knot in S^3 with alternating projection in which there is no nesting among the Seifert cycles. Then either all half-twists are positive and any manifold obtained from S^3 by Dehn surgery along k with nonnegative surgery coefficient contains a taut foliation or else all half-twists are negative and any manifold obtained by Dehn surgery with nonpositive surgery coefficient contains a taut foliation.*

Motivated by the fact that any closed manifold can be realized by Dehn surgery along some knot or link in S^3, the construction proceeds by first yielding taut foliations in certain knot exteriors. Each foliation possesses only noncompact leaves and meets the boundary torus in parallel curves of some slope r; it therefore extends to a foliation without compact leaves in the closed manifold obtained by Dehn filling with coefficient r. In Section 1 we describe the construction for alternating knots. In Section 2 we give a technical description of those knots for which the construction may proceed.

Wanting to promote the theory of essential laminations (see [15]), we hasten to add that the objects first recognized were the underlying laminations and it is on the level of laminations that most of the interesting behaviour occurs (see Theorem 2.3 and Example 2.4). The foliations were in fact discovered as we sought to generalize the essential lamination constructions of Allen Hatcher [19] – an approach which has proved useful on at least two other occasions (see [4, 22]).

This paper is a restatement of results found in [25]. The author would like to extend warmest thanks to Allen Hatcher for his encouragement and guidance as her thesis advisor.

Definitions and basic constructions

Let M be a compact orientable 3-manifold. If $\mathscr{F}$ is a transversely oriented codimension-1 foliation of M meeting ∂M transversely, the $\mathscr{F}$ is said to be *taut* if each leaf of $\mathscr{F}$ intersects a closed transversal to $\mathscr{F}$. The foliations constructed in this paper contain no compact leaves and hence are necessarily taut. (See [16], for example.)

The taut foliations in this paper will be constructed using branched surfaces. A *branched surface* B is a space modelled locally on the object of Figure 1. B intersects ∂M transversely in a *train track* τ, a space modelled locally on the object of Figure 2. When B lies in a 3-manifold M (respectively, τ lies in a surface F), it possesses a regular neighbourhood which is fibred as shown in Figure 3 (respectively, Figure 4). Given such a fibred neighbourhood $N(B)$, we shall denote by $\partial_h N(B)$ that portion of $\partial N(B)$ which lies transverse to the fibres and by $\partial_v N(B)$ that portion of

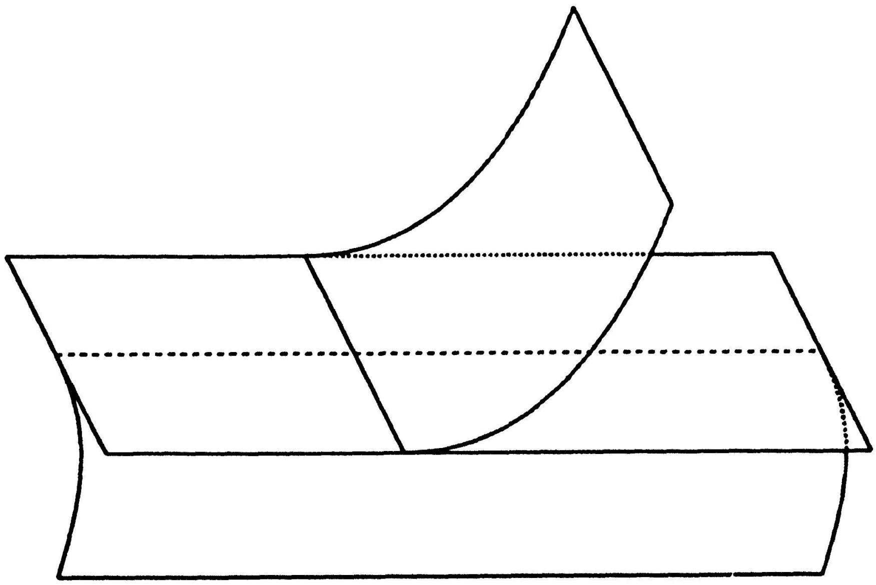


Figure 1. Branched surface model.

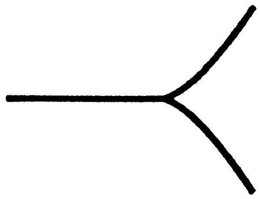


Figure 2. Train track model.

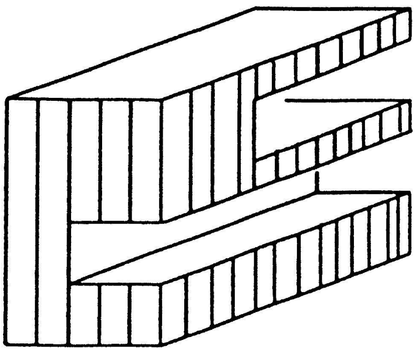


Figure 3. Fibred regular neighbourhood of the branched surface of Figure 1.

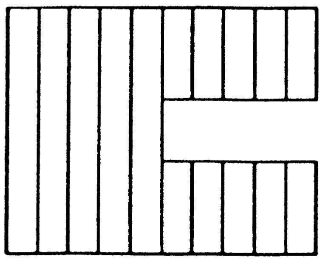


Figure 4. Fibred regular neighbourhood of the train track of Figure 2.

$\partial N(B)$ which contains subarcs of the fibres (see Figure 3). We say that a lamination λ is carried by the branched surface B (respectively, the train track τ) if it can be isotoped to lie in a fibred neighbourhood of **B** (respectively, of τ) transverse to all the fibres.

A *measure* on a train track is an assignment of weights $\alpha_i \geq 0$ to the branches so that the α_i satisfy the branching equations; i.e. if we have weights assigned as in Figure 5, then $a = b + c$. Similarly, a *measure* on a branched surface is an assignment of weights $\alpha_i \geq 0$ to the branches so that the α_i satisfy all branching equations; i.e.: if we have weights assigned as in Figure 6, then $d = b + c$, $e = c + f, f = a + b$. A branched surface or a train track with an assigned measure is said to be *measured.*

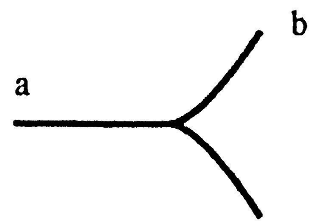


Figure 5. Measured train track.

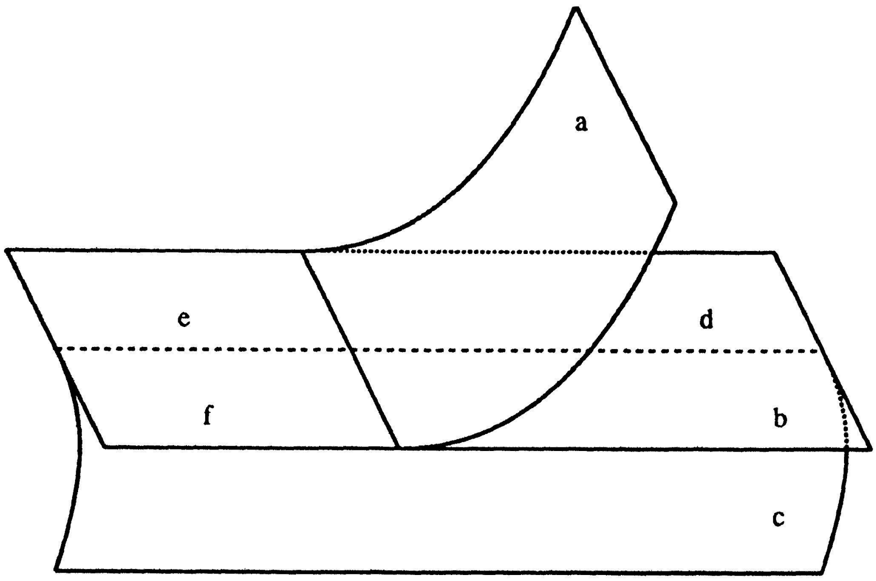


Figure 6. Measured branched surface.

We shall assume the well-known result that a measured branched surface in a 3-manifold M (respectively, a measured train track τ in some surface F) defines a codimension-1 lamination carried by B in M (respectively, by τ in F). (See for example, [1, 23, 21, 18]).

In this paper, we shall construct laminations using a slightly more general object, the *β-measured* branched surface. A branched surface B is said to be *β-measured* if for some finite set $\beta = \{\beta_1, \ldots, \beta_n\}$ of simple arcs, pairwise disjoint and disjoint from the branch loci, the branched surface B' obtained by cutting B open along the arcs of β is measured.

Similarly, a train track τ is said to be *β-measured* if the train track τ' obtained by cutting τ open along the set along the set β of points $\{p_1, \ldots, p_n\}$ is measured.

Notation. Denote $A \backslash \mathring{N}(B)$ by $A \mid B$. In particular, if F is a compact surface properly embedded in a compact 3-manifold M, then $M \mid F$ will denote the compact manifold $M\backslash\mathring{N}(F)$. And if β is a compact 1-manifold properly embedded in F, then $F \mid \beta$ will denote the compact surface $F\backslash\mathring{N}(\beta)$.

CONSTRUCTION 0.3. *β-measured branched surfaces (and β-measured train tracks) generate laminations.*

Let τ be a β-measured train track with $\beta=\{p_1,p_2,\ldots,p_n\}$ and $\tau'=\pi \mid \{p_1,\ldots,p_n\}$. Identify $\mathring{N}(p_i)=(p_i-1,p_i+1)$ for each i, $1\leq i\leq n$. By assigning a measure μ' to τ' we define a lamination λ' lying in a fibred neighbourhood of τ' everywhere transverse to the fibres. In particular, λ' intersects transversely the fibres $I^-(p_i)$ and $I^+(p_i)$ containing p_i-1 and p_i+1 respectively, $1\leq i\leq n$. By attaching the endpoints of leaves $\lambda'\cap I^-(p_i)$ to endpoints $\lambda'\cap I^+(p_i)$ according to some monotonic bijection $\{f_i : I^-(p_i)\to I^+(p_i)\}$ we define laminations carried by τ. (See Figure 7.)

Similarly, if B is a β-measured branched surface with $B'=B\mid(\beta_1\cup\beta_2\cup\cdots\cup\beta_n)$ assigned a measure μ', then μ' determines a lamination fully carried by B' and by gluing products $I^2\times I\to I^2\times I$ across each cutting arc β_i, we obtain a lamination carried by B (see Figure 7).

Sutured manifolds and disc decompositions

We now present those elements of Gabai's theory of sutured manifolds necessary for the construction given in this paper. For a more detailed exposition, see Gabai's trilogy [5, 11, 12].

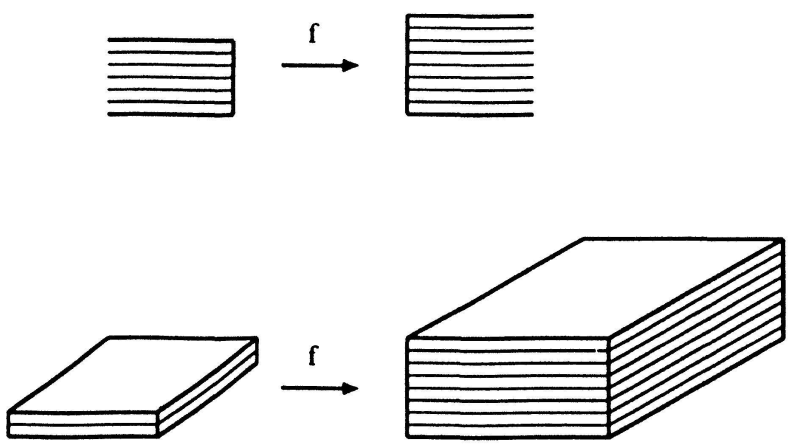


Figure 7. Gluing.

A *sutured manifold* is a pair (M, γ) consisting of a compact, oriented 3-manifold M together with a set $\gamma \subset \partial M$ of pairwise disjoint annuli $A(\gamma)$ and tori $T(\gamma)$. The interior of each component of $A(\gamma)$ contains a homologically nontrivial oriented simple closed curve. Call such a simple closed curve a *suture* and denote the set of sutures by $s(\gamma)$. The components of $R(\gamma) = \partial M \backslash \mathring{\gamma}$ can be oriented so that components lying on opposite sides of a given suture s' are oppositely oriented with corresponding boundary components sharing the orientation of s'. (M, γ) is said to be *taut* if M is irreducible and $R(\gamma)$ is Thurston-norm minimizing in $H_2(M, \gamma)$.

Let (M, γ) be a sutured manifold and let S be an incompressible, ∂-incompressible properly embedded surface in M. Suppose ∂S intersects both γ and $s(\gamma)$ transversely with each arc of $\partial S \cap A(\gamma)$ intersecting the corresponding suture exactly once. Suppose further that no component of ∂S bounds a disc D in $M \backslash \mathring{\gamma}$ and that no component of S is a disc D with ∂D contained in $M \backslash \mathring{\gamma}$. Then S defines a *sutured manifold decomposition*

$$(M, \gamma) \overset{S}{\rightsquigarrow} (M', \gamma')$$

where

$$M' = M \mid S,$$

$$\gamma' = (\gamma \cap M') \cup N(S'_+ \cap R_-(\gamma)) \cup N(S'_- \cap R_+(\gamma)) \backslash \mathring{\gamma}',$$

$$R_+(\gamma') = ((R_+(\gamma) \cap M') \cup S'_+) \backslash \mathring{\gamma}',$$

$$R_-(\gamma') = ((R_-(\gamma) \cap M') \cup S'_-) \backslash \mathring{\gamma}'.$$

The sutured manifold decomposition is said to be *well-groomed* if both (M, γ) and (M', γ') are taut, no subset of toral components of $S \cap R(\gamma)$ is homologically trivial in $H_2(M)$, and for every component V of $R(\gamma)$, either $S \cap V$ is a union of parallel, coherently oriented, nonseparating simple closed curves or $S \cap V$ is a union of parallel arcs such that for each component δ of ∂V, $|\delta \cap \partial S| = |\langle \delta, \partial S \rangle|$ (Definition 0.2, [11]).

A sutured manifold decomposition is called a *disc decomposition* if the cutting surface S is a disc. A sutured manifold (M, γ) is *disc-decomposable* if there exists a sequence of disc decompositions

$$(M, \gamma) \overset{D_1}{\rightsquigarrow} (M_1, \gamma_1) \overset{D_2}{\rightsquigarrow} \cdots \overset{D_n}{\rightsquigarrow} (M_n, \gamma_n)$$

where (M_n, γ_n) is a disjoint union of copies of the sutured manifold $(D^2 \times I, \partial D^2 \times I)$.

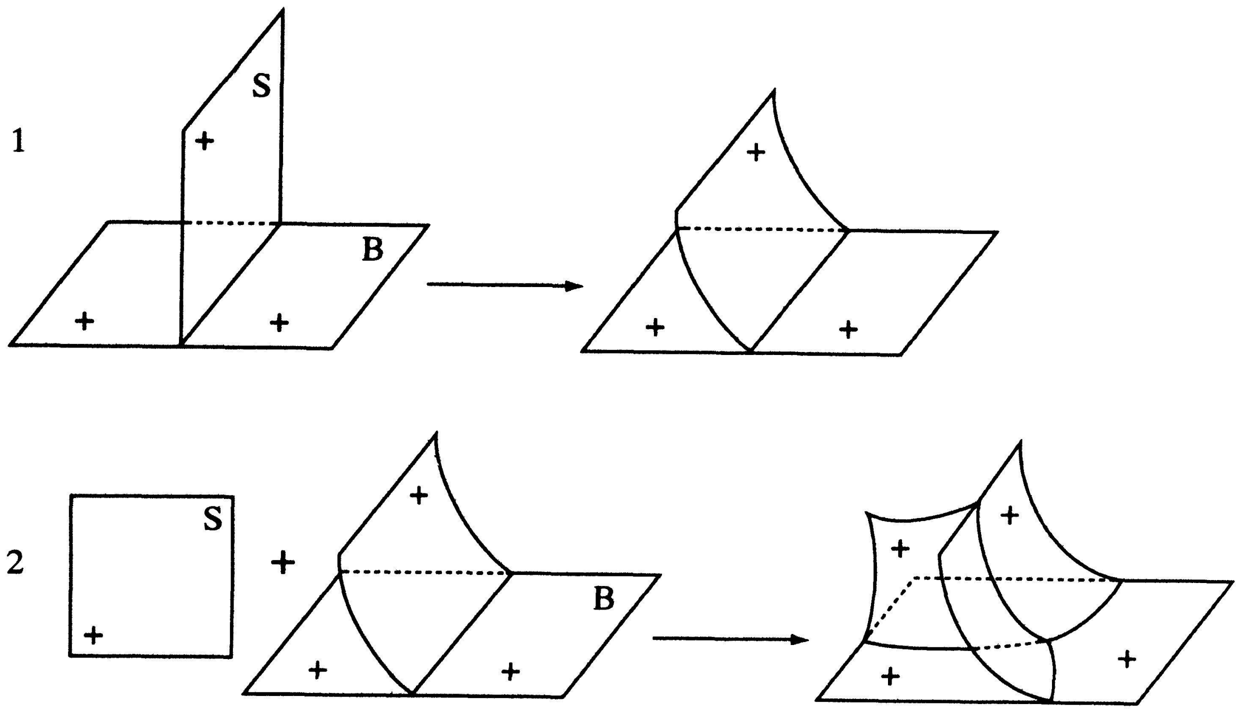


Figure 8. Constructing branched surfaces.

CONSTRUCTION 0.4. *Constructing Branched Surfaces from Sutured Manifold Decompositions.*

Note that if B is a transversely oriented branched surface properly embedded in a compact orientable 3-manifold M, then $(M\backslash \mathring{N}(B), \partial_v N(B))$ is a sutured manifold. Furthermore, given a sutured manifold decomposition

$$(M\backslash \mathring{N}(B), \partial_v N(B)) \overset{S}{\rightsquigarrow} (M', \gamma')$$

where B is a transversely oriented branched surface in an oriented 3-manifold M, we can view S as intersecting B as modelled in Figure 8.

By choosing an orientation on S, we induce branching as shown in Figure 8. The resulting object is again a transversely oriented branched surface B' (denote it also by $\langle B, S\rangle$) with

$$(M', \gamma') = (M\backslash \mathring{N}(B'), \partial_v N(B')).$$

We shall say that B is a branched surface of *depth n* if it can be constructed by applying Construction 0.2 to a sequence of sutured manifold decompositions

$$(M, \partial M) \overset{S_0}{\rightsquigarrow} (M_0, \gamma_0) \overset{S_1}{\rightsquigarrow} (M_1, \gamma_1) \overset{S_2}{\rightsquigarrow} \cdots \overset{S_n}{\rightsquigarrow} (M_n, \gamma_n)$$

for some manifold M with ∂M a (possibly empty) union of tori, where n is minimal over all such sequences.

1. The construction for alternating knots

In [12], Gabai proves that any knot exterior contains taut foliations which intersect the boundary torus in parallel curves of slope 0 and remain taut after longitudinal Dehn filling. In this section we see that when k is an alternating knot, this result generalizes to yield taut foliations in $S^3 \backslash \mathring{N}(k)$ realizing an infinite interval of boundary slopes and remaining taut after the corresponding Dehn fillings.

Let $M = S^3 \backslash \mathring{N}(k)$ where k is an alternating knot. For these knot exteriors, there is a particularly simple construction of β-measured branched surfaces B with nice boundary behaviour. One constructs B as a depth one branched surface $B = B_1 \supset B_0$ where B_0 is the surface obtained by applying Seifert's algorithm to a regular alternating projection of k and B_1 is obtained from B_0 by adding one of the discs properly embedded in $M = S^3 \backslash \mathring{N}(B_0)$.

More precisely, we proceed as follows. Recall that given a regular projection of a link k in the plane, Seifert's algorithm yields a Seifert surface R for k as follows. After choosing an orientation for k, a system of Seifert cycles is obtained by removing crossing points according to orientation as shown in Figure 9. The Seifert cycles bound disjoint discs which can be connected using half-twisted bands as dictated by the original crossings. (See [26], p. 120).

CONVENTION. We call a right-handed half-twist *positive* and a left-handed half-twist *negative*. (See Figure 10.)

As in [7, p. 393), it proves useful to divide the set of alternating links into two classes. We shall say a link k is *nonplanar* if it possesses an alternating projection in

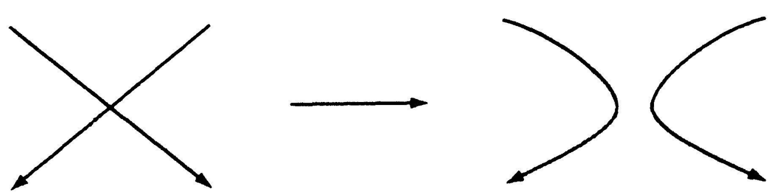


Figure 9. Removing the crossings.

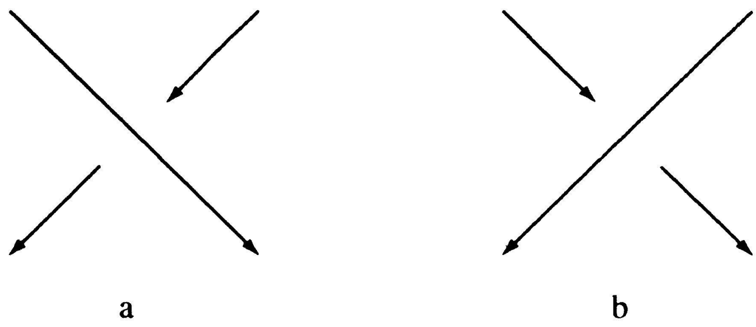


Figure 10. a. Positive half-twist b. Negative half-twist.

which there is nontrivial nesting among the Seifert cycles, i.e., for some Seifert cycle γ, both complementary regions of γ contain Seifert cycles. (By trivial nesting we mean any nesting which may be removed by untwisting half-twisted bands which separate R.) Otherwise, we say that k is planar.

THEOREM 1.1. *Let k be a nonplanar alternating knot. Then $M = S^3\backslash\mathring{N}(k)$ contains a set $\mathcal{F}$ of foliations such that for any finite number r, there is an element of $\mathcal{F}$ which intersects the boundary torus of M in parallel curves of slope r. The foliations realizing the nonzero boundary slopes have no compact leaves.*

COROLLARY 1.2. *Let k be a nonplanar alternating knot. Then any manifold obtained from S^3 by nontrivial Dehn surgery along k contains a taut foliation.*

Proof. Let $\hat{M}$ be the manifold obtained from S^3 by Dehn surgery along k with rational coefficient $r \neq 0$. Theorem 1.1 guarantees the existence of a foliation in $S^3\backslash\mathring{N}(k)$ with only noncompact leaves and meeting $\partial N(k)$ in parallel simple closed curves of slope r. F can therefore be completed to a foliation $\hat{F}$ in $\hat{M}$ by capping off the boundary components of the leaves of F with discs in $\hat{M}\backslash\mathring{M}$. Since $\hat{F}$ can have no compact leaves, it is necessarily taut (see [2]). The corresponding result when $r = 0$ follows similarly and may be found in [12]. □

Recall that a knot k is said to satisfy *Property P* if no nontrivial surgery along k yields a simply-connected manifold.

COROLLARY 1.3. *Nonplanar alternating knots satisfy Property P.*

Proof. Let $\hat{M}$ be a manifold obtained from S^3 by nontrivial Dehn surgery along k. By Theorem 1.1, $\hat{M}$ contains a taut foliation; hence, by [24] or [15], M has infinite fundamental group. □

The corresponding results for k planar are less satisfying. We first note that if k is planar, then it has an alternating projection in which no Seifert cycle separates and hence all half-twists have the same sign. For ease of presentation, we shall say that the alternating knot is *positive planar* if all half-twists are positive and *negative planar* if all half-twists are negative.

THEOREM 1.4. *Let k be a negative (respectively, positive) planar knot. Then $M = S^3 \backslash \mathring{N}(k)$ contains a set $\mathscr{F}$ of foliations such that for any nonpositive (respectively, nonnegative) finite number r, there is a foliation of $\mathscr{F}$ which intersects the boundary torus of M in parallel curves of slope r. The foliations realizing the nonzero boundary slopes have no compact leaves.*

COROLLARY 1.5. *Let k be a negative (respectively, positive) planar alternating knot. Then any manifold obtained from S^3 by Dehn surgery along k with nonpositive (respectively, nonnegative) surgery coefficient contains a taut foliation.*

Proof (*Theorems* 1.1 *and* 1.4).

Case 1. Consider first the case that k is nonplanar. Choose a regular alternating projection in which there is non-trivial nesting and let R be the surface generated by Seifert's construction. Since we can untwist trivial twists as necessary we may assume that no half-twisted band separates R.

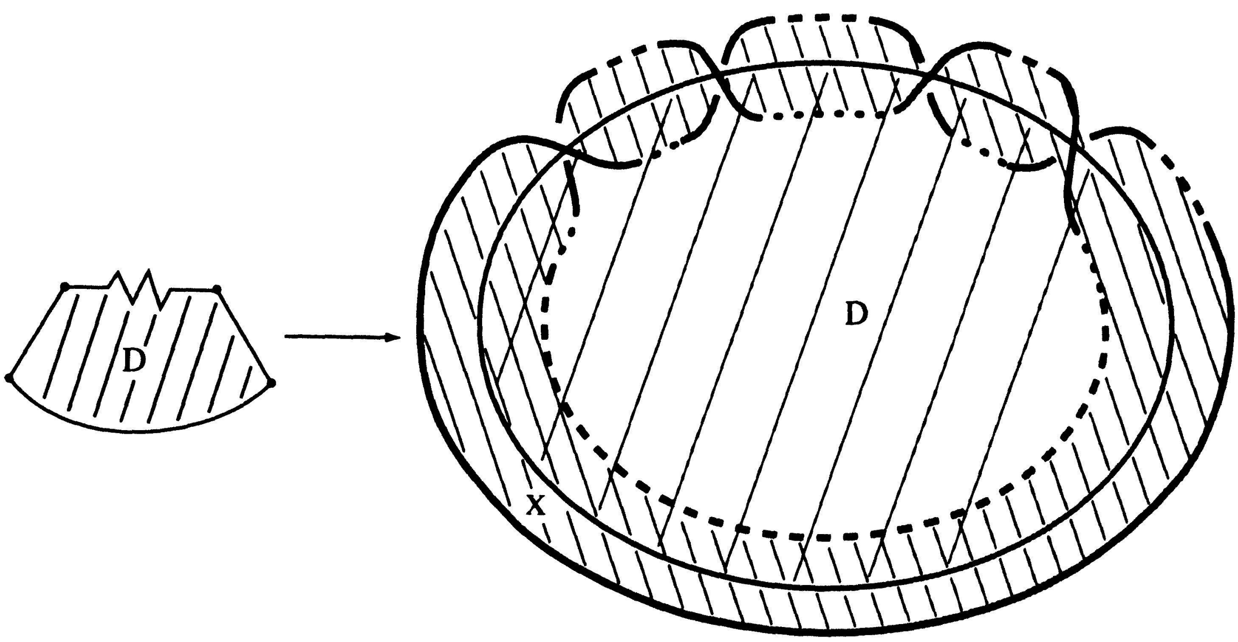


Figure 11. Disc D and annular subsurface.

Let γ be a Seifert cycle separating the plane into two regions A and A', both of which contain Seifert cycles.

Let G be the planar graph obtained by letting vertices correspond to the cycle γ and those Seifert cycles in A which are maximal (not nested within another Seifert cycle of A) and letting edges correspond to the half-twisted bands connecting these maximal cycles.

Since no half-twisted band separates R, no edge of G separates and hence G contains a cycle σ; choose σ innermost. Note that since R is orientable, σ contains an even number of edges. Furthermore, since k is alternating, the edges of G correspond to consistently oriented half-twists. In particular, σ corresponds to an annular subsurface X of R and a properly embedded disc $D \subset M = S^3 \backslash \mathring{N}(k)$ as shown in Figure 11.

Similarly, corresponding to A', define G', σ', X' and D'.

Since k is alternating, the edges of G and those of G' have opposite parity. Without loss of generality, we may assume that the edges of G are negatively twisted while those of G' are positively twisted.

Case 2. If the knot k is negative planar, let G denote the planar graph obtained by letting vertices correspond to the Seifert cycles and letting edges correspond to the half-twisted bands. Then define σ, X and D as above. If k is positive planar, then proceed similarly to construct G', σ', X' and D'.

By choosing orientations on R, D and D', we obtain depth 1 branched surfaces $B = \langle R, D \rangle \supset B_0 = R$ and $B' = \langle R, D' \rangle \supset B_0 = R$. (See Figure 12.)

LEMMA 1.6. *B and B′ are β-measured and there are sets of laminations* $\mathscr{L}$ *and* $\mathscr{L}'$, *fully carried by B and B′ respectively, which intersect* $\partial N(k)$ *in parallel curves. The laminations in* $\mathscr{L}$ *realize boundary slopes* $(-\infty, 0)$ *and those in* $\mathscr{L}'$ *realize* $(0, \infty)$.

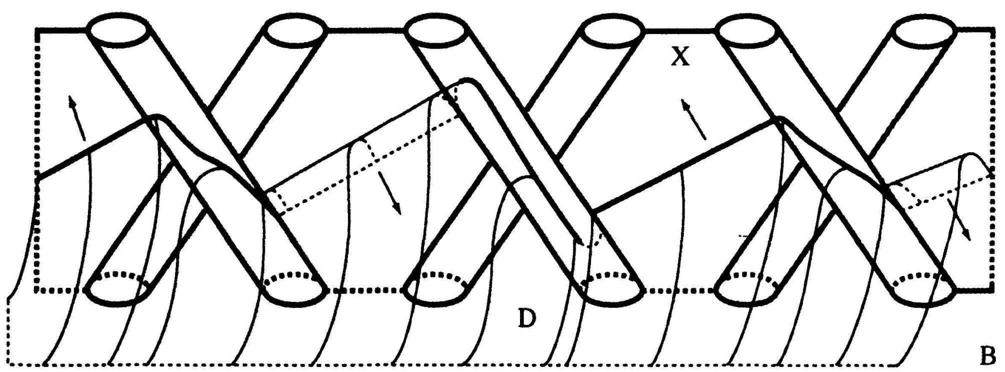


Figure 12. *B*.

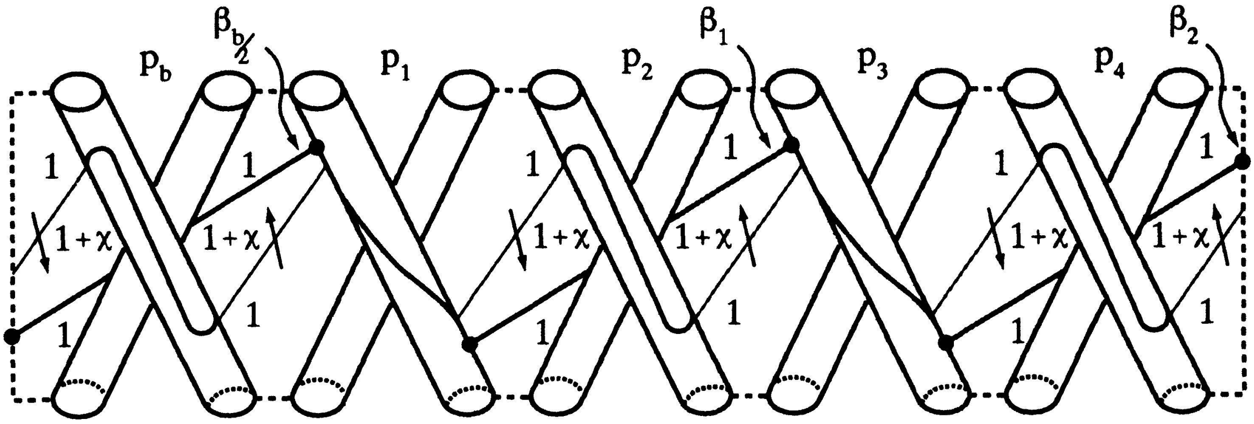


Figure 13. B is β-measured.

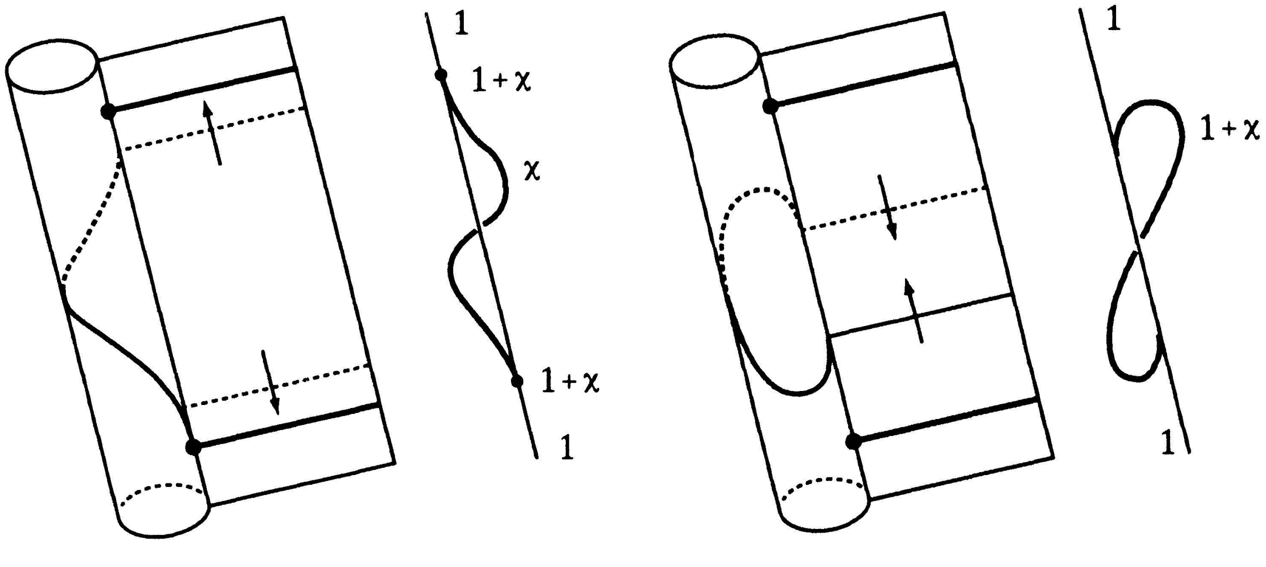


Figure 14. Train track near knot crossings.

Proof. By symmetry it suffices to prove the result for B. Let b denote the (even) length of the cycle σ. As shown in Figure 13, one can choose $b/2$ fusing arcs β_1 and assign weight $x > 0$ to the branch corresponding to D and weights $1, 1 + x$ to the branches corresponding to R in the resulting cut-open branched surface.

So B is β-measured and by choosing gluing functions along the β_i we define laminations λ_x fully carried by B.

Figure 14 depicts the behaviour of B at the crossing points and the corresponding piece of the train track $\tau = B \cap \partial N(k)$.

$\tau = B \cap \partial N(k)$ is obtained by piecing together $b/2$ copies of each of the tracks of Figure 14, in the order dictated by k. Note that by choosing linear gluing functions along the β_i, we induce linear scalings in τ over each subtrack of τ bounded by

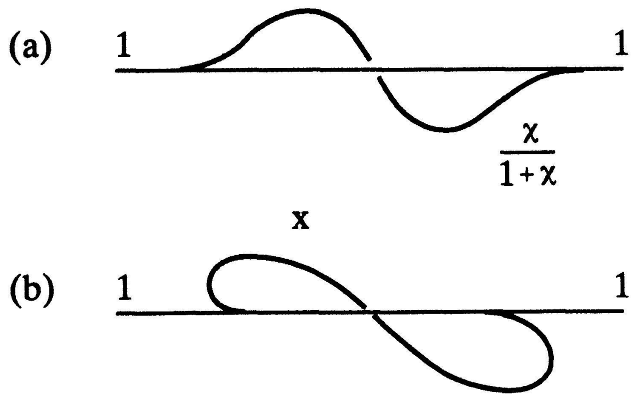


Figure 15. After scaling.

endpoints of β_i. After rescaling, the tracks of Figure 14 have measures as shown in Figure 15.

So τ is measured and for each p_i, i odd (i even), there is a slope contribution of $x/(x+1)$ ($-x$, respectively). Hence, the lamination λ_x intersects $\partial N(k)$ in parallel curves of slope

$$m_\lambda = \frac{b}{2}\left(-x + \frac{x}{x+1}\right) = \frac{b}{2}\left(\frac{-x^2}{x+1}\right).$$

As x ranges over $(0, \infty)$, the boundary slopes $(-\infty, 0)$ are realized. □

LEMMA 1.7. *The laminations λ_x contain no compact leaves.*

Proof. By symmetry it suffices to consider a lamination λ fully carried by the branched surface B. Suppose, by way of contradiction, that the lamination λ contained a compact leaf L. Since L is carried by B it induces a non-negative measure on B. We show that no such measure exists.

Let the measure induced on the branch of B corresponding to D be n. Since the slope of $\partial\lambda$ is nonzero, necessarily $n > 0$. The b arcs of ∂D cut the annular subsurface X into b regions with the branch conditions ensuring that for some nonnegative number m these regions have weights m and $m + n$ as shown in Figure 16.

Let $\partial_- X$ and $\partial_+ X$ denote the two boundary components of X. Since k is connected, there is an arc of k lying outside X and joining $\partial_- X$ to $\partial_+ X$. But this means that there is a region weighted both m and $m + n$. Hence $m = m + n$ and necessarily $n = 0$, contrary to assumption. □

LEMMA 1.8. *Let (M_1, γ_1) be the sutured manifold generated by the decomposition*

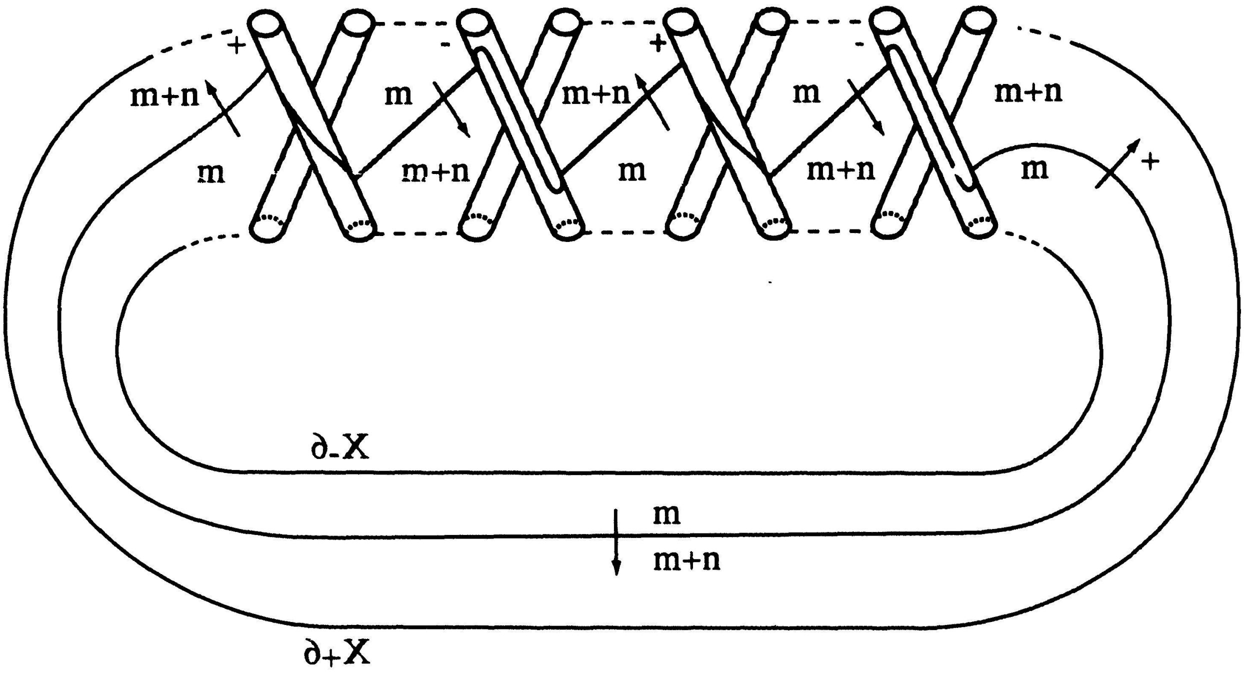


Figure 16. *B*.

$$(S^3\backslash\mathring{N}(R), A(k)) \overset{D_1}{\rightsquigarrow} (M_1, \gamma_1)$$

where D_1 *denotes either D or D'. Then* (M_1, γ_1) *is taut.*

Proof. The following is the argument of [7, p. 393], modified slightly to suit our needs.

Case 1. Consider first the case that k is a *planar* alternating knot. Recall that if we let G denote the planar graph obtained by letting vertices correspond to the Seifert cycles and letting edges correspond to the half-twisted bands, then D_1 arises in a natural way from an innermost cycle σ of G. (See Figure 11.) Figure 17 depicts the disc decomposition arising from one of the two possible choices of orientation of D_1.

We see that

$$(S^3\backslash\mathring{N}(R), A(k)) \overset{D_1}{\rightsquigarrow} (M_1, \gamma_1)$$

where

$$(M_1, \gamma_1) = (S^3\backslash\mathring{N}(R_1), A(l_1))$$

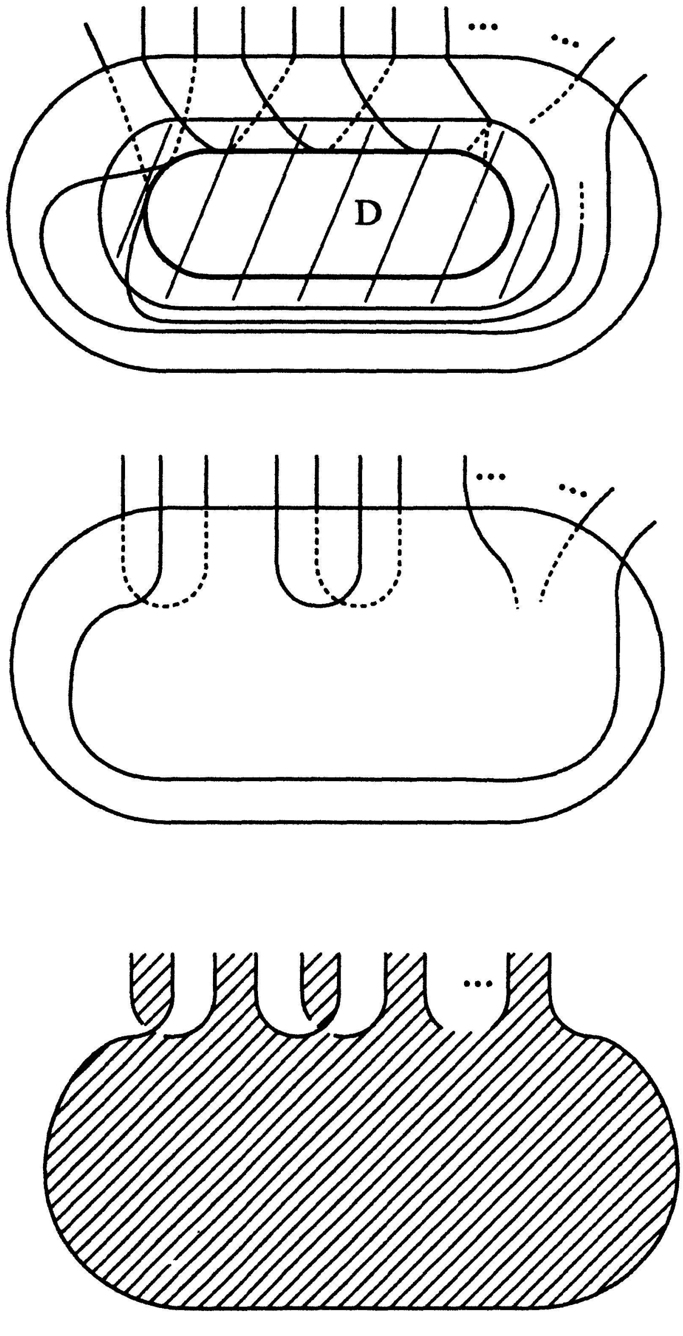


Figure 17. A disc decomposition.

for some planar alternating link l_1 and corresponding Seifert surface R_1. Since $\chi(\partial N(R_1)) = \chi(\partial N(R)) + 2$, $\chi(R_1) = \chi(R) + 1$. So an induction on the Euler characteristic reveals that (M_1, γ_1) is disc decomposable and hence taut.

Case 2. Proceed now to the case that k is *nonplanar*. Let $C_1, \ldots, C_m$ denote the separating Seifert cycles. Focus attention on one such C_i. C_i separates the plane into two regions A_1 and A_2, each of which contains Seifert cycles. Let R_1 denote the

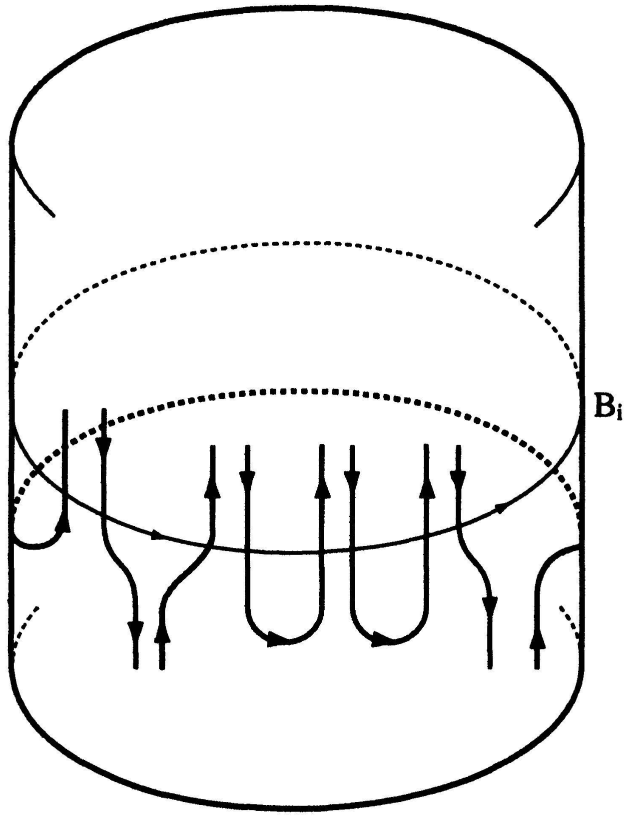


Figure 18. $N(R)$ in a neighbourhood of C_i.

surface generated by C_i and any Seifert cycles contained in A_1, together with the corresponding half-twisted bands. Similarly, define R_2 corresponding to A_2.

Corresponding to C_i there is a natural choice of disc decomposition, which we describe as follows. Isotope R so that the disc B_i' bounded by C_i is planar, lying in some sphere S, with $R_1 \backslash B_i'$ lying above S and $R_2 \backslash B_i'$ lying below S; i.e., R is a *Murasugi sum* of R_1 and R_2. Then $N(R)$ in a neighbourhood of C_i is a tube with the sutures describing 'flaps' hanging up or down (see Figure 18).

Choose $B_i = S^3 \backslash B_i'$, oriented so that ∂B_i is oriented consistently with C_i. Performing the disc decomposition reveals that

$$(S^3\backslash \mathring{N}(R), A(k)) \overset{B_1}{\rightsquigarrow} (S^3\backslash \mathring{N}(R_1), A(\partial R_1) \sqcup (S^3\backslash \mathring{N}(R_2), A(\partial R_2))$$

Repeating the process for each i, $1 \leq i \leq m$, we obtain the sequence

$$(S^3\backslash \mathring{N}(R), A(k)) \overset{B_1}{\rightsquigarrow} (M_1', \gamma_1') \overset{B_2}{\rightsquigarrow} \cdots \overset{B_m}{\rightsquigarrow} (M_m', \gamma_m')$$

where (M_m', γ_m') is a disjoint union of sutured manifolds of the form $(S^3\backslash \mathring{N}(F), A(l))$ for some planar alternating link l and corresponding planar surface

F. In particular, included among the components are the complements of the surfaces corresponding to the planar graphs G and G' respectively. So we may extend the above sequence by the decomposition

$$(M'_m, \gamma'_m) \overset{D_1}{\rightsquigarrow} (M'_{m+1}, \gamma'_{m+1})$$

and by Case 1, $(M'_{m+1}, \gamma'_{m+1})$ is taut.

Finally, we notice that we may choose the discs $D_1, B_1, \ldots, B_m$ to be pairwise disjoint and hence we may rearrange the above sequence to get

$$(S^3 \backslash \mathring{N}(R), A(k)) \overset{D_1}{\rightsquigarrow} (M_1, \gamma_1) \overset{B_1}{\rightsquigarrow} \cdots \overset{B_m}{\rightsquigarrow} (M'_{m+1}, \gamma'_{m+1})$$

Since $(M'_{m+1}, \gamma'_{m+1})$ is taut, so is (M_1, γ_1). □

We next think more carefully about the way in which λ_x lies in $N(B)$. In particular, we are interested in the effect on B of splitting open along compact subsurfaces of leaves of λ_z. We call P a *compact surface of contact for the pair* (λ_x, B) if B splits open along P, a compact subsurface of some leaf of λ_x, so that the components $(\partial P \times I) \backslash \partial N(k)$ bounding the $P \times I$ region created correspond exactly to some subset δ of $\partial_c N(B)$. In the proof of Corollary 1.10 we shall see that planar compact surfaces of contact provide a potential obstruction to extending the laminations λ_x to foliations with nice boundary behaviour. Happily we have the following:

PROPOSITION 1.9. *There are no compact surfaces of contact for* (λ_x, B).

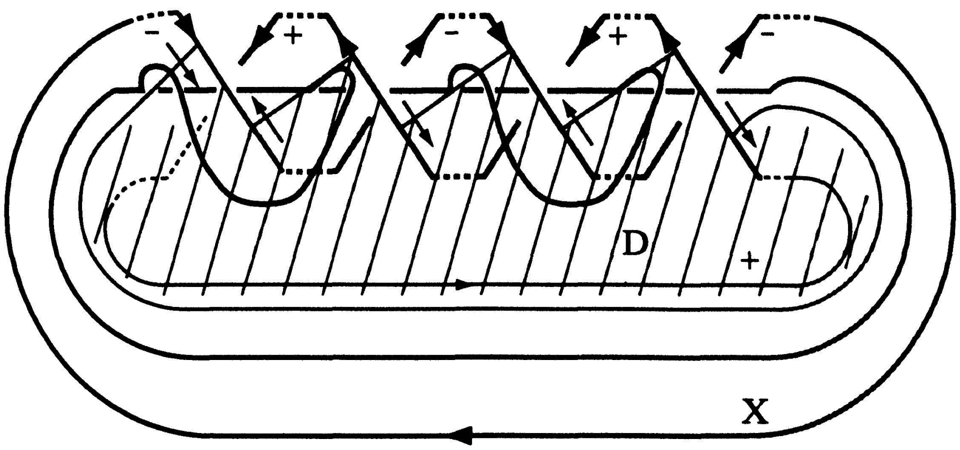
Proof. Suppose that there exists a compact surface P of contact for (λ_x, B). We shall use P to decompose the knot k into multiple components, thereby arriving at a contradiction.

We begin by partitioning the b pieces of $X \backslash \mathring{N}$ (∂D) into two sets. (See Figures 11 and 12 to recall notation. In particular, recall that σ is the core of X.) We do this as follows.

Let $\tau_\sigma \subset B$ be the train track pictured in Figure 19. The cross section of λ_x over τ_σ is a lamination Δ_x carried by τ_σ consisting of parallel curves of 'slope' $(b/2)x$.

CLAIM 0. *x is rational.*

Proof. Since B has a compact surface of contact P, necessarily τ splits open along a closed interval $J \subset P \cap N(\tau)$ such that the components $\partial J \times I$ bounding the $J \times I$ region created correspond exactly to some pair of components of $\partial_v N(\tau)$. But if some pair of components of $\partial_v N(\tau)$ bounds an $I \times I$ region in this way, then by



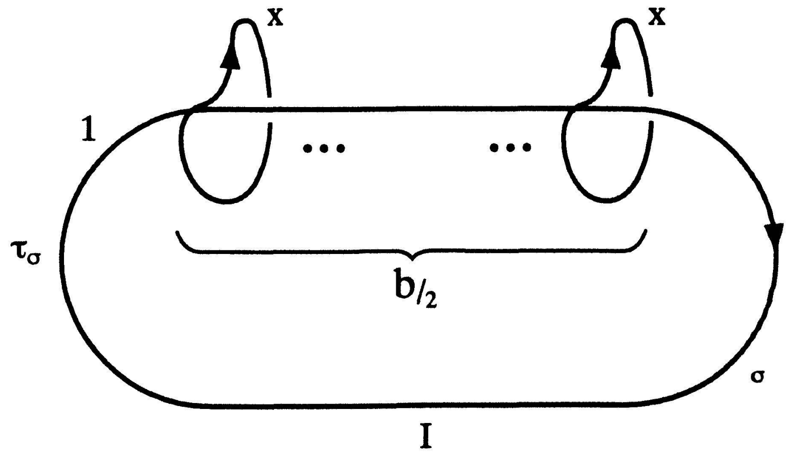


Figure 19. τ_σ.

symmetry each component of $\partial_v N(\tau)$ pairs up to bound such an $I \times I$ region. Therefore Δ_x is obtained from τ_σ by a finite amount of splitting; hence Δ_x is a collection of parallel simple closed curves and x is necessarily rational. $\square$

Let $N(\tau_\sigma)$ denote a 2-dimensional fibred neighbourhood of τ_σ carrying Δ_x. Choose an orientation for σ and denote by π_i, $1 \leq i \leq m$, (ϖ_j, $1 \leq j \leq n$) those arcs of $\partial_v N(\tau_0) \cap \delta$ for which splitting from the corresponding cusp leads one about σ in the positive (respectively, negative) direction. Denote by a_i (b_j) the branch point of τ_σ corresponding to π_i (respectively ϖ_j) We shall sometimes refer to these branch points as *distinguished* branch points. Notice that, after relabelling as necessary, each π_i pairs with ϖ_i to bound a rectangular component of $(P \times I) \cap N(\tau_\sigma)$. In particular, $n = m$.

We now assign a somewhat artificial but useful pairing to the branch points a_i, b_j. Start at one of the branch points a_i and move into and along σ in the positive direction. Define a counter, initially set to 1 and updated each time a distinguished branch point is passed: add $+1$ (-1) if the branch point corresponds to an a_j (respectively, b_j). Let $b_{i'}$ be the point corresponding to the first branch point for which the counter is again zero and let $[a_i, b_{i'}]$ denote the interval of σ which has been traversed.

We use this pairing to partition the components I of $\sigma\backslash\{$branch points of $\tau_\sigma\}$ into two sets. We shall say that I is *marked* if it is contained within some $[a_i, b_{i'}]$ interval. Otherwise, we shall say that I is *unmarked*. Note that when I is marked, the set of branch points in $[a_i, b_{i'}]$ lying to the left of $\mathring{I}$ and corresponding to points in $\{a_m\}$ outnumber those corresponding to points in $\{b_n\}$. So $N(\mathring{I})$ must have nonempty intersection with $P \times I$. We shall use this fact later, in the proof of Claim 2. (Unmarked intervals may or may not have nonempty intersection with $P \times I$.)

This partition of the components of $\sigma\backslash\{$branch points of $\tau_\sigma\}$ induces a partition of the b regions of $X\backslash\partial D$. We shall say that a region Y of $X\backslash\partial D$ is marked if the corresponding arc $Y \cap \sigma$ is marked. Otherwise, we shall say that Y is unmarked.

Recalling that our goal is in fact to partition k into multiple components, we establish the following two facts.

CLAIM 1. *Suppose that an arc of $k\backslash\partial X$ passes between two regions Y_1 and Y_2 of equal weight* (*either* 1 *or* $1+x$). *Then Y_1 and Y_2 are of the same type*: *either both are marked or both are unmarked.*

Proof. Imitating the construction of τ_σ, we construct a train track $\tau_\mu \subset B$ as follows. Let η be an arc of $R\backslash X$ connecting two regions Y_1 and Y_2 of equal weight. Choose a subarc v of σ intersecting $\partial D \cap X$ transversely in a minimal number of points and connecting the endpoints of η and let μ be the simple closed curve obtained by concatenating η and v. Denote by $\tau_\mu \subset B$ the train track obtained from μ by connecting branch points of $\partial D \cap \mu$ by disjoint arcs in D. τ_μ contains some subset $\tau \cup \varpi$ of the distinguished branch points a_i, b_i of τ_σ. Each element of π pairs up with a unique element of ϖ to bound a rectangular component of $(P \times I) \cap N(\tau_\mu)$; so in particular $|\pi| = |\varpi|$. But this is true only if Y_1 and Y_2 are of the same type. For suppose Y_1 is marked. Then $Y_1 \cap [a_i, b_{i'}] \neq \varnothing$ for some a_i. So either $Y_2 \cap [a_i, b_{i'}] \neq \varnothing$ or else one of $a_i, b_{i'}$ lies in v. But if a_i $(b_{i'})$ lies in v, $|\pi| = |\varpi|$ implies that there is a $b_{j'}$ (a_j, respectively) in v such that $Y_2 \cap [a_j, b_{j'}] \neq \varnothing$. In either case, Y_2 is also marked. By symmetry, it follows that Y_1 is marked if and only if Y_2 is marked. Hence, Y_1 and Y_2 are necessarily of the same type. □

CLAIM 2. *Suppose that an arc of $k\backslash\partial X$ passes between two regions Y_1 and Y_2 weighted respectively* $1+x$ *and* 1. *Then Y_1 and Y_2 are unmarked.*

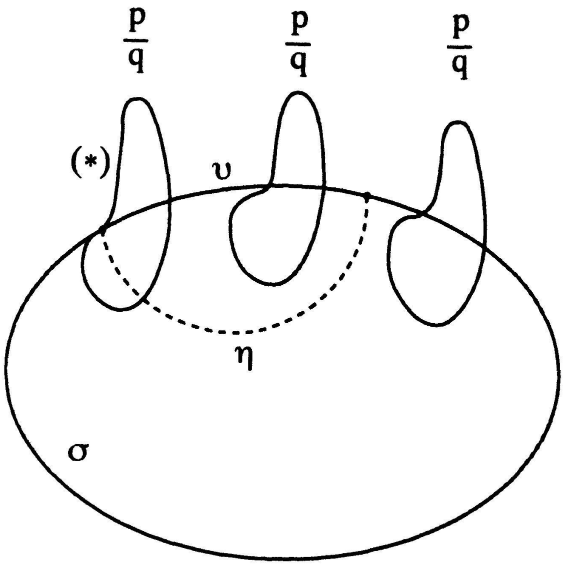


Figure 20. τ_0 and τ_μ.

Proof. Let η be an arc in $R \backslash X$ connecting the regions Y_1 and Y_2. Choose an arc v in X intersecting $\partial D \cap X$ transversely in a minimal number of points and connecting the endpoints of η and let μ be the simple closed curve obtained by concatenating η and v. Denote by $\tau_\mu \subset B$ the train track *with end* obtained from μ by letting the 'end' correspond to the branch point $\partial D \cap v$ having nonempty intersection with $\bar{Y}_1$ and by connecting the remaining branch points in pairs by disjoint arcs in D (see (*) in Figure 20). τ_μ and τ_σ are related as shown in Figure 20.

As noted earlier, the compactness of P implies that x is necessarily rational. So we may write $x = p/q$ where p and q are relatively prime positive integers. We shall show that if at least one of Y_1 and Y_2 is marked then P contains a properly embedded noncompact arc ω.

Case 1. Suppose first that Y_1 is marked. Then, as previously noted, there is a rectangular component of $(P \times I) \cap N(\tau_\sigma)$, corresponding to the pair (π_i, ϖ_i) say, passing through $N(Y_1)$. So we may let ω initially denote a path in $P \times 0$ starting at $\pi_i \cap (P \times 0)$ and following $P \times 0$ into $N(Y_1)$. We next show that it is possible to extend ω so that it passes infinitely about $N(\tau_\mu)$ without encountering either an earlier point of ω or a point $\varpi_j \cap (P \times 0)$.

To do so we first introduce as bookkeeping device a notion of *height* of the leaves of Δ_x in the branches of σ. Recall that associated with the measured track τ_σ there is a singular foliation φ_x obtained by starting with fibred rectangles $I \times$ [0, branch weight] corresponding to branches of τ_σ and identifying these rectangles at branch points using the branch equations ([18], p. 74). Define the *height* of a leaf of φ_x in the fibred rectangle $I \times$ [0, branch weight] to be the image of projection onto the second factor. We define the height of a leaf in Δ_x carried by a branch of τ_σ to be the image of the composition

$$\Delta_x \to \varphi_x \to [0, \text{branch weight}]$$

where the first map is the function given by pinching Δ_x down along the vertical fibres to obtain φ_x.

Returning to the construction of ω, note that τ_σ can be split to simple closed curve(s) of width $1/q$. So the endpoint of ω in $N(Y_1)$ can lie only at one of the heights $1/q, 2/q, \ldots, (p+q)/q$ in $N(Y_1)$. Hence ω enters $N(\tau_\mu)$ at height n_1/q for some integer n_1, $1 \leq n_1 \leq p+q$. If $n_1 = p+q$, we may follow ω along branch (*) of τ_σ (see Figure 20) and reenter $(N(\tau_\mu)$ at level q/q. So we may suppose $1 \leq n_1 < p+q$.

We next show that it is possible to extend ω so that it passes infinitely often about τ_μ without encountering a self-intersection or an endpoint $\varpi_j \cap (P \times 0)$. From $N(Y_1)$, extend ω to follow $P \times 0$ about η. The path thus described exits $N(\eta)$ at level

$$\left(\frac{1}{1+\frac{p}{q}}\right)\left(\frac{n_1}{q}\right) = \frac{n_1}{p+q}$$

Continuing ω about τ_μ until it again enters $N(Y_1)$, we see that ω meets an endpoint $\varpi_j \cap (P \times 0)$ or returns to the initial level only if

$$\frac{n_1}{p+q} = \frac{m}{q}$$

for some integer m. But this would imply

$$(p+q) \mid n_1$$

which is impossible since $1 \leq n_1 < p+q$. We may therefore extend ω so that it reenters η at height $n_1/(p+q) + n_2'/q$ for some integer n_2'.

More generally, after r such passes through η, the path ω is at height

$$\frac{n_1 q^{r-1}}{(p+q)^r} + \frac{n_2 q^{r-2}}{(p+q)^{r-1}} + \cdots + \frac{n_r}{p+q}$$

So ω further extended through τ_μ to $N(Y_1)$ can meet an endpoint $\varpi_j \cap (P \times 0)$ or return to an earlier height only if

$$\frac{n_1 q^{r-1}}{(p+q)^r} + \frac{n_2 1 q^{r-2}}{(p+q)^{r-1}} + \cdots + \frac{n_r}{p+q} = \frac{m_1 q^{r-2}}{(p+q)^{r-1}} + \cdots + \frac{m_{r-1}}{p+q} + \frac{m_r}{q}$$

Again, elementary number theory reveals that this is possible only if $(p+q) \mid n_1$. Since $1 \leq n_1 < p + q$, equality never holds.

Hence, ω may be extended infinitely often to yield a properly embedded noncompact arc in $P \times 0$. Since P is compact, this is impossible; so Y_1 must be unmarked.

Case 2. Case 2 proceeds similarly. We suppose, by way of contradiction, that Y_2 is marked. This implies that there is an arc ω in $P \times 0$ which enters Y_2 at some height n_1/q, $1 \leq n_1 \leq q$. Start at $\pi_i \cap (P \times 0)$ and follow $P \times 0$ into $N(Y_2)$. If $n_1 = q$ then ω may be extended to follow $P \times 0$ about η and along branch (*) of τ_σ (see Figure 14), reentering Y_1 at height q/q. The argument for Case 1 therefore applies. So suppose $n_1 < q$. The argument proceeds as in Case 1 except that we extend ω to follow $P \times 0$ about τ_μ in the opposite direction; so the scaling factor is inverted and $P \times 0$ might lead infinitely often through branch (*) of τ_σ. □

In the proof of Lemma 1.7, we saw that there is a subarc of $k \backslash \partial X$ joining two regions of $X \backslash \partial D$ of unequal weight. Hence, Claim 2 guarantees the existence of a $(1+x)$-weighted region Y which is unmarked.

In Figure 21 we see a subset of X containing the region Y and also the local behaviour of $s(\gamma_1)$. (Notice that we abuse notation and identify k with $A(k) = \partial N(k) \backslash \partial R$.)

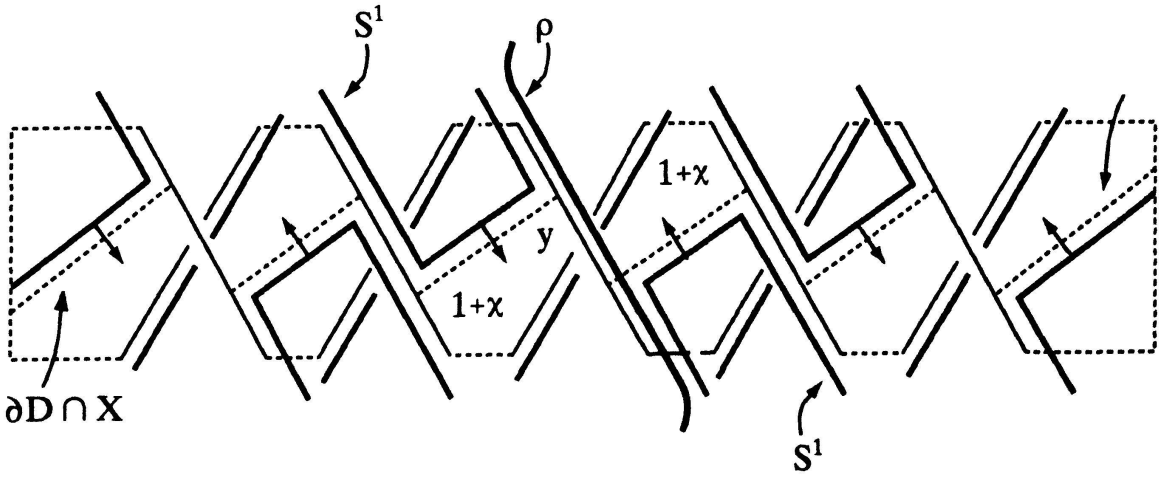


Figure 21. A subset of X.

Recall that δ denotes the subset of $\partial_n N(B) = A(\gamma_1)$ bounding the compact region $P \times I$. Since Y is unmarked, the components of $s(\gamma_1)$ labelled s' in Figure 21 do not belong to δ. So the subarc $\rho \subset k$ shown in Figure 21 is disjoint from both δ and marked regions. Consider starting at a point of ρ and travelling along k. ρ leaves X from an unmarked region. So, by Claim 1, the subarc of $k \backslash \partial X$ passing from ρ back into X leads to an unmarked region. Examining Figure 21, we see that within X, an arc of $k \cap X$ may pass from an unmarked region to a marked region only if it begins as an arc coincident with δ. Furthermore, an arc of $k \cap X$ disjoint from δ may reach an arc coincident with δ only from a marked region.

Hence, the component of k containing ρ is disjoint from δ. Since k is connected, necessarily $\delta = \emptyset$; so $P = \emptyset$. □

COROLLARY 1.10. *The laminations λ_x extend to foliations which meet $\partial N(k)$ in parallel curves and which possess only noncompact leaves.*

Proof. We apply Gabai's inductive construction (Description 2, [12]). Return to the taut sutured manifold (M_1, γ_1). Since taut, it possesses a well-groomed sutured manifold hierarchy (Theorem 4.2, [5]):

$$(M_1, \gamma_1) \overset{S_2}{\rightsquigarrow} (M_2, \gamma_2) \overset{S_3}{\rightsquigarrow} \cdots \overset{S_n}{\rightsquigarrow} (M_n, \gamma_n)$$

Notice that we may assume that the S_i, $i \geq 2$, intersect the sutures $s(\gamma_{i-1})$ away from k. For each i, $1 \leq i \leq n$, let B_i denote the branched surface obtained by applying Construction 0.4 to this sequence. We construct the desired foliation by inductively adding leaves corresponding to $S_2, \ldots, S_n$.

Consider $\partial S_2 \cap M_1$. We first extend the surface S_2 to a surface S_2' properly embedded in the complement of λ_1 and intersecting a finite collection $L_1, \ldots, L_s$ of leaves of λ_1 in properly embedded curves C which are pairwise disjoint and disjoint from $\partial\lambda_1$.

By definition of *well-groomed*, we know that for each component V of $R(\gamma_1)$, either $S_2 \cap V$ is a union of parallel, coherently oriented nonseparating simple closed curves or it is a union of arcs such that for each component δ of ∂V, $|\delta \cap \partial S_2| = |\langle \delta, \partial S_2 \rangle|$.

Consider first an arc σ of $\partial S_2 \cap A(\gamma_1)$. Proposition 1.9 guarantees the existence of an infinite strip extending S_2 at σ; i.e., there exists $V = [0, 1] \times I$ properly embedded in a piece of the complement of λ_1 cut off by $A(\lambda_1)$ and such that σ is the image of $0 \times I$.

Consider next a component γ of ∂S_2 contained in $A(\gamma_1)$. By definition of sutured manifold decomposition, γ and the corresponding suture are consistently oriented. So we extend S_2 by an annulus $\gamma \times I$ where $\gamma \times 0 = \gamma$ and γ_1 lies in a leaf parallel to $\partial_+ \pi_j$ for some j, $1 \leq j \leq m$.

Let S_2' be the leaf obtained by the above extensions. let $\mathscr{C}$ be the collection of curves $\partial S_2'$. To eliminate any intersections $(C_i \times 0) \cap (C_j \times 1) \subset L$, double the leaf L (Operation 2.1.2, [14]). Thicken the resulting leaves L (Operation 2.1.1, [14]), cut each leaf of $L \times I$ open along the corresponding curves of C, and reglue. We thus obtain a lamination λ_2, fully carried by B_2 and intersecting $\partial N(k)$ in coherently oriented curves of slope m_λ. It remains only to show that λ_2 contains no compact leaves. To see this, we note that simple closed curves $\partial S_2 \cap \lambda_1$ are nonseparating and hence give rise to infinite spiralling in the above construction. Similarly, the remaining simple closed curves of $\partial S_2' \cap \lambda_1$ do not bound compact leaves by Proposition 1.9. Arcs $\partial S_2 \cap R(\gamma_1)$ result in leaves containing properly embedded copies of $\mathbf{R}$. Hence, in all cases, there can be no compact leaves.

We now repeat the above construction $n-2$ times to obtain λ_n, fully carried by the branched surface B_n, intersecting $\partial N(k)$ in coherently oriented curves of slope m_λ, and containing no compact leaves. This is possible since at any step of the process, the sutures $A(\gamma_j)\backslash A(\gamma_1)$ cut off an infinite product $(N \times I, \partial N \times I)$ in the complement of λ_j; so infinite strip extension always exist.

Since the complementary regions of B_n are products λ_n extends to a foliation by taking copies of its boundary leaves in the canonical way. □

Hence we have constructed a set $\mathscr{F}$ of foliations without compact leaves such that for any *nonzero* number r in the interval $\mathbf{R}$ (for nonplanar k), $(-\infty, 0]$ (for negative planar k), or $[0, \infty)$ (for positive planar k), there is an element of $\mathscr{F}$ which intersects the boundary torus of M in parallel curves of slope r. Since there are no compact leaves, these foliations remain taut after Dehn filling with coefficient r. As mentioned in the introduction to this section, Gabai ([12]) has constructed taut foliations which intersect the boundary torus in parallel curves of slope 0 and remain taut after longitudinal Dehn filling. □

2. Generalizations

In this section we give a technical description of those knots for which the construction of Section 1 can proceed. We also describe a generalization of this construction which is more widely applicable (see Example 2.4) but which yields essential laminations rather than taut foliations.

The proofs of Section 1 generalize in a straightforward way to give us the following:

THEOREM 2.1. *Let k be a knot for which there exists a sequence*

$$(S^3\backslash \mathring{N}(k), \partial N(k)) \overset{R}{\rightsquigarrow} (M', \gamma) \overset{S}{\rightsquigarrow} (M'', \gamma'')$$

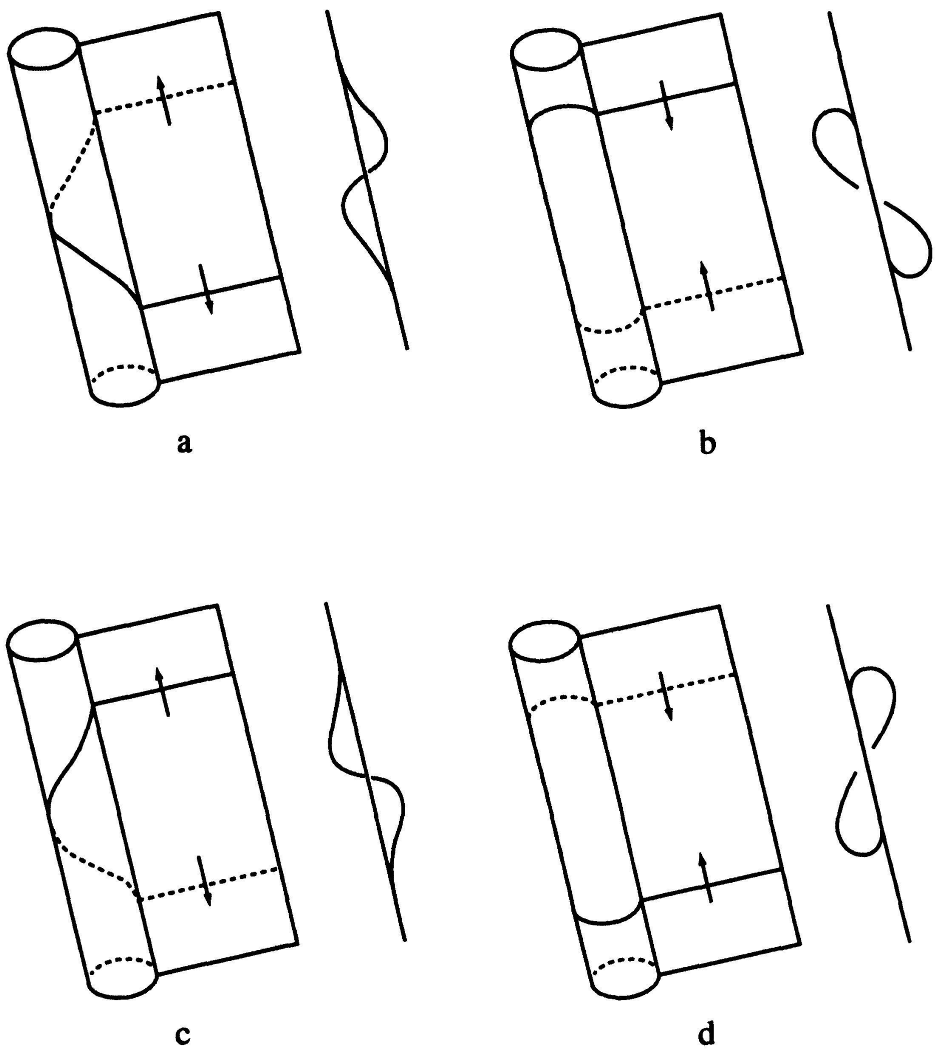


Figure 22.

of sutured manifold decompositions satisfying:

(i) (M', γ') *and* (M'', γ'') *are taut*,

(ii) $B = \langle R, S \rangle$ *contains no planar surfaces of contact*,

(iii) R *is a Seifert surface for* k, *and*

(iv) $\partial S \cap R_+(\gamma')$ *and* $\partial S \cap R_-(\gamma')$ *form a set of pairwise disjoint arcs.*

The arcs of intersection $\partial S \cap A(\gamma')$ *may be modelled on one of the four crossings of Figure 22. Let* p_1, p_2, n_1, n_2 *denote the number of crossings of the type of Figure 22a,b,c,d, respectively. Then* $M = S^3 \backslash N(k)$ *contains a set* $\mathscr{F}$ *of foliations such that for any number* r *in*

$$I = \{(p_1 - n_1)\frac{x}{x+1} + (p_2 - n_2)x \mid x \geq 0\rangle$$

there is a foliation of $\mathscr{F}$ which meets the boundary torus of M transversely in parallel curves of slope r. The foliations realizing the nonzero boundary slopes have no compact leaves.

COROLLARY 2.2. *Let k, I be as described in Theorem 2.1. Then any manifold obtained from S^3 by performing Dehn surgery along k with coefficient in I contains a taut foliation.*

We note that the disc-decomposable knots of Gabai ([7, 10]) provide an obvious source of examples and many yield to this construction. However, an explicit listing of those knots amenable to this method of attack seems difficult to realize.

By relaxing the requirement that (M', γ') and (M'', γ'') be taut, we obtain a more general construction. We shall abuse the terminology of Delman [(3)] as follows: Let λ be an essential lamination in a knot exterior M such that λ meets ∂M transversely in simple closed curves of slope r. Call λ *persistent* if it caps off to yield an essential lamination $\hat{\lambda}$ in the closed manifold $\hat{M}$ obtained from M by performing Dehn filling along k with coefficient r.

THEOREM 2.3. *Let k be a knot for which there exists a sequence*

$$(S^3\backslash\mathring{N}(k), \partial N(k)) \overset{R}{\rightsquigarrow} (M', \gamma') \overset{S}{\rightsquigarrow} (M'', \gamma'')$$

of sutured manifold decompositions such that:

(i) *the branched surface $B = \langle R, S\rangle$ obtained by applying Construction* 0.4 *is essential and contains no planar surfaces of contact,*

(ii) *R is a spanning surface for k, and*

(iii) *$\partial S \cap R_+(\gamma')$ and $\partial S \cap R_-(\gamma')$ form a set of pairwise disjoint arcs. The arcs of intersection $\partial S \cap A(\gamma')$ may be modelled on one of the four crossings of Figure 22. Let p_1, p_2, n_1, n_2 denote the number of crossings of the type of Figure 22a,b,c,d, respectively. Let r denote the slope of ∂R. Then $M = S^3\backslash N(k)$ contains a set $\mathscr{L}$ of persistent laminations such that for any number m in*

$$I = \left\{r + (p_1 - n_1)\frac{x}{x+1} + (p_2 - n_2)x \mid x \geq 0\right\}$$

there is a lamination in $\mathscr{L}$ which meets the boundary torus of M transversely in parallel curves of slope m.

Proof. The proofs of Section 1 generalize in an obvious way to yield a set $\mathscr{L}$ of laminations which meet $\partial N(k)$ in parallel curves. Since the branched surface $\langle R, S\rangle$

is essential, so are the laminations of $\mathscr{L}$. Therefore we need merely show that if λ is an element of $\mathscr{L}$ meeting $\partial N(k)$ in simple closed curves of slope r, then the lamination $\hat{\lambda}$ obtained from λ by capping off with discs is still essential.

We do so by describing a branched surface $\hat{B}$ (see [18]) which fully carries $\hat{\lambda}$. Let T denote the solid torus in $\hat{M}$ bounded by $\partial N(k)$. Choose a tubular neighbourhood $N = F \times (-1, 1)$ for ∂T with $F \times 0 = \partial T$. Note that the complementary regions of $\partial T \cap B$ are digons. Therefore, by choosing N small enough, we may assume that $(F \times (-1, 0)) \backslash B$ consists of product regions of the form *digon* $\times (-1, 0)$. For each such region *digon* $\times (-1, 0)$, add a branch to B by inserting a leaf corresponding to *digon* $\times \{-1/2\}$ and pinching according to some choice of local transverse orientation. Call the resulting branched surface B'. Next extend B' into $F \times (0, 1)$ by first adjoining $(B \cap \partial T) \times (0, 1)$ and then splitting B' near $B' \cap (F \times 1)$ along the train track $(B \cap \partial T) \times 1$ so that one gets a branched surface B'' such that $B'' \cap (F \times 1)$ consists of simple closed curves of slope r and B'' coincides with B' outside T. finally, extend B'' to $\hat{B}$ by capping off the simple closed curves $B'' \cap (F \times 1)$ with disjoing discs in $T \backslash (F \times (0, 1])$.

We note that the only compact leaf carried by B is the spanning surface R. So if $m \neq r$, then $\hat{B}$ can carry no compact leaves. And if $m = r$ but genus $R \geq 2$ then $\hat{B}$ carries no torus or sphere leaves. Finally, if $m = r$ and R caps off to give a torus leaf carried by $\hat{B}$ then necessarily R is a minimal genus Seifert surface and Gabai's construction ([12]) applies. Furthermore, each complementary region of $\hat{B}$ contained in T is a $D^2 \times I$ and the remaining complementary regions of $\hat{B}$ are isotopic to complementary regions of B. Hence, $\hat{B}$ satisfies all conditions of the definition of essential branched surface (see Definition 2.1, [15]) except possibly the condition that it contain no disc of contact.

We may therefore conclude that $\hat{\lambda}$ is essential if we can show that there are no discs of contact for $(\hat{\lambda}, \hat{B})$. Suppose, by way of contradiction, that D were such a disc of contact. If ∂D is contained in the vertical boundary of one of the $D^2 \times I$ complementary components contained in T then it caps off to reveal a sphere carried by $\hat{B}$. We have already noted that no such compact leaf exists. And if ∂D is contained in the vertical boundary of one of the remainder complementary regions then it corresponds to a planar surface of contact for (λ, B). But, by hypothesis, no such surface of contact exists. Hence there are no discs of contact for $(\hat{\lambda}, \hat{B})$. So $\hat{\lambda}$ is essential. □

Again, many examples of knots for which there exists an appealing collection of such persistent laminations may be generated but an explicit listing has thus far proved unrealizable.

We close with an example which demonstrates that the construction of Theorem 2.3 may succeed in situations for which the original construction of Theorem 2.1 fails.

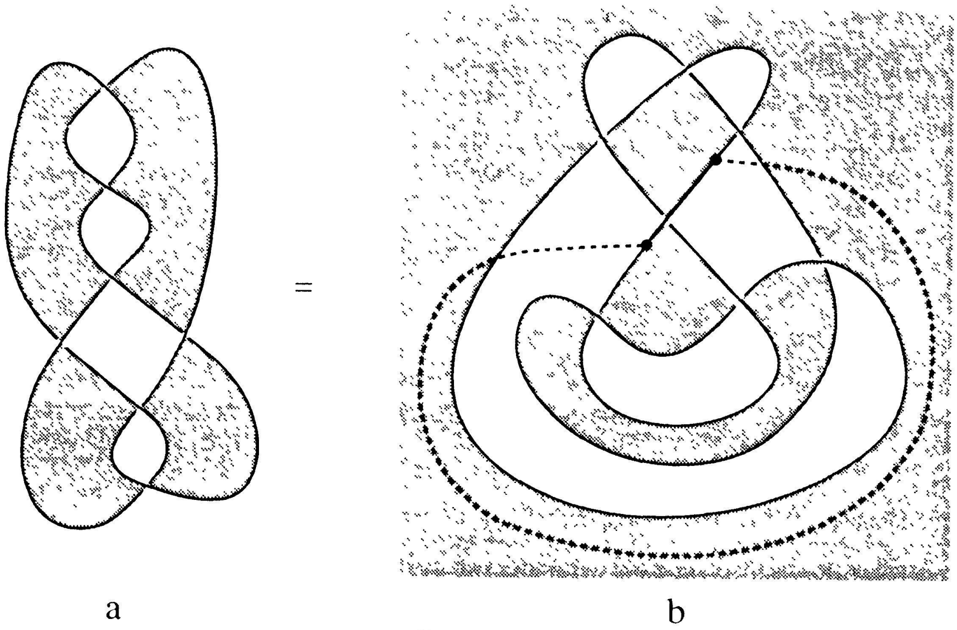


Figure 23. K.

EXAMPLE 2.4. Let k be the 2-bridge knot shown in Figure 23.

By Proposition 1 of [20] we know that there is a unique Seifert surface for k. Since this surface (see Figure 23a) contains only positive half-twists, the construction of Theorem 2.1 yields only taut foliations realizing the nonnegative boundary slopes.

However, the construction of Theorem 2.3 gives persistent laminations realizing all finite boundary slopes as follows. The isotopy of k indicated in Figure 23b yields

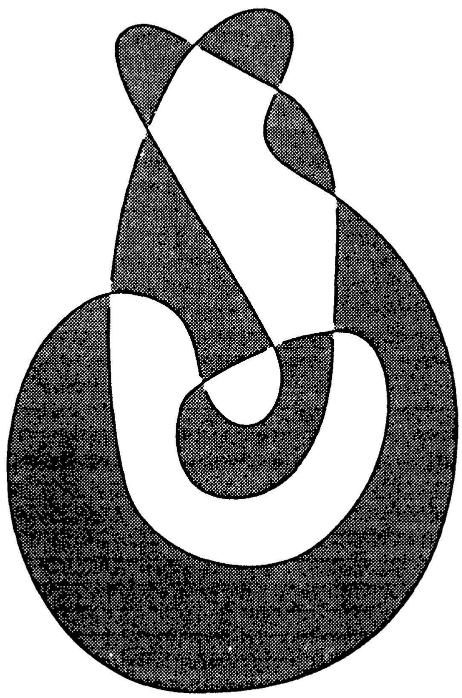


Figure 24. K.

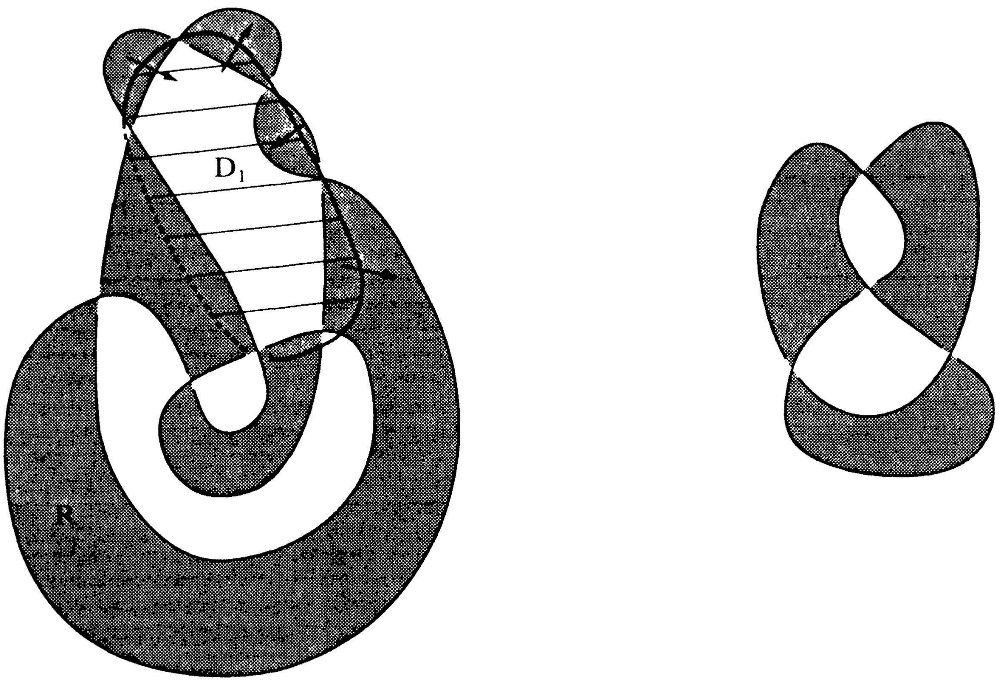


Figure 25. B_1.

the planar projection of Figure 24. Let R denote the spanning surface shown and let r denote the slope of ∂R.

Corresponding to this projection are branched surfaces B_1 and B_2 generating laminations realizing boundary slopes $[r, \infty)$ and $(-\infty, r]$ respectively (see Figures

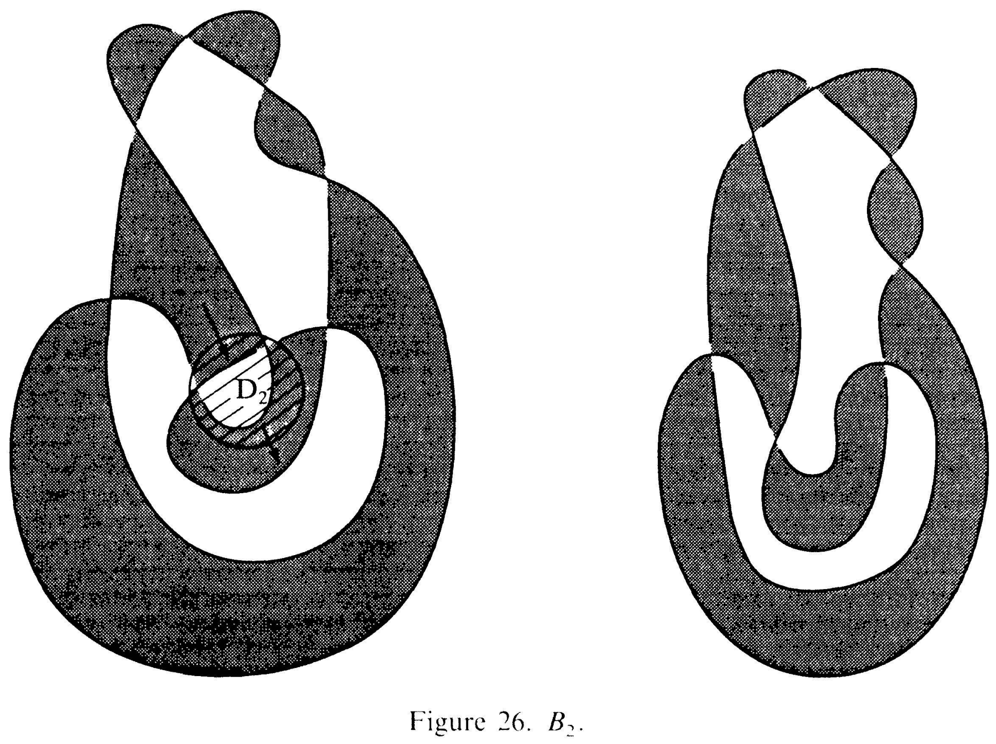


Figure 26. B_2.

25a and 26a). Regarded as a sutured manifold, $N(B)_1$) is homeomorphic to a regular neighbourhood of the incompressible and ∂-incompressible surface of Figure 25b; so B_1 is essential. Furthermore, the proof of Proposition 1.9 guarantees that if λ_1 is one of the laminations generated by B_1 then (λ_1, B_1) contains no planar surface of contact. Hence, the laminations generated by B_1 are persistent. Similarly, B_2 is essential and the laminations generated by B_2 are persistent. □

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