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Deleting-Inserting Theorem for smooth actions of finite nonsolvable groups on spheres*

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Abstract. The paper presents a method which allows to construct smooth finite nonsolvable group actions on spheres with prescribed fixed point data. The idea is to consider an action on a disk with the required fixed point data, and then to apply equivariant surgery to the equivariant double of the disk to remove the second copy of the fixed point data. In this paper, the method is applied to construct smooth group actions on spheres with exactly one fixed point, and more general actions with fixed point set diffeomorphic to any given closed stably parallelizable smooth manifold. The method is expected to be useful for constructions of smooth group actions on spheres with more complicated fixed point data.

Introduction

The main goal of this paper is to prove the Deleting-Inserting Theorem (Theorem 2.2) and to provide some of its applications (Theorems A and B) in the case of smooth G-actions on spheres for a large class of finite nonsolvable groups G. For a given smooth G-action on a homotopy sphere fulfilling some conditions (see Situation 2.1), the Deleting-Inserting Theorem allows us to remove any G-fixed point set connected component, or to create a number of its copies together with the copies of its equivariant normal bundle, so that to obtain a smooth G-action on a new homotopy sphere with the new G-fixed point data (the G-fixed point set and its equivariant normal bundle). In the applications, the idea is usually similar. First, we construct a linear or more generally, a smooth G-action on a disk D^n with prescribed G-fixed point data. Then we take the equivariant double $\partial(D^n \times D^1)$ of D^n, where G acts trivially on D^1, to get a smooth G-action on the sphere $S^n = \partial(D^n \times D^1)$ with "doubled" G-fixed point data. Finally, we apply the Deleting-Inserting theorem to remove the second copy fo the G-fixed point data, and thus to obtain a smooth G-action on S^n with the prescribed G-fixed point data.

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First, we deal with the problem of construction smooth one fixed point actions on the standard spheres. We wish to give a brief history of the problem. In 1946, in connection with their work on fiberings with singularities, D. Montgomery and H. Samelson [MS] made a comment to the effect that when a compact Lie group acts smoothly on the standard n-sphere S^n in such a way as to have one fixed point, it is likely that there must be a second fixed point. In 1977, E. Stein [St] presented the first examples of smooth group actions on a standard sphere with exactly one fixed point. He obtained such G-actions on the sphere S^7 for $G = SL(2, \mathbb{Z}/5)$ and $SL(2, \mathbb{Z}/5) \times \mathbb{Z}/r$, where $(120, r) = 1$. However, note that examples of one fixed point actions on *homology* spheres were obtained earlier (see, e.g., [Br], pp. 55–57, for a description of the Floyd–Richardson construction of a smooth G-action on the Poincaré homology 3-sphere with exactly one point left fixed under the action of the alternating group $G = A_5$). Then, during the period of 1978–1982, T. Petrie ([Pe1]–[Pe3]) constructed smooth one fixed point actions on homotopy spheres of finite odd order abelian groups having at least three noncyclic Sylow subgroups, as well as of the compact connected Lie groups S^3 and $SO(3)$. He also announced that there exist smooth, one fixed point actions on homotopy spheres of $SL(2, F)$ and $PSL(2, F)$ with characteristic F odd and $F \neq \mathbb{Z}/3$. It is well-known that $SL(2, F)$ (resp. $PSL(2, F)$) is a perfect (resp. simple) group when $|F| \geq 4$. We refer the reader to [BKS] and [M1]–[M3] for more details of the history of smooth one fixed point actions on sphere, including the discussions on the existence of such actions on low dimensional spheres. Also, we would like to note that the work [LT] deals with some kind of smooth group actions on homotopy spheres which cannot have exactly one fixed point, confirming the speculation of D. Montgomery and H. Samelson.

As the first application of the Deleting-Inserting Theorem, we show that many finite nonsolvable groups G have smooth actions on the standard spheres with any given finite number of G-fixed points (cf. [Pa1], Section 4 and [Sch], Problems on group actions, 7.16, p. 551). Recall that every group G has a unique maximal perfect subgroup R, and R is a characteristic subgroup of G. If G is finite, then R is the smallest term of the derived series of G (cf. [Ro], Exercises 5.4, p. 151) which amounts to saying that R is the unique smallest normal subgroup of G such that G/R is solvable (cf. [tD], Chapter IV, Proposition (6.7)). Hereafter, we set $G^{sol} = R$. Clearly, G is perfect if and only if $G^{sol} = G$. With the above notation, the following theorem holds.

THEOREM A. *Let G be a finite nonsolvable group such that $|G/G^{sol}|$ is odd. Then, for any even integer $\ell \geq 6$, there exists a smooth action of G on the standard sphere S^n of dimension $n = \ell(|G| - |G/G^{sol}|)$ with any given finite number $k \geq 1$ of G-fixed points.*

For $k = 1$, Theorem A provides further evidence to the conjecture (posed, e.g., in [Pa1], Section 4 or [Sch], Problems on group actions, 7.15, p. 551) that a compact Lie group which acts smoothly in a disk without fixed points, is also able to act smoothly on a sphere with exactly one fixed point (the converse follows from the Slice Theorem). By the work of R. Oliver [O1], [O2], the class of compact Lie groups admitting smooth fixed points free actions on disks is well-known, and the class contains all finite nonsolvable groups. In general, a finite group G has a smooth fixed point free action on a disk if and only if G has no normal series of the form $P \lhd H \lhd G$, where P is of p-power order (possibly $|P| = 1$), G/H is of q-power order (possibly $G = H$), and H/P is cyclic for two (possibly the same) primes p and q.

As the second application of the Deleting-Inserting Theorem, we generalize the result of Theorem A to the effect that we replace each isolated G-fixed point by any closed connected stably parallelizable smooth manifold. More precisely, we obtain the following theorem.

THEOREM B. *Let G be a finite nonsolvable group such that $|G/G^{sol}|$ is odd. Let M be a closed smooth manifold whose connected components all are stably parallelizable and all have the same dimension ≥ 0. Then there exists a smooth action of G on a standard sphere with G-fixed point set diffeomorphic to M.*

A few remarks on the material organization of this paper are in order. In Section 1, for a finite group G, we describe G-connected sums which we form in the paper, and using such a sum, we show that if G is not of prime power order, then smooth G-actions on homotopy spheres can often be converted to smooth G-actions on the standard spheres without changing the G-fixed point data (Proposition 1.3). In Section 2, we state the Deleting-Inserting Theorem (Theorem 2.2) which, using Proposition 1.3, yields a corollary (Corollary 2.3) applied in Section 5 to prove Theorems A and B. The proof of Theorem 2.2 is provided in Sections 3 and 4. The proof employs G-surgery and thus, as ingredients, we need a procedure to construct G-normal maps and a procedure to kill G-surgery obstructions in order to modify a given G-normal map so that to produce a homotopy equivalence. In Section 3, we construct a G-normal map by using a procedure invented by T. Petrie ([Pe1]–[Pe3]). This procedure involves a localization method of the cohomology theory $\omega_G^*()$ and the G-transversality arguments. In Section 4, we apply the technique of G-connected sum to kill the G-surgery obstructions. This technique is expected to be applicable in other situations, and it should be compared with the earlier one used to get explicit evaluation of the surgery obstructions in the construction of smooth, one fixed point A_5-actions on spheres presented by M. Morimoto ([M1]–[M3]).

In Theorems A and B we impose the restriction on G to the effect that $|G/G^{sol}|$ is odd, which is always the case when G is perfect or G is an extension of a finite odd order group by a finite (nontrivial) perfect group (cf. Remark to Proposition 5.1). In Appendix, we show that by using the work [BM], we can prove the Deleting-Inserting Theorem also under the weak gap hypothesis (WGH) imposed in (2.1.4). This allows us to get the result in Theorem A without the restriction that $|G/G^{sol}|$ is odd.

In this paper, even if not stated explicitly, all manifolds with group actions should be understood in the *smooth* category. We refer the reader to [Br] and [tD] for the background material on transformation groups, and to [DP], [PR], and [M4] for basic concepts and facts of equivariant surgery that we use in the paper. For a finite group G, we always denote by $\mathscr{S}(G)$ the set of all subgroups of G, consider the action of G on $\mathscr{S}(G)$ given by conjugation, and put $(H) = \{gHg^{-1} \mid g \in G\}$. Thus, for a G-invariant subset $\mathscr{F}$ of $\mathscr{S}(G)$, $(H) \subset \mathscr{F}$ if and only if $(H) \in \mathscr{F}/G$.

1. G-connected sum construction

Let G be a finite group. We start with the description of G-connected sums which we form in the paper. First, recall that the Burnside ring $\Omega(G)$ of G is the Grothendieck ring on the set of isomorphism classes of all finite G-sets with addition and multiplication established, respectively, by disjoint union and cartesian product. Additively, $\Omega(G)$ is the free abelian group on the set of isomorphism classes $[G/H]$ of the transitive G-sets G/H for all $(H) \subset \mathscr{S}(G)$. Moreover, $\Omega(G)$ is a commutative ring with unit $[G/G]$ (see, e.g., [tD], Chapter I, (2.18)).

Let X be an orientable G-manifold (with a chosen invariant Riemannian metric) such that the G-action is orientation preserving. Choose an orientation of X and write $+X$ (resp. $-X$) for X with the specified (resp. reversed) orientation and the same G-action. Let ω be an element of $\Omega(G)$, so that using the free abelian group presentation,

$$\omega = \sum_{(H) \subset \mathscr{S}(G)} a(H)[G/H]$$

for some integers $a(H)$. Put $\mathscr{F}(\omega) = \{H \in \mathscr{S}(G) \mid a(H) \neq 0\}$. Consider the disjoint union

$$\omega X = \coprod_{(H) \subset \mathscr{F}(\omega)} \coprod_{i=1}^{|a(H)|} G/H \times \operatorname{sign} a(H) X_i, \qquad X_i = X,$$

where each product $G/H \times (\pm X)$ with the diagonal G-action is defined for a chosen subgroup H within the given isotropy type in $\mathscr{F}(\omega)$. Now, from the disjoint union of $+X$ (denoted also by X) and ωX, we will form a G-connected sum using connected sum data

$$\mathscr{D}(H) = (x_i(H), \varphi_i(H))_{i=1}^{|a(H)|}, \qquad (H) \subset \mathscr{F}(\omega),$$

given as follows: each $x_i(H)$ is a point in X with isotropy subgroup H, and each $\varphi_i(H)$ is an orthogonal H-automorphism of the H-module determined on the tangent space $T_{x_i(H)}X$, such that

$$\deg \varphi_i(H) = \begin{cases} +1 & \text{when } a(H) < 0 \\ -1 & \text{when } a(H) > 0. \end{cases}$$

Moreover, we assume that the orbits of points $x_i(H)$ all are different. In order to form the G-connected sum, for all $(H) \subset \mathscr{F}(\omega)$ and all $i = 1, \ldots, |a(H)|$, choose sufficiently small H-invariant closed disk neighborhoods $D(x_i(H))$ of $x_i(H)$ in X so that H acts orthogonally on $D(x_i(H))$ and the G-invariant closed tubular neighborhoods $GD(x_i(H)) \cong G \times_H D(x_i(H))$ of the orbits $G(x_i(H))$ all are disjoint. By identifying each $D(x_i(H))$ with the H-invariant closed (unit) disk in $T_{x_i(H)}X$, we obtain the G-embedding

$$\begin{aligned} \Phi_i(H) : GD(x_i(H)) &\to G/H \times \operatorname{sign} a(H)X_i, \qquad X_i = X \\ gx &\mapsto (gH, g(\varphi_i(H) \cdot x)). \end{aligned}$$

From the disjoint union of $X \setminus \bigcup_{(H) \subset \mathscr{F}(\omega)} \bigcup_{i=1}^{|a(H)|} G(x_i(H))$ and

$$\coprod_{(H) \subset \mathscr{F}(\omega)} \coprod_{i=1}^{|a(H)|} (G/H \times \operatorname{sign} a(H)X_i) \setminus (G/H \times \{x_i(H)\}),$$

we form the quotient space by identifying $G(tx)$ with $\Phi_i(H)(G((1-t)x))$ for all $(H) \subset \mathscr{F}(\omega)$, $i = 1, \ldots, a(H)$, $x \in \partial D(x_i(H))$, and $0 < t < 1$. As the result, we obtain an orientable G-manifold (and the G-action is orientation preserving) which we denote by

$$X \#_G \omega X \quad \operatorname{rel}\{\mathscr{D}(H) \mid (H) \subset \mathscr{F}(\omega)\}$$

and call the *G-connected sum of X and ωX along orbits of types (H) with connected sum data $\mathscr{D}(H)$ for all $(H) \subset \mathscr{F}(\omega)$.*

Remark. The G-space $G/H \times (\pm X)$ with the diagonal G-action and the twisted product $G \times_H \operatorname{Res}_H^G(\pm X)$ are naturally G-diffeomorphic, so that in our construction, we can replace the cartesian products by the corresponding twisted products.

Now, using such a G-connected sum, we wish to show that G-actions on homotopy spheres can often be converted to G-actions on the standard spheres without changing the G-fixed point data (cf. [St], Section 4, as well as [M2], Section 3 and [M3], Section 3). Here, we say that a G-invariant subset $\mathscr{F}$ of $\mathscr{S}(G)$ is *efficient* if for all $(H) \subset \mathscr{F}$, the representatives H are proper subgroups of G and the integers $|G/H|$ are relatively prime, where (H) runs through the classes in $\mathscr{F}$, which amounts to saying that for some integers $b(H)$ given for all $(H) \subset \mathscr{F}$,

$$\sum_{(H) \subset \mathscr{F}} b(H)|G/H| = 1. \tag{1.1}$$

Clearly, if such a (nonempty) subset $\mathscr{F}$ of $\mathscr{S}(G)$ exists, G is not of prime power order.

EXAMPLE 1.2. Let G be a finite group not of prime power order. Then the set of all Sylow subgroups of G is an efficient G-invariant subset of $\mathscr{S}(G)$.

PROPOSITION 1.3. *Let G be a finite group not of prime power order, let $\mathscr{F}$ be a (nonempty) efficient G-invariant subset of $\mathscr{S}(G)$, and let Σ^n be a homotopy sphere of dimension $n \geq 5$. Suppose there exists a smooth, orientation preserving action of G on Σ^n with G-fixed point set M, such that for each $(H) \subset \mathscr{F}$, the submanifold $\Sigma^n_{(H)}$ of all orbits in Σ^n of type (H) has a connected component of positive dimension. Then there exists a smooth, orientation preserving action of G on the standard sphere S^n containing M as the G-fixed point set, such that the equivariant normal bundles $\nu(M, \Sigma^n)$ and $\nu(M, S^n)$ are equivalent as G-vector bundles.*

Proof. Consider Σ^n with a chosen orientation as an element of Θ_n, the group of oriented homotopy n-spheres with addition given via connected sum. In Θ_n, $\Sigma^n \#(-\Sigma^n) = 0$ and more general, for any integer k, $\Sigma^n \# (k\Sigma^n) = (1+k)\Sigma^n$. Moreover, Θ_n is a finite group (see [KM]). Assume $|\Theta_n| > 1$ (otherwise, there is nothing to prove).

Let $\{b(H) \mid (H) \subset \mathscr{F}\}$ be a set of integers $b(H)$ such that (1.1) holds. For each $(H) \subset \mathscr{F}$, choose an integer $c(H)$ so that

$$a(H) = (|\Theta_n| - 1)b(H) + |\Theta_n| c(H) < 0.$$

Since for each $H \in \mathscr{F}$, Σ^n has infinitely many points with isotropy subgroup H, we can form the following G-connected sum:

$$\Sigma^n \#_G \Big(\sum_{(H) \subset \mathscr{F}} a(H)[G/H] \Big) \Sigma^n \operatorname{rel}\{\mathscr{D}(H) \mid (H) \subset \mathscr{F}\},$$

where in all connected sum data $\mathscr{D}(H) = (x_i(H), \varphi_i(H))_{i=1}^{|a(H)|}$, each $\varphi_i(H)$ is the identity on $T_{x_i(H)}\Sigma^n$. We claim that the underlying space S^n of this G-connected sum is the standard n-sphere. In fact, in Θ_n,

$$\begin{aligned} S^n &= \Big(1 + \sum_{(H)\subset\mathscr{F}} a(H)|G/H|\Big)\Sigma^n \\ &= \Big(1 + \sum_{(H)\subset\mathscr{F}} (|\Theta_n| - 1)b(H)|G/H|\Big)\Sigma^n \\ &= \Big(1 - \sum_{(H)\subset\mathscr{F}} b(H)|G/H|\Big)\Sigma^n \\ &= 0, \end{aligned}$$

proving the claim. Clearly, for the G-action on S^n (which is orientation preserving), M occurs as the G-fixed point set and $\nu(M, \Sigma^n) \cong \nu(M, S^n)$. □

Remark. As the proof of Proposition 1.3 shows, in order to get the result, we do not need to assume that for each $(H) \subset \mathscr{F}$, $\Sigma^n_{(H)}$ has a connected component of positive dimension. What we need is that for each $H \in \mathscr{F}$, Σ^n has sufficiently many points with isotropy subgroup H.

2. Deleting-Inserting Theorem

In the current paper, we use three kinds of dimension conditions, and the usage does not necessarily coincide with that of other papers. The dimension conditions are the *strong gap hypothesis*, the *gap hypothesis*, and the *weak gap hypothesis*. Let G be a finite group and let X be a G-manifold. As usual, such hypotheses, put some dimension conditions on the connected components $X_1^H, X_2^H, X_3^H, \ldots$ of the H-fixed point sets X^H for subgroups H of G. More precisely, let $\mathscr{F}$ be a subset of $\mathscr{S}(G)$. We say that X satisfies the *strong gap hypothesis for* $\mathscr{F}$, if

$$(\mathrm{SGH})\begin{cases} \dim X_i^H \geq 5 \textit{ for all } H \in \mathscr{F}, i = 1, 2, 3, \ldots, \textit{and} \\ \dim X_i^H > 2(\dim X_j^K + 1) \textit{ for } K \in \mathscr{S}(G), \textit{provided} \\ H \subsetneqq K \textit{ and } X_i^H \supset X_j^K. \end{cases}$$

We say that X satisfies the *gap hypothesis for* $\mathscr{F}$, if

$$(\mathrm{GH})\begin{cases} \dim X_i^H \geq 5 \textit{ for all } H \in \mathscr{F}, i = 1, 2, 3, \ldots, \textit{and} \\ \dim X_i^H > 2 \dim X_j^K \textit{ for } K \in \mathscr{S}(G), \textit{provided} \\ H \subsetneqq K \textit{ and } X_i^H \supset X_j^K. \end{cases}$$

We say that X satisfies the *weak gap hypothesis for* $\mathscr{F}$, if

$$(\mathrm{WGH})\begin{cases} \dim X_i^H \geq 5 \textit{ for all } H \in \mathscr{F}, i = 1, 2, 3, \ldots, \textit{and} \\ \dim X_i^H \geq 2 \dim X_j^K \textit{ for } K \in \mathscr{S}(G), \textit{provided} \\ H \subsetneqq K \textit{ and } X_i^H \supset X_j^K, \textit{ where the equality} \\ \textit{may happen only when } |K/H| = 2. \end{cases}$$

Following Oliver [O2], p. 92, we say that a G-invariant subset $\mathscr{F}$ of $\mathscr{S}(G)$ is a *separating family*, if for any pair $H \lhd K$ of subgroups of G such that K/H is of prime (power) order, either both H and K are in $\mathscr{F}$ or neither is. For a finite nonsolvable group G, we write $\mathscr{F}(G)$ for the smallest separating family containing G. Note that $H \in \mathscr{F}(G)$ if and only if $H \supseteq G^{sol}$. Now we set

$$\mathscr{I}(G) = \mathscr{S}(G) \backslash \mathscr{F}(G).$$

For example, if G is perfect, $\mathscr{F}(G) = \{G\}$ so that $\mathscr{I}(G) = \mathscr{S}(G)\backslash\{G\}$. In this paper, for a finite nonsolvable group G, we say that a separating family $\mathscr{I}$ of subgroups of G is *proper* if $\mathscr{I}$ is closed under taking subgroups, contains the trivial subgroup of G, and does not contain G. Clearly, such an $\mathscr{I}$ contains all solvable subgroups of G, and $\mathscr{I}$ is contained in $\mathscr{I}(G)$. Thus, in $\mathscr{S}(G)$, the smallest proper separating family consists of all solvable subgroups of G and $\mathscr{I}(G)$ is the largest proper separating family.

SITUATION 2.1. Let G be a finite nonsolvable group and let $\mathscr{I}$ be a proper separating family in $\mathscr{S}(G)$. Let Y be a homotopy sphere with a smooth action of G such that the G-fixed point set Y^G is nonempty and the following four conditions hold.

(2.1.1) $\mathscr{I} = \mathrm{Iso}(G, Y\backslash Y^G)$, the set of all isotropy subgroups appearing for the G-action on $Y\backslash Y^G$.

(2.1.2) For each $H \in \mathscr{I}$, Y^H is connected and simply connected.

(2.1.3) For each $H \in \mathscr{I}$, each element of $N_G(H)$, the normalizer of H in G, acts on Y^H via an orientation preserving transformation.

(2.1.4) Y satisfies the strong gap hypothesis (SGH) (resp. gap hypothesis (GH)) for $\mathscr{I}$.

Furthermore, let M be a manifold obtained from all connected components $M_1, M_2, \ldots, M_k$ of Y^G by deleting some of them and, perhaps, inserting copies of some of the maintaining connected components. More precisely, for a sequence $d_1, d_2, \ldots, d_k$ of integers $d_i \geq 0$, define M as the disjoint union

$$M = \coprod_{i=1}^{k} M_i \overbrace{\amalg \cdots \amalg}^{d_i\text{-times}} M_i,$$

so that $d_i = 0$ means that we have deleted the M_i connected component. Finally, let ν be the G-vector bundle over M which restricts to $\nu(M_i, Y)$ over each copy of the connected component M_i of M.

THEOREM 2.2. (Deleting-Inserting Theorem) *Let* $G, \mathcal{I}, Y, M$, *and* ν *be as in Situation* 2.1. *Then there exists a smooth action of G on a homotopy sphere X with G-fixed point set* X^G *diffeomorphic to M and the equivariant normal bundle* $\nu(X^G, X)$ *equivalent to* ν, *such that the following four conditions hold.*

(2.2.1) $\mathrm{Iso}(G, X \backslash X^G) = \mathcal{I}$.

(2.2.2) *For each* $H \in \mathcal{I}$, X^H *is connected and simply connected.*

(2.2.3) *For each* $H \in \mathcal{I}$, *each element of* $N_G(H)$ *acts on* X^H *via an orientation preserving transformation.*

(2.2.4) *For each* $H \in \mathcal{I}$, $\dim X^H = \dim Y^H$.

The proof of Theorem 2.2 will be provided in Sections 3 and 4. Clearly, in the condition (2.1.4), the restriction (SGH) implies the restriction (GH) on Y for $\mathcal{I}$. The reason we distinguish the two cases is to stress that in the proof of Theorem 2.2, the equivariant surgery theories (in particular, the surgery obstruction groups) that we use are different for the two cases. When (SGH) (resp. (GH)) is satisfied, we may apply the equivariant surgery from [DP], [PR], or [LüMa] (resp. [M4]). The case of (SGH) is sufficient to use Theorem 2.2 in order to prove Theorems A and B. Now, Theorem 2.2, the proof of Proposition 1.3, and Example 1.2 yield immediately the following corollary.

COROLLARY 2.3. *Let* $G, \mathcal{I}, Y, M$, *and* ν *be as in Situation* 2.1. *Then there exists a smooth action of G on the standard sphere* S^n *of dimension* $n = \dim Y$, *such that* S^n *contains M as the G-fixed point set with equivariant normal bundle* ν, *and* $\mathrm{Iso}(G, S^n \backslash M) = \mathcal{I}$.

Now we wish to recall (e.g. from [tD], Chapter IV) that for a finite group G, the Burnside ring $\Omega(G)$ of G (originally defined to be the Grothendieck group on all finite G-sets) can be defined as the set of equivalence classes $[X]$ for all finite G-CW complexes X, where $[X] = [Y]$ if and only if the Euler characteristics $\chi(X^H)$ and $\chi(Y^H)$ are equal for all $H \in \mathcal{S}(G)$. Addition and multiplication in $\Omega(G)$ are defined by using (again) disjoint union and cartesian product, respectively. Clearly, $[X] = 0$ in $\Omega(G)$ if and only if $\chi(X^H) = 0$ for all $H \in \mathcal{S}(G)$, and $-[X] = [X \times F]$, where F is a finite CW complex with trivial G-action and $\chi(F) = -1$. Moreover, $[X] = 1$ in $\Omega(G)$ if and only if $\chi(X^H) = 1$ for all $H \in \mathcal{S}(G)$. In the proof of Theorem 2.2, we will make use of the following ring homomorphisms defined on $\Omega(G)$ for any $H \in \mathcal{S}(G)$,

$$\chi_H : \Omega(G) \to \mathbb{Z}, \qquad [X] \mapsto \chi(X^H),$$

$$\mathrm{Res}^G_H : \Omega(G) \to \Omega(H), \qquad [X] \mapsto [\mathrm{Res}^G_H(X)],$$

$$\mathrm{Fix}^G_H : \Omega(G) \to \Omega(N/H), \qquad [X] \mapsto [X^H],$$

where $N = N_G(H)$, so that X^H has the canonical action of N/H. Clearly, for an element $\mu \in \Omega(G)$, $\mathrm{Res}^G_H(\mu) = 0$ in $\Omega(H)$ if and only if $\chi_K(\mu) = 0$ for all subgroups K of H. Now, recall also that there is an embedding of rings,

$$\Omega(G) \longrightarrow \bigoplus_{(H) \subset \mathcal{S}(G)} \mathbb{Z}$$

$$\omega \mapsto (\chi_H(\omega) \mid (H) \subset \mathcal{S}(G)),$$

and an element $(a(H) \mid (H) \subset \mathcal{S}(G))$ lies in the image of this embedding if and only if

$$\sum_{(K)} n(H, K)a(K) \equiv 0 \qquad \mathrm{mod}|N_G(H)/H|$$

for all $(H) \subset \mathcal{S}(G)$, where $n(H, K)$ are some integers with $n(H, H) = 1$, and the sum is over the G-conjugacy classes (K) such that $H \lhd K$ and K/H is cyclic (see, e.g., [tD], Chapter IV, Theorem (5.7)). In particular, we get the following idempotent $\iota_{\mathcal{J}}$ (different than 0 and 1) in the Burnside ring of G (cf. [tD], Chapter IV, Proposition (7.1) and Theorem (7.7)).

PROPOSITION 2.4. *Let G be a finite nonsolvable group and let $\mathcal{J}$ be as in Situation* 2.1. *Then there exists an idempotent $\iota_{\mathcal{J}}$ in the Burnside ring $\Omega(G)$ such that for any $H \in \mathcal{S}(G)$,*

$$\chi_H(\iota_{\mathcal{J}}) = \begin{cases} 0 & \text{when } H \in \mathcal{J}, \\ 1 & \text{when } H \notin \mathcal{J}. \end{cases}$$

COROLLARY 2.5. *Let G be a finite nonsolvable group and let $\mathcal{J}$ be as in Situation* 2.1. *Then there exists an element μ in the Burnside ring $\Omega(G)$ such that the following two conditions hold.*

(2.5.1) *For the homomorphism $\chi_G : \Omega(G) \to \mathbb{Z}$, $\chi_G(\mu) = 1$.*

(2.5.2) *For all $H \in \mathcal{J}$, $\mathrm{Res}^G_H(\mu) = 0$ in $\Omega(H)$.*

3. Construction of G-normal maps

For a finite group G and a G-module V, the product G-vector bundle $X \times V$ over a G-space X we denote by $\varepsilon(V)$ provided the base space X is obvious from the context and we use the notation $\varepsilon(\mathbb{R})$ when V is just $\mathbb{R}$ with the trivial action of G. Moreover, if ξ is a G-vector bundle over X (with a chosen invariant inner product), then for a subgroup H of G and a subspace A of X^H, we get the decomposition of $\xi \mid A$ into the Whitney sum,

$$\xi \mid A \cong (\xi \mid A)_H \oplus (\xi \mid A)^H,$$

of the H-nontrivial summand $(\xi \mid A)_H$ and the H-trivial summand $(\xi \mid A)^H$.

In the current paper, for a finite group G, by a *G-normal map* (*with source manifold X and target manifold Y*), we mean a triple $\mathbf{w} = (f; b; c)$ defined as follows.

Source-Target Data. $f: X \to Y$ is a G-map between two closed G-manifolds X and Y with $\mathscr{I} = \mathrm{Iso}(G, X \backslash X^G) = \mathrm{Iso}(G, Y \backslash Y^G)$, and for each $H \in \mathscr{I}$, X^H and Y^H are oriented and (usually) connected, $\dim X^H = \dim Y^H$, and $f^H : X^H \to Y^H$ has degree one.

Bundle Data. b is a stable G-vector bundle isomorphism between the tangent bundle TX and the induced bundle $f^*(TY)$; that is,

$$b : TX \oplus \varepsilon(V) \xrightarrow{\cong} f^*(TY) \oplus \varepsilon(V)$$

is a G-vector bundle isomorphism for some real G-module V, and c is a collection of H-vector bundle isomorphisms,

$$c_H : (TX \mid X^H)_H \xrightarrow{\cong} f^*(TY \mid Y^H)_H, \qquad H \in \mathscr{I},$$

such that on each

$$((TX \mid X^H) \oplus \varepsilon(V))_H = (TX \mid X^H)_H \oplus \varepsilon(V_H),$$

b restricts to c_H on $(TX \mid X^H)_H$ and the identity on $\varepsilon(V_H)$.

If X^H and Y^H are not connected, we assume that f^H induces a bijection $f^H_* : \pi_0(X^H) \to \pi_0(Y^H)$ between the sets of the connected components of X^H and Y^H, respectively, and define c similarly as above by restricting the bundles to the connected components, so that c occurs as "a part" of b. More precisely, c is a $\Pi_G(X)$-bundle map related with b as described in [Pe3], p. 9, by using the notion of stabilization (see [DP], Definitions 4.11 and 4.12 for the notions of $\Pi_G(X)$-bundle map and stabilization in more general circumstances).

The goal of this section is to prove the following theorem.

THEOREM 3.1. *Let G, $\mathcal{I}$, Y, M, and ν be as in Situation* 2.1, *and assume that for each $H \in \mathcal{I}$, Y^H has a specified orientation. Then there exists a G-normal map $\mathbf{w} = (f; b; c)$ with source G-manifold X such that X^G and M are diffeomorphic, and the G-vector bundles $\nu(X^G, X)$ and ν are equivalent.*

For a finite group G, we will make use of the G-equivariant cohomology theory defined as follows. Let (X, A) be a pair of finite G-CW complexes (X contains A as a G-invariant subcomplex) with a base point in A. For a real G-module U, we let $U^\bullet = U \cup \{\infty\}$ to be the one-point compactification of U with ∞ as base point, and note that there is a canonical pointed G-homeomorphism $U^\bullet \cong S(U \oplus \mathbb{R})$ with 0 in $U^\bullet$ corresponding to $(0, -1)$ in $S(U \oplus \mathbb{R})$ and ∞ in $U^\bullet$ corresponding to $(0, 1)$ in $S(U \oplus \mathbb{R})$. For any integer $q \geq 0$, put

$$\omega_G^q(X, A) = \varinjlim_n [(n\mathbb{R}[G])^\bullet \wedge (X/A), (n\mathbb{R}[G])^\bullet \wedge S^q]_G^0,$$

where $[\ ,\ ]_G^0$ denotes the set of all G-homotopy classes of base point preserving G-maps between the occuring smash products and $S^q = \mathbb{R}^q \cup \{\infty\}$ has the trivial G-action.

For a finite G-CW complex Y, we put

$$\omega_G^q(Y) = \omega_G^q(Y \amalg pt, pt),$$

where the isolated point pt is the base point of $Y \amalg pt$. Assume U is the realification of a complex G-module including $\mathbb{C}[G]$. Then one has the isomorphisms

$$\Omega(G) \cong \omega_G^0(pt) \cong [U^\bullet, U^\bullet]_G^0. \tag{3.2}$$

Via the first isomorphism (noticed by Segal [Se]), $\omega_G^0(X, A)$ is a module over the Burnside ring $\Omega(G)$; cf. [Pe1], Chapter II, Section 6. For the description of the second isomorphism, see (e.g.) [H] and [Ru]. In particular, it follows from [Ru], Sections 4 and 7 (especially the proof of Theorem 7.2) that the following lemma holds.

LEMMA 3.3. (Rubinsztein Lemma) *Via the isomorphisms in* (3.2), *each element*

$$\sum_{(H) \subset \mathcal{S}(G)} z(H)[G/H] \quad \text{in } \Omega(G)$$

corresponds to an element in $[U^\bullet, U^\bullet]^0_G$ *which has a representative* $\varphi : U^\bullet \to U^\bullet$, *a base point preserving* G*-map, such that the following four conditions hold.*

(3.3.1) φ *is* G*-transverse regular to* 0 *in* $U^\bullet$.

(3.3.2) *The number of points in* $\varphi^{-1}(0)$ *with isotropy subgroup* H *is equal to* $|N_G(H)/H| \cdot |z(H)|$.

(3.3.3) φ *is orientation preserving (resp. reversing) at each point in* $\varphi^{-1}(0)$ *with isotropy subgroup* H *provided* $z(H) > 0$ *(resp.* $z(H) < 0$*).*

(3.3.4) *For each point* $x \in \varphi^{-1}(0)$ *with isotropy subgroup* H, *the normal derivative*

$$d\varphi_x : U_H = (T_x U^\bullet)_H \to (T_0 U^\bullet)_H = U_H$$

is the identity map.

Hereafter, for a G-manifold X and a G-module U, we identify X with $X \times \{0\}$ in $X \times U^\bullet$, and similarly as in [Pe3], p. 11, for a smooth map $\varphi : X \times U^\bullet \to X \times U^\bullet$, by the *normal derivative of* φ *at* $x \in \varphi^{-1}(X)$ with $H = G_x$, we mean the H-endomorphism $pr \circ \mathrm{d}\varphi_x \circ in$ is the composition

$$U \overset{in}{\hookrightarrow} T_x(X \times U^\bullet) \xrightarrow{d\varphi_x} T_{\varphi(x)}(X \times U^\bullet) \xrightarrow{pr} N_{\varphi(x)}(X, X \times U^\bullet) = U.$$

Now, with the hypothesis of Theorem 3.1, consider an element μ in the Burnside ring $\Omega(G)$ provided by Corollary 2.5. Then it follows from (2.5.1) and (2.5.2) that the following lemma holds (cf. [Pe1], Localization Lemma 1.8, [Pe2], Lemma 3.9, and [Pe3], Lemma 1.6).

LEMMA 3.4. (Localization Lemma) *Let* $j : Y^G \hookrightarrow Y$ *be the inclusion map and let* $S \subset \Omega(G)$ *be the multiplicatively closed subset consisting of the powers* μ^m *for all* $m \geq 0$. *Then the localized restriction homomorphism*

$$S^{-1}j^* : S^{-1}\omega^0_G(Y) \to S^{-1}\omega^0_G(Y^G)$$

is an isomorphism.

Proof of Theorem 3.1. For a finite G-CW complex B, $\omega^0_G(B)$ has a special element $\mathbf{1}_B$ which is represented by the map

$$U^\bullet \wedge (B \amalg pt) \longrightarrow U^\bullet, \qquad [x, y] \mapsto x \quad \text{and} \quad [x, pt] \mapsto \infty$$

for all $x \in U^\bullet$ and $y \in B$. From Situation 2.1, recall that

$$Y^G = \coprod_{i=1}^{k} M_i \quad \text{and} \quad M = \coprod_{i=1}^{k} M_i \overbrace{\amalg \cdots \amalg}^{d_i\text{-times}} M_i$$

for some integers $d_i \geq 0$. Clearly, $\omega_G^0(Y^G) = \bigoplus_{i=1}^k \omega_G^0(M_i)$. Set

$$\delta = ((1 - d_i)\mathbf{1}_{M_i})_{i=1}^k \in \bigoplus_{i=1}^k \omega_G^0(M_i).$$

Using Lemma 3.4, take an element $\gamma \in S^{-1}\omega_G^0(Y)$ such that $S^{-1}j^*(\gamma) = \mu\delta$ in $S^{-1}\omega_G^0(Y^G)$. Then, for a sufficiently large integer m, there exists an element $\beta \in \omega_G^0(Y)$ such that

$$j^*(\beta\mu) = \mu^{m+1}\delta \quad \text{in } \omega_G^0(Y^G).$$

Now, take a base point preserving G-map $U^\bullet \wedge (Y \amalg pt) \to U^\bullet$ which represents $\mathbf{1}_Y - \beta\mu$. This map induces a G-map $\varphi : U^\bullet \times Y \to U^\bullet$ such that $\varphi(\infty, y) = \infty$ for all $y \in Y$. Now we get a G-map $\psi : Y \times U^\bullet \to Y \times U^\bullet$ by setting

$$\psi(y, x) = (y, \varphi(x, y)) \qquad \text{for all } x \in U^\bullet \text{ and } y \in Y.$$

Here, U is a sufficiently large real G-module (which may be suppose to be a realification of a complex G-module) with G-invariant inner product. As before, we identify Y with $Y \times \{0\}$ in $Y \times U^\bullet$. The restriction ψ_i of ψ to $M_i \times U^\bullet$ may be supposed to be a product of the identity map on M_i with a G-map $U^\bullet \to U^\bullet$, since the i-th component of $j^*(\mathbf{1}_Y - \beta\mu)$ is equal to $(1 - \mu^{m+1}(1 - d_i))\mathbf{1}_{M_i}$. By assumption,

$$\chi_G(1 - \mu^{m+1}(1 - d_i)) = d_i \qquad \text{for } i = 1, \ldots, k.$$

We can suppose that ψ_i is transverse regular to M_i,

$$\psi_i^{-1}(M_i) = \overbrace{M_i \amalg \cdots \amalg M_i}^{d_i\text{-times}},$$

and the normal derivative of ψ at any point in $\psi^{-1}(Y^G)$ is the identity map (cf. Lemma 3.3, the conditions (3.3.1) and (3.3.4)).

Deform ψ by a G-homotopy relative to $Y^G \times U^\bullet$ so that the resulting map $\theta : Y \times U^\bullet \to Y \times U^\bullet$ is transverse regular to Y (see [Pe3], Lemma 2.7). Set $X = \theta^{-1}(Y)$ and, as $f : X \to Y$, take $\theta \mid X$. Then $X \cap (Y \times \{\infty\}) = \emptyset$. Moreover, it follows from the construction that X^G and M are diffeomorphic, and the G-vector bundles $\nu(X^G, X)$ and ν are equivalent. Write $p_Y : Y \times U^\bullet \to Y$ and $p_{U^\bullet} : Y \times U^\bullet \to U^\bullet$ for the projection maps. Let ν_X and ν_Y be the equivariant normal bundles of X and Y, respectively, in $Y \times U^\bullet$. Note that ν_X and the pull-back bundle $f^*(\nu_Y)$ are equivalent as G-vector bundles. Clearly, ν_Y is just the product bundle $\varepsilon(U)$ over Y.

Now observe that, as G-vector bundles,

$$\begin{aligned}
TX \oplus \varepsilon(U) \oplus \varepsilon(\mathbb{R}) &= TX \oplus f^*(\varepsilon(U)) \oplus \varepsilon(\mathbb{R}) \\
&= TX \oplus f^*(\nu_Y) \oplus \varepsilon(\mathbb{R}) \\
&\simeq TX \oplus \nu_X \oplus \varepsilon(\mathbb{R}) \\
&= T(Y \times U^\bullet) \mid X \oplus \varepsilon(\mathbb{R}) \\
&= (p_Y^*(TY) \oplus p_{U^\bullet}^*(TU^\bullet)) \mid X \oplus p_{U^\bullet}^*(\varepsilon(\mathbb{R})) \mid X \\
&= p_Y^*(TY) \mid X \oplus p_{U^\bullet}^*(TU^\bullet \oplus \varepsilon(\mathbb{R})) \mid X \\
&= p_Y^*(TY) \mid X \oplus p_{U^\bullet}^*(\varepsilon(U) \oplus \varepsilon(\mathbb{R})) \mid X \\
&= \psi^*(p_Y^*(TY)) \mid X \oplus \psi^*(p_{U^\bullet}^*(\varepsilon(U) \oplus \varepsilon(\mathbb{R}))) \mid X \\
&\cong \theta^*(p_Y^*(TY)) \mid X \oplus \theta^*(p_{U^\bullet}^*(\varepsilon(U) \oplus \varepsilon(\mathbb{R}))) \mid X \\
&= \theta^*(p_Y^*(TY)) \mid X \oplus \theta^*(p_{U^\bullet}^*(TU^\bullet \oplus \varepsilon(\mathbb{R}))) \mid X \\
&= \theta^*(p_Y^*(TY) \oplus p_{U^\bullet}^*(TU^\bullet)) \mid X \oplus \theta^*(p_{U^\bullet}^*(\varepsilon(\mathbb{R}))) \mid X \\
&= \theta^*(T(Y \times U^\bullet)) \mid X \oplus \varepsilon(\mathbb{R}) \\
&= f^*(T(Y \times U^\bullet) \mid Y) \oplus \varepsilon(\mathbb{R}) \\
&= f^*(TY) \oplus f^*(\nu_Y) \oplus \varepsilon(\mathbb{R}) \\
&= f^*(TY) \oplus \varepsilon(U) \oplus \varepsilon(\mathbb{R}),
\end{aligned}$$

where $=$ (resp. $\cong$, resp. $\simeq$) indicates that there exists a standard isomorphism (resp. a metric-preserving isomorphism, resp., an isomorphism). Therefore, we obtain a stable isomorphism $b : TX \to f^*(TY)$. Since we may assume that the normal derivative of θ at any point in X is the identity map, we also obtain the required collection c_H of H-vector bundle isomorphisms, $H \in \mathcal{I}$. As a result, we get the triple $\mathbf{w} = (f; b; c)$. Moreover, for $H \in \mathcal{I}$,

$$\operatorname{Res}_H^G(\mathbf{1}^Y - \beta\mu) = \operatorname{Res}_H^G(\mathbf{1}_Y) \quad \text{in } \omega_H^0(Y).$$

Thus, we can obtain an H-normal cobordism between $\operatorname{Res}_H^G(\mathbf{w})$ and the identity H-normal map on $\operatorname{Res}_H^G(Y)$. This allows us to orient X^H so that $f^H : X^H \to Y^H$ is of degree one, producing the required G-normal map $\mathbf{w} = (f; b; c)$. □

4. Modification of G-normal maps

Let $\mathbf{w} = (f; b; c)$ be the G-normal map constructed in Theorem 3.1. By performing G-surgery on $\mathbf{w}$ of types $(H) \subset \mathscr{I} = \mathrm{Iso}(G, X \backslash X^G) = \mathrm{Iso}(G, Y \backslash Y^G)$, we may assume that

$$f^H : X^H \to Y^H \quad \text{is } [\dim X^H/2]\text{-connected for each } H \in \mathscr{I}. \tag{4.1}$$

In particular, X^H is connected and simply connected, and so is the set of all points in X with isotropy subgroup H.

As usual, by $-\mathbf{w} = (-f; -b; -c)$ we mean a copy of $\mathbf{w}$ (denoted also by $+\mathbf{w}$) with the orientations of all underlying spaces reversed. For each $H \in \mathscr{I}$, we will consider the obvious G-normal maps

$$\mathrm{Ind}_H^G(\mathrm{Res}_H^G(\pm\mathbf{w})) = G \times_H \mathrm{Res}_H^G(\pm\mathbf{w}).$$

Let $\mathscr{H}$ be a proper separating family in $\mathscr{S}(G)$ with $\mathscr{H} \subseteq \mathscr{I}$. Then it follows from Proposition 2.4 that the following corollary holds.

COROLLARY 4.2. *For the idempotent* $\iota_{\mathscr{I}} \in \Omega(G)$ *occuring in Proposition* 2.4, *the idempotent* $\kappa = 1 - \iota_{\mathscr{I}}$ *in the Burnside ring* $\Omega(G)$ *fulfills the following two conditions.*

(4.2.1) *For some integers* $a(H)$ *with* $(H) \subset \mathscr{I}$,

$$\kappa = \sum_{(H) \subset \mathscr{I}} a(H)[G/H].$$

(4.2.2) *For each* $H \in \mathscr{H}$, $\mathrm{Res}_H^G(\kappa) = 1$ *in* $\Omega(H)$.

Now, for the element $\kappa \in \Omega(G)$ occuring in Corollary 4.2 and any integer t, we define a G-normal map

$$\mathbf{w} \#_G t\kappa\mathbf{w} = (f \#_G t\kappa f, b \#_G t\kappa b, c \#_G t\kappa c)$$

by forming a G-connected sum of $\mathbf{w}$ and (simultaneously) a number of copies of $\mathrm{Ind}_H^G(\mathrm{Res}_H^G(\pm\mathbf{w}))$ for types $(H) \subset \mathscr{I}$. More precisely, first we consider the following G-connected sum:

$$X \#_G t\kappa X \ \mathrm{rel}\{\mathscr{D}(H) \mid (H) \subset \mathscr{I}, t \cdot a(H) \neq 0\}$$

(cf. Section 1), where in all connected sum data $\mathscr{D}(H) = (x_i(H), \varphi_i(H))_{i=1}^{|t \cdot a(H)|}$, each

H-automorphism $\varphi_i(H)$ of $T_{x_i(H)}X$ is the identity on the H-nontrivial summand of $T_{x_i(H)}X$, and it restricts to $\varphi_i(H)^H$ on $(T_{x_i(H)}X)^H$ with

$$\deg \varphi_i(H)^H = \begin{cases} +1 & \text{when } t \cdot a(H) < 0, \\ -1 & \text{when } t \cdot a(H) > 0. \end{cases}$$

Remark. It follows immediately from the condition (4.1) that the G-connected sum $X \#_G t\kappa X$ rel$\{\mathscr{D}(H) \mid (H) \subset \mathscr{I}, t \cdot a(H) \neq 0\}$ is unique (up to G-diffeomorphism); that is, it does not depend on the choice of points $x_i(H)$, as well as the choice of H-automorphisms $\varphi_i(H)$ with $\deg \varphi_i(H)^H = \pm 1$. Therefore, in the notation of the G-connected sum, we drop the connected sum data.

Now, similarly as $X \#_G t\kappa X$, we form $Y \#_G t\kappa Y$, and by G-homotopy deformations of f, using Whitney's trick on the H-fixed point sets X^H, $H \in \mathscr{I}$, (recall $\dim X^H \geq 5$), we obtain the degree one G-map

$$f \#_G t\kappa f : X \#_G t\kappa X \to Y \#_G t\kappa Y$$

whose bundle data $b \#_G t\kappa b$ and $c \#_G t\kappa c$ are obtained by forming the appropriate G-connected sums of bundle data in $\mathbf{w}$ and $\operatorname{Ind}_H^G(\operatorname{Res}_H^G(\pm \mathbf{w}))$.

Now, with the hypothesis in Theorem 2.2, assume $\mathscr{K}$ is a G-invariant subset of $\mathscr{H}$ such that if $K \in \mathscr{K}$, $L \in \mathscr{H}$, and $K \subseteq L$, then $L \in \mathscr{K}$. In the following theorem, we deal with $\mathbf{w} = (f; b; c)$ modified by performing G-surgery of type $\mathscr{K}$ (i.e., G-surgery operations of isotropy types in $\mathscr{K}$) and taking G-connected sums of the form described above, which we simply refer to as G-connected sums of type $\mathscr{I}$. The modified $\mathbf{w}$ is a G-normal map rel $\mathscr{K}$; that is, a G-normal map such that for the modified degree one G-map $f : X \to Y$, $f^K : X^K \to Y^K$ is a homotopy equivalence for all $K \in \mathscr{K}$. The theorem may be compared, e.g., with [PR], Chapter 3, Theorem 12.4 and Chapter 4, Theorem 2.2.

THEOREM 4.3. *Let H be an element in $\mathscr{H} \backslash \mathscr{K}$ such that $K \in \mathscr{H}$, $H \subsetneq K$ implies $K \in \mathscr{K}$. If the G-normal map $\mathbf{w} = (f; b; c)$ can be modified by G-surgery of type $\mathscr{K}$ and G-connected sums of type $\mathscr{I}$ to a G-normal map* rel $\mathscr{K}$, *then $\mathbf{w}$ can be modified by G-surgery of type $\mathscr{K} \cup (H)$ and G-connected sums of type $\mathscr{I}$ to a G-normal map* rel $\mathscr{K} \cup (H)$.

Using Theorem 4.3, we complete the proof of Theorem 2.2 inductively on $\mathscr{K}$. We start our induction with $\mathscr{K} = \emptyset$, do the inductive step by adding one isotropy type at a time as described in Theorem 4.3, and end the induction with $\mathscr{K} \cup (\{e\}) = \mathscr{H}$. As the result, after all modifications of $\mathbf{w} = (f; b; c)$, we obtain a

G-normal map $\mathbf{w}' = (f'; b'; c')$ with a degree one G-map $f' : X' \to Y'$ such that

$$f'^H : X'^H \to Y'^H \qquad \text{is a homotopy equivalence for all } H \in \mathscr{H}. \tag{4.4}$$

Sincy Y is a homotopy n-sphere, so is Y', and hence by (4.4) applied for $H = \{e\}$, X' is a homotopy n-sphere. It follows from the construction that X'^G and M are diffeomorphic, and the G-vector bundles $\nu(X'^G, X')$ and ν are equivalent. Moreover, the conditions (2.2.1)–(2.2.4) all hold, proving Theorem 2.2.

Proof of Theorem 4.3. Suppose $\mathbf{w} = (f; b; c)$ is modified by G-surgery of type $\mathscr{K}$ and G-connected sums of type $\mathscr{I}$ to a G-normal map rel $\mathscr{K}$. Observe that in Situation 2.1, (SGH) (resp. (GH)) holds for the modified target Y. Also, for the modified X, $\mathrm{Iso}(G, X \backslash X^G) = \mathscr{I}$ and X satisfies the strong gap. (resp. gap) hypothesis for $\mathscr{I}$.

Let H be an element in $\mathscr{H} \backslash \mathscr{K}$ such that $K \in \mathscr{H}, H \subsetneq K$ implies $K \in \mathscr{K}$. Set $N = N_G(H)$. By performing G-surgery of type (H), we may assume that

$$f^H : X^H \to Y^H \qquad \text{is } [\dim X^H/2]\text{-connected.}$$

In particular, X^H is connected and simply connected. Note that if $K \in \mathscr{S}(G)$, $H \lhd K$, and K/H is a nontrivial hyperelementary group, then $K \in \mathscr{K}$. Therefore, the surgery kernel (in the sense of [M4], Definitions 3.1 and 4.1) of f^H is a stably free $\mathbb{Z}[N/H]$-module. We denote by $\sigma_H(\mathbf{w})$ the obstruction for G-surgery of type (H) to changing $\mathbf{w}$ so that to make $f^H : X^H \to Y^H$ a homotopy equvalence. If X satisfies the strong gap hypothesis for (H), then by [DP] or [PR], the obstruction $\sigma_H(\mathbf{w})$ lies in the Wall group $L_q^h(\mathbb{Z}[N/H])$, where $q = \dim X^H$. If X satisfies the gap hypothesis for (H), then by [M4], $\sigma_H(\mathbf{w})$ lies in the Bak group $W_q(\mathbb{Z}[N/H], \Gamma(N/H, X^H))$, where $\Gamma(N/H, X^H)$ is the form parameter on $\mathbb{Z}[N/H]$ generated by all elements g in N/H such that $g^2 = 1$ and $\dim(X^H)^g = [(q-1)/2]$. We have mentioned above two kinds of surgery obstruction groups; that is, the Wall groups and the Bak groups. In both cases, we denote the surgery obstruction groups by the same notation $\mathcal{O}(N/H, X^H)$, and consider the assignment:

$$S \mapsto \mathcal{O}(S, \mathrm{Res}_S^{N/H}(X^H)), \qquad S \in \mathscr{S}(N/H).$$

It is well-known that in the case of Wall groups, as well as Bak groups, this assignment defines a Mackey functor from the category of subgroups of N/H to the category of abelian groups. Therefore, in both cases,

$$\{\mathcal{O}(S, \mathrm{Res}_S^{N/H}(X^H)) \mid S \in \mathscr{S}(N/H)\} \qquad \text{is a Mackey functor.}$$

Moreover, this functor has the defect set consisting of all solvable subgroups of N/H. That is,

$$\mathcal{O}(N/H, X^H) \xrightarrow{\text{Res}} \bigoplus_{S \text{ solvable}} \mathcal{O}(S, \operatorname{Res}_S^{N/H}(X^H)) \qquad \text{is injective.} \tag{4.5}$$

This is well-known in the case of Wall groups, as well as Bak groups. In fact, in these two cases, the defect sets consist of all 2-hyperelementary subgroups of N/H (see [Dr] and [Ba], Section 12). Now, for an integer t, we form $\mathbf{w} \#_G t\kappa\mathbf{w}$, and we wish to compute the corresponding G-surgery obstruction $\sigma_H(\mathbf{w} \#_G t\kappa\mathbf{w})$. First, we claim that for each $L \in \mathcal{I}$,

$$\sigma_H(\operatorname{Ind}_L^G(\operatorname{Res}_L^G(\pm\mathbf{w}))) = [(G/L)^H]\sigma_H(\pm\mathbf{w}), \tag{4.6}$$

where $[(G/L)^H]$ lies in $\Omega(N/H)$. Indeed, for $N = N_G(H)$ and $L \in \mathcal{I}$, choose a complete set $\mathcal{A}$ of (N, L)-double coset representatives in G (say, with $e \in G$ representing NL). For each $a \in \mathcal{A}$, let $\operatorname{Res}_L^G(X)_a$ be the $(N \cap aLa^{-1})$-space

$$(N \cap aLa^{-1}) \times \operatorname{Res}_L^G(X) \longrightarrow \operatorname{Res}_L^G(X), \qquad (g, x) \mapsto a^{-1}gax.$$

Then we get the following N-equivalences (cf. [tD], Chapter I, Section 4, (4.6)):

$$\begin{aligned}\operatorname{Res}_N^G(G \times_L \operatorname{Res}_L^G(X)) &\cong \coprod_{a \in \mathcal{A}} NaL \times_L \operatorname{Res}_L^G(X)\\ &\cong \coprod_{a \in \mathcal{A}} N \times_{N \cap aLa^{-1}} \operatorname{Res}_L^G(X)_a.\end{aligned}$$

Moreover,

$$(G \times_L \operatorname{Res}_L^G(X))^H = \coprod_a N \times_{N \cap aLa^{-1}} \operatorname{Res}_L^G(X)_a^{a^{-1}Ha},$$

where the disjoint union runs over all $a \in \mathcal{A}$ with $H \subseteq aLa^{-1}$. Thus, we get the following equalities, where the sums run over all $a \in \mathcal{A}$ with $H \subseteq aLa^{-1}$:

$$\begin{aligned}\sigma_H(\operatorname{Ind}_L^G(\operatorname{Res}_L^G(\pm\mathbf{w}))) &= \sum_a \operatorname{Ind}_{(N \cap aLa^{-1})/H}^{N/H} \sigma_H(\operatorname{Res}_{N \cap aLa^{-1}}^G(\pm\mathbf{w}))\\ &= \sum_a \operatorname{Ind}_{(N \cap aLa^{-1})/H}^{N/H} \operatorname{Res}_{(N \cap aLa^{-1})/H}^{N/H}(\sigma_H(\pm\mathbf{w}))\\ &= [(G/L)^H]\sigma_H(\pm\mathbf{w}),\end{aligned}$$

proving the claim.

Now consider the Burnside ring homomorphism $\mathrm{Fix}_H^G : \Omega(G) \to \Omega(N/H)$, where $N = N_G(H)$. If $H \subseteq L \subseteq N$ and L/H is solvable, then $L \in \mathscr{H}$. Thus, for such an L, $\mathrm{Res}_L^G(\kappa) = 1$, by (4.2.2), and hence $\mathrm{Res}_{L/H}^{N/H}(\mathrm{Fix}_H^G(\kappa)) = 1$. Now, by the injectivity in (4.5), $\mathrm{Fix}_H^G(\kappa)\sigma_H(\mathbf{w}) = \sigma_H(\mathbf{w})$, so that

$$\mathrm{Fix}_H^G(1 + t\kappa)\sigma_H(\mathbf{w}) = (1 + t)\sigma_H(\mathbf{w}).$$

On the other hand, it follows from (4.6) and the definition of G-surgery obstruction that

$$\sigma_H(\mathbf{w} \#_G t\kappa\mathbf{w}) = \mathrm{Fix}_H^G(1 + t\kappa)\sigma_H(\mathbf{w}).$$

Consequently, we obtain

$$\sigma_H(\mathbf{w} \#_G t\kappa\mathbf{w}) = (1 + t)\sigma_H(\mathbf{w}). \tag{4.7}$$

completing the calculation of $\sigma_H(\mathbf{w} \#_G t\kappa\mathbf{w})$. Since the modified $\mathbf{w}$ is a G-normal map rel $\mathscr{H}$, so is $\mathbf{w} \#_G t\kappa\mathbf{w}$. Now, for $t = -1$, the obstruction $\sigma_H(\mathbf{w} \#_G t\kappa\mathbf{w})$ vanish by (4.7). Thus, we can perform G-surgery on $\mathbf{w} \#_G(-1)\kappa\mathbf{w}$ of type (H) to make

$$(f \#_G(-1)\kappa f)^H : (X \#_G(-1)\kappa X)^H \to (Y \#_G(-1)\kappa Y)^H$$

a homotopy equivalence, to produce the required G-normal map rel $\mathscr{H} \cup (H)$. □

5. Proofs of Theorems A and B

First, recall from the introduction that for a finite group G, we let G^{sol} be the unique smallest normal subgroup of G such that G/G^{sol} is solvable. In other words, G^{sol} is the smallest term of the derived series of G. Moreover, the separating family $\mathscr{F}(G)$ defined in Section 2 can be characterized as follows: $H \in \mathscr{F}(G)$ if and only if $H \supseteq G^{sol}$. Finally, recall that $\mathscr{I}(G) = \mathscr{S}(G) \backslash \mathscr{F}(G)$.

PROPOSITION 5.1. *Let G be a finite nonsolvable group. Then, for any integer $\ell \geq 5$, the real G-module $V = \ell(\mathbb{R}[G] - \mathbb{R}[G/G^{sol}])$ fulfills the following three conditions.*

(5.1.1) $\mathrm{Iso}(G, V\backslash\{0\}) = \mathscr{I}(G)$.

(5.1.2) *V satisfies the weak gap hypothesis for $\mathscr{I}(G)$.*

(5.1.3) *For any automorphism $\varphi : G \to G$, $\varphi^* V$ and V are equivalent as G-modules, where $\varphi^* V$ has the same underlying vector space as does V, and the G-action given by $(g, v) \mapsto \varphi(g)v$.*

Proof. Consider the canonical G-module embedding of $\mathbb{R}[G/G^{sol}]$ into $\mathbb{R}[G]$ and take the orthogonal complement

$$W = \mathbb{R}[G] - \mathbb{R}[G/G^{sol}]$$

of $\mathbb{R}[G/G^{sol}]$ in $\mathbb{R}[G]$. Then, for each $H \in \mathscr{S}(G)$,

$$\dim W^H = |G/H| - |G/HG^{sol}|. \tag{5.2}$$

Thus, $\dim W^H = 0$ if and only if $|H| = |HG^{sol}|$, if and only if $H = HG^{sol}$, if and only if $H \supseteq G^{sol}$, if and only if $H \in \mathscr{F}(G)$. In other words,

$$\dim W^H \geq 1 \qquad \text{if and only if } H \in \mathscr{I}(G). \tag{5.3}$$

We claim that the following condition holds for any $H, K \in \mathscr{I}(G)$.

$$\begin{cases} \textit{If } H \subsetneq K, \textit{ then } \dim W^H \geq 2 \dim W^K \\ \textit{and the equality holds if and only if } |K/H| = |KG^{sol}/HG^{sol}| = 2. \end{cases} \tag{5.4}$$

In fact, since $|KG^{sol}/HG^{sol}| \leq |K/H|$ for $H \subsetneq K$, thus

$$\begin{aligned} \dim W^H &= |G/H| - |G/HG^{sol}| \\ &= |K/H||G/K| - |KG^{sol}/HG^{sol}||G/KG^{sol}| \\ &\geq |K/H|(|G/K| - |G/KG^{sol}|) \\ &\geq 2 \dim W^K, \end{aligned}$$

proving the claim. Now, the conditions (5.1.1) and (5.1.2) follow immediately from (5.3) and (5.4); remember $\ell \geq 5$. The condition (5.1.3) follows from the fact that G^{sol} is a characteristic subgroup of G. □

Remark. In Proposition 5.1, suppose further $|G/G^{sol}|$ is odd (for example, G is perfect or G is an extension of a finite odd order group by a perfect group). Then $|KG^{sol}/HG^{sol}|$ is odd, too, whenever $H \subset K$ for $H, K \in \mathscr{I}(G)$. According to (5.4), the G-module $W = \mathbb{R}[G] - \mathbb{R}[G/G^{sol}]$ satisfies the gap hypothesis for $\mathscr{I}(G)$. Therefore, it follows that for $\ell \geq 5$, the G-module ℓW satisfies the strong gap hypothesis for $\mathscr{I}(G)$. Note also that if X is any G-manifold (or a G-CW complex) with nonempty G-fixed point set, then the cartesian product $X \times \ell W$ (with the diagonal G-action) satisfies the strong gap hypothesis for $\mathscr{I}(G)$ provided ℓ is sufficiently large.

If ℓ is even, the real G-module $\ell(\mathbb{R}[G] - \mathbb{R}[G/G^{sol}])$ is the realification of the complex G-module $\ell/2(\mathbb{C}[G] - \mathbb{C}[G/G^{sol}])$. Therefore, Proposition 5.1 yields immediately the following corollary.

COROLLARY 5.5. *Let G be a finite nonsolvable group, let $W = \mathbb{R}[G] - \mathbb{R}[G/G^{sol}]$, and let $V = \ell W$ for an even integer $\ell \geq 6$. Then the G-module V of dimension $\ell(|G| - |G/G^{sol}|)$ fulfills the conditions* (5.1.1)–(5.1.3) *in Proposition* 5.1, *and for $H \in \mathcal{I}(G)$, each element of the normalizer $N_G(H)$ acts on V^H via an orientation preserving transformation.*

Proof of Theorem A. Let G be a finite nonsolvable group such that $|G/G^{sol}|$ is odd. Let $\mathcal{I} = \mathcal{I}(G)$. For any G-module $V = \ell W$ occurring in Corollary 5.5, let Y be the double of the G-invariant unit disk $D(V)$ in V; i.e., $Y = S(V \oplus \mathbb{R})$, the G-invariant unit sphere in $V \oplus \mathbb{R}$, where G acts trivially on $\mathbb{R}$. Since $V^G = \{0\}$, thus $Y^G = \{x, y\}$. Moreover, $\mathrm{Iso}(G, Y \backslash Y^G) = \mathcal{I}(G)$. Set $M = \{x\}$. Then, for $G, \mathcal{I}, Y, M$, and $\nu = V$, Situation 2.1 is fulfilled with Y satisfying the strong gap hypothesis for $\mathcal{I}$ (see Remark to Proposition 5.1). Therefore, Corollary 2.3 yields a smooth action of G on S^n with exactly one G-fixed point x at which the tangent G-module is equivalent to V. In particular, $n = \dim V$. The "inserting part" of Theorem 2.2 and Corollary 2.3 complete the proof of Theorem A. □

Proof of Theorem B. Let G be a finite nonsolvable group such that $|G/G^{sol}|$ is odd. Let $\mathcal{I} = \mathcal{I}(G)$. According to [O2], Section 2, Theorem 4 and Proposition 8, there exists a finite contractible G-CW complex X such that X^H is nonempty and contractible for each solvable subgroup H of G, and X^H is empty for each nonsolvable subgroup H of G. Let M be a closed smooth manifold whose connected components all are stably parallelizable and all have the same dimension $m \geq 0$. Consider the join $X * M$ with the join G-action. Then $X * M$ is a finite contractible G–CW complex with $(X * M)^G = M$. For a G-module $V = \ell W$ occurring in Corollary 5.5, take the product G-vector bundle $\varepsilon(\mathbb{R}^m \oplus V)$ over $X * M$, where G acts trivially on $\mathbb{R}^m$. By using this bundle for sufficiently large ℓ, the equivariant thickening ([Pa2], Theorem 2.4 and Corollary 3.3) converts $X * M$ into a disk D^n with a smooth G-action such that $(D^n)^G = M$, $\mathrm{Iso}(G, D^n \backslash M) = \mathcal{I}$, and as G-vector bundle, the tangent bundle $T(D^n)$ is equivalent to the product G-vector bundle $\varepsilon(\mathbb{R}^m \oplus V)$ over D^n. Now the equivariant double of D^n yields a smooth action of G on the sphere $Y = S^n$ such that $Y^G = M \amalg M$. Moreover, it follows from the equivariant thickening that for $G, \mathcal{I}, Y, M$, and the product G-vector bundle $\nu = \varepsilon(V)$ over M, Situation 2.1 is fulfilled with Y satisfying the strong gap hypothesis for $\mathcal{I}$ (provided ℓ is sufficiently large) because $\dim Y^H = m + \dim V^H$ for any $H \in \mathcal{I}$ (cf. Remark to Proposition 5.1). Thus, the "deleting part" of Theorem 2.2 and Corollary 2.3 completes the proof of Theorem B. □

Added in proof. Recently, Laitinen and Morimoto [LM] proved that for each finite group G admitting a smooth fixed point free action on a disk, there exists a smooth action of G on a standard sphere with any given finite number $k \geq 1$ of G-fixed points. However, the dimensions of the spheres occurring in Theorem A of this paper differ from the dimensions of the corresponding spheres in [LM]. Also, for a class of groups G, the result in [LM] depends on the work [BM] by making use of a G-module satisfying the weak gap hypothesis.

Appendix

The goal of this appendix is to point out that by using the work [BM] and [M5], our arguments show that the Deleting-Inserting Theorem (Theorem 2.2) holds for any finite nonsolvable group G, when the target G-manifold Y satisfies *only the weak gap hypothesis* for $\mathscr{I} = \text{Iso}(G, Y \backslash Y^G)$. Consequently, by argueing as in the proof of Theorem A, it follows that Theorem A is true without assuming that $|G/G^{sol}|$ is odd.

In order to show that Theorem 2.2 holds when Y satisfies the weak gap hypothesis for $\mathscr{I}$, it suffices to point out that Theorem 4.3 is true under the weak gap hypothesis. To do it, we just repeat the proof of Theorem 4.3 with Wall (or Bak) surgery obstruction groups replaced by the Witt groups whose definition follows. Assume G is a finite group, $\lambda = (-1)^k$, $w : G \to \{1, -1\}$ is a group homomorphism, $\Theta(G)$ is a finite G-set, and $\rho = \rho_G : \Theta(G) \to \mathscr{S}(G)$ is a G-map. The involution $-$ on $A = \mathbb{Z}[G] = \text{Map}(G, \mathbb{Z})$ is given so that $g \mapsto w(g)g^{-1}$ for $g \in G$. We fix G-invariant (with respect to conjugation) subsets $S_{-\lambda} = S_{-\lambda}(G)$ and $S_\lambda = S_\lambda(G)$ of G such that $S_{-\lambda} \subseteq \{g \in G \mid g = -\lambda \bar{g} \text{ in } A\}$ and $S_\lambda \subseteq \{g \in G \mid g = \lambda \bar{g} \text{ in } A\}$, and we set

$$A_{-\lambda} = \mathbb{Z}[G \backslash S_\lambda] = \text{Map}(G \backslash S_\lambda, \mathbb{Z}) \quad \text{and} \quad A_\lambda = \mathbb{Z}[S_\lambda] = \text{Map}(S_\lambda, \mathbb{Z}).$$

Clearly, $A = A_{-\lambda} \oplus A_\lambda$. Let Λ is the smallest form parameter on A containing all elements of $S_{-\lambda}$.

By a *quadratic module* we mean the quadruple $\mathbf{M} = (M, \langle\ ,\ \rangle, q, \alpha)$, where

- M is a finitely generated, stably free A-module,
- $\langle\ ,\ \rangle : M \times M \to A$ is a nonsingular λ-hermitian form over A,
- $q : M \to A_{-\lambda}/\Lambda$ is a quadratic map, and
- $\alpha : \Theta(G) \to M$ is a G-map (which will be called a *positioning map*).

Here we assume the following properties

(Q1) $\langle\ ,\ \rangle$ is biadditive,
(Q2) $\langle ax, by\rangle = b\langle x, y\rangle\bar{a}$,
(Q3) $\langle x, y\rangle = \lambda\overline{\langle y, x\rangle}$,
(Q4) $q(gx) = gq(x)\bar{g}$ in $A_{-\lambda}/\Lambda = A/(\Lambda + A_{\lambda})$,
(Q5) $q(x+y) - q(x) - q(y) = \langle x, y\rangle$ in $A_{-\lambda}/\Lambda = A/(\Lambda + A_{\lambda})$, and
(Q6) $\widetilde{q(x)} + \lambda\overline{\widetilde{q(x)}} = \langle x, x\rangle$ in $A_{-\lambda} = A/A_{\lambda}$,

where $x, y \in M$, $a, b \in A$, $g \in G$, and $\widetilde{q(x)}$ is a lifting of $q(x)$.

Let $\mathscr{Q}(A)$ be the category of quadratic modules of this kind. A morphism $f: \mathbf{M} \to \mathbf{M}'$ in $\mathscr{Q}(A)$ is a homomorphism between the underlying A-modules, preserving hermitian form, quadratic form, and positioning map.

For $\mathbf{M} = (M, \langle\ ,\ \rangle, q, \alpha)$, we define $\nabla : M \to \mathbb{Z}/2[S_{\lambda}]$ by

$$\nabla(x)(g) = [\varepsilon(\langle \Sigma^{\alpha}(g) - x, gx\rangle)] \in \mathbb{Z}/2, \qquad \text{for } x \in M \quad \text{and} \quad g \in S_{\lambda},$$

where

$$\Sigma^{\alpha}(g) = \sum_{\gamma} \{\alpha(\gamma) \mid \gamma \in U(G) \text{ and } \rho(\gamma) \ni g\}$$

and $\varepsilon : A \to \mathbb{Z}$ is the map defined by

$$\varepsilon\left(\sum_{g \in G} a_g g\right) = a_e \qquad \text{for } a_g \in \mathbb{Z}, \text{ where } e \text{ is the neutral element of } G.$$

We denote by $\mathscr{S}\mathscr{Q}(A)$ the full subcategory of $\mathscr{Q}(A)$ consisting of all $\mathbf{M}$ with trivial ∇. If $\mathbf{M} \in \mathscr{S}\mathscr{Q}(A)$ has a laglangian $L \subset M$ such that $L \supseteq \alpha(\Theta(G))$, then we call $\mathbf{M}$ a *null module* in the category $\mathscr{S}\mathscr{Q}(A)$.

Now we define the Witt group:

$$W_{2k}(\mathbb{Z}, G, S_{-\lambda}, S_{\lambda}, \Theta(G)) = K_0(\mathscr{S}\mathscr{Q}(A))/\langle \text{null modules in } \mathscr{S}\mathscr{Q}(A)\rangle,$$

where $K_0(\mathscr{S}\mathscr{Q}(A))$ is the Grothendieck group of the category $\mathscr{S}\mathscr{Q}(A)$.

APPENDIX LEMMA. *Let Z be a finite G-set and let $\Theta : \mathscr{S}(G) \to \mathscr{P}(Z)$ be a G-map, where $\mathscr{P}(Z)$ is the power set of Z. For $H \in \mathscr{S}(G)$, suppose $\Theta(H) \subset \Theta(G)$, put*

$$S_{-\lambda}(H) = H \cap S_{-\lambda}, \qquad S_{\lambda}(H) = H \cap S_{\lambda},$$

and consider $\rho_H : \Theta(H) \to \mathscr{S}(H)$, $\gamma \mapsto H \cap \rho(\gamma)$. *Then the assignment*

$$H \mapsto W_{2k}(\mathbb{Z}, H, S_{-\lambda}(H), S_\lambda(H), \Theta(H))$$

is a Mackey functor from the category of all subgroups of G to the category of abelian groups if the following three conditions are satisfied.

(C1) $\Theta(\{e\}) = \varnothing$.

(C2) $\Theta(H) \cap \Theta(H') = \Theta(H \cap H')$ *for all* $H, H' \in \mathscr{S}(G)$.

(C3) γ *lies in* $\Theta(\langle g \rangle)$, *whenever* $\rho_G(\gamma) \ni g$ *for* $\gamma \in \Theta(G)$ *and* $g \in S_\lambda(G)$.

Furthermore, for a set $\mathscr{H}$ *of subgroups of G,*

$$W_{2k}(\mathbb{Z}, G, S_{-\lambda}(G), S_\lambda(G), \Theta(G)) \xrightarrow{\text{Res}} \bigoplus_{H \in \mathscr{H}} W_{2k}(\mathbb{Z}, H, S_{-\lambda}(H), S_\lambda(H), \Theta(H))$$

is injective if the conditions (C1)–(C3), *as well as the next three conditions all are satisfied.*

(C4) $\mathscr{H}$ *contains any 2-hyperelementary subgroup of G.*

(C5) $\bigcup_{H \in \mathscr{H}}(\Theta(H) \times \Theta(H)) = \Theta(G) \times \Theta(G)$.

(C6) *There exists an element* $\beta \in \Omega(G)$ *such that* $\beta = \Sigma_{H \in \mathscr{H}} a(H)[G/H]$ *for some integers* $a(H)$, *and* $\text{Res}^G_H(\beta) = 1$ *in* $\Omega(H)$ *for all* $H \in \mathscr{H}$.

The proof of Appendix Lemma can be provided by straightforward arguments using induction of Dress type. For the details, we refer the reader to the work [BM] or [M5].

An explanation about the definition of the surgery obstruction $\sigma_H(\mathbf{w})$ in the Witt group is in order. Consider the context in the proof of Theorem 4.3 with $H = \{e\}$. Since the G-action on X is orientation preserving, the homomorphism $w : G \to \{1, -1\}$ is given as the trivial map. Performing G-surgery of the G-normal map $\mathbf{w} = (f; b; c)$ until the middle dimension, we can assume that $f : X \to Y$ is k-connected, where $2k = n = \dim X$. Then

$$K_x(f) = \text{Ker}[f_* : H_k(X) \to H_k(Y)]$$

is a finitely generated, stably free A-module. Set

$$S_{-\lambda} = \{g \in G \mid g^2 = 1, \dim X^g = k - 1\},$$

and

$$S_\lambda = \{g \in G \mid g^2 = 1, \dim X^g = k\}.$$

Let Θ be the set of all k-dimensional connected components of L-fixed point sets,

$L \in \mathcal{S}(G)$. For every member X_γ of Θ, there is a unique subgroup L of G such that L is of order 2 and $X_\gamma = X^L$. Furthermore, such X^L must be 1-connected. Let $\rho : \Theta \to \mathcal{S}(G)$ be the map sending $X_\gamma = X^L$ to L. It may be assumed that each member X^L of Θ is oriented in such a way that $g : X^L \to X^{gLg^{-1}}$ is orientation preserving for any $g \in G$. By the same argument as in ordinary surgery theory, we can obtain the λ-hermitian form $\langle\ ,\ \rangle : K_k(f) \times K_k(f) \to A$ from intersection numbers, and the quadratic form $q : K_k(f) \to A_{-\lambda}/A$ from selfintersection numbers. The positioning map $\alpha : \Theta \to K_k(f)$ is given by

$$\alpha(X^L) = proj \circ incl_*[X^L],$$

where $[X^L]$ is the orientation class of X^L, $incl : X^L \to X$ is the canonical inclusion map, and $proj : H_k(X) \to K_k(f)$ is the standard projection. Then, $\mathbf{M} = (K_k(f), \langle\ ,\ \rangle, q, \alpha)$ belongs to the category $\mathcal{S}\mathcal{Q}(A)$. For $H = \{e\}$, we define the element $\sigma_H(\mathrm{w})$ in

$$W_{2k}(\mathbb{Z}, G, S_{-\lambda}, S_\lambda, \Theta)$$

as the equivalence class containing this $\mathbf{M}$. For an arbitrary $H \in \mathcal{S}(G)$, we replace G by N/H with $N = N_G(H)$ in the definition above, and apply the same definition for the trivial subgroup of N/H to get the element $\sigma_H(\mathrm{w})$ in

$$W_{2k}(\mathbb{Z}, N/H, N/H(X^H, k-1), N/H(X^H, k), \Theta(N/H, X^H))$$

with $N/H(X^H, k-1)$, $N/H(X^H, k)$ and $\Theta(N/H, X^H)$ defined in (D2)–(D4) and (A3) below, where $2k = \dim X^H$. In order to apply the G-surgery theory presented in [BM] and [M5], we should note that the following four assertions hold.

(A1) For each subgroup K of N/H such that $\dim(X^H)^K = k$, $|K| = 2$.
(A2) For such a K, $(X^H)^K$ is connected and orientable.
(A3) The set $\Theta(N/H, X^H)$ of all k-dimensional connected components of K-fixed point sets $(X^H)^K$, where K runs over all subgroups of N/H, can be identified with the set $\{K \in \mathcal{S}(N/H) \mid \dim(X^H)^K = k\}$ via the map $\rho_{N/H} : (X^H)^K \mapsto K$.
(A4) One can orient $(X^H)^K \in \Theta(N/H, X^H)$ so that $g : (X^H)^K \to (X^H)^{gKg^{-1}}$ is an orientation preserving map for any $g \in N/H$.

The assertions (A1) and (A3) both obviously hold. The assertion (A2) is seen as follows. Take a subgroup L of N such that $L \rhd H$ and $L/H = K$. Since $H \in \mathcal{H}$, L also belongs to $\mathcal{H}$, as well as to $\mathcal{K}$. Thus, Y^L is connected and simply connected by the assumption (2.1.2). Since $f^L: X^L \to Y^L$ is a homotopy equivalence, $(X^H)^K = X^L$ is connected and simply connected. The assertion (A4) follows from the assumption (2.1.3).

With this background, the Witt group depends only on the following four data.

(D1) The group ring $\mathbb{Z}[N/H]$ with involution – and symmetry $(-1)^k$.
(D2) The set $N/H(X^H, k-1) = \{g \in N/H \mid g^2 = 1, \dim(X^H)^g = k-1\}$.
(D3) The set $N/H(X^H, k) = \{g \in N/H \mid g^2 = 1, \dim(X^H)^g = k\}$.
(D4) The set $\Theta(N/H, X^H)$ with map $\rho_{N/H} : \Theta(N/H, X^H) \to \mathcal{S}(N/H)$.

As we did in the proof of Theorem 4.3 for the Wall and Bak groups, we denote the Witt surgery obstruction groups by $\mathcal{O}(N/H, X^H)$, and consider the assignment:

$$S \mapsto \mathcal{O}(S, \operatorname{Res}_S^{N/H}(X^H)),\ S \in \mathcal{S}(N/H).$$

If X satisfies only the weak gap hypothesis for (H), then also in the case of Witt groups, this assignment defines a Mackey functor by [BM] or [M5], Theorem 3.4 (cf. Appendix Lemma) because

$$\Theta(S, H^H) \cap \Theta(S', X^H) = \Theta(S \cap S', X) \qquad \text{for all } S \text{ and } S' \text{ in } \mathcal{S}(N/H).$$

Therefore, also in the case of Witt groups,

$$\{\mathcal{O}(S, \operatorname{Res}_S^{N/H}(X^H)) \mid S \in \mathcal{S}(N/H)\} \text{ is a Mackey functor.}$$

Moreover, this functor has the defect set consisting of all solvable subgroups of N/H. That is, as in (4.5),

$$\mathcal{O}(N/H, X^H) \xrightarrow{\operatorname{Res}} \bigoplus_{S \text{ solvable}} \mathcal{O}(S, \operatorname{Res}_S^{N/H}(X^H)) \textit{ is injective.}$$

In the case of Witt groups, this injectivity follows from [BM], and [M5], Theorem 3.4 (cf. Appendix Lemma) because the following three properties hold.

(P1) Any finite 2-hyperelementary group is solvable.
(P2) $\bigcup_{S \text{ solvable}} (\Theta(S, X^H) \times \Theta(S, X^H)) = \Theta(N/H, X^H) \times \Theta(N/H, X^H)$.
(P3) $\kappa = 1 - \iota_{\mathcal{S}}$ satisfies the conditions (4.2.1) and (4.2.2).

The second property follows from the fact that any finite group generated by two elements or order 2 is a cyclic group or a dihedral group.

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