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Embedding punctured lens spaces in four-manifolds

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1. Introduction

The general problem of classifying embeddings of a given compact 3-manifold into a given 4-manifold has a rich history, intrinsically tied to broader developments in low-dimensional topology. It is our goal here to investigate the embedding problem in light of the significant advances in 4-manifold theory of recent years. The focus will be on one special case of the problem, the determination of the least integer n for which the lens space $L(p, q)$, $p > 1$, or the punctured lens space $L(p, q)_0$, embeds in $\#_n \mathbf{C}P^2$. Except for some elementary preliminary results we will restrict attention to the lens spaces $L(p, 1)$ that arise as total spaces of circle bundles over the 2-sphere. We will consider both the topological locally flat and smooth categories, for which the results differ significantly. Although this is certainly a special case, it is nonetheless sufficient to illustrate the basic methods, to demonstrate the impact of the recent developments in the field, and finally to indicate the limitations of these methods and to point to problems calling for further research.

It is a well known fact, due originally to Hantzsche [1938] and based upon duality, that no lens spaces (other than S^3 and $S^1 \times S^2$) can embed in 4-space. Zeeman [1965] proved as a by-product of his work on twist-spun knots that a punctured lens space $L(p, q)_0$ embeds in S^4 provided p is odd. (For an alternative approach see Hosokawa and Suzuki [1981].) Meanwhile, Epstein [1965] observed that a homotopy-theoretic result of D. Puppe [1958] implies that the condition that p be odd is in fact necessary. He concluded that, "The situation with regard to embeddings of lens spaces and punctured lens spaces is therefore completely solved." He was, of course, implicitly referring to embeddings in euclidean space. In fact, as we shall see, there are many other interesting questions involving just which lens space or punctured lens space might embed in which 4-manifolds.

We note below that any lens space $L(p, q)$ embeds smoothly in $\#_n \mathbf{C}P^2$, for some positive integer n, a fact not true for all 3-manifolds. In the topological category we give a complete analysis of the minimum n such that $L(p, 1)$ so embeds, and a nearly complete analysis for the punctured lens spaces $L(p, 1)_0$. We contrast this with some non-embedding results for $L(p, 1)_0$ in the smooth case, in the light of recent work on gauge theory. Both the cases of $L(p, q)$, $q \neq \pm 1$, and the smooth situation merit further study. In addition we handle the cases of $S^2 \times S^2$ and $-\mathbf{C}P^2 \# \mathbf{C}P^2$. While the detailed results are too complicated to summarize in the introduction, they are laid out in a series of tables in the final section of the paper.

2. Some elementary embeddings

We begin with some link-calculus style embeddings of certain lens spaces and punctured lens spaces in connected sums of copies of $S^2 \times S^2$, the twisted S^2-bundle over S^2, and $\mathbf{C}P^2$.

2.1 PROPOSITION. *Any lens space embeds smoothly in* $\#_n S^2 \times S^2$ *for some positive integer* n.

Proof. Every lens space bounds a standard plumbing manifold with all even framings, as in Hirzebruch et al. [1971]. The double of such a plumbing manifold is necessarily of the form $\#_n S^2 \times S^2$.

2.2 REMARK. By taking a further connected sum with a copy of the twisted S^2-bundle over S^2, we get an embedding in $\#_m S^2 \tilde{\times} S^2$.

In fact, of course, every closed, orientable 3-manifold embeds smoothly in some $\#_n S^2 \times S^2$, by essentially the same proof. If one punctures the lens space it turns out that $n = 1$ always suffices.

2.3 PROPOSITION. *For all integers* $p \geq 0$ *and* q, $(q, p) = 1$, *the punctured lens space* $L(p, q)_0$ *embeds smoothly in* $S^2 \times S^2$.

Proof. Without loss of generality we may assume that pq is even. Start with a standard handlebody description of $L(p, q)_0 \times I$, coming from a handlebody description of $L(p, q)$ and consisting of a 0-handle, a 1-handle, and a 2-handle attached along a "(p, q)-curve" with framing pq. Add two 0-framed 2-handles as small linking circles to the 1-handle and to the 2-handle. Then the whole framed link falls apart into a cancelling 1-handle and 2-handle pair, and a Hopf link with one component with framing 0 and the other with framing pq. Since pq is even, this is a framed link picture of $S^2 \times S^2$, as required.

2.4 REMARK. If p is odd, and one repeats the construction above using an odd q, then the resulting 4-manifold is $S^2 \tilde{\times} S^2$. But, as we shall see, punctured lens spaces with even order fundamental group do not always embed here.

2.5 PROPOSITION. *Every lens space $L(p, q)$ embeds smoothly in $\#_n CP^2$ for some positive integer n.*

Proof. We give a link calculus approach to this. We may assume that p and q have opposite parity. Expand p/q in a continued fraction decomposition

$$a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \cdots - \cfrac{1}{a_k}}}$$

with all entries ≥ 2. Take a collection of $n \gg k$ unknotted unlinked circles with framing $+1$, describing some $\#_n CP^2$. Start with k of the circles. Sliding one over the next creates a simple chain with framings $1, 2, \ldots, 2$. By sliding links of this chain over other framed components, it is easy to create a linked chain of unknotted curves with framings corresponding to the terms of the continued fraction (many of the curves that are slid over are tangled in with the chain). The chain then describes the lens space sitting in $\#_n CP^2$.

This gives another interesting, nontrivial, genus-like invariant of lens spaces. It follows easily that $L(p, 1)$ embeds smoothly in $\#_p CP^2$, with simply connected complementary domains with second betti numbers 1 and $p - 1$. We shall see that $L(p, 1)$ always embeds topologically in $\#_n CP^2$, with $n \leq 5$. It is an interesting question whether n can be chosen independent of p and q in general. This is almost surely false in the smooth category.

2.6 THEOREM. *If $S^2 \subset X^4$ and $S^2 \cdot S^2 = \pm p$, then $L(p, 1) \subset X^4$.*

Proof. The lens space arises as the boundary of a tubular neighborhood of the 2-sphere. In the smooth category this is standard elementary material involving the existence of normal bundles and the classification of oriented 2-plane bundles by their euler classes. In the topological category we must appeal to the much deeper existence of topological normal bundles in the case of codimension two embeddings in topological 4-manifolds, due to Freedman. See Freedman and Quinn [1990], Section 9.3.

2.7 REMARK. The only homologically nontrivial embedded 2-spheres in CP^2 represent the generator and twice the generator and yield smooth embeddings of $L(1, 1) \cong S^3$ and $L(4, 1)$. In $CP^2 \# CP^2$ we also obtain a smooth 2-sphere representing 3 times the first generator, by the "Boardman construction" (see Boardman [1964]), and hence a smooth $L(9, 1)$. In this case one takes three 2-spheres, each representing the first generator, and two 2-spheres respectively representing the second generator and its negative. Among these five embedded 2-spheres there are three points of positive intersection and one point of negative intersection. Appropriate tubing eliminates the intersection points and creates an embedded sphere representing the class $(3, 0)$. As Boardman notes, the same technique allows one to represent the class $(3, 2, 1)$ in $\#_3 CP^2$ by an embedded sphere: Start with three spheres representing the first generator, two spheres representing the second generator, two spheres representing the third generator, and one sphere representing the negative of the third generator. Appropriate tubing creates the desired 2-sphere. We note one more example of this technique. The class $(3, 3, 1, 0)$ in $\#_4 CP^2$ is represented by a smoothly embedded 2-sphere: Take three spheres representing the first generator, three spheres representing the second generator, two spheres representing the third generator, and one sphere representing the negative of the third generator, and one sphere representing the fourth generator and one sphere representing the negative of the fourth generator. Once again, appropriate tubing creates the desired smoothly embedded 2-sphere. All other known smoothly embedded 2-spheres in $\#_n CP^2$, n small, are "formal" consequences of these.

2.8 REMARK. Not every 3-manifold embeds *smoothly* in some $\#_n CP^2$, however. For example the Poincaré homology 3-sphere Σ does not.

Proof. If it did, then both $+\Sigma$ and $-\Sigma$ would bound positive definte 4-manifolds. But Σ also bounds a smooth E_8-manifold. Putting these together yields a smooth positive definite 4-manifold with a nonstandard form, contradicting early work of Donaldson.

3. The basic construction

The essential idea here is to show that an embedded lens space or punctured lens space gives rise to a family of embedded surfaces, which, in favorable cases, can be chosen to be 2-spheres, but perhaps in a different 4-manifold. We begin with several standard lemmas, whose proofs will be omitted.

3.1 LEMMA. *In any simply connected spin 4-manifold X^4 there are exactly four isotopy classes of embeddings $S^1 \times D^3 \to X^4$, distinguished by orientation and spin parity. If X^4 has odd type then there are exactly two isotopy class of embeddings, distinguished by orientation.*

3.2 LEMMA. *In any simply connected spin 4-manifold X^4 there are exactly two isotopy classes of embeddings $S^1 \times D^2 \to X^4$, distinguished by spin parity. If X^4 has odd type then there is exactly one isotopy class of embeddings.*

3.3 LEMMA. *A (p, q) curve on the boundary of $S^1 \times D^2$ under a change of framing homeomorphism $(z, w) \mapsto (z, z^n w)$ becomes a $(p, q+np)$ curve. Under the change of orientation homeomorphism $(z, w) \mapsto (z, \bar{w})$ the (p, q) curve becomes a $(p, -q)$ curve.*

The first two lemmas above show that all of the self-homeomorphisms of $S^1 \times D^2$ described in the preceding lemma can be realized by isotopies of the ambient 4-manifold, assuming it has odd type.

3.4 LEMMA. *A (p, q) curve $K \subset S^1 \times S^1 \subset S^1 \times D^2 \subset S^1 \times S^2 \subset B^2 \times S^2$ bounds an orientable surface in $B^2 \times S^2$ of genus less than or equal to $\frac{1}{2}(|p|-1)(|q|-1)$. This surface can be chosen to be disjoint from $B^2 \times \{\text{point}\}$.*

3.5 THEOREM. *If p is even and $L(p, q)_0 \subset X^4$, where X^4 is a simply connected 4-manifold of odd type, then there are embedded surfaces $\Sigma_\pm$ in $S^2 \times S^2 \# X^4$, of genus less than or equal to $\frac{1}{2}(|p|-1)(|q|-1)$, representing homology classes of the form $(p, k_\pm; \alpha)$ in $H_2(S^2 \times S^2 \# X^4) = \mathbf{Z} \oplus \mathbf{Z} \oplus H_2(X^4)$ with self-intersection $\Sigma_\pm \cdot \Sigma_\pm = 2k_\pm p + \alpha^2 = \pm pq$ where k_+ and k_- have opposite parities.*

Proof. Write $L(p, q)_0 = S^1 \times D^2 \cup$2-handle, where the 2-handle, $D^2 \times I$, is attached along a (p, q)-curve on $S^1 \times \partial D^2$. Isotope $S^1 \times D^2$ into standard position on the boundary of a standard 4-ball inside a slightly larger 4-ball. In particular we have in X^4 a copy of $D^2 \times D^2$ containing our $S^1 \times D^2$ in a standard way. In the usual longitude-meridian coordinates induced from the 3-sphere, the attaching curve becomes a (p, q_1)-curve for some q_1 of the form $q_1 = \pm q + np$ for some integer n. Using the preceding lemmas we can arrange that $q_1 = q$.

Remove the interior of a small regular neighborhood of $S^1 \times D^2$ of the form $S^1 \times D^3$, replacing it with a copy of $B^2 \times S^2$. We do this in such a way that the ambient manifold becomes $S^2 \times S^2 \# X^4$. The attaching (p, q) curve for the 2-handle bounds a surface F_- in $B^2 \times S^2$ missing $B^2 \times \{\text{point}\}$. This surface, together with the core C of the 2-handle of $L(p, q)_0$, defines the surface Σ_-. Moreover, the self-intersection $\Sigma_- \cdot \Sigma_- = -pq$, as one sees if one traces through standard sign conventions carefully.

One can understand k_- in the statement of the theorem as the intersection number $\Sigma_- \cdot (B^2 \times \{\text{point} \cup D^2 \times \{\text{point}\}) = C \cdot (D^2 \times \{\text{point}\})$ since the first factor of the $S^2 \times S^2$ summand is made up of the added $B^2 \times \{\text{point}\}$ and the $D^2 \times \{\text{point}\}$. Here the $D^2 \times \{\text{point}\}$ comes from the product $D^2 \times D^2$ structure inside X^4.

To construct surface Σ_+, change the attaching map for the $B^2 \times S^2$ by an isotopy of the identity that moves the $(p, -q)$ curve to a (p, q) curve. We cap off the core C of the 2-handle of $L(p, q)_0$ with a different surface F_+ in $B^2 \times S^2$ missing $B^2 \times \{\text{point}\}$ with boundary the $(p, -q)$ curve. Note that the resulting surface represents a homology class of the form $(p, k_+; \alpha)$, since the only change from Σ_- is supported inside the added $B^2 \times S^2$. Thus $\Sigma_+ \;\; \Sigma_+ = 2k_+p + \alpha^2 = +pq$. The equations $2k_+p + \alpha^2 = +pq$ and $2k_-p + \alpha^2 = -pq$ then imply that $2(k_+ - k_-)p = 2(pq)$, so that $k_+ - k_- = q$, which is odd. Thus, k_+ and k_- have opposite parities, as required.

3.6 COROLLARY. *If p is even and $L(p, 1)_0 \subset X^4$, where X^4 is a simply connected 4-manifold of odd type, then there is an embedded 2-sphere Σ in $S^2 \times S^2 \# X^4$, representing a homology class of the form $(p, k; \alpha)$ in $H_2(S^2 \times S^2 \# X^4) = \mathbf{Z} \oplus \mathbf{Z} \oplus H_2(X^4)$ with self-intersection $\Sigma \cdot \Sigma = 2kp + \alpha^2 = \pm p$, where k is even.*

3.7 REMARK. In our main applications the 4-manifold X^4 will be of the form $\#_n \mathbf{C}P^2$, with $n \leq 5$. In this case we can adjust the homology class of the surface Σ by diffeomorphisms of $S^2 \times S^2 \#_n \mathbf{C}P^2$ to normalize the homology class α as much as possible, while fixing the self-intersection number and the first coordinate. In this case a homology class can be represented in the form $(p, k; r_1, \ldots, r_n)$. According to C. T. C. Wall [1964] every automorphism of $H_2(S^2 \times S^2 \#_n \mathbf{C}P^2)$ (respecting the intersection form) is realizable by a diffeomorphism, since $n \leq 8$. Let $(a, b; e_1, \ldots, e_n)$ be a standard basis for the homology. Consider changes of basis of the form $a' = a \pm e_i \pm e_j \mp b$, $b' = b$, $e_i' = e_i \mp b$, $e_j' = e_j \mp b$, and $e_k' = e_k$ for $k \neq i$ or j. (It is allowed that $i = j$.) The effect of such an automorphism on $(p, k; r_1, \ldots, r_n)$ is to leave the first coordinate alone, adding or subtracting p, respectively $2p$, from two, respectively one, of the last n coordinates. In addition we have the automorphisms that change one basis element e_i to its negative. (On a single $\mathbf{C}P^2$ this is induced by complex conjugation.) In this way we can arrange that all $r_i \geq 0$, $r_1 \leq p$, $r_i \leq p/2$ for $i \geq 2$. By further permutation of basis elements we can further assume that $r_2 \geq r_3 \geq \cdots \geq r_n$.

Note also that if the original punctured lens space $L(p, 1)_0$ was smoothly embedded in a smooth 4-manifold X^4, then the resulting 2-sphere $\Sigma \subset S^2 \times S^2 \# X^4$ may be chosen to be a smoothly embedded 2-sphere.

4. Two topological applications of the Basic Construction

As a quick application of these ideas we obtain an elementary proof of the result of Epstein on punctured lens spaces in S^4.

4.1 COROLLARY (Epstein). *If $L(p, q)_0 \subset S^4$ then p is odd.*

Proof. If p is even, then by the preceding observations we have the relation $2kp + 0 = \pm pq$. But this would imply that q is even, contradicting the fact that p and q must be relatively prime.

4.2 COROLLARY. *If $L(p, q)_0 \subset \mathbf{C}P^2$ then the exponent of 2 in p is even.*

Proof. The Basic Construction yields a surface in $S^2 \times S^2 \# \mathbf{C}P^2$ representing a homology class of the form $(p, k; r)$, where $2pk + r^2 = \pm pq$. Then $r^2 = p(\pm q - 2k)$. If p is even, so that there is actually something to prove, then q must be odd, and the result follows immediately from the latter equation.

5. Classical results of Rochlin, Hsiang-Szczarba, and Kervaire-Milnor

The first results use work of Rochlin and Kervaire-Milnor and of Rochlin and Hsiang-Szczarba: Suppose $\Sigma \subset X^4$ is a 2-sphere. Recall that Σ, or its underlying homology class, is said to be *characteristic* if $\Sigma \cdot F \equiv F \cdot F \bmod 2$ for any surface or homology class F in X^4.

5.1 ROCHLIN CONGRUENCE (as formulated by Rochlin [1952], generalized by Kervaire and Milnor [1961] and suitably extended to the topological case). If X^4 is a spin manifold, then $sign(X^4) \equiv 0 \bmod 16$. If $[\Sigma]$ is a characteristic homology class in a (stably smoothable), not necessarily spin, 4-manifold X^4, then $\Sigma \cdot \Sigma \equiv sign(X^4) \bmod 16$. More generally,

$$\Sigma \cdot \Sigma \equiv sign(X^4) + 8\mathrm{KS}(X^4) \bmod 16$$

for X^4 an arbitrary closed, oriented topological 4-manifold, where $\mathrm{KS}(X^4) \in \mathbf{Z}_2$ denotes the Kirby-Siebenmann triangulation obstruction and $8\colon \mathbf{Z}_2 \to \mathbf{Z}_{16}$ is the standard inclusion.

5.2 ROCHLIN INEQUALITY (as formulated by Rochlin [1971] (compare also similar results of Hsiang and Szczarba [1970]) and extended to the topological category). If $[\Sigma]$ is divisible by 2, then

$$|\Sigma \cdot \Sigma/2 - sign(X^4)| \leq b_2(X^4)$$

If $[\Sigma]$ is divisible by $k = p^r$, where p is an odd prime, then

$$\left|\frac{k^2-1}{k^2}\Sigma \cdot \Sigma/2 - sign(X^4)\right| \leq b_2(X^4).$$

6. Two topological applications of the Rochlin Conditions to unpunctured lens spaces in certain 4-manifolds

Here we give a couple of quick applications.

6.1 THEOREM. *If $L(p, 1)$ embeds topologically in a closed, simply connected 4-manifold X^4 in such a way that one complementary domain is rationally acyclic, then $p = 4$.*

Proof. Suppose $X^4 = U \cup_L V$ with U rationally acyclic. Replace V with the euler class p 2-disk bundle E_p over the 2-sphere, forming a new 4-manifold $Y^4 = U \cup_L E_p$. Then Y^4 is a homotopy $\pm CP^2$ containing the 0-section of E_p, a topologically embedded 2-sphere of self-intersection p. It follows that $p = d^2$ where d is the divisibility of the homology class represented by the 2-sphere. The Rochlin Inequalities then imply that further $p \leq 4$.

6.2 PROPOSITION. *If $L(p, 1)$ can be embedded in a positive definite 4-manifold X^4 with $b_2 = n$, then p can be written as a sum of at most n squares.*

Proof. Since by number theory every positive integer p can be written as a sum of at most 4 squares (see Grosswald [1985], for example) we may as well assume $n \leq 3$. We may also assume that $p \neq 4$. The algebraic classification of definite forms of low rank implies that the intersection form of X^4 is a standard diagonal form. Let $X^4 = U \cup_L V$. Both U and V are positive definite in their induced orientations, with nonzero second betti number, by Theorem 6.1. One of U and V, say V, has the same oriented boundary as the disk bundle E_p. Replace V by E_p, forming a new simply connected 4-manifold $Y^4 = U \cup_L E_p$. Now Y^4 is also positive definite by our choice of which complementary domain to replace with E_p, and has $b_2(Y^4) \leq n$. Since n is assumed to be small Y^4 also has a standard diagonal form. The homology class represented by the core 2-sphere Σ in E_p is a linear combination of the elements of an orthonormal basis. It follows that $p = \Sigma \cdot \Sigma$ is a sum of at most n squares.

6.3 COMPLEMENT. *If $L(p, 1)$ can be embedded in a positive definite 4-manifold X^4 with $b_2 = n \leq 4$ and p cannot be written as a primitive sum of n squares (i.e. with* gcd $= 1$), *then* $(p, n) = (4, m)$, $1 \leq m \leq 3$; $(8, m)$, $2 \leq m \leq 4$; $(9, 2)$; $(12, 3)$; *or* $(16, 4)$.

Proof. As in the proof of Proposition 6.2 we can assume that $L(p, 1)$ is the boundary of a tubular neighborhood of an embedded 2-sphere Σ with $\Sigma \cdot \Sigma = p$. Moreover, expressing Σ as a linear combination of an orthonormal basis displays p as a sum of $\leq n$ squares. By the Rochlin Inequality, if this expression is divisible, then $p \leq 9n/2 \leq 18$. A case by case analysis of when p and n are small completes the argument.

In the rest of this paper we will follow the notational convention established by Freedman and Quinn [1990], Section 10.4: if X^4 is a simply connected topological 4-manifold with odd intersection pairing, then $*X^4$ denotes the unique topological 4-manifold homotopy equivalent to X^4 but not homeomorphic to X^4.

6.4 COROLLARY. *If p is odd, $p \neq 9$, and $L(p, 1) \subset \#_2\mathbf{C}P^2$ or $* \#_2\mathbf{C}P^2$, then p is not divisible by any prime congruent to* 3 *mod* 4.

Proof. According to the above, $p = r^2 + s^2$ is a primitive sum of two squares. Then $r^2 + s^2 \equiv 0 \bmod q$ for any prime q dividing p. This means -1 is a square mod q, hence the claim.

6.5 COROLLARY. *If $L(p, 1) \subset \#_3\mathbf{C}P^2$ or $* \#_3\mathbf{C}P^2$, then $p \not\equiv 7$ mod* 8.

Proof. We can express p as a sum of three squares. The squares mod 8 are 0, 1, and 4. No three of these add up to 7 mod 8.

In Section 9 we shall make use of number-theoretic analyses of sums of squares to reverse the implications in the above propositions.

7. Topological applications of the Rochlin Conditions to punctured lens spaces in definite 4-manifolds

Here we combine the Rochlin Conditions with the Basic Construction. Throughout this section we assume that p is even, since Zeeman has shown that all punctured lens spaces with odd order fundamental group embed in S^4.

7.1 PROPOSITION. *If p is even and $L(p, 1)_0$ embeds topologically in $\mathbf{C}P^2$ or in $*\mathbf{C}P^2$ then $p = 4$.*

Proof. Let $X^4 = \mathbf{C}P^2$ or $*\mathbf{C}P^2$. The basic construction yields a 2-sphere Σ in $S^2 \times S^2 \# X^4$ representing a homology class of the form $(p, k; r)$, where $2pk + r^2 = \pm p$, and we may assume that k is even. The latter equation implies that r also must be even. Hence Σ represents a nontrivial, 2-divisible homology class. By the Rochlin Inequality $p \leq 8$. By Corollary 4.2 we know that $p \neq 2$ or 6. It remains to rule out $p = 8$. But if $p = 8$, then the fundamental equation reads $16k + r^2 = \pm 8$. Reduction mod 8 shows that $r \equiv 0 \bmod 4$. But then $r^2 \equiv 0 \bmod 16$, which is a contradiction.

7.2 REMARK. $L(4, 1)$ does embed in $\mathbf{C}P^2$ as the boundary of a tubular neighborhood of a smooth quadric curve. By the embedding theorem of Lee-Wilczyński and Hambleton-Kreck (see Section 8 and Theorem 8.1 below), there is also an embedded 2-sphere Σ in $*\mathbf{C}P^2$ such that $\Sigma \cdot \Sigma = 4$. So again $L(4, 1)$ embeds in $*\mathbf{C}P^2$ as the boundary of a tubular neighborhood of Σ.

7.3 PROPOSITION. *If p is even, and $L(p, 1)_0$ embeds topologically in $\mathbf{C}P^2 \# \mathbf{C}P^2$ then $p = 4$ or 8, or $p \equiv \pm 2 \bmod 16$.*

Proof. As before, we obtain a 2-sphere Σ in $S^2 \times S^2 \# \mathbf{C}P^2 \# \mathbf{C}P^2$, representing a homology class of the form $(p, k; r, s)$, where we can assume k is even, and $\Sigma \cdot \Sigma = 2kp + r^2 + s^2 = \pm p$. Since p is even, either both r and s are odd, or both r and s are even. In the former case, Σ is a characteristic 2-sphere and the condition $p \equiv \pm 2 \bmod 16$ follows from the Rochlin Congruence. In the latter case, Σ represents a 2-divisible homology class, and it follows from the Rochlin Inequality that $p \leq 12$. Since Σ is divisible by 2, we see that $\Sigma \cdot \Sigma = \pm p$ is divisible by 4, thus ruling out $p = 6$ or 10. It remains to rule out the possibility that $p = 12$. In this case our basic equation says that $24k + r^2 + s^2 = \pm 12$, with k, r, and s all even. The even squares mod 48 are 0, 4, 16, and 36. Reasoning mod 48, it follows that $\Sigma \cdot \Sigma = -12$. But this contradicts the Rochlin Inequality.

7.4 REMARK. If $p = 2$, 4, or 8, then $L(p, 1)$ (unpunctured) embeds smoothly in $\mathbf{C}P^2 \# \mathbf{C}P^2$ as the boundary of a tubular neighborhood of a smooth curve representing the homology class $(1, 1)$, $(2, 0)$, or $(2, 2)$, respectively.

7.5 PROPOSITION. *If p is even, and $L(p, 1)_0$ embeds topologically in $*\mathbf{C}P^2 \# \mathbf{C}P^2$ then $p = 4$, or 8, or $p \equiv \pm 6 \bmod 16$.*

Proof. The proof is the same as above, only taking into account the term $8\mathrm{KS} \equiv 8 \bmod 16$ appearing in the Rochlin Congruence.

7.6 PROPOSITION. *If p is even, and $L(p, 1)_0$ embeds topologically in $\#_3\mathbf{C}P^2$ or in $* \#_3\mathbf{C}P^2$, then $p \equiv 2 \bmod 4$ or $p \leq 16$.*

Proof. We suppose that $p \equiv 0 \bmod 4$. Let $X^4 = \#_3\mathbf{C}P^2$ or $* \#_3 \mathbf{C}P^2$. The basic construction yields an embedded 2-sphere $\Sigma \subset S^2 \times S^2 \# X^4$ representing a homology class of the form $(p, k; r, s, t)$, with $2kp + r^2 + s^2 + t^2 = \pm p$, where we may assume that k is even. Since $p \equiv 0 \bmod 4$ we see that r, s, and t are all even, and the homology class represented by Σ is 2-divisible. The Rochlin Inequality then implies that $p \leq 4 \times 4 = 16$.

7.7 REMARK. Reasoning as before we see that $L(p, 1)$ embeds smoothly in $\#_3\mathbf{C}P^2$ for $p = 2$, 4, 6, 8, 10, or 14 as the boundary of a tubular neighborhood of a smooth 2-sphere representing a class of the form $(1, 1, 0)$, $(2, 0, 0)$, $(1, 1, 2)$, $(2, 2, 0)$, $(1, 3, 0)$, or $(1, 2, 3)$, respectively. Compare the summary of the Boardman construction in Remark 2.7.

7.8 PROPOSITION. *If p is even, and $L(p, 1)_0$ embeds topologically in $\#_4\mathbf{C}P^2$ then $p \not\equiv 0$ mod 8, or $p \leq 20$.*

Proof. Suppose that $p \equiv 0 \bmod 8$. The basic construction yields an embedded 2-sphere $\Sigma \subset S^2 \times S^2 \#_4 \mathbf{C}P^2$ representing a homology class of the form $(p, k; r, s, t, u)$, with k even and $2kp + r^2 + s^2 + t^2 + u^2 = \pm p$. Since $p \equiv 0 \bmod 8$ we see that r, s, t and u are all even, and the homology class represented by Σ is 2-divisible. The Rochlin Inequality then implies that $p \leq 4 \times 5 = 20$.

7.9 COROLLARY. *$L(8k, 1)_0$ does not embed topologically in $\#_4\mathbf{C}P^2$ for $k \geq 3$.*

Our primary remaining need is to decide whether $L(p, 1)_0$ embeds in $\#_4\mathbf{C}P^2$ when $p \equiv -4 \bmod 16$. We will see below that it cannot embed smoothly.

8. Topological embeddings of spheres

R. Lee and D. Wilcyzński [1990] have investigated the problem of representing a homology class by a topological locally flat embedded 2-sphere. The end result is that the Rochlin-Kervaire-Milnor Congruence and the Rochlin Inequality provide necessary and sufficient conditions for the existence of a *simple* 2-sphere – that is, a 2-sphere whose complement has abelian fundamental group.

8.1 THEOREM. *Let X^4 be a closed, simply connected topological 4-manifold and $\xi \in H_2(X^4)$. Then ξ is represented by a simple, topological, locally flat, embedded 2-sphere in X^4 if and only if*

- $\xi \cdot \xi \equiv \mathrm{sign}(X^4) + 8KS(X^4)$ *mod* 16 *if ξ is characteristic and*
- $|\Sigma \cdot \Sigma/2\text{-}\mathrm{sign}(X^4)| \leq b_2(X^4)$ *if $[\Sigma]$ is divisible by* 2 *and*
  $|((d^2 - 1)/d^2)\Sigma \cdot \Sigma/2 - \mathrm{sign}(X^4)| \leq b_2(X^4)$ *if the divisibility d of ξ is odd.*

We note that Lee and Wilczyński originally proved this in the case d is odd; subsequently they [1993] and, independently, I. Hambleton and M. Kreck [1993] completed the earlier program to give the full result in the case d is even. We have slightly modified the statement into a form different from but equivalent to that given by these authors. We primarily need the case of divisibility one here. But we do need to refer occasionally to the general case including especially even divisibility.

As an application of this result we have the following statement about certain embeddings, which will be used to indicate the limitations of the Basic Construction in Section 3 as a necessary condition for embedding punctured lens spaces.

8.2 COROLLARY. *The following homology classes are represented by topologically embedded 2-spheres in* $S^2 \times S^2 \#_n CP^2$:

- $p \equiv \pm 2 \bmod 16$: *The classes* $(p, \frac{1}{4}(-p \pm 2); \frac{1}{2}p, \frac{1}{2}p)$ *in* $S^2 \times S^2 \#_2 CP^2$
- $p \equiv 12 \bmod 16$: *The classes* $(p, \frac{1}{8}(-p \pm 4); \frac{1}{4}p, \frac{1}{4}p, \frac{1}{4}p)$ *in* $S^2 \times S^2 \#_4 CP^2$
- $p = 16$: *The classes* $(16, 0; 4, 0, 0)$ *and* $(16, -1; 4, 0, 0)$ *in* $S^2 \times S^2 \#_3 CP^2$.

9. Topological embeddings of unpunctured lens spaces in positive definite 4-manifolds

In this section we realize topologically embedded lens spaces $L(p, 1)$ by using number-theoretic information about writing p as a sum of squares and applying the embedding theorem above to produce an embedded 2-sphere with euler class p. There will be many cases to consider.

9.0 FACTS ABOUT SUMS OF SQUARES. Here we summarize some basic facts. References are Grosswald [1985] and Dickson [1971].

- Any positive integer is a sum of 4 squares. For odd integers such a decomposition can always be chosen to be primitive. But not all even integers can be expressed as a primitive sum of 4 squares e.g., 32.
- Any positive integer is a primitive sum of 5 squares. *Proof*: Express $p-1$ as a sum of 4 squares and then add 1.
- A positive integer p is a sum of 3 squares if and only if p is not of the form $4^a(8b+7)$. Such a decomposition can be chosen to be primitive if $p \not\equiv 0$ mod 4.
- A positive integer p is a sum of 2 squares if and only if all prime divisors q of n such that $q \equiv 3 \pmod 4$ occur to an even power in p. Such a decomposition can be chosen to be primitive if and only if p is not divisible by any prime congruent to 3 mod 4 and is not divisible by 4.

9.1 PROPOSITION. *If p is even, then $L(p, 1)$ embeds topologically in* $\#_5 CP^2$ *and* $* \#_5 CP^2$.

Proof. Writing $p-1$ as a sum of 4 squares, we see that p can always be written as a sum of five relatively prime squares. These five numbers, before squaring, determine a homology class in $H_2(\#_5 CP^2)$. This class is primitive by the relatively prime condition. It is ordinary, because not all five integers can be odd when p is even. Therefore the Embedding Theorem 8.1 of Lee and Wilczyński and Hambleton and Kreck shows that this class is represented by a topologically embedded 2-sphere with self-intersection p. The boundary of the tubular neighborhood of this 2-sphere is $L(p, 1)$. The same proof applies to $* \#_5 CP^2$.

9.2 PROPOSITION. *If p is odd, then $L(p, 1)$ embeds topologically in $\#_4 CP^2$ and $* \#_4 CP^2$.*

Proof. Represent p as a primitive sum of 4 squares. Since p is odd the summands cannot all be odd. Therefore this decomposition corresponds to a primitive ordinary homology class in $\#_4 CP^2$. By Theorem 8.1 it is represented by a topologically embedded 2-sphere in $\#_4 CP^2$. The boundary of the tubular neighborhood of this 2-sphere is $L(p, 1)$. The same proof applies to $* \#_4 CP^2$.

9.3 PROPOSITION. *If $p \equiv 4 \bmod 16$, then $L(p, 1)$ embeds topologically in $\#_4 CP^2$.*

Proof. One can write the odd integer $p - 1$ as a sum of three squares, since it is not of the form $4^e(8b + 7)$. Thus p can be expressed as a sum of four squares, one of which is 1. Reducing mod 8 shows that all four of the squares are odd. Thus we have $p = r_1^2 + \cdots + r_4^2$ where $r_1, \ldots, r_4$ are odd, with gcd 1. Consider the corresponding homology class $\xi = (r_1, \ldots, r_4)$ in $H_2(\#_4 CP^2)$. The class ξ is primitive and characteristic with $\xi \cdot \xi = p \equiv \mathrm{sign}(\#_4 CP^2) \bmod 16$. Theorem 8.1 shows that ξ is represented by an embedded 2-sphere. The boundary of the tubular neighborhood of this 2-sphere is $L(p, 1)$.

9.4 PROPOSITION. *If $p \equiv -4 \bmod 16$, then $L(p, 1)$ embeds topologically in $* \#_4 CP^2$.*

Proof. The proof is the same as that of Proposition 9.3 except that the application of Theorem 8.1 in this case would provide the embedded 2-sphere in $* \#_4 CP^2$.

9.5 PROPOSITION. *If $p \equiv -4 \bmod 16$, then $L(p, 1)$ does not embed (unpunctured) topologically in $\#_4 CP^2$.*

Proof. Suppose that $L = L(p, 1)$ does embed in $X^4 = \#_4 CP^2$. Then $X = U \cup_L V$. It follows from Theorem 6.1 that $b_2 U \neq 0 \neq b_2 V$. Suppose that $b_2 U = 2 = b_2 V$. Then $U \cup E_p$ or $V \cup E_p$ is a positive definite simply connected 4-manifold with $b_2 = 3$ and containing a copy of $L(p, 1)$. This contradicts Proposition 7.6. Therefore we may assume without loss of generality that $b_2 U = 1$ and $b_2 V = 3$.

We claim that the boundary of U as an oriented 4-manifold is the same as ∂E_p. If not consider $U \cup E_p$, which would be a positive definite 4-manifold with $b_2 = 2$

containing $L(p, 1)$. Note futher that $U \cup E_p$ is simply connected, since E_p is simply connected and the Seifert-Van Kampen theorem applied to $X = U \cup V$ shows that $\pi_1 U$ is normally generated by $\pi_1(\partial U)$. This contradicts Propositions 7.3 and 7.5.

We claim that V has vanishing relative Kirby-Siebenmann stable triangulation obstruction $\mathrm{KS}(V) \in H^4(V, \partial V; \mathbf{Z}_2) \approx \mathbf{Z}_2$. By additivity of top classes, $\mathrm{KS}(U) + \mathrm{KS}(V) = \mathrm{KS}(X) = 0$. Therefore it suffices to see that $\mathrm{KS}(U) = 0$. Consider $Z = U \cup -E_p$. In this case the intersection form is indefinite of rank 2. Again, by Seifert-Van Kampen we have $\pi_1(U \cup -E_p) = 0$. Both U and E_p have even intersection pairings. It follows that Z is homotopy equivalent to $S^2 \times S^2$. By Freedman's classification of closed simply connected 4-manifolds, $Z \cong S^2 \times S^2$, so that $\mathrm{KS}(Z) = 0$. By additivity $\mathrm{KS}(U) = 0$ and then as well $\mathrm{KS}(V) = 0$.

Since $\mathrm{KS}(V) = 0$, $\mathrm{KS}(E_p \cup V) = 0$. The Rochlin Congruence applied to the core 2-sphere in E_p implies $p \equiv 4 \bmod 16$.

9.6 PROPOSITION. *If* $p \equiv 2 \bmod 4$, *then* $L(p, 1)$ *embeds topologically in* $\#_3\mathbf{C}P^2$ *and in* $* \ \#_3\mathbf{C}P^2$.

Proof. Such an integer p can be written as a sum of 3 relatively prime squares: $p = r^2 + s^2 + t^2$. Consider the corresponding homology class $(r, s, t) \in H_2(\#_3\mathbf{C}P^2)$. Since p is even, not all of r, s, and t can be odd. Thus this homology class is primitive and ordinary. By work of Lee and Wilczyński it is represented by a topological locally flat 2-sphere. The boundary of a tubular neighborhood of this 2-sphere is $L(p, 1)$. The same argument applies to $* \ \#_3\mathbf{C}P^2$.

9.7 REMARK. If p is even, and $L(p, 1)_0$ embeds topologically in $\#_4\mathbf{C}P^2$, but $L(p, 1)_0$ does not embed topologically in $\#_3\mathbf{C}P^2$, then $p \equiv 4 \bmod 8$.

It remains to resolve whether $L(16, 1)_0$ embeds topologically in $\#_3\mathbf{C}P^2$. (It cannot embed unpunctured by the Rochlin Inequality.)

9.8 PROPOSITION. *If* $p = 1$, 3, *or* $5 \bmod 8$, *then* $L(p, 1)$ *embeds topologically* $\#_3\mathbf{C}P^2$ *and in* $* \ \#_3\mathbf{C}P^2$.

Proof. We can write $p = r^2 + s^2 + t^2$ as a primitive sum of 3 squares. Then (r, s, t) represents a primitive ordinary class ξ in $H_2(\mathbf{C}P^2)$ and in $H_2(*\mathbf{C}P^2)$.

First suppose that $p \equiv 1$ or $5 \bmod 8$. Since odd squares are congruent to $1 \bmod 8$, it follows that not all three squares are odd. As such it is represented in both of these 4-manifolds by a topologically embedded 2-sphere by Theorem 8.1. The boundary of a tubular neighborhood of the 2-sphere provides the required lens space.

On the other hand, if $p \equiv 3 \bmod 8$, then all three squares must be odd. Then the homology class ξ is primitive and characteristic with self-intersection p. It satisfies $\xi \cdot \xi \equiv sign(X^4) + 8\mathrm{KS}(X^4) \pmod{16}$ for exactly one choice of $X^4 = \#_3\mathbf{C}P^2$ or $* \#_3\mathbf{C}P^2$, depending on whether $p \equiv 3$ or $11 \bmod 16$. Theorem 8.1 implies that for this choice of X^4 there is an embedded 2-sphere with self-intersection p yielding an embedding of $L(p, 1)$. We must show that we can also embed $L(p, 1) \subset *X^4$. Write $X^4 = E_p \cup_L V$. Replace E_p by $*E_p$, which exists since p is odd, to create the desired $*X^4$.

9.9 REMARK. *When* $p \equiv 11 \bmod 16$, *we obtain an embedding* $L(p, 1) \subset \#_3\mathbf{C}P^2$, *but there is no 2-sphere* $\Sigma \subset \#_3\mathbf{C}P^2$ *with* $\Sigma \cdot \Sigma = p$. *This is in fact our only example of an embedding of a lens space* $L(p, 1)$ *in a situation in which there is no 2-sphere with self-intersection* p.

9.10 PROPOSITION. *If* $p \equiv 2 \bmod 16$ *and not divisible by a factor congruent to* $3 \bmod 4$, *then* $L(p, 1)$ *embeds topologically in* $\#_2\mathbf{C}P^2$.

Proof. Let $p = r^2 + s^2$, where r and s are relatively prime integers. Since $p \equiv 2 \bmod 16$ it follows that r and s are odd. Consider the corresponding homology class $\xi = (r, s) \in H_2(\mathbf{C}P^2)$. Then ξ is a primitive characteristic homology class, and $\xi \cdot \xi = r^2 + s^2 = p \equiv 2 = \mathrm{sign}(\#_2\mathbf{C}P^2) \bmod 16$. Therefore, by Theorem 8.1 ξ is represented by a topologically embedded 2-sphere. The boundary of a tubular neighborhood of that 2-sphere gives the required embedding of $L(p, 1)$.

9.11 PROPOSITION. *If* $p \equiv 10 \bmod 16$ *and is not divisible by a factor congruent to* $3 \bmod 4$, *then* $L(p, 1)$ *embeds topologically in* $* \#_2\mathbf{C}P^2$.

Proof. One proceeds as above. The only difference is in the Rochlin-Kervaire-Milnor Congruence: $\xi \cdot \xi = r^2 + s^2 = p \equiv 10 = \mathrm{sign}(*\#_2\mathbf{C}P^2) + 8\mathrm{KS}(*\#_2\mathbf{C}P^2) \bmod 16$.

9.12 PROPOSITION. *If* p *is odd and not divisible by a prime congruent to* $3 \bmod 4$, *then* $L(p, 1)$ *embeds topologically in* $\#_2\mathbf{C}P^2$ *and in* $* \#_2\mathbf{C}P^2$.

Proof. Proceed as in the preceding two propositions. The only difference is that one seeks to embed a primitive ordinary class, which can always be done with no further obstruction.

10. Fundamental results of Donaldson

For use below we quote a basic a result from Donaldson [1987].

10.1 THEOREM. (*A*) *A simply connected smooth 4-manifold with a definite intersection pairing has a diagonalizable intersection form, and hence is homotopy equivalent to a connected sum of copies of* $\mathbf{C}P^2$. (*B*) *A simply connected smooth spin 4-manifold with* $b_2^+ \leq 2$ *or* $b_2^- \leq 2$ *has* 0 *signature.*

11. Further restrictions on smooth embeddings of punctured lens spaces in definite 4-manifolds

Here we apply Donaldson's theorems to find further restrictions on the existence of smooth punctured lens spaces.

11.1 PROPOSITION. *If p is even, and $L(p, 1)_0$ embeds smoothly in* $\mathbf{C}P^2 \# \mathbf{C}P^2$ *then* $p = 2$, 4, *or* 8.

Proof. Referring to Proposition 7.3 we need to show that if $p \equiv \pm 2 \bmod 16$, then $p = 2$. As above, we obtain a smooth 2-sphere Σ in $X^4 = S^2 \times S^2 \# \mathbf{C}P^2 \# \mathbf{C}P^2$, representing a homology class of the form $(p, k; r, s)$, where we can assume k is even, and $\Sigma \cdot \Sigma = 2kp + r^2 + s^2 = \pm p$. Since $p \equiv 2 \bmod 4$, we see that r and s must both be odd. Thus we have a characteristic homology class represented by a *smoothly* embedded 2-sphere Σ, with $\Sigma \cdot \Sigma = \pm p$. Taking connected sum with $p - 1$ copies of $\mp(\mathbf{C}P^2, \mathbf{C}P^1)$ we obtain a smoothly embedded 2-sphere S representing a characteristic homology class and satisfying $S \cdot S = \pm 1$. Blowing down S we obtain a smooth, simply connected spin 4-manifold with $b_2 = p + 2$ and signature $2 \mp (p - 1) \mp 1$ which equals $2 - p$ or $p + 2$. By Donaldson's Theorem 10.1, we must have signature 0, so that $p = 2$.

11.2 PROPOSITION. *If p is even, and $L(p, 1)_0$ embeds smoothly in* $\#_4\mathbf{C}P^2$ *and* $p \equiv 4 \bmod 8$, *then* $p \equiv 4 \bmod 16$, *or* $p = 12$.

Proof. Suppose that $p \equiv 4 \bmod 8$. The basic construction yields an embedded 2-sphere $\Sigma \subset S^2 \times S^2 \#_4\mathbf{C}P^2$ representing a homology class of the form $(p, k; r, s, t, u)$, with k even and $\Sigma \cdot \Sigma = 2kp + r^2 + s^2 + t^2 + u^2 = \pm p$. We see that if r, s, t and u are all even, then the homology class represented by Σ is 2-divisible. The Rochlin Inequality then implies that $-2 \leq \Sigma \cdot \Sigma \leq 18$. This provides for the cases that $p = 4$ or $p = 12$.

If not all the squares are even, then, since $p \equiv 4 \bmod 8$, we see that all of r, s, t and u must be odd, so that $[\Sigma]$ is characteristic. The Rochlin Congruence states that $\Sigma \cdot \Sigma \equiv 4 \bmod 16$. If $\Sigma \cdot \Sigma = +p$, then $p \equiv 4 \bmod 16$, as required.

Suppose that $\Sigma \cdot \Sigma = -p$. Blow up $p - 1$ $(+1)$'s to produce a characteristic 2-sphere S with self-intersection -1 in a 4-manifold with $b_2^+ = p + 4$ and $b_2^- = 1$.

Blow down the 2-sphere S to obtain a smooth, positive definite, spin 4-manifold, contradicting Donaldson's Theorem 10.1.

11.3 REMARK. We know that $L(12, 1)$ embeds smoothly in $\#_3 CP^2$ as the boundary of a tubular neighborhood of a curve representing the homology class (2, 2, 2). We also know that $L(12, 1)$ does not embed topologically in $\#_2 CP^2$, since 12 cannot be written as a sum of 2 squares.

11.4 PROPOSITION. *If $p \equiv 3 \bmod 4$ and $p > 3$, then $L(p, 1)$ does not embed smoothly in $\#_3 CP^2$.*

Proof. The lens space L would separate $X^4 = \#_3 CP^2$ into two closed complementary domains U and V. We may replace U, say, with the disk bundle E_p to form $Y^4 = E_p \cup V$, a manifold with definite intersection pairing. By Corollary 6.4, Y^4 has the homotopy type of X^4, not $\#_2 CP^2$. Now Y^4 contains a smooth 2-sphere Σ as the core of E_p, with self-intersection number $\Sigma \cdot \Sigma = p$. In terms of a standard orthonormal basis for the homology of Y^4, the homology class represented by Σ has the form (r_1, r_2, r_3), where $r_1^2 + r_2^2 + r_3^2 = p$. Since $p \equiv 3 \bmod 4$, we see that r_1, r_2, and r_3 are all odd, so that Σ is characteristic. Now, blowing up $p - 1$ (-1)'s and then blowing down the resulting $(+1)$, we obtain a smooth spin 4-manifold with $b_2^+ = 2$. According to Donaldson's Theorem 10.1, this manifold has signature $0 = 3 - (p - 1) - 1$, which implies that $p = 3$, a contradiction.

12. Further restrictions on embeddings of punctured lens spaces in indefinite 4-manifolds

Here we complete our picture to include an analysis of which punctured lens spaces $L(p, 1)_0$ embed in which standard indefinite 4-manifolds. Recall that an indefinite simply connected 4-manifold is homeomorphic to $-CP^2 \# CP^2$ or $*(-CP^2 \# CP^2)$ or $S^2 \times S^2 \# Y^4$ for some 4-manifold Y^4. Since all punctured lens spaces embed in $S^2 \times S^2$, we need only investigate further the first two cases.

12.1 PROPOSITION. *If p is even, and $L(p, 1)_0$ embeds topologically in $-CP^2 \# CP^2$ then $p \leq 8$ or $p \equiv 0 \bmod 16$.*

Proof. The existence of such an embedding gives rise via the Basic Construction to an embedded 2-sphere Σ in $S^2 \times S^2 \# -CP^2 \# CP^2$ representing a homology class of the form $(p, k; r, s)$, with self-intersection $2kp - r^2 + s^2 = \pm p$, where we may assume that k is even. Since p is even, r and s have the same parity. If they are both even, Σ represents a 2-divisible class and the Rochlin Inequality implies that

$p \leq 8$. If they are both odd, then the 2-sphere Σ represents a characteristic element in homology, and the Rochlin Congruence yields $p \equiv 0 \bmod 16$.

12.2 PROPOSITION. *If p is even, and $L(p,1)_0$ embeds topologically in $*(-CP^2 \# CP^2)$ then $p \leq 8$ or $p \equiv 8 \bmod 16$.*

Proof. The proof is the same as that for Proposition 12.1, except for the last sentence, where we conclude instead that $p \equiv 8\mathrm{KS}(*(-CP^2 \# CP^2)) \bmod 16$.

12.3 PROPOSITION. *If p is even, and $L(p,1)_0$ embeds smoothly in $-CP^2 \# CP^2$ then $p \leq 8$.*

Proof. If $p > 8$ then, as above, we obtain a smoothly embedded characteristic 2-sphere Σ in $S^2 \times S^2 \# -CP^2 \# CP^2$, with self-intersection $\pm p$. The blow up/blow down trick, blowing up $(p-1)$ (∓ 1)'s and blowing down a (± 1), produces a smooth spin 4-manifold with $b_2 = 4 + (p-1) - 1 = p + 2$ and $b_2^{\pm} = 1$, contradicting Donaldson's Theorem 10.1.

12.4 PROPOSITION. *If $p \equiv 0 \bmod 16$ then $L(p,1)$ embeds topologically in $-CP^2 \# CP^2$. If $p \equiv 8 \bmod 16$ then $L(p,1)$ embeds topologically in $*(-CP^2 \# CP^2)$.*

Proof. Let $p = 8k$. We can express $8k = r^2 - s^2$, where $r = 2k+1$ and $s = 2k-1$. Since r and s are relatively prime and odd, we see that (r,s) represents a primitive characteristic homology class in $H_2(-CP^2 \# CP^2)$, with $(r,s) \cdot (r,s) = 8k$. It follows from Theorem 8.1 that (r,s) is represented by a topologically embedded 2-sphere in $-CP^2 \# CP^2$ when k is even, i.e., $p \equiv 0 \bmod 16$, and represented by a topologically embedded 2-sphere in $*(-CP^2 \# CP^2)$ when k is odd, i.e., $p \equiv 8 \bmod 16$. The boundary of a tubular neighborhood of this 2-sphere yields the required lens space in either case.

12.5 REMARK. In the concrete specific case above one can find the 2-sphere by a more direct argument depending only on the early work of Freedman, as follows. Consider any framed link of 2 unknotted components with framings $16k$ and $4k-1$, with linking number $8k-1$. This defines a simply connected smooth 4-manifold containing a 2-sphere with self-intersection $16k$ and with boundary a homology 3-sphere, since the determinant of the linking matrix is -1. By Freedman this homology 3-sphere bounds a contractible topological 4-manifold. Freedman's classification of topological 4-manifolds shows that the result X^4 of capping off with the contractible 4-manifold is homotopy equivalent to $-CP^2 \# CP^2$. Since the component of the framed link with framing $16k$ is characteristic and unknotted the Rochlin Congruence implies that $16k \equiv 8\mathrm{KS}(X^4) \bmod 16$. Therefore

$KS(X^4)=0$ and we can conclude that X^4 is homeomorphic to $-CP^2 \# CP^2$. Similar constructions can often be used in place of our previous appeals to Theorem 8.1, the embedding theorem of Lee-Wilczyński and Hambleton-Lee.

13. Tables summarizing the main results

The charts below summarize what is known concerning the embedding of lens spaces and punctured lens spaces in a connected sum of complex projective planes. Designations of "yes", "no", and "?" indicate that such an embedding exists, does not exist, or that the embeddability is unknown. For a few of the cases in which p is only given mod 16, the result depends on the actual value of p. In these cases all possible answers are given in the chart, with details appearing in a footnote.

The remainder of the notation is as follows. If a result derives directly from a result presented elsewhere on the chart, it is written in lower case. This can occur in three ways: smooth embeddings yield topological embeddings, closed embeddings yield punctured embeddings, and embeddings in $\#_n CP^2$ yield embeddings in $\#_{n+1} CP^2$. Of course, the contrapositive of each of these statements gives a nonembedding result as well. Other results are presented with uppercase "YES" and "NO". These are labelled in one of two ways: either the proposition number that yields the result is given in parentheses, or a homology class in $\#_n CP^2$ that is representable by an embedded sphere of prescribed self-intersection is listed as a subscript. In the second case, the realizability of that class follows from either the Embedding Theorem 8.1 in the topological setting, or from the smooth realizability of certain basic classes using the general Boardman construction as described in Remark 2.7.

Table 1. p even, large, TOP, unpunctured: $L(p, 1) \overset{TOP}{\subset} \#_n CP^2$, $p>20$

p mod 16	CP^2	$\#_2 CP^2$	$\#_3 CP^2$	$\#_4 CP^2$	$\#_5 CP^2$
0	no	no	no	no	YES (9.1)
2	no	YES & NO[a]	YES (9.6)	yes	yes
4	no	no	no	YES (9.3)	yes
6	no	no	YES (9.6)	yes	yes
8	no	no	no	no	YES (9.1)
10	no	no	YES (9.6)	yes	yes
12	no	no	no	NO (9.5)	YES (9.1)
14	no	NO (6.2)	YES (9.6)	yes	yes

Table 2. p even, large, TOP, punctured: $L(p,1)_0 \overset{TOP}{\subset} \#_n CP^2$, $p > 20$

p mod 16	CP^2	$\#_2CP^2$	$\#_3CP^2$	$\#_4CP^2$	$\#_5CP^2$
0	no	no	no	NO (7.8)	yes
2	NO (4.2)	yes & ?[b,c]	yes	yes	yes
4	no	no	NO (7.6)	yes	yes
6	no	NO (7.3)	yes	yes	yes
8	no	no	no	NO (7.8)	yes
10	no	NO (7.3)	yes	yes	yes
12	no	no	NO (7.6)	?[c]	yes
14	NO (4.2)	?[c]	yes	yes	yes

Table 3. p even, large, DIFF, unpunctured: $L(p,1) \overset{DIFF}{\subset} \#_n CP^2$, $p > 20$

p mod 16	CP^2	$\#_2CP^2$	$\#_3CP^2$	$\#_4CP^2$	$\#_5CP^2$
0	no	no	no	no	?
2	no	no	?	?	?
4	no	no	no	?	?
6	no	no	?	?	?
8	no	no	no	no	?
10	no	no	?	?	?
12	no	no	no	no	?
14	no	no	?	?	?

Table 4. p even, large, DIFF, punctured: $L(p,1)_0 \overset{DIFF}{\subset} \#_n CP^2$, $p > 20$

p mod 16	CP^2	$\#_2CP^2$	$\#_3CP^2$	$\#_4CP^2$	$\#_5CP^2$
0	no	no	no	no	?
2	no	NO (11.1)	?	?	?
4	no	no	no	?	?
6	no	no	?	?	?
8	no	no	no	no	?
10	no	no	?	?	?
12	no	no	no	NO (11.2)	?
14	no	NO (11.1)	?	?	?

Table 5. p even, small, TOP, unpunctured: $L(p,1) \overset{TOP}{\subset} \#_n CP^2$, $p \le 22$

p	CP^2	$\#_2CP^2$	$\#_3CP^2$	$\#_4CP^2$	$\#_5CP^2$
2	no	yes	yes	yes	yes
4	yes	yes	yes	yes	yes
6	no	no	yes	yes	yes
8	no	yes	yes	yes	yes
10	no	no	yes	yes	yes
12	no	no	yes	yes	yes
14	no	NO (6.2)	$\text{YES}_{(3,2,1)}$	yes	yes
16	no	no	NO	yes	yes
18	no	NO (6.3)	$\text{YES}_{(4,1,1)}$	yes	yes
20	no	no	no	$\text{YES}_{(3,3,1,1)}$	yes
22	no	no	$\text{YES}_{(3,3,2)}$	yes	yes

Table 6. p even, small, TOP, punctured: $L(p,1)_0 \overset{TOP}{\subset} \#_n CP^2$, $p \le 22$

p	CP^2	$\#_2CP^2$	$\#_3CP^2$	$\#_4CP^2$	$\#_5CP^2$
2	NO (7.1)	yes	yes	yes	yes
4	yes	yes	yes	yes	yes
6	no	NO (7.3)	yes	yes	yes
8	NO (7.1)	yes	yes	yes	yes
10	no	NO (7.3)	yes	yes	yes
12	no	NO (7.3)	yes	yes	yes
14	NO (7.1)	?[c]	yes	yes	yes
16	no	NO (7.3)	?[c]	yes	yes
18	NO (7.1)	?[c]	yes	yes	yes
20	no	no	NO (7.6)	yes	yes
22	no	NO (7.3)	yes	yes	yes

Table 7. p even, small, DIFF, unpunctured: $L(p,1) \overset{DIFF}{\subset} \#_n CP^2$, $p \le 22$

p	CP^2	$\#_2CP^2$	$\#_3CP^2$	$\#_4CP^2$	$\#_5CP^2$
2	no	$\text{YES}_{(1,1)}$	yes	yes	yes
4	$\text{YES}_{(2)}$	yes	yes	yes	yes
6	no	no	$\text{YES}_{(2,1,1)}$	yes	yes
8	no	$\text{YES}_{(2,2)}$	yes	yes	yes
10	no	no	$\text{YES}_{(3,0,1)}$	yes	yes
12	no	no	$\text{YES}_{(2,2,2)}$	yes	yes
14	no	no	$\text{YES}_{(3,2,1)}$	yes	yes
16	no	no	no	$\text{YES}_{(2,2,2,2)}$	yes
18	no	no	?	$\text{YES}_{(3,0,3,0)}$	yes
20	no	no	no	?	$\text{YES}_{(2,2,2,2,2)}$
22	no	no	?	?	$\text{YES}_{(3,0,3,0,2)}$

Table 8. p even, small, DIFF, punctured: $L(p, 1)_0 \overset{DIFF}{\subset} \#_n CP^2$, $p \leq 22$

p	CP^2	$\#_2 CP^2$	$\#_3 CP^2$	$\#_4 CP^2$	$\#_5 CP^2$
2	no	yes	yes	yes	yes
4	yes	yes	yes	yes	yes
6	no	no	yes	yes	yes
8	no	yes	yes	yes	yes
10	no	no	yes	yes	yes
12	no	no	yes	yes	yes
14	no	NO (11.1)	?	yes	yes
16	no	no	?	yes	yes
18	no	NO (11.1)	?	yes	yes
20	no	no	no	?	yes
22	no	no	?	?	yes

Table 9. p odd, TOP, unpunctured: $L(p, 1) \overset{TOP}{\subset} \#_n CP^2$, $p > 1$

p mod 16	CP^2	$\#_2 CP^2$	$\#_3 CP^2$	$\#_4 CP^2$
1	NO (6.3)	YES & NO[a]	YES (9.8)	yes
3	no	NO (6.4)	YES (9.8)	yes
5	NO (6.3)	YES & NO[a]	YES (9.8)	yes
7	no	no	NO (6.2)	YES (9.2)
9	NO (6.3)	YES & NO[a]	YES (9.8)	yes
11	no	no	YES (9.8)	yes
13	NO (6.3)	YES & NO[a]	YES (9.8)	yes
15	no	no	NO (6.2)	YES (9.2)

Table 10. p odd, DIFF, unpunctured: $L(p, 1) \overset{DIFF}{\subset} \#_n CP^2$, $p > 1$

p mod 18	CP^2	$\#_2 CP^2$	$\#_3 CP^2$	$\#_4 CP^2$
1	no	NO & ?[e]	?	?
3	no	no	NO (11.4)[d]	?[d]
5	no	NO (11.4)[d]	?	?
7	no	no	no	?[d]
9	no	NO (11.4)[d]	?	?
11	no	no	no	?[d]
13	no	NO & ?[e]	?	?[d]
15	no	no	no	?[d]

[a]With only three exceptions, $L(p, 1)$ embeds topologically in $\#_2 CP^2$ if and only if p can be written as a primitive sum of two squares and, if p is even, $p \equiv 2 \bmod 16$. An integer n is a primitive sum of two squares if and only if 4 does not divide n and no prime $q \equiv 3 \bmod 4$ divides n. The three exceptions arise from the smooth embeddings of $L(p, 1)$ in $\#_2 CP^2$ when $p = 4, 8$, or 9. See Section 9.0 and Proposition 9.10 and 9.12.
[b]The only known "yes" answers derive from embeddings of the corresponding unpunctured lens space.
[c]By Remark 8.2, the known obstructions to embedding punctured lens spaces do not apply in this situation.
[d]Isolated exceptions to these occur because of the special smooth embeddings of spheres in $\#_n CP^2$. The exceptions in the realm of this chart are described in Remark 2.7.
[e]The only known "no" answers that occur here are those that follow from the topological case.

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