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The classification of compact hyperbolic Coxeter d-polytopes with $d+2$ facets

FRANK ESSELMANN

Abstract. This paper provides a list of all compact hyperbolic Coxeter polytopes the combinatorial type of which is the product of two simplices of dimension greater than 1. Combined with results of Kaplinskaja ([Ka]) this completes the classification of compact hyperbolic Coxeter d-polytopes with $d+2$ facets.

1. Introduction

Let $\mathbb{H}^d$ denote the d-dimensional hyperbolic space.

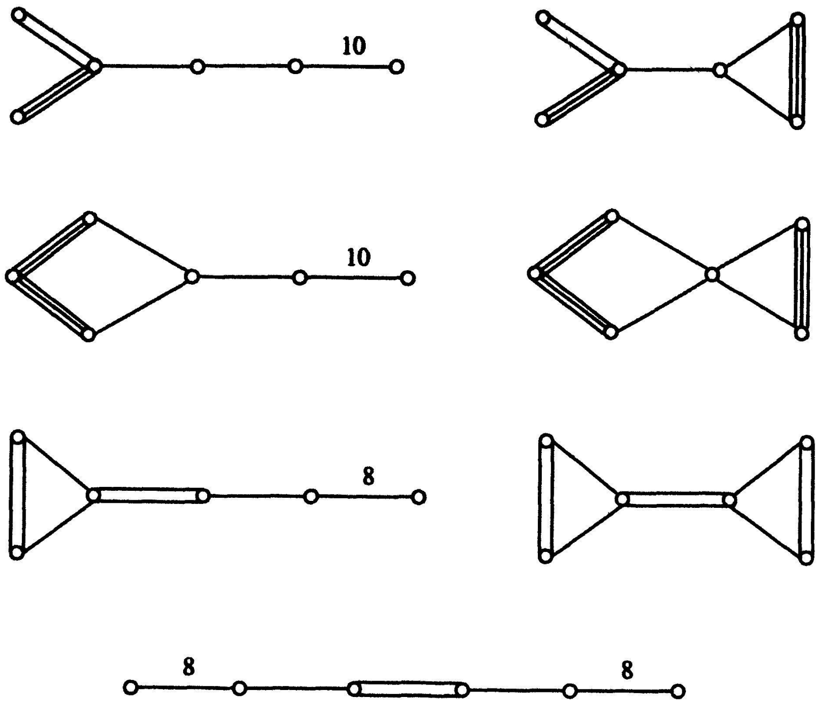
A polytope $P \subset \mathbb{H}^d$ bounded by hyperplanes $H_1, \ldots, H_n$ is said to be a Coxeter polytope if for each pair H_i, H_j the intersection is either empty, or the angle $\angle(H_i, H_j)$ is of the form π/k for some $k \in \mathbb{N}$, $k \geq 2$. (By $\angle(H_i, H_j)$ we always mean the angle lying in P.) Such a polytope is a fundamental domain of the discrete group generated by the reflections in the bounding hyperplanes. Conversely, each discrete finitely generated reflection group has a Coxeter polytope as a fundamental domain.

Of special interest are Coxeter polytopes of finite volume. In contrast to the spherical and euclidian cases where complete descriptions were obtained, in the hyperbolic space only particular cases have been treated successfully. For example, in dimension 2 and 3 Coxeter polytopes of finite volume are completely characterized ([Po], [vD], [An]).

Furthermore, the polytopes with the lowest possible number of bounding hyperplanes, the simplices, are classified ([La], [Ko], [Ch]). For polytopes with a few more than $d+1$ bounding hyperplanes ($d = \dim P$) there are already gaps in the classification. Besides some examples, the following is known: Kaplinskaja 1975 described all hyperbolic Coxeter polytopes of finite volume the combinatorial type of which is a product of a segment and a simplex ([Ka], see also [Vi2]). Im Hof classified hypberbolic Coxeter polytopes of finite volume which can be described by Napier cycles ([ImH]). Their bounding hyperplanes satisfy orthogonality conditions limiting their number to be at most $d+3$.

In particular, compact hyperbolic Coxeter polytopes are of finite volume. In this paper, we will complete the classification of compact hyperbolic Coxeter polytopes bounded by $d+2$ hyperplanes, $d \geq 3$, by proving:

THEOREM 1.1. *In $\mathbb{H}^4$ there exist exactly the following compact Coxeter polytopes the combinatorial type of which is the product of two simplices of dimension greater than* 1:



In $\mathbb{H}^d$, $d \geq 5$, no such polytopes exist.

Compact hyperbolic polytopes with angles $\leq \pi/2$ are simple, and the combinatorial type of a simple polytope with $d+2$ facets equals the product of two simplices. Thus within the set of compact hyperbolic Coxeter polytopes bounded by $d+2$ hyperplanes the set of polytopes considered in the theorem is complementary to that considered by Kaplinskaja.

The methods used in this paper can easily be extended to investigate compact hyperbolic Coxeter polytopes bounded by $d+3$ hyperplanes. Because of the great number of combinatorial types there is so far no complete classification. Nevertheless, it is proved in [Es] that such polytopes do not exist in $\mathbb{H}^d$, $d > 8$, and that the example found by Bugaenko ([Bu]) in $\mathbb{H}^8$ is unique. This result is of interest with respect to the maximal dimension: Vinberg has shown that compact hyperbolic Coxeter polytopes only exist in dimension $d \leq 29$ ([Vi1]). Examples in $\mathbb{H}^d$, $d \geq 9$, are not known.

2. Polytopes and Gale diagrams

For details and proofs see [Gr]. Let us recall some facts from polytope theory: A *polytope* $P \subset \mathbb{R}^d$ is the convex hull of finitely many points in $\mathbb{R}^d$. A *supporting* *hyperplane* of P is a hyperplane H which intersects P but does not cut it. A subset $f \subseteq P$ is a *face* of P if either $f \in \{\emptyset, P\}$ or there exists a supporting hyperplane of P such that $f = H \cap P$. In the latter case f is called a *proper face* of P. A k-*face* is a face of dimension k. The 0, 1, $(d-1)$-faces are respectively its *vertices*, *edges* and *facets*. With respect to the ordering given by set inclusion, the set of all faces of P is a lattice, the *face-lattice* of P.

A *simplex* is the convex hull of $d+1$ affinely independent points. A polytope is said to be *simplicial* if each proper face is a simplex. A d-Polytope is said to be *simple* if each k-face is contained in exactly $d-k$ facets.

Two polytopes P and P^* are said to be *dual* to each other provided there exists an inclusion reversing bijection between their face-lattices.

Let $P \subseteq \mathbb{R}^d$ be a polytope with vertices $\{x_j \mid j \in J\}$, $J = \{1, \ldots, n\}$. It is well known that the face-lattice of P can be read off from the *Gale transform* $\{\bar{x}_1, \ldots, \bar{x}_n\} \subset \mathbb{R}^{n-d-1}$ of the vertices of P. It has the property that $\operatorname{conv}\{x_i \mid i \in I \subseteq J\}$ is a face of P if and only if either $I = J$ or 0 is contained in the relative interior of $\operatorname{conv}\{x_j \mid i \in J \backslash I\}$. After normalizing, we may assume $|\bar{x}_i| = 1$ for all $x_i \neq 0$. A normalized Gale transform is called a *Gale diagram*. The polytope P is simplicial if and only if $0 \in \operatorname{relint} \operatorname{conv}\{\bar{x}_i \mid i \in I\}$ implies $\dim \operatorname{conv}\{\bar{x}_i \mid i \in I\} = n-d-1$.

Now let P be a simplicial d-polytope with $d+2$ vertices. Then each point in the Gale diagram of $\{x_1, \ldots, x_{d+2}\}$ is equal to $+1$ or -1. The *multiplicity of* $+1$, -1 is defined to be the number of points in the Gale diagram equal to $+1$, -1 respectively. There exists a polytope to a Gale diagram with multiplicities x and y if and only if $x, y \geq 2$ holds.

Let P^* be dual to P with facets $\{f_1, \ldots, f_{d+2}\}$, and let φ denote the inclusion reversing bijection of the face-lattices with $\varphi(x_i) = f_i$. Since P is simplicial and hence P^* is simple, we have $\varphi(\operatorname{conv}\{x_i \mid i \in I\}) = \bigcap_{i \in I} f_i$ for each face $\operatorname{conv}\{x_i \mid i \in I\}$ of P. Therefore, the face-lattice of P^* can be read off from the Gale transform of $\{x_1, \ldots, x_{d+2}\}$ in the following way: The intersection $\bigcap_{i \in I} f_i$ is a proper face of P^* (i.e. unequal $\emptyset$) if and only if 0 is contained in the relative interior of $\operatorname{conv}\{\bar{x}_i \mid i \notin J \backslash I\}$.

Let x, y be the multiplicity of $+1$, -1 respectively. In the following we denote by $P_{x,y}$ the class of combinatorially equivalent simple polytopes with $d+2$ facets which is determined in the above way by the Gale diagram. The reader should bear in mind that $P_{x,y}$ is determined by the existence of two sets of facets, say F_1, F_2, of order x, y respectively, which are minimal with respect to the property that $\bigcap_{f_i \in F_j} f_i = \emptyset$ for $j = 1, 2$.

3. Hyperbolic Coxeter polytopes

For details and proofs see [Vi2]. We choose the Kleinian model of hyperbolic space:

Let $\mathbb{R}^{d,1}$ denote the vector space $\mathbb{R}^{d+1}$ endowed with the quadratic form $-x_0^2+x_1^2+\cdots+x_d^2$ and the corresponding bilinear form $(\,,\,)$. The set $C=\{v\in\mathbb{R}^{d,1}|(v,v)<0\}$ consists of two connected components C_+ and C_-. We identify $\mathbb{H}^d$ with the set $C_+/\mathbb{R}_{>0}$. Let π denote the canonical map $\mathbb{R}^{d,1}\backslash\{0\}\to\mathbb{R}^{d,1}/\mathbb{R}_{>0}$. A hyperplane in $\mathbb{H}^d$ is of the form $H_v=\{\pi(w)|w\in C_+,(v,w)=0\}$ for a vector v of length >0. Denote by H_v^- the halfspace given by $\{\pi(w)|w\in C_+,(v,w)\leq 0\}$. A Coxeter polytope is of the form

$$P=\bigcap_{i\in I}H_{v_i}^-.$$

We assume that in this representation of P none of the halfspaces contains the intersection of the others, and that the v_i are normalized by the condition $(v_i,v_i)=1$. The Gram matrix $G(P)$ of P is defined to be the Gram matrix of the v_i, $i\in I$. The mutual disposition of H_{v_i} and H_{v_j} can be read off from (v_i,v_j):

H_{v_i} and H_{v_j} intersect if and only if $-1<(v_i,v_j)\leq 0$. In this case we have $(v_i,v_j)=-\cos(\angle(H_{v_i},H_{v_j}))$.

H_{v_i} and H_{v_j} are parallel if and only if $(v_i,v_j)=-1$.

H_{v_i} and H_{v_j} diverge if and only if $(v_i,v_j)<-1$. In this case we have $(v_i,v_j)=-\cosh(\rho)$ where ρ denotes the distance between H_{v_i} and H_{v_j}.

Since P has only angles $\leq\pi/2$ (lying in P), we have $(v_i,v_j)\leq 0$.

A Coxeter polytope P can be described most conveniently by a *Coxeter diagram* $S=S(P)$: The vertices of S correspond to the vectors v_i, $i\in I$. If $\angle(H_{v_i},H_{v_j})=\pi/k$ the vertices v_i and v_j are joined by a k-labeled edge or a $(k-2)$-fold edge. In this case the multiplicity $\mathrm{m}[v_i,v_j]$ of the edge is defined to be $k-2$. If the corresponding hyperplanes are parallel they are joined by a bold edge, and if they diverge by a dotted edge labeled, if necessary, by $\cosh(\rho)$.

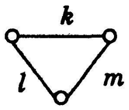
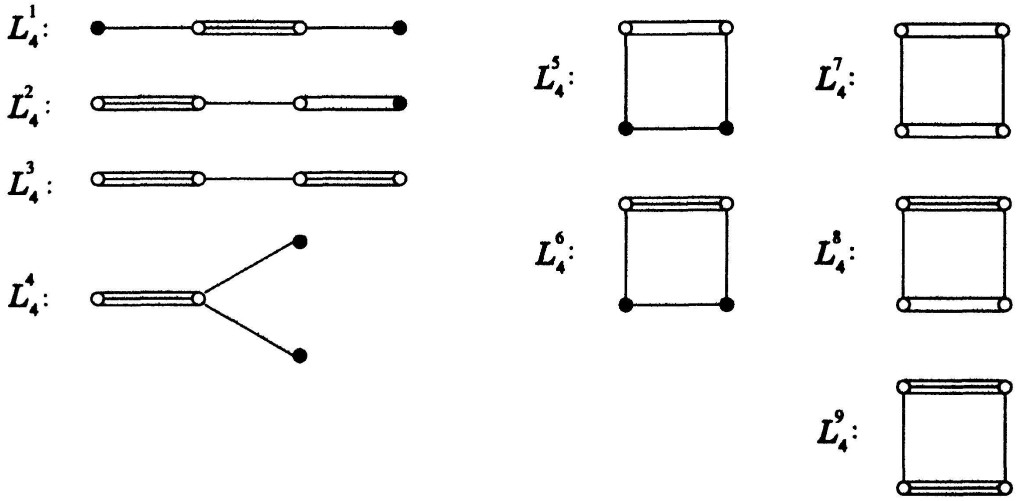
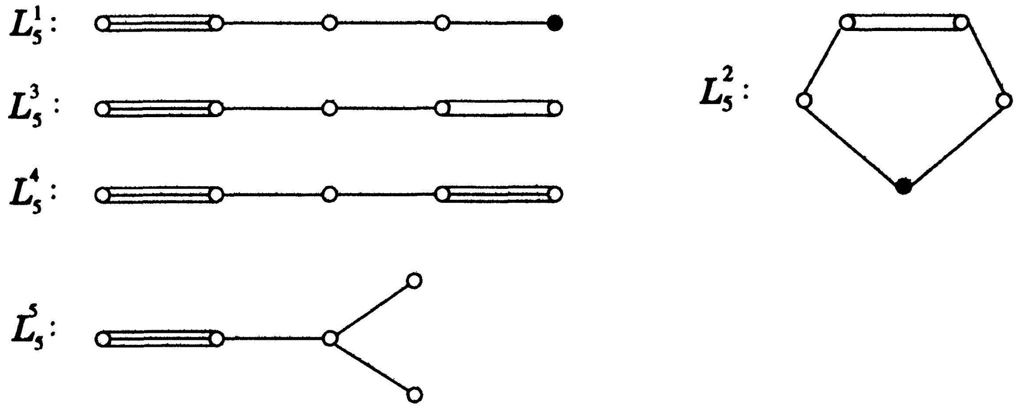
To each Coxeter diagram there corresponds in the obvious way a symmetric matrix with 1's along the diagonal and entries ≤ 0 off it. So we may use the terms describing Gram matrices to describe the corresponding Coxeter diagrams. We denote by $S'\subseteq S$, $v\in S$ and $|S|$ a subdiagram, a vertex and the order (i.e. the number of vertices) of S, respectively.

A Coxeter diagram is said to be *spherical* if the corresponding Gram matrix is positive definite. A connected Coxeter diagram is said to be *parabolic* if it has determinant 0 but every proper subdiagram is spherical. An arbitrary Coxeter diagram is said to be parabolic if every connected component is parabolic.

The spherical and parabolic Coxeter diagrams are classified, see for example [Bou].

A *Lannér diagram* is the Coxeter diagram of a compact hyperbolic Coxeter simplex. It is characterized by the fact that it is neither spherical nor parabolic, but each proper subdiagram is spherical. Table 1 contains all Lannér diagrams. For

Table 1

order	Lannér diagrams
2	
3	 $(2 \leq k, l, m < \infty,$ $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} < 1)$
4	
5	

later references we have given names to the diagrams of order greater than 3. The black vertices are explained later.

We recall the following fundamental facts:

A Coxeter diagram of a compact hyperbolic Coxeter polytope does not contain a parabolic subdiagram ([Vi2], Th. 4.1).

For each Coxeter diagram S of signature $(d, 1, n_0)$ (i.e. with d, 1, n_0 eigenvalues $+1$, -1, 0, respectively), there exists a Coxeter polytope $P \subset \mathbb{H}^d$ (possibly not of finite volume) with $S(P) = S$ ([Vi2], Th. 2.1).

Let P be a compact hyperbolic Coxeter polytope. The intersection $f = P \cap \bigcap_{i \in J} H_i$ is a proper face of P if and only if the Coxeter diagram generated by the $\{v_i | i \in J\}$ is spherical. In this case $\dim f = d - |J|$ ([Vi2], Th. 3.1).

4. Technical tools

By the facts mentioned above and the characterization of $P_{x,y}$ by the sets of facets F_1, F_2 (cf. end of section 2), we now can reformulate our classification problem: For each $x, y \geq 3$ we have to find every Coxeter diagram S with the following properties:

(C1) S does not contain parabolic subdiagrams,

(C2) S contains exactly two disjoint Lannér diagrams L_1, L_2 of order x and y,

(C3) the signature of S is $(x + y - 2, 1, 1)$.

(The compactness of the polytope described by S follows from the fact that the face-lattice coincides with the face-lattice of a compact polytope in $P_{x,y}$.) From condition (C3) it follows immediately that

(C3′) L_1 and L_2 are adjacent.

To fulfil (C3′) instead of (C3) alread means a strong restriction on S. For that reason, we fix some notations and make some remarks helpful to verify (C1), (C2) and (C3′):

A Coxeter diagram satisfying (C1), (C2) and (C3′) is said to be a *Coxeter realization* of $P_{x,y}$. Unless otherwise stated, the Lannér diagrams of order x and y in a Coxeter realization of $P_{x,y}$ are denoted respectively by L_1 and L_2.

Since decreasing the multiplicity of an edge in a spherical diagram always leads to a spherical diagram, we get:

LEMMA 4.1. *Let S be a Coxeter realization of $P_{x,y}$ containing the Lannér diagrams L_1 and L_2. Let S' be a connected Coxeter diagram we can get from S by decreasing the multiplicity of some edges joining L_1 and L_2. Then S' is also a Coxeter realization of $P_{x,y}$.*

For a Coxeter diagram S, we denote by $C_k(S)$ the set of all Coxeter diagrams $S' \supseteq S$, $|S'| = |S| + k$, satisfying

—S' is connected,

—S' does not contain a parabolic subdiagram, and

—S' contains exactly the Lannér diagrams already contained in S.

If S is a Coxeter realization of $P_{x,y}$, then for a suitable $v \in S$ we have $S \backslash v \in C_{x-1}(L_2)$.

Let S be a Coxeter diagram. We call a vertex $v \in S$ an *open vertex* of S if there exists a Coxeter diagram $S' \in C_1(S)$ such that v is adjacent to the vertex $w \in S' \backslash S$.

Obviously each Lannér diagram in a Coxeter realization of $P_{x,y}$, $x, y \geq 2$, contains an open vertex.

By the following lemma, open vertices are easy to detect:

LEMMA 4.2. *Let v be an open vertex of S. Then there exists a Coxeter diagram $S' \in C_1(S)$ such that* $\mathrm{m}[w, v] = 1$ *and* $\mathrm{m}[w, u] = 0$ *for $w \in S' \backslash S$ and each $u \in S \backslash v$.*

The proof is obvious. In Table 1 the open vertices of Lannér diagrams of order 4 and 5 are marked black.

To check condition (C3) we will use the following technical tools of Vinberg ([Vi1]):

A Coxeter diagram is called *superhyperbolic* if the corresponding Gram matrix has more than one negative eigenvalue. The *local determinant* of a Coxeter diagram S on a subdiagram T is defined to be the ratio

$$\det(S, T) = \frac{\det(S)}{\det(S \backslash T)}.$$

LEMMA 4.3. (*cf.* [*Vi*1], *Prop.* 12) *If a Coxeter diagram S is generated by subdiagrams S_1, S_2 having a unique vertex v in common, then*

$$\det(S, v) = \det(S_1, v) + \det(S_2, v) - 1.$$

LEMMA 4.4. ([*Vi*1], *Prop.* 13) *If a Coxeter diagram S is generated by disjoint subdiagrams S_1 and S_2 joined by a single edge $[v_1, v_2]$, then*

$$\det(S, \langle v_1, v_2 \rangle) = \det(S_1, v_1)\det(S_2, v_2) - (v_1, v_2)^2,$$

where $\langle v_1, v_2 \rangle$ denotes the subdiagram generated by v_1 and v_2.

LEMMA 4.5. ([*Vi*1], *Prop.* 15) *Suppose the Coxeter diagram S is generated by two disjoint hyperbolic subdiagrams S_1 and S_2 joined by a unique edge $[v_1, v_2]$ and that the subdiagrams $S_1 \backslash v_1$ and $S_2 \backslash v_2$ are spherical diagrams. Assume that one of the following conditions holds*:

(i) $\mathrm{m}[v_1, v_2] = 1$ *and* $\det(S_1, v_1)\det(S_2, v_2) > \frac{1}{4}$.

(ii) $\mathrm{m}[v_1, v_2] = 2$ *and* $\det(S_1, v_1)\det(S_2, v_2) > \frac{1}{2}$.

Then the diagram S is superhyperbolic.

The following table contains some local determinants $\det(S, v)$ used for later calculations. The function $d(k, l, m)$ is defined to be

$$d(k,l,m) = \frac{\cos^2(\pi/k) + \cos^2(\pi/l) + \cos(\pi/k)\cos(\pi/l)\cos(\pi/m)}{\sin^2(\pi/m)} - 1.$$

Notice that $d(k, l, m)$ is for $k, l, m \geq 2$, an increasing function of k, l, m.

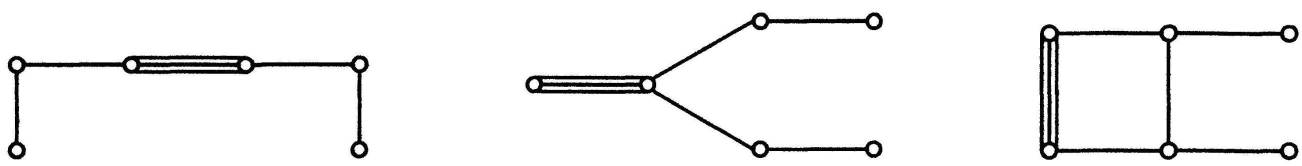
4.1. *Proof of the theorem*

Step 1) Let us first consider the Coxeter realizations of $P_{x,y}$ for $x \geq 4$ and $y \geq 3$.

Since L_1 contains an open vertex, we have $L_1 \in \{L_4^1, L_4^2, L_4^4, L_4^5, L_4^6, L_5^1, L_5^2\}$. Assume $L_1 = L_4^2$. Obviously the only diagram in $C_1(L_4^2)$ is given by

It contains no open vertex, hence $C_2(L_4^2) = \emptyset$, and thus $L_1 \neq L_4^2$.

If we have $L_1 \in \{L_4^1, L_4^4, L_4^6\}$, it is easy to see by inductive construction that $C_3(L_1)$ contains exactly one diagram. It is given by



respectively. None of these contains an open vertex, so we can conclude $y = 3$. It is easy to see that under these conditions there exists no Coxeter realization of $P_{x,y}$. Altogether for the Coxeter realizations of $P_{x,y}$, $x \geq 4$, $y \geq 3$, only $L_1 \in \{L_4^5, L_5^1, L_5^2\}$ is possible.

Step 2) Let us assume $x, y \geq 4$. By step 1) we have $L_1, L_2 \in \{L_4^5, L_5^1, L_5^2\}$. In this set, L_4^5 is the only diagram with two open vertices. If S is a Coxeter realization of $P_{x,y}$ such that both open vertices are adjacent to the second Lannér diagram, then S contains a cycle S' and $L_i \subsetneq S'$ for $i = 1, 2$. This is a contradiction to condition (C1) or (C2).

Therefore, in each Coxeter realization of $P_{x,y}$, the Lannér diagrams are joined by a single edge, say $[v_1, v_2]$, such that v_1 and v_2 are open vertices of L_1 and L_2 respectively. For the pairs (L_4^5, L_4^5), (L_4^5, L_5^2) and (L_5^2, L_5^2) a multiplicity $m[v_1, v_2] = 1$ provides parabolic subdiagrams. The same is true for the pairs (L_5^1, L_5^1), (L_5^1, L_5^2),

Table 2

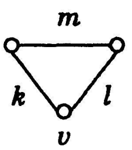
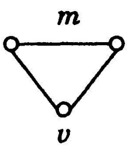
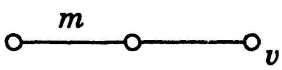
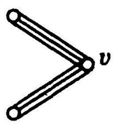
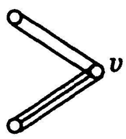
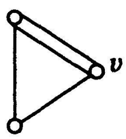
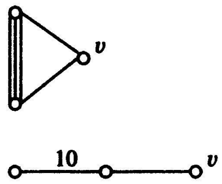
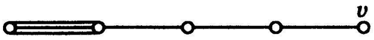
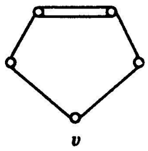
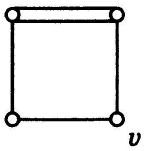
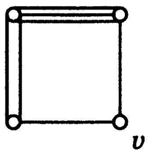
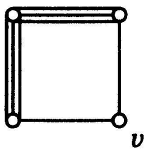
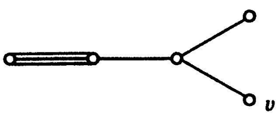
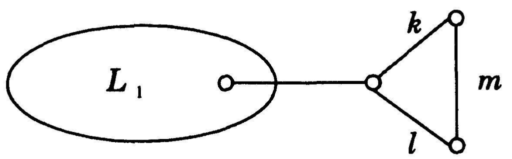
graphs	$\det(S,v)$	values
	$-\mathrm{d}(k,l,m)$	$< -\frac{1}{2}$ for $k,l \geq 3, m \geq 4$; $k=2, l \geq 4, m \geq 6$; $k,l \geq 4, m=3$; $k=3, l \geq 5, m=3$; $k=2, l \geq 5, m=5$;
	$\frac{-1}{4\sin^2\frac{\pi}{2m}}+1$	$< -\frac{1}{\sqrt{2}}$ for $m \geq 5$; $= -\frac{1}{\sqrt{2}}$ for $m=4$;
 $m \geq 7$	$\frac{-1}{4\sin^2\frac{\pi}{m}}+1$	$< -\frac{1}{\sqrt{2}}$ for $m \geq 9$; $= -\frac{1}{\sqrt{2}}$ for $m=8$; ≈ -0.328 for $m = 7$;
 	$\frac{1-\sqrt{5}}{4}$	$\approx -0,309$
	$\frac{1-\sqrt{5}}{8}$	$\approx -0,156$
	$\frac{-\sqrt{2}}{3}$	$\approx -0,471$
	$\frac{-\sqrt{5}}{5}$	$\approx -0,447$

Table 2 (Cont.)

	$\frac{-1-\sqrt{5}}{2}$	$\approx -1,618$
	$\frac{-1-\sqrt{5}}{2}$	$\approx -1,618$
	$-1-\sqrt{2}$	$\approx -2,414$
	$\frac{-1-2\sqrt{2}}{4}$	≈ -0.957
	$\frac{5+2\sqrt{5}+3\sqrt{2}+\sqrt{10}}{4}$	≈ 4.219
	$1+\frac{1+\sqrt{5}}{2}$	≈ 2.618
	$-2(\sqrt{5}+2)$	≈ -8.472

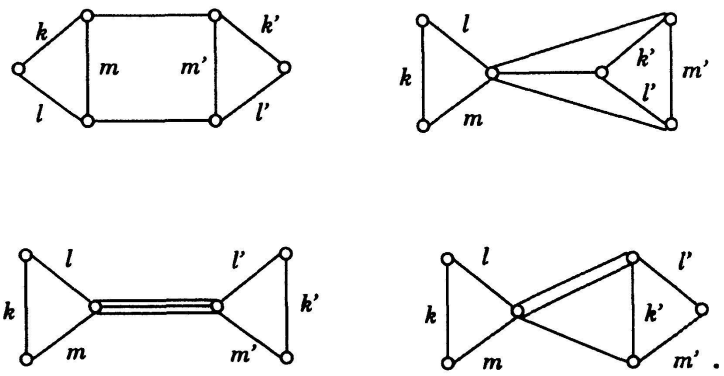
(L_5^1, L_4^5) and multiplicity $\mathrm{m}[v_1, v_2] = 2$. Hence only the last three pairs with $\mathrm{m}[v_1, v_2] = 1$ can be Coxeter realizations of $P_{x,y}$. In fact they are, but they are all superhyperbolic (Lemma 4.5, Table 2).

Step 3) Let us assume $x \geq 4$, $y = 3$. It is easy to see that there are no Coxeter realizations of $P_{x,3}$ such that L_1 and L_2 are joined by a single edge of multiplicity 2 or two edges of multiplicity 1. So the Coxeter realizations are of the following form:



They are exactly given by $L_1 = L_5^1$, $k, l \leq 4$ and $L_1 \in \{L_5^2, L_4^5\}$, $k, l \leq 3$. Each of these diagrams is superhyperbolic (Lemma 4.5, Table 2).

Step 4) Only the case $x = y = 3$ is left. Obviously, the following diagrams are no Coxeter realizations of $P_{3,3}$:



For this reason, we only have the three following possibilities:

1) L_1 and L_2 are joined by a single edge of multiplicity 1,

2) L_1 and L_2 are joined by a single edge of multiplicity 2,

3) L_1 and L_2 are joined by two edges of multiplicity 1 having a common vertex in L_1.

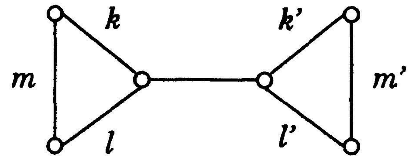
Each Coxeter realization S of $P_{3,3}$ describes a compact hyperbolic Coxeter polytope if and only if the signature of S is equal to $(4, 1, 1)$. Therefore, a necessary condition is $\det(S) = 0$. For the diagrams listed in the theorem this is already sufficient because in each case there exists a vertex v such that $S \backslash v$ splits up into a spherical and a Lannér diagram.

Let $[v, u]$ denote the unique edge joining L_1 and L_2 in the cases 1) and 2).

Case 1: $\mathrm{m}[v, u] = 1$. Since $\det(S) = 0$, we have by Lemma 4.4:

(*) $\det(L_1, v)\det(L_2, u) - \frac{1}{4} = 0.$

Therefore, without loss of generality, we may assume $|\det(L_1, v)| \le \frac{1}{2}$. Fixing the following notations



we have $k, l, k', l' \le 5$.

a) Assume $m = 2$. Then only the values $k = 4$, $l = 5$ and $k = l = 5$ are possible. In the first case we have $\det(L_1, v) = (1 - \sqrt{5})/8$, and in the second $\det(L_1, v) = (1 - \sqrt{5})/4$. Since $l = 5$, it follows $k', l' \le 3$. We have $\det(L_2, u) = 1 - 1/(4 \sin^2(\pi/2m'))$ if $k' = l' = 3$ and $\det(L_2, u) = 1 - 1/(4 \sin^2(\pi/m'))$ if $k' = 2$, $l' = 3$. Equation (*) can only be solved if in the first case $\mathbb{Q}(\cos(2\pi/2m')) = \mathbb{Q}(\sqrt{5})$, or in the second case $\mathbb{Q}(\cos(2\pi/m')) = \mathbb{Q}(\sqrt{5})$. Hence we must have $m' = 5$, $m' = 5, 10$, respectively (whereby $k' = 2$, $l' = 3$, $m' = 5$ does not describe a Lannér diagram). By Table 2, $\det(S) = 0$ exactly for the first two diagrams listed in the theorem.

b) Assume $m = 3$. Then we have $k, l \ge 3$. For $k = 3$, $l = 4$ we compute $\det(L_1, v) = -\sqrt{2}/3$ and this is the only case where we have $|\det(L_1, v)| \le \frac{1}{2}$ (Table 2, first row). Like above, we can conclude $k' = 2$, $l' = 3$ or $k' = l' = 3$. In the first case (*) implies $\mathbb{Q}(\cos(2\pi/m')) = \mathbb{Q}(\sqrt{2})$ and in the second $\mathbb{Q}(\cos(2\pi/2m')) = \mathbb{Q}(\sqrt{2})$. Thus we have $m' = 8$ or 4. In both cases $\det(L_2, u) = -1/\sqrt{2}$, and (*) is not solved.

c) Assume $m = 4$. Since we have $|\det(L_1, v)| \le 1/2$, we may assume $k = 2$. Then only $l = 5$ is possible. Hence we have $\det(L_1, v) = (1 - \sqrt{5})/4$ and by the same calculations as in a) we can conclude that (*) is not solvable.

d) Assume $m = 5$. Only in the case $k = 2$, $l = 4$ we have $|\det(L_1, v)| \le \frac{1}{2}$ (Table 2). Hence $\det(L_1, v) = -\sqrt{5}/5$. Since $l = 4$, it follows $k', l' \le 3$. Therefore, we have for L_2 the same multiplicities as in case a), they obviously do not solve (*) in this case.

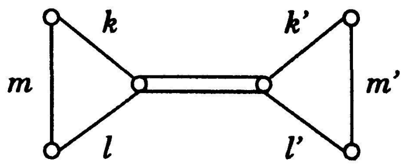
e) Assume $m = 6$. For multiplicities $k = 2$, $l = 3$ the diagram L_1 is parabolic. In no other case we have $|\det(L_1, v)| \le 1/2$.

f) Assume $m = 7$. Only for $k = 2$, $l = 3$ we have $|\det(L_1, v)| \le 1/2$. By (*) follows $\sin^2(\pi/7) \in \mathbb{Q}(\cos(\pi/k'), \cos(\pi/l'), \cos(\pi/m'))$, hence $7 \mid m'$. Due to the fact that $d(k', l', m')$ is an increasing function of $k', l', m' \ge 2$, the following values show that there is no Coxeter diagram with these multiplicities and determinant 0:

m'	k'	l'	$\lvert\det(L_2, v_2)\rvert$
7	2	3	≈ 0.328
7	2	4	> 1.6
7	3	3	> 4
14	2	3	> 4

For $m \geq 8$ we have $|\det(L_1, v)| > 1/2$.

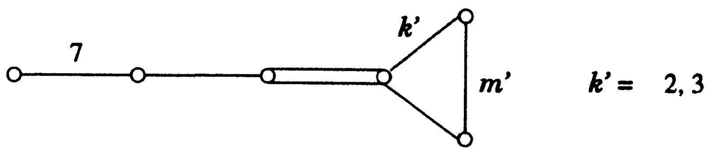
Case 2: $\mathrm{m}[v, u] = 2$. Fixing the following notations for S



we have $k, l, k', l' \leq 3$. By Lemma 4.4 we have the equation

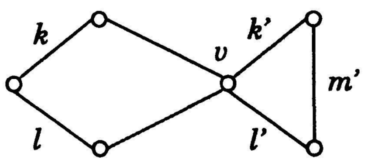
$$(*) \quad \det(L_1, v)\det(L_2, u) - \tfrac{1}{2} = 0,$$

hence we may assume $|\det(L_1, v)| \leq 1/\sqrt{2}$. Therefore, the only possibilities for L_1 are $(m, k, l) = (4, 3, 3)$, $(8, 3, 2)$ and $(7, 3, 2)$. In the first two cases, the local determinant equals $-1/\sqrt{2}$, thus we get $\det(S) = 0$ if and only if $L_2 \in \{(4, 3, 3), (8, 3, 2)\}$. For the diagram



$(*)$ implies $m' = 7, 14$ for $k' = 2$ and $m' = 7$ for $k' = 3$. In each case we have $\det(S) \neq 0$.

Case 3: Fixing the following notations



we have $4 \leq k, l \leq 5$ and $k', l' \leq 3$. By Lemma 4.3 the equation

$$\det(S, v) = \det(L_1 \cup \{v\}, v) + \det(L_2, v) - 1 = 0$$

is to solve. We have $\det(L_1 \cup \{v\}, v) = (5 + 2\sqrt{5} + 3\sqrt{2} + \sqrt{10})/4$ if $k = 4$, $l = 5$ and $\det(L_1 \cup \{v\}, v) = (3 + \sqrt{5})/2$ if $k = l = 5$ (cf. Table 2). Only in the second case $\det(L_2, v) - 1$ can generate the same groundfield as $\det(L_1 \cup \{v\}, v)$, where m' has to be 5 or 10. It is easy to see now that exactly in the two cases listed in the theorem we have $\det(S) = 0$.

The theorem is proved.

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