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The marked length spectrum vs. the Laplace spectrum on forms on Riemannian nilmanifolds

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Abstract. The subject of this paper is the relationships among the marked length spectrum, the length spectrum, the Laplace spectrum on functions, and the Laplace spectrum on forms on Riemannian nilmanifolds. In particular, we show that for a large class of three-step nilmanifolds, if a pair of nilmanifolds in this class has the same marked length spectrum, they necessarily share the same Laplace spectrum on functions. In contrast, we present the first example of a pair of isospectral Riemannian manifolds with the same marked length spectrum but not the same spectrum on one-forms. Outside of the standard spheres vs. the Zoll spheres, which are not even isospectral, this is the only example of a pair of Riemannian manifolds with the same marked length spectrum, but not the same spectrum on forms. This partially extends and partially contrasts the work of Eberlein, who showed that on two-step nilmanifolds, the same marked length spectrum implies the same Laplace spectrum both on functions and on forms.

Section 1: Introduction

The *spectrum* of a closed Riemannian manifold (M, g), denoted spec(M, g), is the collection of eigenvalues with multiplicities of the associated Laplace–Beltrami operator acting on smooth functions. Two Riemannian manifolds (M, g) and (M', g') are said to be *isospectral* if spec(M, g) = spec(M', g').

The Laplace–Beltrami operator may be extended to act on smooth p-forms by $\Delta = d\delta + \delta d$, where δ is the adjoint of d and p is a positive integer. We call its eigenvalue spectrum the *p-form spectrum.*

The *length spectrum* of a Riemannian manifold is the set of lengths of smoothly closed geodesics, counted with multiplicity. The multiplicity of a length is defined as the number of distinct free homotopy classes of loops that contain a closed geodesic of that length. We denote the length spectrum of (M, g) by $[L]$-spec(M, g). This is a natural notion, since the geodesic of shortest length in a free homotopy class is just the shortest loop representing that class. (Note that other definitions of multiplicity appear in the literature.)

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Two Riemannian manifolds (M_1, g_1) and (M_2, g_2) have the same *marked length spectrum* if there exists an isomorphism between the fundamental groups of M_1 and M_2 such that corresponding free homotopy classes contain smoothly closed geodesics of the same length. Manifolds with the same marked length spectrum necessarily have the same length spectrum.

The purpose of this paper is to study the relationships among the marked length spectrum, the length spectrum, the Laplace spectrum on functions and the Laplace spectrum on forms on Riemannian nilmanifolds.

The relationship between the Laplace spectrum and lengths of closed geodesics arises from the study of the wave equation (see [DGu], [GuU]), and in the case of compact, hyperbolic manifolds, from the Selberg Trace Formula (see [C], Chapter XI). Colin de Verdiere [CdV] has shown that generically, the Laplace spectrum determines the length spectrum. On Riemann surfaces, Huber showed that the length spectrum and the Laplace spectrum are equivalent notions (see [Bu] for an exposition).

The Poisson formula gives the relationship between the Laplace spectrum and length spectrum of flat tori, with the result that pairs of flat tori are isospectral if and only if they share the same length spectrum (see [CS], [G3]). Pesce [P2] has computed a Poisson-type formula relating the Laplace spectrum and length spectrum of Heisenberg manifolds and has also shown that pairs of Heisenberg manifolds that are isospectral must have the same lengths of closed geodesics. Previously, Gordon [G1] exhibited the first examples of isospectral manifolds that do not have the same length spectrum. These Heisenberg manifolds have the same lengths of closed geodesics. However, the length spectra often differ in the multiplicities that occur. All known examples of manifolds that are isospectral have the same lengths of closed geodesics.

The marked length spectrum contains significantly more geometric information than the length spectrum. Croke [Cr] and Otal [Ot1], [Ot2] independently showed that if a pair of compact surfaces with negative curvature have the same marked length spectrum, they are necessarily isometric. The same is true for flat tori (see [G3]). In the cases studied by Croke and Otal, the marked length spectrum and the geodesic flow are, roughly speaking, equivalent notions. On two-step nilmanifolds, Gordon, Mao, and Schueth [GM], [GMS] showed that the geodesic flow is significantly stronger. Recently Eberlein [E1] showed that for two-step nilmanifolds, the same marked length spectrum implies the same Laplace spectrum both on functions and on p-forms for all p. (See Section 3 for more details.)

In contrast, the standard sphere and the Zoll sphere (see [Bes]) have the same marked length spectrum (trivially so, as they are both simply connected and by definition have the same lengths of closed geodesics), yet they are not even isospectral on functions. Indeed, any manifold isospectral to a standard sphere of dimension less than or equal to six must be isometric to it (see [B2]).

Examples of pairs of Riemannian manifolds that are isospectral on functions but not on forms are sparse. Most constructions for producing pairs of isospectral manifolds can be explained by Sunada's method [S] or its generalizations [DG], [GW1], [B3]. Pairs of manifolds constructed by the Sunada techniques necessarily have the same p-form spectrum for all p.

For any choice of $P \in \mathbf{Z}^+$, Ikeda [I2] has constructed examples of isopectral lens spaces that are isospectral on p-forms for $p = 0, 1, \ldots, P$ but not isospectral on $(P+1)$-forms. A straightforward argument shows that for the family of lens spaces considered by Ikeda, if a pair of lens spaces in this family has the same marked length spectrum, they are necessarily isometric. Gordon [G2] has constructed pairs of Heisenberg manifolds that are isospectral on functions, but not isospectral on one-forms. A consequence of Eberlein's theorem is that Heisenberg manifolds with the same marked length spectrum are necessarily isometric. (See Section 3 for more details.) The only other known examples of manifolds that are isospectral on functions but not isospectral on forms are pairs of isospectral three-step nilmanifolds presented and studied in [Gt3]. These examples are studied further here.

This paper focuses almost exclusively on three-step nilmanifolds. The main results are a partial extension and a partial converse to Eberlein's theorem for three-step nilmanifolds.

MAIN THEOREM 3.2.2. *For a large class of three-step nilmanifolds, if a pair of nilmanifolds in this class has the same marked length spectrum, they necessarily share the same Laplace spectrum on functions.*

MAIN EXAMPLE. *Example V in the table below exhibits the first example of a pair of isospectral Riemannian manifolds with the same marked length spectrum, but not the same spectrum on one-forms.*

These results have led to the following.

CONJECTURE. *Pairs of Riemannian nilmanifolds with the same marked length spectrum are necessarily isospectral on functions.*

Background ideas and notation are established and explained in Section 2. In [Gt3], we presented a new construction for producing pairs of isospectral nilmanifolds of arbitrary-step. In Section 3, this construction together with Eberlein's theorem and techniques from Riemannian geometry are used to prove the Main Theorem.

Also in [Gt3], we presented new examples of isospectral three-step nilmanifolds with combinations of properties described in the table below. For consistency, the

Table I. New examples of isospectral manifolds

Pair of 3-step isospectral nilmanifolds	$\forall p$ same p-form spectrum	Rep. equiv. fundamental groups	Isomorphic fundamental groups	Same length spectrum	Same marked length spectrum
I (7 dim)	Yes	Yes	No	No	No
II (5 dim)	Yes	Yes	Yes	Yes	No
III\IV (7\5 dim)	No	No	No	No	No
V (7 dim)	No	No	Yes	Yes	Yes

numbering of the examples in this paper coincides with the numbering of the examples in [Gt3]. Note that Example V is also the Main Example.

The spectrum on functions, spectrum on forms, quasi-regular representations, and fundamental groups of these examples were examined in [Gt3]. In Sections 4 and 5 we compare the length spectrum and marked length spectrum of these examples. The pairs of isospectral manifolds described in Table I have the same lengths of closed geodesics. However, the length spectra often differ in the multiplicities that occur.

All of the examples described in the above table are of the form $(\Gamma\backslash G, g)$, where G is a three-step nilpotent Lie group, Γ is a cocompact, discrete subgroup of G (i.e. $\Gamma\backslash G$ compact), and g arises from a left invariant metric on G. Two cocompact, discrete subgroups Γ_1 and Γ_2 of a Lie group G are called *representation equivalent* if the associated quasi-regular representations are unitarily equivalent. If Γ_1 and Γ_2 are representation equivalent, then $(\Gamma_1\backslash G, g)$ and $(\Gamma_2\backslash G, g)$ are necessarily isospectral on functions and on smooth p-forms for any choice of left invariant metric g on G.

REMARK. Example I provided the first example of a pair of representation equivalent subgroups of a solvable Lie group producing nilmanifolds with unequal length spectra. This cannot happen in the two-step nilpotent case. The relationships among the quasi-regular representation, the length spectrum, and marked length spectrum of nilmanifolds are studied in [Gt2], where we also present the first examples of pairs of representation equivalent subgroups of two-step nilpotent Lie groups that do not produce nilmanifolds with the same marked length spectrum. Example I is also the first example of a pair of nonisomorphic, representation equivalent subgroups of a solvable Lie group. See [Gt1] for more details. Note that nilpotent Lie groups are necessarily solvable.

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Section 2: Background and notation

Section 2.1: *Definitions*

Let G be a simply connected Lie group with Lie algebra $\mathfrak{g}$. A metric on G is *left invariant* if left translations are isometries. Note that a left invariant metric is determined by a choice of orthonormal basis of the Lie algebra $\mathfrak{g}$ of G. We denote the corresponding inner product on $\mathfrak{g}$ by $\langle , \rangle$.

Let Γ be a cocompact, discrete subgroup of G. A left invariant metric g on G descends to a Riemannian metric on $\Gamma \backslash G$, which we also denote by g. This paper focuses exclusively on manifolds of the form $(\Gamma \backslash G, g)$, where g arises from a left invariant metric on G.

As G is unimodular, the Laplace–Beltrami operator on $(\Gamma \backslash G, g)$ is

$$\Delta = - \sum_{i=1}^{n} E_i^2,$$

where $\{E_1, \ldots, E_n\}$ is an orthonormal basis of the Lie algebra $\mathfrak{g}$ of G.

Recall that free homotopy classes of loops of a manifold $\Gamma \backslash G$ correspond to the conjugacy classes in the fundamental group Γ. We will denote by $[\gamma]_\Gamma$ the free homotopy class of $\Gamma \backslash G$ represented by $\gamma \in \Gamma$. That is, $[\gamma]_\Gamma = \{\hat{\gamma}\gamma\hat{\gamma}^{-1} : \hat{\gamma} \in \Gamma\}$.

For real numbers $\lambda > 0$, we write $\lambda \in [\gamma]_\Gamma$ if there exists a closed geodesic of length λ in the free homotopy class of loops $[\gamma]_\Gamma$ of $(\Gamma \backslash G, g)$.

Let γ be an element of Γ. We say a geodesic σ of (G, g) is *translated by the element* γ *with period* $\lambda > 0$ if

$$\gamma\sigma(s) = \sigma(s + \lambda) \qquad \forall s \in \mathbf{R}.$$

If σ is a unit speed geodesic on (G, g), then σ projects to a smoothly closed geodesic on $(\Gamma \backslash G, g)$ of length λ, and the projection of σ is contained in the free homotopy class $[\gamma]_\Gamma$. Note that we do not assume that λ is the smallest period of σ.

As the projection $(G, g) \to (\Gamma \backslash G, g)$ is a Riemannian covering, all closed geodesics of $(\Gamma \backslash G, g)$ must arise in this fashion. So to study the closed geodesics of $(\Gamma \backslash G, g)$, it is enough to study the γ-translated geodesics of (G, g).

Let $\sigma(s)$ be a geodesic of G through $p = \sigma(0)$. Let $\hat{\sigma}(s) = p^{-1}\sigma(s)$. As left translations are isometries, $\hat{\sigma}$ is a geodesic of G through e. If σ is translated by γ

with period λ, then $\hat{\sigma}$ is translated by $(p^{-1}\gamma p)$, also with period λ. To see this, note that if $\gamma\sigma(s) = \sigma(s+\lambda)$, then

$$(p^{-1}\gamma p)\hat{\sigma}(s) = (p^{-1}\gamma p)p^{-1}\sigma(s) = p^{-1}\gamma\sigma(s) = p^{-1}\sigma(s+\lambda) = \hat{\sigma}(s+\lambda).$$

2.1.1 NOTATION. In summary, the length $\lambda \in [\gamma]_\Gamma$ if and only if there exists $x = p^{-1}\gamma p \in [\gamma]_G$ and a unit speed geodesic $\sigma(s)$ on (G, g) through $e = \sigma(0)$ such that $x\sigma(s) = \sigma(s+\lambda)$, $\forall s \in \mathbf{R}$. That is, x translates σ with period λ. Here $[\gamma]_G$ denotes the conjugacy class of γ in G.

With this notation, a pair of manifolds $(\Gamma_1\backslash G_1, g_1)$ and $(\Gamma_2\backslash G_2, g_2)$ share the same marked length spectrum if and only if there exists an isomorphism $\Phi: \Gamma_1 \to \Gamma_2$ such that for all $\gamma \in \Gamma_1$ and for all $\lambda > 0$,

$$\lambda \in [\gamma]_{\Gamma_1} \text{ if and only if } \lambda \in [\Phi(\gamma)]_{\Gamma_2}.$$

We say that the isomorphism Φ *marks* the length spectrum between $(\Gamma_1\backslash G_1, g_1)$ and $(\Gamma_2\backslash G_2, g_2)$.

Section 2.2: *Nilmanifolds*

Let $\mathfrak{g}$ be a Lie algebra. We denote by $\mathfrak{g}^{(1)}$ the derived algebra $[\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$. That is, $\mathfrak{g}^{(1)}$ is the Lie subalgebra of $\mathfrak{g}$ generated by all elements of the form $[X, Y]$ for X, Y in $\mathfrak{g}$. Inductively, define $\mathfrak{g}^{(k+1)} = [\mathfrak{g}, \mathfrak{g}^{(k)}]$. The Lie algebra $\mathfrak{g}$ is said to be *k-step nilpotent* if $\mathfrak{g}^{(k)} \equiv 0$ but $\mathfrak{g}^{(k-1)} \not\equiv 0$. A Lie group G is called *k-step nilpotent* if its Lie algebra is.

If G is a nilpotent Lie group with cocompact, discrete subgroup Γ, the locally homogeneous space $\Gamma\backslash G$ is called a *nilmanifold.* If G is an abelian Lie group, then Γ is merely a lattice of rank n in G, where n is the dimension of G. In this case, $\log \Gamma$ is also a lattice in $\mathfrak{g}$.

Let exp denote the Lie algebra exponential from $\mathfrak{g}$ to G. The Campbell-Baker-Hausdorff formula gives us the group operation of G in terms of $\mathfrak{g}$. Namely, for $X, Y \in \mathfrak{g}$:

$$\exp(X)\exp(Y) = \exp\left(X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [Y, X]] + \cdots\right),$$

where the remaining terms are higher-order brackets. Note that for two-step nilpotent Lie groups, only the first three terms in the right-hand side are nonzero.

For three-step groups, only the first five terms are nonzero. If $\mathfrak{g}$ is nilpotent and G is simply connected, then exp is a diffeomorphism from $\mathfrak{g}$ onto G. Denote its inverse by log.

If G_1 and G_2 are nilpotent Lie groups with cocompact, discrete subgroups Γ_1 and Γ_2, respectively, any abstract group isomorphism $\Phi: \Gamma_1 \to \Gamma_2$ lifts uniquely to a Lie group isomorphism $\Phi: G_1 \to G_2$.

For details of cocompact, discrete subgroups of nilpotent Lie groups, see [Ra].

2.2.1 DEFINITION. *Let Φ be a Lie group automorphism of G. Let Γ be a cocompact, discrete subgroup of G.*

(i) *We call Φ an almost inner automorphism if for all elements x of G there exists a_x in G such that $\Phi(x) = a_x x a_x^{-1}$.*

(ii) *We say Φ is a Γ-almost inner automorphism if for all elements γ of Γ there exists a_γ in G such that $\Phi(\gamma) = a_\gamma \gamma a_\gamma^{-1}$.*

Denote by IA(G) (respectively, AIA(G), Γ-AIA(G)) the group of inner automorphisms (respectively, almost inner automorphisms, Γ-almost inner automorphisms) of G. Note that IA(G) $\subset$ AIA(G) $\subset$ Γ-AIA(G).

2.2.2 THEOREM (Gordon and Wilson, Gordon [GW1], [G1]). *Let G be an exponential solvable Lie group, and let Γ_1 and Γ_2 be cocompact, discrete subgroups of G. Let Φ be a Γ_1-almost inner automorphism of G such that $\Phi(\Gamma_1) = \Gamma_2$. Then $(\Gamma_1 \backslash G, g)$ and $(\Gamma_2 \backslash G, g)$ are isospectral on functions and on forms for any choice of left invariant metric g on G. Moreover, the automorphism Γ marks the length spectrum between $(\Gamma_1 \backslash G, g)$ and $(\Gamma_2 \backslash G, g)$.*

Note that a nilpotent Lie group is necessarily exponential solvable.

Section 3: The marked length spectrum vs. the Laplace spectrum on functions of three-step nilmanifolds

Throughout this section, G is a simply connected, k-step nilpotent Lie group with Lie algebra $\mathfrak{g}$, Γ is a cocompact, discrete subgroup of G, and g is a left invariant metric on G, which descends to a metric on $\Gamma \backslash G$, also denoted by g. We denote the center of $\mathfrak{g}$ by $\mathfrak{z}$ and the center of G by $Z(G)$. Let L_x denote left multiplication by $x \in G$. As g is left invariant, L_x is always an isometry of (G, g). Let $G^{(k)} = \exp(\mathfrak{g}^{(k)})$ denote the kth derived subgroup of G. Note that if G is k-step nilpotent, then $G^{(k-1)} \subset Z(G)$.

Section 3.1: *Preliminaries*

3.1.1 THEOREM. *Let G be a three-step nilpotent Lie group with left invariant metric g. Let σ be a geodesic on (G, g) that is translated by the element $\gamma \in G$ with period $\lambda > 0$. Let $p = \sigma(0)$. Then*

$$\langle L_{p*}([\log(p^{-1}\gamma p), \mathfrak{g}]), \dot{\sigma}(0) \rangle_p \equiv 0.$$

REMARK. This is the three-step generalization of a result due to Eberlein [E1]. Recently Dorothee Schueth [Sch] has given an elegant proof, which generalizes the result to nilpotent Lie groups of *arbitrary* step.

Outline of Proof of 3.1.1. We briefly describe the basic steps in the original three-step proof. For more details, see [Gt4], Chapter 4.

Let G be a simply connected, three-step nilpotent Lie group with Lie algebra $\mathfrak{g}$ and left invariant metric g. Let $\mathfrak{g} = v \oplus \mathfrak{g}^{(1)}$, where v is the orthogonal complement of $\mathfrak{g}^{(1)}$ in $\mathfrak{g}$. Let $\mathfrak{g}^{(1)} = \zeta \oplus \mathfrak{g}^{(2)}$, where ζ is the orthogonal complement of $\mathfrak{g}^{(2)}$ in $\mathfrak{g}^{(1)}$. Thus $\mathfrak{g} = v \oplus \zeta \oplus \mathfrak{g}^{(2)}$.

Let $\{X_1, X_2, \ldots, X_J\}$ be an orthonormal basis of v. Let $\{Z_1, Z_2, \ldots, Z_K\}$ be an orthonormal basis of ζ, and let $\{W_1, W_2, \ldots, W_T\}$ be an orthonormal basis of $\mathfrak{g}^{(2)}$. Throughout this proof the indices i, j, and l run from 1 to J, the indices h and k run from 1 to K, and the indices t and r run from 1 to T.

Define A^k_{ij}, B^t_{ij}, C^t_{ik} by

$$[X_i, X_j] = \sum_k A^k_{ij} Z_k + \sum_t B^t_{ij} W_t$$

$$[X_i, Z_k] = -[Z_k, X_i] = \sum_t C^t_{ik} W_t.$$

As $[X_i, X_j] = -[X_j, X_i]$, we have $A^k_{ij} = -A^k_{ji}$ and $B^t_{ij} = -B^t_{ji}$. By the Jacobi equation $[\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] \subset [\mathfrak{g}, \mathfrak{g}^{(2)}] \equiv 0$. Thus $[Z_k, Z_h] = 0$. Finally, by applying the Jacobi equation to X_i, X_j, X_k and examining the W_t coefficient, we obtain:

$$0 = \sum_k (A^k_{jl} C^t_{ik} + A^k_{ij} C^t_{lk} + A^k_{li} C^t_{jk}).$$

For Lie algebras with a left invariant metric, the covariant derivatives can be calculated via

$$\langle \nabla_V Y, U \rangle = \frac{1}{2}\langle [U, V], Y \rangle + \frac{1}{2}\langle [U, Y], V \rangle + \frac{1}{2}\langle [V, Y], U \rangle$$

for U, V, Y in $\mathfrak{g}$. We obtain the covariant derivatives:

$$\nabla_{X_i} X_j = \frac{1}{2}\sum_k A_{ij}^k Z_k + \frac{1}{2}\sum_t B_{ij}^t W_t,$$
$$\nabla_{X_i} Z_k = \frac{1}{2}\sum_j A_{ji}^k X_j + \frac{1}{2}\sum_t C_{ik}^t W_t,$$
$$\nabla_{Z_k} X_i = \frac{1}{2}\sum_j A_{ji}^k X_j - \frac{1}{2}\sum_t C_{ik}^t W_t,$$
$$\nabla_{X_i} W_t = \nabla_{W_t} X_i = \frac{1}{2}\sum_j B_{ji}^t X_j - \frac{1}{2}\sum_k C_{ik}^t Z_k,$$
$$\nabla_{Z_k} Z_h = \nabla_{W_t} W_r = 0,$$
$$\nabla_{Z_k} W_t = \nabla_{W_t} Z_k = \frac{1}{2}\sum_j C_{jk}^t X_j.$$

For $x \in G$, $x = \exp(\Sigma_j x_j X_j + \Sigma_k z_k Z_k + \Sigma_t w_t W_t)$ gives us a global coordinate system on G. With this coordinate system, a straightforward computation shows us that

$$X_j = \frac{\partial}{\partial x_j} + \sum_k \left(\frac{1}{2}\sum_i x_i A_{ij}^k\right)\frac{\partial}{\partial z_k}$$
$$+ \sum_t \left(\frac{1}{2}\sum_i x_i B_{ij}^t - \frac{1}{2}\sum_k C_{jk}^t z_k + \frac{1}{12}\sum_{i,l,k} x_i C_{ik}^t x_l A_{lj}^k\right)\frac{\partial}{\partial w_t},$$
$$Z_k = \frac{\partial}{\partial z_k} + \sum_t \left(\frac{1}{2}\sum_i x_i C_{ik}^t\right)\frac{\partial}{\partial w_t},$$
$$W_t = \frac{\partial}{\partial w_t}.$$

Let $\sigma(s) = \exp(\Sigma_j x_j(s) X_j + \Sigma_k z_k(s) Z_k + \Sigma_t w_t(s) W_t)$ be a geodesic of (G, g) with initial velocity $\dot\sigma(0) = \Sigma_j \bar{x}_j X_j + \Sigma_k \bar{z}_k Z_k + \Sigma_t \bar{w}_t W_t$. A straightforward computation of $\nabla_{\dot\sigma(s)}\dot\sigma(s) \equiv 0$ produces the following geodesic equations for a three-step nilpotent Lie group, reduced to a system of n-ordinary differential equations.

$$\dot{x}_j(s) = -\sum_{l,k} x_l(s) A_{jl}^k \bar{z}_k - \sum_{l,t} x_l(s) B_{jl}^t \bar{w}_t - \sum_{k,t} z_k(s) C_{jk}^t \bar{w}_t$$
$$-\frac{1}{2}\sum_{i,l,k,t} x_i(s) x_l(s) \bar{w}_t C_{ik}^t A_{jl}^k + \bar{x}_j$$
$$\dot{z}_k(s) = \frac{1}{2}\sum_{i,j} x_i(s)\dot{x}_j(s) A_{ij}^k + \sum_{j,t} x_j(s)\bar{w}_t C_{jk}^t + \bar{z}_k$$
$$\dot{w}_t(s) = \frac{1}{2}\sum_{i,j} x_i(s)\dot{x}_j(s) B_{ij}^t - \frac{1}{2}\sum_{j,k}\dot{x}_j(s) z_k(s) C_{jk}^t + \frac{1}{2}\sum_{j,k} x_j(s)\dot{z}_k(s) C_{jk}^t$$
$$-\frac{1}{6}\sum_{i,j,k,l} x_i(s)\dot{x}_j(s) x_l(s) C_{ik}^t A_{lj}^k + \bar{w}_t.$$

If we assume that a geodesic $\sigma(s)$ starts at the identity and is translated by the element γ, then a lengthy but straightforward (brute-force) calculation yields

$$\langle[\log(\gamma), \mathfrak{g}], \dot{\sigma}(0)\rangle_e \equiv 0.$$

Here one uses extensively the fact that if $\gamma\sigma(s) = \sigma(s+\lambda)$, then $L_{\gamma*}(\dot{\sigma}(s)) = \dot{\sigma}(s+\lambda)$.

In the general case, let $\sigma(s)$ be a geodesic of G through $p = \sigma(0)$. Let $\alpha(s) = p^{-1}\sigma(s)$. Then α is a geodesic of G through e. If σ is translated by γ with period λ, then α is translated by $p^{-1}\gamma p$, also with period λ. Thus

$$\langle[\log(p^{-1}\gamma p), \mathfrak{g}], \dot{\alpha}(0)\rangle_e \equiv 0.$$

But $\dot{\alpha}(0) = (L_{p^{-1}})_*(\dot{\sigma}(0))$. As our metric is left invariant, we obtain

$$\langle L_{p*}([\log(p^{-1}\gamma p), \mathfrak{g}]), \dot{\sigma}(0)\rangle_p = 0,$$

as desired. □

REMARK. Ron Karidi [K] has recently given a formulation of the geodesic equations for an arbitrary nilpotent Lie group with a left invariant metric. As above, this formulation is in terms of an orthonormal basis and structure constants of the Lie algebra.

3.1.2 NOTATION. Let π denote the canonical projection from G onto $\bar{G} = G/G^{(k-1)}$. For Γ a cocompact, discrete subgroup of G, denote by $\bar{\Gamma}$ the image of Γ under π. The group $\bar{\Gamma}$ is then a cocompact, discrete subgroup of $\bar{G}$. Let $\bar{g}$ denote the metric on $\bar{G}$ defined by restricting the left invariant metric g to an orthogonal complement of $\mathfrak{g}^{(k-1)} \subset \mathfrak{z}$, where $\mathfrak{g}$ is the Lie algebra of G. With this choice of metric $\bar{g}$ on $\bar{G}$, the mapping

$$\pi: (G, g) \to (\bar{G}, \bar{g})$$

is a Riemannian submersion with totally geodesic fibers.

If $\Phi: G_1 \to G_2$ is a Lie group homomorphism, then necessarily $\Phi: G_1^{(k-1)} \to G_2^{(k-1)}$. Let $\bar{\Phi}$ denote the canonical projection of Φ onto $\bar{\Phi} = \pi \circ \Phi: \bar{G}_1 \to \bar{G}_2$.

The Lie algebra of $\bar{G}$ is $\bar{\mathfrak{g}} = \mathfrak{g}/\mathfrak{g}^{(k-1)}$. We denote elements of $\bar{\mathfrak{g}}$ by $\bar{U}$ where $\bar{U}$ is the image of $U \in \mathfrak{g}$ under the canonical projection from $\mathfrak{g}$ onto $\bar{G}$. Similarly, we will denote elements of $\bar{G}$ by $\bar{x}$ where $\bar{x}$ is the image of $x \in G$ under the canonical projection from G onto $\bar{G}$.

All of the nilpotent Lie groups studied here have the following property.

3.1.3 DEFINITION. *Let G be a simply connected, k-step nilpotent Lie group. We say G is strictly nonsingular if the following property holds: for all z in $Z(G)$ and for all noncentral x in G there exists a in G such that the commutator*

$$axa^{-1}x^{-1} = z.$$

The Lie algebra $\mathfrak{g}$ is strictly nonsingular if for all X in $\mathfrak{g} - \mathfrak{z}$ and all Z in $\mathfrak{z}$ there exists Y in $\mathfrak{g}$ such that $[X, Y] = Z$, that is

$$\mathfrak{z} \subset ad(X)(\mathfrak{g}).$$

One easily sees that the Lie group G is strictly nonsingular if and only if its Lie algebra $\mathfrak{g}$ is strictly nonsingular. Note that for strictly nonsingular nilpotent Lie algebras, $\mathfrak{z} = \mathfrak{g}^{(k-1)}$.

3.1.4 COROLLARY. *Let G be a simply connected, strictly nonsingular three-step nilpotent Lie group with left invariant metric g. Consider the Riemannian submersion $(G, g) \to (\bar{G}, \bar{g})$. If σ is a geodesic on G such that $\gamma\sigma(s) = \sigma(s + \lambda)$ for some noncentral γ in G and some $\lambda > 0$, then σ is a horizontal geodesic. That is,*

$$\langle L_{\sigma(s)*}(\mathfrak{z}), \dot{\sigma}(s) \rangle \equiv 0 \qquad \forall s \in \mathbf{R}.$$

Before proving Corollary 3.1.4, recall the following properties of Riemannian submersions.

3.1.5 PROPOSITION (see [GHL]). *Let $(M, g) \to (\bar{M}, \bar{g})$ be a Riemannian submersion.*

(i) *Let α be a geodesic of (M, g). If the vector $\dot{\alpha}(0)$ is horizontal, then $\dot{\alpha}(s)$ is horizontal for all s, and the curve $\pi \circ \alpha$ is a geodesic of $(\bar{M}, \bar{g})$ of the same length as σ.*

(ii) *Conversely, let $p \in M$ and let σ be a geodesic of $(\bar{M}, \bar{g})$ with $\sigma(0) = \pi(p)$. Then there exists a unique local horizontal lift $\hat{\sigma}$ of σ through $p = \hat{\sigma}(0)$, and $\hat{\sigma}$ is also a geodesic of (M, g).*

Proof of Corollary 3.1.4. By Theorem 3.1.1

$$\langle L_{p*}([\log(p^{-1}\gamma p), \mathfrak{g}]), \dot{\sigma}(0) \rangle_p \equiv 0,$$

where $p = \sigma(0)$. By strict nonsingularity

$$\mathfrak{z} = \mathfrak{g}^{(2)} \subset [\log(p^{-1}\gamma p), \mathfrak{g}].$$

Thus

$$\langle L_{p*}(\mathfrak{z}), \dot{\sigma}(0)\rangle_p \equiv 0.$$

Thus $\dot{\sigma}(0)$ is horizontal. By Proposition 3.1.5, we know that $\dot{\sigma}(s)$ is horizontal for all $s \in \mathbf{R}$. □

Section 3.2: *Main Theorem*

On two-step nilmanifolds, we have the following relationship between the marked length spectrum and the p-form spectrum.

3.2.1 THEOREM (Eberlein [E1]). *Let Γ_1, Γ_2 be cocompact, discrete subgroups of simply connected, two-step nilpotent Lie groups G_1, G_2 with left invariant metrics g_1, g_2 respectively. Assume that $(\Gamma_1\backslash G_1, g_1)$ and $(\Gamma_2\backslash G_2, g_2)$ have the same marked length spectrum, and let $\Phi\colon \Gamma_1 \to \Gamma_2$ be an isomorphism inducing this marking. Then $\Phi = (\Phi_1 \circ \Phi_2)|_{\Gamma_1}$, where Φ_2 is a Γ_1-almost inner automorphism of G_1, and Φ_1 is an isomorphism of (G_1, g_1) onto (G_2, g_2) that is also an isometry. Moreover, this factorization is unique. In particular, $(\Gamma_1\backslash G_1, g_1)$ and $(\Gamma_2\backslash G_2, g_2)$ have the same spectrum of the Laplacian on functions and on p-forms for all p.*

REMARK. Note that if Γ-AIA$(G) =$ IA(G), then the elements of Γ-AIA(G) are isometries of (G, g), where g is any choice of left invariant metric g of G. So by Theorem 3.2.1, any two-step nilmanifold with the same marked length spectrum as $(\Gamma\backslash G, g)$ is necessarily isometric to it. This property applies to Heisenberg groups. Thus pairs of Heisenberg manifolds with the same marked length spectrum are necessarily isometric.

We may now state the main result of this paper.

3.2.2 MAIN THEOREM. *Let G be a simply connected, strictly nonsingular, three-step nilpotent Lie group. Let Γ_1 and Γ_2 be cocompact, discrete subgroups of G such that $\Gamma_1 \cap Z(G) = \Gamma_2 \cap Z(G)$. If $(\Gamma_1\backslash G, g)$ and $(\Gamma_2\backslash G, g)$ have the same marked length spectrum, then $(\Gamma_1\backslash G, g)$* and *$(\Gamma_2\backslash G, g)$* are isospectral on functions.

To prove Theorem 3.2.2, we need the following.

3.2.3 THEOREM [Gt3, Theorem 3.2]. *Let G be a simply connected, strictly nonsingular nilpotent Lie group with left invariant metric g. If Γ_1 and Γ_2 are* *cocompact, discrete subgroups of G such that*

$$\Gamma_1 \cap Z(G) = \Gamma_2 \cap Z(G) \quad \textit{and} \quad spec(\bar{\Gamma}_1 \backslash \bar{G}, \bar{g}) = spec(\bar{\Gamma}_2 \backslash \bar{G}, \bar{g}),$$

then

$$spec(\Gamma_1 \backslash G, g) = spec(\Gamma_2 \backslash G, g).$$

3.2.4 THEOREM. *Let G be a simply connected, strictly nonsingular three-step nilpotent Lie group with cocompact, discrete subgroup Γ and left invariant metric g. Let γ be a noncentral element of Γ. Then for all $\lambda > 0$ we have the following condition:*

$$\lambda \in [\gamma]_\Gamma \quad \textit{if and only if} \quad \lambda \in [\pi(\gamma)]_{\bar{\Gamma}}.$$

Assume for the moment that Theorem 3.2.4 is true.

3.2.5 COROLLARY. *Let G_1 and G_2 be simply connected, strictly nonsingular, three-step nilpotent Lie groups with cocompact, discrete subgroups Γ_1 and Γ_2 and left invariant metrics g_1 and g_2, respectively. Let Φ mark the length spectrum between $(\Gamma_1 \backslash G_1, g_1)$ and $(\Gamma_2 \backslash G_2, g_2)$. Then $\bar{\Phi}$ must mark the length spectrum between $(\bar{\Gamma}_1 \backslash \bar{G}_1, \bar{g}_1)$ and $(\bar{\Gamma}_2 \backslash \bar{G}_2, \bar{g}_2)$.*

Proof of Corollary 3.2.5. For noncentral $\gamma \in \Gamma_1$, let the length $\lambda \in [\pi(\gamma)]_{\bar{\Gamma}_1}$. By (3.2.4) $\lambda \in [\gamma]_{\Gamma_1}$. By hypothesis $\lambda \in [\Phi(\gamma)]_{\Gamma_2}$. By (3.2.4) again, $\lambda \in [\pi(\Phi(\gamma))]_{\bar{\Gamma}_2} = [\bar{\Phi}(\pi(\gamma))]_{\bar{\Gamma}_2}$.

Reversing the roles of $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$, we obtain the desired result. □

Proof of Main Theorem 3.2.2. Let Φ mark the length spectrum between $(\Gamma_1 \backslash G, g)$ and $(\Gamma_2 \backslash G, g)$. By (3.2.5) we know that $\bar{\Phi}$ must mark the length spectrum between $(\bar{\Gamma}_1 \backslash \bar{G}, \bar{g})$ and $(\bar{\Gamma}_2 \backslash \bar{G}, \bar{g})$. By Theorem 3.2.1 $\mathrm{spec}(\bar{\Gamma}_1 \backslash \bar{G}, \bar{g}) = \mathrm{spec}(\bar{\Gamma}_2 \backslash \bar{G}, \bar{g})$.

The result now follows directly from Theorem 3.2.3. □

It remains only to prove Theorem 3.2.4, which is immediate from the following two lemmas.

3.2.6 LEMMA. *Let G be a simply connected, strictly nonsingular three-step nilpotent Lie group with cocompact, discrete subgroup Γ and left invariant metric g. Let γ be a noncentral element of Γ. With the above notation, if the length $\lambda \in [\gamma]_\Gamma$ then $\lambda \in [\pi(\gamma)]_{\bar{\Gamma}}$.*

Proof of Lemma 3.2.6. If the length $\lambda \in [\gamma]_\Gamma$, then there exists a unit speed geodesic $\sigma(s)$ of G through e such that

$$p^{-1}\gamma p\sigma(s) = \sigma(s + \lambda)$$

for some $p \in G$.

By (3.1.4), $\sigma(s)$ is a horizontal geodesic, and by (3.1.5), $\pi \circ \sigma(s)$ is a unit speed geodesic of $(\bar{G}, \bar{g})$.

But $\pi(p^{-1}\gamma p\sigma(s)) = \pi(p^{-1})\pi(\gamma)\pi(p)\pi(\sigma(s)) = \pi(\sigma(s + \lambda))$. Thus $\pi(\sigma)$ is a unit speed geodesic translated by $\pi(p^{-1})\pi(\gamma)\pi(p)$ with period λ. That is, $\lambda \in [\pi(\gamma)]_{\bar{\Gamma}}$, as desired. □

3.2.7 LEMMA. *Let G be a simply connected, strictly nonsingular k-step nilpotent Lie group. Using the above notation, let the length $\lambda \in [\bar{\gamma}]_{\bar{\Gamma}}$, where $\bar{\gamma} \neq e$. Then $\lambda \in [\gamma]_{\Gamma}$ for all $\gamma \in \pi^{-1}(\bar{\gamma})$.*

Proof of Lemma 3.2.7. Let σ be a unit speed geodesic of $(\bar{G}, \bar{g})$ through $\bar{e} = \sigma(0)$ and translated by $p^{-1}\bar{\gamma}p$ for some $p \in \bar{G}$ with period λ.

By (3.1.5), the unique horizontal lift $\hat{\sigma}$ of σ with $\hat{\sigma}(0) = e$ is a geodesic of (G, g).

As both G and $\bar{G}$ are complete, we see that $\hat{\sigma}$ is defined for all $s \in \mathbf{R}$. We also have $\pi \circ \hat{\sigma}(s) = \sigma(s)$ for all $s \in \mathbf{R}$. To see this, note that the set S of all such s is nonempty as $0 \in S$, open by completeness, and closed by uniqueness and smoothness. Thus, $S = \mathbf{R}$.

Now $\pi(\hat{\sigma}(\lambda)) = p^{-1}\bar{\gamma}p$. Let $\hat{p}$ be such that $\pi(\hat{p}) = p$.

Let $\gamma \in \pi^{-1}(\bar{\gamma})$. Then $\pi(\hat{p}^{-1}\gamma\hat{p}) = p^{-1}\bar{\gamma}p = \pi(\hat{\sigma}(\lambda))$. Thus $(\hat{\sigma}(\lambda))(\hat{p}^{-1}\gamma\hat{p})^{-1}$ is a central element of G.

By strict nonsingularity, there exists $x \in G$ such that

$$x^{-1}(\hat{p}^{-1}\gamma\hat{p})x(\hat{p}^{-1}\gamma\hat{p})^{-1} = \hat{\sigma}(\lambda)(\hat{p}^{-1}\gamma\hat{p})^{-1},$$

that is $x^{-1}(\hat{p}^{-1}\gamma\hat{p})x = \hat{\sigma}(\lambda)$.

If we let $p' = \hat{p}x$, then $\hat{\sigma}(\lambda) = p'^{-1}\gamma p'$. Note that $\pi(p'^{-1}\gamma p') = \pi(\hat{\sigma}(s)) = p^{-1}\bar{\gamma}p$.

We now show that $p'^{-1}\gamma p'\hat{\sigma}(s) = \hat{\sigma}(s + \lambda)$ for all $s \in \mathbf{R}$. Let

$$\alpha(s) = (p'^{-1}\gamma p')^{-1}\hat{\sigma}(s + \lambda).$$

Now $\alpha(0) = (p'^{-1}\gamma p')^{-1}\hat{\sigma}(\lambda) = e$. Also, $\alpha(s)$ is horizontal since g is left invariant and α is just a left translate of the horizontal curve $\hat{\sigma}$. Moreover,

$$\begin{aligned}\pi(\alpha(s)) &= \pi((p'^{-1}\gamma p')^{-1}\hat{\sigma}(s + \lambda)) = p^{-1}\bar{\gamma}^{-1}p\sigma(s + \lambda)\\ &= p^{-1}\bar{\gamma}^{-1}pp^{-1}\bar{\gamma}p\sigma(s) = \sigma(s).\end{aligned}$$

Thus α is a horizontal geodesic through $e \in G$ whose projection agrees with σ. By uniqueness in Proposition 3.1.5, $\alpha(s) = \hat{\sigma}(s)\ \forall s \in \mathbf{R}$.

Consequently,

$$p'^{-1}\gamma p'\hat{\sigma}(s) = \hat{\sigma}(s + \lambda)$$

for all $s \in \mathbf{R}$. Thus the length

$$\lambda \in [\gamma]_{\Gamma},$$

as desired. □

Section 3.3: *Three-step nilmanifolds with a one-dimensional center*

3.3.1 THEOREM. *Let G be a simply connected, strictly nonsingular, three-step nilpotent Lie group with a one-dimensional center. Let Γ_1 and Γ_2 be cocompact, discrete subgroups of G such that $\Gamma_1 \cap Z(G) = \Gamma_2 \cap Z(G)$. Let g be any left invariant metric on G. Then $(\Gamma_1\backslash G, g)$ and $(\Gamma_2\backslash G, g)$ have the same marked length spectrum if and only if there exists an isomorphism $\Phi: \Gamma_1 \to \Gamma_2$ such that $\bar{\Phi}: \bar{\Gamma}_1 \to \bar{\Gamma}_2$ marks the length spectrum between $(\bar{\Gamma}_1\backslash\bar{G}, \bar{g})$ and $(\bar{\Gamma}_2\backslash\bar{G}, \bar{g})$.*

Proof of Theorem 3.3.1. The forward direction follows immediately from Corollary 3.2.5.

For the converse direction, assume that there exists an isomorphism $\Phi: \Gamma_1 \to \Gamma_2$ such that $\bar{\Phi}$ marks the length spectrum between $(\bar{\Gamma}_1\backslash\bar{G}, \bar{g})$ and $(\bar{\Gamma}_2\backslash\bar{G}, \bar{g})$.

We need to show that for all $\gamma \in \Gamma_1$ and for all $\lambda > 0$, the length $\lambda \in [\gamma]_{\Gamma_1}$ if and only if $\lambda \in [\Phi(\gamma)]_{\Gamma_2}$.

We consider two cases:

Case 1. $\gamma \in \Gamma_1 \cap Z(G)$.

If the length $\lambda \in [\gamma]_{\Gamma_1}$, then there exists a unit speed geodesic $\sigma(s)$ of G such that $\gamma\sigma(s) = \sigma(s + \lambda)$.

As Φ is an isomorphism, we known that $\Phi(\Gamma_1 \cap Z(G)) = \Gamma_2 \cap Z(G) = \Gamma_1 \cap Z(G)$, and hence Φ must map a generator of $\Gamma_1 \cap Z(G)$ to a generator of $\Gamma_1 \cap Z(G)$. There are only two such generators. Thus for all $\gamma \in \Gamma_1 \cap Z(G)$, either $\Phi(\gamma) = \gamma$ or $\Phi(\gamma) = \gamma^{-1}$. Hence $[\Phi(\gamma)]_{\Gamma_2} = [\gamma]_{\Gamma_2}$ or $[\Phi(\gamma)]_{\Gamma_2} = [\gamma^{-1}]_{\Gamma_2}$.

If $[\Phi(\gamma)]_{\Gamma_2} = [\gamma]_{\Gamma_2}$, then the geodesic $\sigma(s)$ of G projects to a closed geodesic of $(\Gamma_2\backslash G, g)$ of length λ in the free homotopy class $[\gamma]_{\Gamma_2}$.

If $[\Phi(\gamma)]_{\Gamma_2} = [\gamma^{-1}]_{\Gamma_2}$, then the geodesic $\alpha(s) = \sigma(-s)$ of G projects to a closed geodesic of $(\Gamma_2\backslash G, g)$ of length λ in the free homotopy class $[\gamma^{-1}]_{\Gamma_2}$.

This argument also works for $\Phi^{-1}\colon \Gamma_2 \to \Gamma_1$, which must necessarily mark the length spectrum. Consequently, for all $\gamma \in \Gamma_1 \cap Z(G)$ and for all $\lambda > 0$,

$$\lambda \in [\gamma]_{\Gamma_2} \text{ if and only if } \lambda \in [\Phi(\gamma)]_{\Gamma_2}.$$

Case 2. $\gamma \notin Z(G)$.

Let the length $\lambda \in [\gamma]_{\Gamma_1}$. By strict nonsingularity and Theorem 3.2.4, we know that $\lambda \in [\pi(\gamma)]_{\bar{\Gamma}_1}$. By assumption, we know that $\lambda \in [\bar{\Phi}(\pi(\gamma))]_{\bar{\Gamma}_2}$. Now $\pi(\Phi(\gamma)) = \bar{\Phi}(\pi(\gamma))$. Thus by Theorem 3.2.4 again we know $\lambda \in [\Phi(\gamma)]_{\Gamma_2}$. Reversing the roles of Γ_1 and Γ_2 in the above, we see that for all noncentral $\gamma \in \Gamma_1$ and for all $\lambda > 0$,

$$\lambda \in [\gamma]_{\Gamma_1} \text{ if and only if } \lambda \in [\Phi(\gamma)]_{\Gamma_2},$$

as desired. □

Section 4. The marked length spectrum vs. the one-form spectrum

The example below is the first example of a pair of isospectral Riemannian manifolds with the same marked length spectrum, but not the same spectrum on one-forms. Outside of the standard vs. Zoll spheres, which are not even isospectral for dimension less than or equal to six, this is the only example of a pair of Riemannian manifolds that have the same marked length spectrum but not the same spectrum on one-forms.

Example V

We use the notation of Section 3.

Consider the simply connected, strictly nonsingular, three-step nilpotent Lie group G with Lie algebra

$$\mathfrak{g} = \operatorname{span}_{\mathbb{R}}\{X_1, X_2, Y_1, Y_2, Z_1, Z_2, W\}$$

and Lie brackets

$$[X_1, Y_1] = [X_2, Y_2] = Z_1$$
$$[X_1, Y_2] = Z_2$$
$$[X_1, Z_1] = [X_2, Z_2] = [Y_1, Y_2] = W$$

and all other basis brackets zero.

We fix a left invariant metric on G by letting $\{E_1, E_2, E_3, E_4, E_5, E_6, E_7\}$ be an orthonormal basis of $\mathfrak{g}$ where

$$E_1 = X_1 - \frac{1}{2}X_2 - \frac{1}{4}Y_2,$$

$$E_2 = X_2 - \frac{1}{4}Y_1,$$

$$E_3 = Y_1,$$

$$E_4 = Y_1 + Y_2,$$

$$E_5 = Z_1,$$

$$E_6 = \frac{1}{2}Z_1 + Z_2,$$

$$E_7 = W.$$

Let Φ be the automorphism of G defined on the Lie algebra level by

$$X_1 \to -X_1 + X_2 + \frac{1}{4}Y_1 + \frac{1}{2}Y_2,$$

$$X_2 \to X_2 - \frac{1}{2}Y_1 + \frac{1}{4}Z_1,$$

$$Y_1 \to -Y_1,$$

$$Y_2 \to 2Y_1 + Y_2 + Z_2,$$

$$Z_1 \to Z_1 + \frac{1}{2}W,$$

$$Z_2 \to -Z_1 - Z_2 + \frac{1}{4}W,$$

$$W \to -W.$$

A straightforward calculation shows that $\Phi_*([U, V]) = [\Phi_*(U), \Phi_*(V)]$ for all U, V in $\mathfrak{g}$. Thus Φ is indeed a Lie group automorphism.

Let Γ_1 be the cocompact, discrete subgroup of G generated by

$$\{\exp(2X_1), \exp(2X_2), \exp(Y_1), \exp(Y_2), \exp(Z_1), \exp(Z_2), \exp(W)\},$$

and let $\Gamma_2 = \Phi(\Gamma_1)$. Note that $\Gamma_1 \cap Z(G) = \Gamma_2 \cap Z(G) = \{\exp(jW) : j \in \mathbf{Z}\}$.

Let $\bar{\Phi}$ be the projection of Φ onto $\bar{G}$. Then $\bar{\Phi}$ factors as $\bar{\Phi} = \Psi_1 \circ \Psi_2$ where Ψ_1 is the automorphism of $\bar{G}$ given on the Lie algebra level by

$$\bar{X}_1 \to -\bar{X}_1 + \bar{X}_2 + \frac{1}{4}\bar{Y}_1 + \frac{1}{2}\bar{Y}_2,$$

$$\bar{X}_2 \to \bar{X}_2 - \frac{1}{2}\bar{Y}_1,$$

$$\bar{Y}_1 \to -\bar{Y}_1,$$

$$\bar{Y}_2 \to 2\bar{Y}_1 + \bar{Y}_2,$$

$$\bar{Z}_1 \to \bar{Z}_1,$$

$$\bar{Z}_2 \to -\bar{Z}_1 - \bar{Z}_2,$$

and Ψ_2 is the automorphism of $\bar{G}$ given on the Lie algebra level by

$$\bar{X}_1 \to \bar{X}_1,$$

$$\bar{X}_2 \to \bar{X}_2 + \frac{1}{4}\bar{Z}_1,$$

$$\bar{Y}_1 \to \bar{Y}_1,$$

$$\bar{Y}_2 \to \bar{Y}_2 - \bar{Z}_1 - \bar{Z}_2,$$

$$\bar{Z}_1 \to \bar{Z}_1,$$

$$\bar{Z}_2 \to \bar{Z}_2.$$

By rewriting Ψ_1 in terms of the orthonormal basis $\{\bar{E}_1, \bar{E}_2, \bar{E}_3, \bar{E}_4, \bar{E}_5, \bar{E}_6\}$ of $\bar{g}$, one easily sees that $\Psi_1(\bar{E}_i) = \pm\bar{E}_i$ for $i = 1, \ldots, 6$. Thus the automorphism Ψ_1 is also an isometry of $\bar{\Gamma}$. A simple calculation shows that Ψ_2 is an almost inner automorphism of $\bar{G}$. Thus by (3.2.1), $\bar{\Phi}$ marks the length spectrum between $(\bar{\Gamma}_1\backslash\bar{G}, \bar{g})$ and $(\bar{\Gamma}_2\backslash\bar{G}, \bar{g})$. By (3.3.1), Φ marks the length spectrum between $(\Gamma_1\backslash G, g)$ and $(\Gamma_2\backslash G, g)$.

By (3.2.2), the manifolds $(\Gamma_1\backslash G, g)$ and $(\Gamma_2\backslash G, g)$ must be isospectral on functions.

In contrast, we have the following.

4.1 THEOREM [Gt3, Proposition 4.11]. *The manifolds* $(\Gamma_1\backslash G, g)$ *and* $(\Gamma_2\backslash G, g)$ *are not isospectral on one-forms.*

Section 5: The (marked) length spectrum and previous examples

We now compare the length spectra and marked length spectra of Example I through Example IV described in Table I. The spectrum on functions, spectrum on one-forms, quasi-regular representations, and fundamental groups of these examples were studied in [Gt3].

We use the notation of Section 3. All of these examples can be constructed by Theorem 3.2.3, in particular, $\Gamma_1 \cap Z(G) = \Gamma_2 \cap Z(G)$.

Let the length $\lambda \in [L]$-spec$(\Gamma_i \backslash G, g)$. Let $m_i(\lambda)$ denote the multiplicity of the length λ in $[L]$-spec$(\Gamma_i \backslash G, g)$. We decompose $m_i(\lambda)$ as

$$m_i(\lambda) = m_i'(\lambda) + m_i''(\lambda) \tag{5.1}$$

where $m_i''(\lambda)$ is the number of central free homotopy classes in which λ occurs, and $m_i'(\lambda)$ is the number of noncentral free homotopy classes in which λ occurs.

5.2 PROPOSITION. *For pairs of isospectral manifolds constructed using Theorem* 3.2.3, *the central multiplicities are equal*; *that is,* $m_1''(\lambda) = m_2''(\lambda)$.

Proof of Proposition 5.2. If $\gamma \in \Gamma_1 \cap Z(G) = \Gamma_2 \cap Z(G)$, then by (2.1.1), the length $\lambda \in [\gamma]_{\Gamma_1}$ if and only if $\lambda \in [\gamma]_{\Gamma_2}$. As the conjugacy classes of γ in Γ_1 and Γ_2 respectively contain only the element γ, we have a natural correspondence between the central conjugacy classes in Γ_1 containing a closed geodesic of length λ and the central conjugacy classes in Γ_2 containing a closed geodesic of length λ. □

So for the examples below, we need only compare $m_1'(\lambda)$ and $m_2'(\lambda)$.

Example I: Remarks

Let

$$\mathfrak{g} = \mathrm{span}_{\mathbf{R}}\{X_1, X_2, Y_1, Y_2, Z_1, Z_2, W\}$$

with Lie brackets

$$[X_1, Y_1] = [X_2, Y_2] = Z_1$$
$$[X_1, Y_2] = Z_2$$
$$[X_1, Z_1] = [X_2, Z_2] = [Y_1, Y_2] = W$$

and all other basis brackets zero.

Clearly $\mathfrak{g}$ is a strictly nonsingular, three-step nilpotent Lie algebra.

Let Γ_1 be the cocompact, discrete subgroup of G generated by

$$\{\exp(2X_1), \exp(2X_2), \exp(Y_1), \exp(Y_2), \exp(Z_1), \exp(Z_2), \exp(W)\},$$

and let Γ_2 be the cocompact, discrete subgroup of G generated by

$$\left\{\exp(2X_1), \exp(2X_2), \exp(Y_1), \exp\left(Y_2 + \frac{1}{2}Z_2\right), \exp(Z_1), \exp(Z_2), \exp(W)\right\}.$$

The fundamental groups and the quasi-regular representations of Example I are studied extensively in [Gt1]. There we showed that Γ_1 and Γ_2 are not abstractly isomorphic, hence $(\Gamma_1\backslash G, g)$ and $(\Gamma_2\backslash G, g)$ cannot possibly have the same marked length spectrum for any choice of left invariant metric.

Let g be the left invariant metric on G defined by letting

$$\{X_1, X_2, Y_1, Y_2, Z_1, Z_2, W\}$$

be an orthonormal basis of $\mathfrak{g}$.

In [Gt2], we showed that $(\Gamma_1\backslash G, g)$ and $(\Gamma_2\backslash G, g)$ do not even have the same length spectrum. Although the same lengths of closed geodesics occur, the multiplicities of certain lengths differ.

Example I provided the first example of a pair of representation equivalent subgroups of a solvable Lie group producing manifolds with unequal length spectra. Note that nilpotent Lie groups are necessarily solvable.

Example II: The (marked) length spectrum

Let

$$\mathfrak{g} = \mathrm{span}_{\mathbf{R}}\{X_1, Y_1, Y_2, Z, W\}$$

with Lie brackets given by

$$[X_1, Y_1] = Z$$
$$[X_1, Z] = [Y_1, Y_2] = W$$

and all other basis brackets zero.

Clearly $\mathfrak{g}$ is a strictly nonsingular, three-step nilpotent Lie algebra.

Let Γ_1 be the cocompact, discrete subgroup of G generated by

$$\{\exp(2X_1), \exp(Y_1), \exp(Y_2), \exp(Z), \exp(W)\}$$

and let Γ_2 be the cocompact, discrete subgroup of G generated by

$$\left\{\exp(2X_1), \exp\left(Y_1 + \frac{1}{2}Z\right), \exp(Y_2), \exp(Z), \exp(W)\right\}.$$

Note that these generating sets are canonical in the sense that every element of Γ_1 can be written in the form $\exp(2n_1 X_1)\exp(m_1 Y_1)\exp(m_2 Y_2)\exp(kZ)\exp(jW)$ for some integers n_1, m_1, m_2, k, j. Likewise for Γ_2.

5.3 PROPOSITION. *The above nilmanifolds have the same length spectrum, that is*

$$[L]\text{-}spec(\Gamma_1\backslash G, g) = [L]\text{-}spec(\Gamma_2\backslash G, g)$$

for any choice of left invariant metric g of G.

We showed in [Gt3] that Γ_1 and Γ_2 are isomorphic as groups. Thus a natural question to ask is, if a pair of nilmanifolds have the same length spectrum and have isomorphic fundamental groups, must they necessarily have the same marked length spectrum? In [Gt2] we exhibited examples of two-step nilmanifolds that answer this question in the negative. Example II is a higher-step example with similar properties.

5.4 PROPOSITION. *The manifolds $(\Gamma_1\backslash G, g)$ and $(\Gamma_2\backslash G, g)$ do not have the same marked length spectrum for any choice of left invariant metric g on G.*

Proof of Proposition 5.4. Let g be any left invariant metric on G, and assume $\Psi: \Gamma_1 \to \Gamma_2$ marks the length spectrum between $(\Gamma_1\backslash G, g)$ and $(\Gamma_2\backslash G, g)$. Extend Ψ to the Lie group isomorphism $\Psi: G \to G$ such that $\Psi(\Gamma_1) = \Gamma_2$.

We showed in [Gt3], Proposition 4.6, that any isomorphism $\Psi: \Gamma_1 \to \Gamma_2$ must be given at the Lie algebra level by:

$$\Psi_*(W) = \pm W,$$
$$\Psi_*(Z) = \pm Z + h_0 W$$
$$\Psi_*(Y_2) = \pm Y_2 \bmod \mathfrak{g}^{(1)}$$
$$\Psi_*(Y_1) = \pm\left(Y_1 + \frac{1}{2}Z\right) + h_1 Y_2 + h_2 Z \bmod \mathfrak{g}^{(2)}$$
$$\Psi_*(X_1) = \pm X_1 + \frac{1}{2}h_3 Y_1 + \frac{1}{2}h_4 Y_2 \bmod \mathfrak{g}^{(1)}$$

where h_0, h_1, h_2, h_3, and h_4 are integers and $h_3^2 + h_4^2 \neq 0$.

By Corollary 3.2.5 and Theorem 3.2.1, $\bar{\Psi} = \Phi_1 \circ \Phi_2$, where $\Phi_1 : \bar{G} \to \bar{G}$ is an isomorphism that is also an isometry of $(\bar{G}, \bar{g})$, and $\Phi_2 \in \bar{\Gamma}_1$-AIA$(\bar{G})$. As $\bar{Y}_1$ and $\bar{Y}_2$ are not in $[\bar{X}_1, \bar{\mathfrak{g}}]$, we must have

$$\Phi_{1*}(\bar{X}_1) = \pm \bar{X}_1 + \tfrac{1}{2}h_3 \bar{Y}_1 + \tfrac{1}{2}h_4 \bar{Y}_2 + z_1 \bar{Z},$$

$$\Phi_{1*}(\bar{Y}_1) = \pm \bar{Y}_1 + h_1 \bar{Y}_2 + z_2 \bar{Z},$$

$$\Phi_{1*}(\bar{Y}_2) = \pm \bar{Y}_2 + z_3 \bar{Z},$$

$$\Phi_{1*}(\bar{Z}) = \pm \bar{Z},$$

for some $z_1, z_2, z_3 \in \mathbf{R}$.

Now Φ_1 an isometry implies that for all $\bar{U}$, $\bar{V}$ in $\bar{\mathfrak{g}}$,

$$\langle \bar{U}, \bar{V} \rangle = \langle \Phi_{1*}(\bar{U}), \Phi_{1*}(\bar{V}) \rangle. \qquad (*)$$

Setting $\bar{U} = \bar{Z}$ and $\bar{V} = \bar{Y}_2$ in $(*)$, we see that $z_3 = 0$. Setting $\bar{U} = \bar{Y}_1$ in $(*)$ and letting $\bar{V} = \bar{Y}_2$, and then $\bar{V} = \bar{Z}$, we obtain $h_1 = z_2 = 0$. Finally, by setting $\bar{U} = \bar{X}_1$ in $(*)$ and letting $\bar{V} = \bar{Z}$, then $\bar{V} = \bar{Y}_2$, and then $\bar{V} = \bar{Y}_1$, we see that $z_1 = h_3 = h_4 = 0$, which contradicts $h_3^2 + h_4^2 \neq 0$. □

Before proving Proposition 5.3, we need the following.

5.5. PROPOSITION (see [Gt3, Proposition 2.1]). *Let Γ_1 and Γ_2 be cocompact, discrete subgroups of the Lie group G with left invariant metric g. If for each x in G we have*

$$\#\{[\gamma]_{\Gamma_1} \subset [x]_G\} = \#\{[\gamma]_{\gamma_2} \subset [x]_G\},$$

then

$$[L]\text{-}spec(\Gamma_1 \backslash G, g) = [L]\text{-}spec(\Gamma_2 \backslash G, g).$$

Here $\#\{[\gamma]_{\Gamma_i} \subset [x]_G\}$ denotes the number of distinct conjugacy classes in Γ_i contained in the conjugacy class of x in G.

Proof of Proposition 5.3. Let $x \in G$. We count the number of distinct conjugacy classes in Γ_1 and Γ_2 contained in $[x]_G$.

Let $\gamma_1 = \exp(2n_1 X_1)\exp(m_1 Y_1)\exp(m_2 Y_2)\exp(kZ)\exp(jW) \in \Gamma_1$ for n_1, m_1, m_2, $k \in \mathbf{Z}$. Define the mapping $F: \Gamma_1 \to \Gamma_2$ by

$$F(\gamma_1) = \exp(2n_1 X_1)\exp\left(m_1\left(Y_1 + \frac{1}{2}Z\right)\right)\exp(m_2 Y_2)\exp(kZ)\exp(jW).$$

The mapping F gives us a correspondence between the elements of Γ_1 and the elements of Γ_2. Note that F is *not* a Lie group isomorphism.

Now γ_1 and $F(\gamma_1) = \gamma_2$ are conjugate in G. In particular, $F(\gamma_1) = a\gamma_1 a^{-1}$ where $a = e$ if $m_1 = 0$, and $a = \exp(\frac{1}{2}X_1)\exp((\frac{1}{8} + (k/2m_1))Y_2)$ if $m_1 \neq 0$. Thus $[\gamma_1]_{\Gamma_1} \subset [x]_G$ if and only if $[F(\gamma_1)]_{\Gamma_2} \subset [x]_G$.

To use Proposition 5.5, we must now compare the number of distinct conjugacy classes in Γ_1 and Γ_2 respectively that are contained in a fixed $[x]_G$.

Using the Campbell-Baker-Hausdorff formula, two elements

$$\gamma_1 = \exp(2n_1 X_1)\exp(m_1 Y_1)\exp(m_2 Y_2)\exp(kZ)\exp(jW)$$

and

$$\gamma_1' = \exp(2n_1' X_1)\exp(m_1' Y_1)\exp(m_2' Y_2)\exp(k'Z)\exp(j'W),$$

of Γ_1 are conjugate in Γ_1 if and only if there exist integers $\bar{n}_1, \bar{m}_1, \bar{m}_2, \bar{k}$ such that

$$n_1' = n_1, \qquad m_1' = m_1, \qquad m_2' = m_2, \qquad k' = k + 2m_1\bar{n}_1 - 2n_1\bar{m}_1,$$
$$j' = j + m_2\bar{m}_1 - m_1\bar{m}_2 + 2k\bar{n}_1 - 2n_1\bar{k} + 2m_1\bar{n}_1^2 - 4n_1\bar{n}_1\bar{m}_1 + 2n_1^2\bar{m}_1.$$

Let $K = gcd(2n_1, 2m_1)$. From the above, we see that every conjugacy class in Γ_1 contains at least one representative such that $k \in \{1, 2, \ldots, K\}$. We call such a representative *nice*. Two nice representatives are in the same conjugacy class in Γ_1 if and only if $k = k'$ and there exist integers $\bar{n}_1, \bar{m}_1, \bar{m}_2, \bar{k}$ such that $m_1\bar{n}_1 - n_1\bar{m}_1 = 0$ and

$$j' = j + m_2\bar{m}_1 - m_1\bar{m}_2 + 2k\bar{n}_1 - 2n_1\bar{k} + 2m_1\bar{n}_1^2 - 4n_1\bar{n}_1\bar{m}_1 + 2n_1^2\bar{m}_1.$$

Similarly, two elements of Γ_2

$$\gamma_2 = \exp(2n_1 X_1)\exp\left(m_1\left(Y_1 + \frac{1}{2}Z\right)\right)\exp(m_2 Y_2)\exp(kZ)\exp(jW),$$

and

$$\gamma_2' = \exp(2n_1' X_1)\exp\left(m_1'\left(Y_1 + \frac{1}{2}Z\right)\right)\exp(m_2' Y_2)\exp(k'Z)\exp(j'W),$$

are conjugate in Γ_2 if and only if there exists integers $\bar{n}_1, \bar{m}_1, \bar{m}_2, \bar{k}$ so that

$$n_1' = n_1, \qquad m_1' = m_1, \qquad m_2' = m_2, \qquad k' = k + 2m_1\bar{n}_1 - 2n_1\bar{m}_1,$$
$$j' = j + (m_1\bar{n}_1 - n_1\bar{m}_1) + m_2\bar{m}_1 - m_1\bar{m}_2 + 2k\bar{n}_1 - 2n_1\bar{k}$$
$$+ 2m_1\bar{n}_1^2 - 4n_1\bar{n}_1\bar{m}_1 + 2n_1^2\bar{m}_1.$$

Again we see that every conjugacy class in Γ_2 contains at least one nice representative, that is, a representative such that $k \in \{1, 2, \ldots, K\}$, where $K = gcd(2n_1, 2m_1)$ as above. Again, two nice representatives are in the same conjugacy class in Γ_2 if and only if $k = k'$ and there exist integers $\bar{n}_1, \bar{m}_1, \bar{m}_2, \bar{k}$ such that $m_1\bar{n}_1 - n_1\bar{m}_1 = 0$ and

$$j' = j + m_2\bar{m}_1 - m_1\bar{m}_2 + 2k\bar{n}_1 - 2n_1\bar{k} + 2m_1\bar{n}_1^2 - 4n_1\bar{n}_1\bar{m}_1 + 2n_1^2\bar{m}_1.$$

Note that the correspondence $F\colon \Gamma_1 \to \Gamma_2$ sends nice representatives to nice representatives. Thus if we restrict ourselves in nice representatives, the conjugacy conditions are equivalent. That is, two nice representatives γ_1 and γ_1' are in the same conjugacy class in Γ_1 if and only if the corresponding elements $F(\gamma_1)$ and $F(\gamma_2)$ are in the same conjugacy class in Γ_2.

Let $\gamma_1, \gamma_2, \ldots, \gamma_L$ be nice representatives of the L distinct conjugacy classes in Γ_1 contained in $[x]_G$. Then $F(\gamma_1)$, $F(\gamma_2), \ldots, F(\gamma_L)$ are nice representatives of L distinct conjugacy classes in Γ_2. The same applies to $F^{-1}\colon \Gamma_2 \to \Gamma_1$.

Thus

$$\#\{[\gamma]_{\Gamma_1} \subset [x]_G\} = \#\{[\gamma]_{\Gamma_2} \subset [x]_G\},$$

as desired. □

Example III: *The length spectrum*

Let

$$\mathfrak{g} = \mathrm{span}_{\mathbf{R}}\{X_1, X_2, Y_1, Y_2, Z_1, Z_2, W\}$$

with Lie brackets

$$[X_1, Y_1] = [X_2, Y_2] = Z_1$$

$$[X_1, Y_2] = Z_2$$

$$[X_1, Z_1] = [X_2, Z_2] = [Y_1, Y_2] = W$$

and all other basis brackets zero.

Clearly $\mathfrak{g}$ is a strictly nonsingular, three-step nilpotent Lie algebra.

Let Γ_1 be the cocompact, discrete subgroup of G generated canonically by

$$\{\exp(2X_1), \exp(2X_2), \exp(Y_1), \exp(Y_2), \exp(Z_1), \exp(Z_2), \exp(W)\},$$

and let Γ_2 be the cocompact, discrete subgroup of G generated canonically by

$$\{\exp(X_1), \exp(X_2), \exp(2Y_1), \exp(2Y_2), \exp(Z_1), \exp(Z_2), \exp(W)\}.$$

Let g be the left invariant metric on G defined by letting

$$\{X_1, X_2, Y_1, Y_2, Z_1, Z_2, W\}$$

be an orthonormal basis of $\mathfrak{g}$.

5.6 PROPOSITION. *The nilmanifolds $(\Gamma_1\backslash G, g)$ and $(\Gamma_2\backslash G, g)$ do not have the same length spectrum. In particular, the multiplicity of the length* 1 *in* $[L]$-*spec*$(\Gamma_1\backslash G, g)$ *is greater than its multiplicity in* $[L]$-*spec*$(\Gamma_2\backslash G, g)$.

Proof of Proposition 5.6. By Proposition 5.2 we need only consider the noncentral free homotopy classes. That is, we need only show $m_1'(1) > m_2'(1)$.

Let $\gamma =$

$$\exp(A_1 n_1 X_1)\exp(A_2 n_2 X_2)\exp(B_1 m_1 Y_1)\exp(B_2 m_2 Y_2)\exp(k_1 Z_1)\exp(k_2 Z_2)\exp(jW)$$

for integers $n_1, n_2, m_1, m_2, k_1, k_2, j$ and $A_1, A_2, B_1, B_2 \in \{1, 2\}$. Note that $\gamma \in \Gamma_1$ if and only if

$$A_1 = A_2 = 2, \qquad B_1 = B_2 = 1 \qquad (*)$$

and if $\gamma \in \Gamma_2$ if and only if

$$A_1 = A_2 = 1, \qquad B_1 = B_2 = 2. \tag{**}$$

By Theorem 3.2.4, to determine if $1 \in [\gamma]_{\Gamma_i}$ for noncentral $\gamma \in \Gamma_i$, we need only determine if $1 \in [\bar{\gamma}]_{\bar{\Gamma}_i}$. That is, rather than looking at the lengths of closed geodesics on the three-step nilmanifolds $(\Gamma_i \backslash G, g)$, we instead look at the lengths of closed geodesics on the quotient two-step nilmanifolds $(\bar{\Gamma}_i \backslash \bar{G}, \bar{g})$ for $i = 1, 2$.

The Lie algebra of $\bar{G} = \bar{\mathfrak{g}} = \mathfrak{g}/\mathfrak{g}^{(2)} = \operatorname{span}_{\mathbf{R}}\{\bar{X}_1, \bar{X}_2, \bar{Y}_1, \bar{Y}_2, \bar{Z}_1, \bar{Z}_2\}$ with Lie brackets

$$[\bar{X}_1, \bar{Y}_1] = [\bar{X}_2, \bar{Y}_2] = \bar{Z}_1$$
$$[\bar{X}_1, \bar{Y}_2] = \bar{Z}_2,$$

and all other basis brackets zero.

We may now use the following result due to Eberlein.

5.7 THEOREM [E1]. *Let N be a simply connected, two-step nilpotent Lie group with Lie algebra $\mathfrak{n}$ and left invariant metric g. Let Γ be a cocompact, discrete subgroup of N. Let $\mathfrak{z}$ be the center of $\mathfrak{n}$ and $\mathfrak{v}$ the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{n}$. Any element $\gamma \in \Gamma$ may be expressed uniquely as* $\exp(V^* + Z^*)$ *where $V^* \in \mathfrak{v}$ and $Z^* \in \mathfrak{z}$. Let Z^{**} be the component of Z^* orthogonal to $[V^*, \mathfrak{n}]$. Let $\lambda > 0$.*

(1) *If the length $\lambda \in [\gamma]_\Gamma$, then $[V^*] \leq \lambda \leq \sqrt{|V^*|^2 + |Z^{**}|^2}$.*

(2) *The length $\lambda = |V^*| \in [\gamma]_\Gamma$ if and only if $|Z^{**}| = 0$.*

(3) *The length $\lambda = \sqrt{|V^*|^2 + |Z^{**}|^2} \in [\gamma]_\Gamma$.*

Here $\mathfrak{n} = \bar{\mathfrak{g}}$ and the metric $\bar{g}$ is determined by the orthonormal basis of $\bar{\mathfrak{g}}$

$$\{\bar{X}_1, \bar{X}_2, \bar{Y}_1, \bar{Y}_2, \bar{Z}_1, \bar{Z}_2\}.$$

By Theorem 5.7, to find $\bar{\gamma}$ such that $1 \in [\bar{\gamma}]_{\bar{\Gamma}_i}$, we need $\bar{\gamma} = \exp(\bar{V}^* + \bar{Z}^*)$ such that $|\bar{V}^*|^2 \leq 1 \leq |\bar{V}^*|^2 + |\bar{Z}^{**}|^2$, where $\bar{V}^* \in \operatorname{span}_{\mathbf{R}}\{\bar{X}_1, \bar{X}_2, \bar{Y}_1, \bar{Y}_2\}$ and $\bar{Z}^* \in \operatorname{span}_{\mathbf{R}}\{\bar{Z}_1, \bar{Z}_2\}$.

For both Γ_1 and Γ_2, $\bar{V}^* = A_1 n_1 \bar{X}_1 + A_2 n_2 \bar{X}_2 + B_1 m_1 \bar{Y}_1 + B_2 m_2 \bar{Y}_2$, where $n_1, n_2, m_1, m_2 \in \mathbf{Z}$. Note that if $|\bar{V}^*| \neq 0$, $|\bar{V}^*|^2 = A_1^2 n_1^2 + A_2^2 n_2^2 + B_1^2 m_1^2 + B_2^2 m_2^2 \geq 1$. So $|\bar{V}^*|^2 \leq 1$ if and only if $|\bar{V}^*|^2 = 1$. By Theorem 5.4, $\lambda = 1 = |\bar{V}^*| \in [\bar{\gamma}]_{\bar{\Gamma}_i}$ if and only if $|\bar{Z}^{**}| = 0$.

So if $\bar{\gamma} = \exp(\bar{V}^* + \bar{Z}^*)$ with $|\bar{V}^*| \neq 0$, then $1 \in [\bar{\gamma}]_{\bar{\Gamma}_i}$ if and only if $|\bar{V}^*| = 1$ and $|\bar{Z}^{**}| = 0$.

We consider two cases.

Case 1. $(n_1)^2+(m_2)^2\neq 0$.

In this case, $\bar{\mathfrak{z}}=[\log\bar{\gamma},\bar{\mathfrak{g}}]$, so $\bar{Z}^{**}$ is automatically zero. Applying the condition $|\bar{V}^*|=1$ and lifting to the three-step level, we have $1\in[\gamma_1]_{\Gamma_1}$ if and only if (see (*))

$$\gamma_1=\exp(\pm Y_2)\exp(k_1Z_1)\exp(k_2Z_2)\exp(jW),$$

and $1\in[\gamma_2]_{\Gamma_2}$ if and only if (see (**))

$$\gamma_2=\exp(\pm X_1)\exp(k_1Z_1)\exp(k_2Z_2)\exp(jW).$$

We must now compare the number of distinct free homotopy classes of Γ_1 and Γ_2 that take on one of these forms.

Another element $\gamma_1'=\exp(\pm Y_2)\exp(k_1'Z_1)\exp(k_2'Z_2)\exp(j'W)$ of Γ_1 is conjugate to γ_1 in Γ_1 if and only if there exist integers $\bar{n}_1$, $\bar{n}_2$, $\bar{m}_1$ and $\bar{k}_1$ such that

$$k_1'=k_1\pm 2\bar{n}_2;\qquad k_2'=k_2\pm 2\bar{n}_1;\qquad j'=j\pm\bar{m}_1+2k_1\bar{n}_1+2k_2\bar{n}_2\pm 4\bar{n}_1\bar{n}_2.$$

Another element $\gamma_2'=\exp(\pm X_1)\exp(k_1'Z_1)\exp(k_2'Z_2)\exp(j'W)$ of Γ_2 is conjugate to γ_2 in Γ_2 if and only if there exist integers $\bar{n}_1,\bar{n}_2,\bar{m}_1$, and $\bar{m}_2$ such that

$$k_1'=k_1\mp 2\bar{m}_1;\qquad k_2'=k_2\mp 2\bar{m}_2;$$

$$j'=j\mp\bar{k}_1+\bar{m}_1+k_1\bar{n}_1+k_2\bar{n}_2\mp 2\bar{m}_1\bar{n}_1\mp 2\bar{m}_2\bar{n}_2.$$

For Γ_1 we have two choices $\{-1,+1\}$ for the coefficient of Y_2, two choices for k_1, two choices for k_2, and one choice for j for a total of 8 distinct free homotopy classes. For Γ_2 we have two choices $\{-1,+1\}$ for the coefficient of X_1, two choices for k_1, two choices for k_2 and one choice for j for a total of 8 distinct free homotopy classes. Thus, the multiplicities of 1 coming from this case are equal.

Case 2. $n_1^2+m_2^2=0$ but $n_2^2+m_1^2\neq 0$.

In this case, $[\log\bar{\gamma},\bar{\mathfrak{g}}]=\mathrm{span}_{\mathbf{R}}\{\bar{Z}_1\}$, so $\bar{Z}^{**}=0$ if and only if $k_2=0$. Applying the condition $|\bar{V}^*|=1$ and lifting to the three-step level, we have the length $1\in[\gamma_1]_{\Gamma_1}$

if and only if (see (*))

$$\gamma_1 = \exp(\pm Y_1)\exp(k_1 Z_1)\exp(jW),$$

and the length $1 \in [\gamma_2]_{\Gamma_2}$ if and only if (see (**))

$$\gamma_2 = \exp(\pm X_2)\exp(k_1 Z_1)\exp(jW).$$

We must now count the number of distinct free homotopy classes of Γ_1 and Γ_2 that take on one of these forms.

Another element $\gamma_1' = \exp(\pm Y_1)\exp(k_1' Z_1)\exp(j' W)$ of Γ_1 is conjugate to γ_1 in Γ_1 if and only if there exist integers $\bar{n}_1$, $\bar{m}_2$ such that

$$k_1' = k_1 \pm 2\bar{n}_1; \qquad j' = j \mp \bar{m}_2 + 2k_1\bar{n}_1 \pm 2\bar{n}_1^2.$$

Another element $\gamma_2' = \exp(\pm X_2)\exp(k_1' Z_1)\exp(j' W)$ in Γ_2 is conjugate to γ_2 in Γ_2 if and only if there exist integers $\bar{n}_1$, $\bar{m}_2$ and $\bar{k}_2$ such that

$$k_1' = k_1 \mp 2\bar{m}_2; \qquad j' = j \mp \bar{k}_2 + k_1\bar{n}_1 \mp 2\bar{n}_1\bar{m}_2.$$

For Γ_1 we have two choices $\{-1, +1\}$ for the coefficient of Y_1, two choices for k_1, and one choice for j for a total of 4 distinct free homotopy classes. For Γ_2 we have two choices $\{-1, +1\}$ for the coefficient of X_2, two choices for k_1, and one choice for j for a total of 4 distinct free homotopy classes. Again the multiplicities of 1 coming from this case are equal.

Case 3. $|\bar{V}^*| = 0$, $|\bar{Z}^*| \neq 0$.

Let $\gamma = \exp(k_1 Z_1)\exp(k_2 Z_2)\exp(jW)$, for $k_1, k_2, j \in \mathbf{Z}$. Note that $\gamma \in \Gamma_1 \cap \Gamma_2$. Thus by (2.1.1), any length occurring in $[\gamma]_{\Gamma_1}$ will also occur in $[\gamma]_{\Gamma_2}$. Let $\gamma' = \exp(k_1' Z_1)\exp(k_2' Z_2)\exp(j' W)$ be another element of $\Gamma_1 \cap \Gamma_2$, where $k_1', k_2', j' \in \mathbf{Z}$.

Now γ' is conjugate to γ in Γ_1 if and only if there exist integers $\bar{n}_1$, $\bar{n}_2$ such that

$$k_1' = k_1; \qquad k_2' = k_2; \qquad j' = j + 2(k_1\bar{n}_1 + k_2\bar{n}_2).$$

However γ' is conjugate to γ in Γ_2 if and only if there exist integers $\bar{n}_1$, $\bar{n}_2$ such that

$$k_1' = k_1; \qquad k_2' = k_2; \qquad j' = j + (k_1\bar{n}_1 + k_2\bar{n}_2).$$

Note that there are twice as many distinct conjugacy classes represented by elements of the form $\gamma = \exp(k_1 Z_1)\exp(k_2 Z_2)\exp(jW)$ for Γ_1 as for Γ_2. Thus, to show the multiplicities are not equal here, we need to exhibit a closed geodesic of length 1 in just one free homotopy class of this form.

Note that $|\bar{Z}^{**}|^2 = |\bar{Z}^*|^2 = k_1^2 + k_2^2$. By Theorem 5.7(3) and lifting to the three-step level, we see $\sqrt{k_1^2+k_2^2} \in [\gamma]_{\Gamma_1}$ and $\sqrt{k_1^2+k_2^2} \in [\gamma]_{\Gamma_2}$. Thus, the length $1 \in [\exp(\pm Z_i)]_{\Gamma_j}$, $i, j = 1, 2$.

Therefore, for Case 3, the length 1 occurs with twice the multiplicity in $[L]$-spec$(\Gamma_1\backslash G, g)$ as it does in $[L]$-spec$(\Gamma_2\backslash G, g)$.

As the multiplicities of the length 1 are equal in all of the other cases, the multiplicities of the length 1 are *not* equal, as claimed. $\square$

Example IV: The length spectrum

Here the Lie algebra is the same Lie algebra as Example II, that is

$$\mathfrak{g} = \operatorname{span}_{\mathbf{R}}\{X_1, Y_1, Y_2, Z, W\}$$

with Lie brackets

$$[X_1, Y_1] = Z$$

$$[X_1, Z] = [Y_1, Y_2] = W$$

and all other basis brackets zero.

Let Γ_1 be the cocompact, discrete subgroup of G generated canonically by

$$\{\exp(2X_1), \exp(Y_1), \exp(Y_2), \exp(Z), \exp(W)\},$$

and let Γ_2 be the cocompact, discrete subgroup of G generated canonically by

$$\{\exp(X_1), \exp(2Y_1), \exp(Y_2), \exp(Z), \exp(W)\}.$$

Let g be the left invariant metric on G defined by letting

$$\{X_1, Y_1, Y_2, Z, W\}$$

be an orthonormal basis of $\mathfrak{g}$.

5.8 PROPOSITION. *The nilmanifolds* $(\Gamma_1\backslash G, g)$ *and* $(\Gamma_2\backslash G, g)$ *do not have the same length spectrum. In particular, the multiplicity of the length* $\lambda = \sqrt{4\pi(7-\pi)}$ *in* $[L]$*-spec*$(\Gamma_1\backslash G, g)$ *is greater than its multiplicity in* $[L]$*-spec*$(\Gamma_2\backslash G, g)$.

Proof of Proposition 5.8. By Proposition 5.2, we only consider the noncentral free homotopy classes. In particular, we show $m_1'(\lambda) > m_2'(\lambda)$ where $\lambda = \sqrt{4\pi(7-\pi)}$.

By Theorem 3.2.4 if we wish to determine if $\lambda \in [\gamma]_{\Gamma_i}$ for noncentral $\gamma \in \Gamma_i$, we need only determine if $\lambda \in [\bar{\gamma}]_{\bar{\Gamma}_i}$. That is, rather than looking at the lengths of closed geodesics on the three-step nilmanifolds $(\Gamma_i\backslash G, g)$, we instead look at the lengths of closed geodesics on the quotient two-step nilmanifolds $(\bar{\Gamma}_i\backslash\bar{G}, \bar{g})$ for $i = 1, 2$.

However, for this example, $\bar{\mathfrak{g}} \cong \mathfrak{h}_1 \oplus \mathbf{R}$ where $\mathfrak{h}_1$ denotes the three-dimensional Heisenberg algebra. To see this, note that

$$\mathfrak{h}_1 \cong \{\bar{X}_1, \bar{Y}_1, \bar{Z}\}, \quad \text{and} \quad [\bar{X}_1, \bar{Y}_1] = \bar{Z}.$$

This is an ideal in $\bar{\mathfrak{g}}$. And

$$\mathbf{R} \cong \{\bar{Y}_2\}$$

which is also an ideal in $\mathfrak{g}$. Let H_1 be the three-dimensional Heisenberg group. Note that

$$H_1 \cong \{\exp(x_1\bar{X}_1)\exp(y_1\bar{Y}_1)\exp(z\bar{Z}) : x_1, y_1, z \in \mathbf{R}\}.$$

This direct sum is actually a *Riemannian* direct sum, as the metric may also be written as

$$\bar{g} = \bar{g}_1 \oplus \bar{g}_2$$

where $\bar{g}_1$ is the left invariant metric on $\mathfrak{h}_1$ given by the orthonormal basis $\{\bar{X}_1, \bar{Y}_1, \bar{Z}\}$ and $\bar{g}_2$ is the left invariant metric on $\mathbf{R}$ given by the unit vector $\{\bar{Y}_2\}$.

Furthermore, as $\bar{\Gamma}_i = (\bar{\Gamma}_i \cap H_1) \oplus (\bar{\Gamma}_i \cap \mathbf{R})$, we also have the Riemannian direct sum

$$(\bar{\Gamma}_i\backslash\bar{G}, \bar{g}) \cong ((\bar{\Gamma}_i \cap H_1)\backslash H_1, \bar{g}_1) \oplus ((\bar{\Gamma}_i \cap \mathbf{R})\backslash\mathbf{R}, \bar{g}_2).$$

Using rescaling of geodesics, it is not difficult to show that the length $\lambda \in [\bar{\gamma}]_{\bar{\Gamma}_i}$ if and only if

$$\lambda^2 = \lambda_1^2 + \lambda_2^2$$

where the length $\lambda_1 \in [\bar{\gamma}_1]_{\bar{\Gamma}_i \cap H_1}$ and the length $\lambda_2 \in [\bar{\gamma}_2]_{\bar{\Gamma}_i \cap \mathbf{R}}$. Here $\bar{\gamma} = (\bar{\gamma}_1, \bar{\gamma}_2)$ with respect to the direct product $\bar{\Gamma}_i = (\bar{\Gamma}_i \cap H_1) \oplus (\bar{\Gamma}_i \cap \mathbf{R})$.

Now, the length spectrum of $((\bar{\Gamma}_i \cap \mathbf{R})\backslash\mathbf{R}, \bar{g}_2)$ is easily seen to be $|\log(\bar{\gamma}_2)|$ for all $\bar{\gamma}_2 \in \bar{\Gamma}_i \cap \mathbf{R}$. Thus the length spectrum here (not counting multiplicities) is precisely the positive integers.

The length spectrum of $((\bar{\Gamma}_i \cap H_1)\backslash H_1, \bar{g}_1)$ has been calculated by both Gordon and Eberlein (see [E], [G1]) and is known to be

(i) $|\log(\bar{\gamma}_1)|$ if $\bar{\gamma}_1 \in \bar{\Gamma}_i \cap H_1$, for $\bar{\gamma}_1 \notin Z(H_1)$.

(ii) $\{|\log(\bar{\gamma}_1)|, \sqrt{(4\pi k)(|\log(\bar{\gamma}_1)| - \pi k)}\colon 1 \le k < ((1/2\pi)|\log(\bar{\gamma}_1)|), k \in \mathbf{Z}\}$, for $\bar{\gamma}_2 \in \bar{\Gamma}_i \cap Z(H_1)$.

Nonintegral lengths occur in $(\bar{\Gamma}_i \cap H_1)\backslash H_1$ only when $|\log(\bar{\gamma}_1)| \ge 2\pi > 6$.

Also note that $\sqrt{4\pi(7-\pi)} \in [\bar{\gamma}_1]_{\bar{\Gamma}_i \cap H_1}$ if and only if $\bar{\gamma}_1 = \exp(\pm 7\bar{Z}) \in \bar{\Gamma}_i \cap H_1$. This is the smallest possible nonintegral length.

Thus

$$4\pi(7-\pi) = \lambda^2 = \lambda_1^2 + \lambda_2^2$$

if and only if $\lambda_2^2 = 0$ and $\lambda_1^2 = 4\pi(7-\pi)$ if and only if $\bar{\gamma} = \exp(\pm 7\bar{Z}) \in \bar{\Gamma}_i$.

By lifting to $(\Gamma_i\backslash G, g)$, we see $\sqrt{4\pi(7-\pi)} \in [\gamma]_{\Gamma_i}$ if and only if

$$\gamma = \exp(\pm 7Z)\exp(jW) \in \Gamma_i.$$

We now count the number of distinct free homotopy classes represented by a γ of this form.

Let $\gamma' = \exp(\pm 7Z)\exp(j'W)$.

Now γ' is conjugate to γ in Γ_1 if and only if there exists integer $\bar{n}_1$ such that

$$j' = j \pm 14\bar{n}_1.$$

However, γ' is conjugate to γ in Γ_2 if and only if there exists integer $\bar{n}_1$ such that

$$j' = j \pm 7\bar{n}_1.$$

Thus there are 14 choices for j in Γ_1 and there are 7 choices for j in Γ_2. So the multiplicity of the length $\sqrt{4\pi(7-\pi)}$ in $(\Gamma_1\backslash G, g)$ is 28, (14 for each of $\exp(+7Z)\exp(jW)$ and $\exp(-7Z)\exp(jW)$), and likewise the multiplicity in $(\Gamma_2\backslash G, g)$ is 14.

Thus the multiplicities of $\sqrt{4\pi(7-\pi)}$ are *not* equal here, as claimed. □

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