Triangle subgroups of Kleinian groups.

Autor(en): **Martin, Gaven J.**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **71 (1996)**

PDF erstellt am: **01.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-53847

Nutzungsbedingungen
Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.
Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.
Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss
Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Ein Dienst der *ETH-Bibliothek*
ETH Zürich, Rämistrasse 101, 8092 Zürich, Schweiz, www.library.ethz.ch

http://www.e-periodica.ch

Comment. Math. Helvetici 71 (1996) 339–361
0010–2571/96/030339–23\$1.50 + 0.20/0
© 1996 Birkhäuser Verlag, Basel

Triangle subgroups of Kleinian groups

GAVEN J. MARTIN*

(for J. A. Kalman on the occasion of his 65th birthday)

Abstract. We exhibit an interesting new phenomenon concerning certain triangle subgroups Δ of Kleinian groups Γ. Namely the hyperbolic plane Π stabilized by Δ has a precisely invariant tubular neighbourhood. Thus the corresponding 2-orbifold $F^2 = \Pi/\Gamma_\Pi$ is always embedded in the hyperbolic 3-orbifold $\mathbf{M}^3 = \mathbb{H}^3/\Gamma$. We deduce that any two such triangle groups can algebraically intersect only in a finite cyclic subgroup.

We give sharp estimates for the radius of these tubular neighbourhoods and present applications concerning the estimation of co-volumes of Kleinian groups containing these triangle subgroups.

1. Introduction

In this paper we exhibit a seemingly new phenomenon concerning certain triangle subgroups of Kleinian groups. Our results as presented here only apply to the $(2, 3, p)$-triangle groups, $\Delta(2, 3, p)$. We are aware that our results hold in more generality for certain other infinite families of triangle groups and the methods of proof we use here apply in those situations too. However the necessary proofs seem to require a case by case analysis which would considerably lengthen this paper. We are therefore content to present the results in this special case in the hope that some general geometric explanation might later be found which identifies all triangle groups with the properties we describe.

There are many reasons for studying Kleinian groups containing triangle subgroups. First, such Kleinian groups are extremal for many geometric problems. For instance a Kleinian group which contains elements of finite order $p \geq 7$ whose fixed point sets are as close as possible always contain a $(2, 3, p)$-triangle subgroup [5]. As another example, the smallest limit volume hyperbolic 3-orbifold, see [1], can be obtained as the limit $(p \to \infty)$ of a sequence of orbifolds which are the orbit

* Research supported in part by grants from the Australian Research Council, the New Zealand Foundation for Research Science and Technology and the U.K. Scientific and Engineering Research Council.

spaces of Kleinian groups containing $(2, 3, p)$-triangle groups [3]. The limit Kleinian group contains a $(2, 3, \infty)$-triangle group.

Secondly, the torsion free finite index subgroups of such a Kleinian group contain surface groups. If the hyperbolic plane stabilized by a triangle subgroup has a precisely invariant neighbourhood of given radius, then the torsion free subgroups yield quotients which are hyperbolic 3-manifolds with geodesically embedded surfaces having an embedded tubular neighbourhood of at least the same radius. (It is relevant to note here that the $(2, 3, p)$-triangle groups contain among them all surface groups). Often the solutions to various extremal problems for hyperbolic 3-manifolds concerning geodesically embedded surfaces are realized as the orbit spaces of torsion free subgroups of Kleinian groups with triangle subgroups. For instance the smallest volume hyperbolic 3-manifold containing a geodesically embedded surface is the orbit space of a subgroup of a Kleinian group containing a $(2, 3, 12)$-triangle group [3]. These examples are partly described in §4 and other related examples of higher genus surfaces are described in [3] and [13].

And finally, rigid Kleinian groups (those admitting no nontrivial deformations) with nonempty domains of discontinuity on the Riemann Sphere have triangle subgroups as component stabilizers. For instance certain extremal two generator web groups with simple branch set, which are in a sense close to arithmetic groups, contain triangle groups with the properties we describe [10].

Before stating our main results let us recall some basic definitions and notation. We shall use the books of Beardon [2] and Maskit [12] as standard references for facts concerning Kleinian groups.

The complex plane is denoted by $\mathbb{C}$ and its one point compactification is the Riemann Sphere $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We let Möb(2) denote the group of all Möbius transformations of $\bar{\mathbb{C}}$. Each $f \in$ Möb(2) is of the form

$$f(z) = \frac{az + b}{cz + d}, \qquad ad - bc = 1, \tag{1.1}$$

where a, b, c, $d \in \mathbb{C}$. Each such f extends uniquely to an isometry of hyperbolic 3-space $\mathbb{H}^3$ via the Poincaré extension [2]. Here hyperbolic 3-space is identified with the upper-half space $\mathbb{H}^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$ with the Riemannian metric $ds^2 = |dx|^2/x_3^2$ of constant negative curvature equal to -1. We shall denote the hyperbolic distance between a pair of points x and y in $\mathbb{H}^3$ by $\rho(x, y)$. If A and B are subsets of $\mathbb{H}^3$, then

$$\rho(A, B) = \inf_{x \in A, y \in B} \rho(x, y). \tag{1.2}$$

Given a set $A \subset \mathbb{H}^3$ *a tubular neighbourhood* of radius ε about A is the set

$$N_\varepsilon(A) = \{x \in \mathbb{H}^3 : \rho(x, A) < \varepsilon\}. \tag{1.3}$$

When A is a hyperbolic line, a tubular neighbourhood is usually called a *collar*.

A subgroup of Möb(2) is *discrete* if the identity is isolated in the topology of uniform convergence of $\bar{\mathbb{C}}$. See [2] and [4] for a discussion of discreteness and various reformulations of the definition. A *Kleinian group* Γ is a discrete nonelementary subgroup of Möb(2). Here *nonelementary* means that the group Γ does not contain an abelian subgroup of finite index. The elementary Kleinian groups are completely classified and it is the nonelementary groups which are of primary geometric interest.

Because of the Poincaré extension we can, and shall, view a Kleinian group as a group of orientation preserving isometries of hyperbolic 3-space. The *orbit space* $\mathcal{Q} = \mathbb{H}^3/\gamma$ is called a *hyperbolic 3-orbifold.* The space $\mathcal{Q}$ is a *hyperbolic 3-manifold* if the group Γ is torsion free. The *co-volume* of a Kleinian group Γ is the volume of the orbit space $\mathcal{Q}$ in the induced hyperbolic metric, equivalently it is the hyperbolic volume of a measurable fundamental domain for the action of Γ on $\mathbb{H}^3$.

A *hyperbolic plane* $\Pi \cup \mathbb{H}^3$ is that subset of $\mathbb{H}^3$ meeting a plane or sphere of $\mathbb{R}^3$ which is perpendicular to $\partial\mathbb{H}^3 \approx \bar{\mathbb{C}}$. A hyperbolic plane with the metric induced from $\mathbb{H}^3$ is isometric to the usual 2-dimensional hyperbolic space $\mathbb{H}^2$ with metric of constant Guassian curvature -1. A *triangle subgroup* Δ of a Kleinian group Γ is a subgroup isomorphic to some standard hyperbolic (p, q, r)-triangle group, that is the index two subgroup of the group generated by the reflections in the sides of a 2-dimensional hyperbolic triangle with vertex angles π/p, π/q, and π/r in hyperbolic 2-space. Here we must have $1/p + 1/q + 1/r < 1$. Since the triangle groups are rigid in Möb(2), if Δ is a triangle subgroup of a Kleinian group, then there is a hyperbolic plane $\Pi_\Delta \subset \mathbb{H}^3$ which is invariant under Δ, that is $f(\Pi_\Delta) = \Pi_\Delta$ for each $f \in \Delta$. Furthermore the action of Δ on the invariant hyperbolic plane Π_Δ is conjugate to the standard action of a triangle group on hyperbolic 2-space. Thus we see the usual picture of a tesselation by hyperbolic triangles of the invariant hyperbolic plane. For $p \geq 7$ we shall usually denote the $(2, 3, p)$-triangle subgroup by $\Delta(p)$.

A set $A \subset \mathbb{H}^3$ is said to be *precisely invariant* under a Kleinian group Γ if

$$g(A) = A \quad \text{or} \quad g(A) \cap A = \varnothing \qquad \text{for all } g \in \Gamma. \tag{1.4}$$

The *stabilizer* of A is the subgroup Γ_A defined by

$$\Gamma_A = \{g \in \Gamma : g(A) = A\}. \tag{1.5}$$

We can now state our main theorems. The first theorem describes the possible algebraic intersections of different triangle subgroups of a Kleinian group.

THEOREM 1.6. *Let $\Delta(2, 3, p)$ and $\Delta'(2, 3, q)$ be distinct triangle subgroups of a Kleinian group.*

- *If $p \neq q$, then $\Delta(2, 3, p) \cap \Delta'(2, 3, q)$ is a finite cyclic subgroup of order 2 or 3;*
- *If $p = q$, then $\Delta(2, 3, p) \cap \Delta'(2, 3, q)$ is a finite cyclic subgroup of order 2, 3 or p.*

It is a routine matter to verify that the intersections described in Theorem 1.6 can actually occur. The next theorem shows that the hyperbolic planes stabilized by $(2, 3, p)$-triangle subgroups are precisely invariant.

THEOREM 1.7. *Let Δ_1 and Δ_2 be distinct $(2, 3, p)$-triangle subgroups of a Kleinian group stabilizing hyperbolic planes Π_1 and Π_2 respectively. Then*

$$\rho(\Pi_1, \Pi_2) \geq \operatorname{arccosh}\left(1 + \frac{2\sin^2(\pi/p)}{1 - 4\sin^2(\pi/p)}\right). \tag{1.8}$$

The lower bound is sharp in the sense that for every $p \geq 7$ there is a Kleinian group Γ_p containing distinct $(2, 3, p)$-triangle subgroups whose invariant hyperbolic planes are separated by exactly the distance given in the right hand side of equation (1.8).

We shall show in addition that the extremal case exhibiting the sharpness of Theorem 1.7 occurs only when there is a common elliptic element of order p. We then establish the following variant of Theorem 1.7.

THEOREM 1.9. *Let $\Delta(p)$ and $\Delta'(q)$ be $(2, 3, p)$ and $(2, 3, q)$-triangle subgroups of a Kleinian group stabilizing hyperbolic planes Π_p and Π'_q respectively. If $p = q$, assume in addition that the common perpendicular between Πp and Π'_q does not coincide with an elliptic axis of order p. Then*

$$\rho(\Pi_p, \Pi'_q) \geq \operatorname{arcsinh}(\sqrt{3}) \approx 1.3170\ldots. \tag{1.10}$$

We shall actually give an explicit bound for the right hand side of equation (1.10) for each value of p and q in Corollary 3.20. We believe this bound to be of roughly the correct magnitude although we have no good examples to verify this.[1)]

We next turn our attention to estimating the co-volume of Kleinian groups with $(2, 3, p)$-triangle subgroups.

[1)] Added in proof. This bound has recently proven to be sharp in joint work with F. W. Gehring and T. H. Marshall.

THEOREM 1.11. *Let Γ be a Kleinian group containing a $(2, 3, p)$-triangle subgroup. Then*

$$Vol_{hyp}(\mathbb{H}^3/\Gamma) \geq \frac{\pi}{2}\left(\frac{1-4\sin^2(\pi/p)}{4\sin^2(\pi/p)}\right)\operatorname{arccosh}\left(\frac{1-2\sin^2(\pi/p)}{1-4\sin^2(\pi/p)}\right). \tag{1.12}$$

We give a different formula which is somewhat better in the case $p \in \{7, 8\}$. A table of the estimates we obtain compared with the conjectured sharp examples can be found at the end of §5. We see from this that the volume estimate of Theorem 1.11 is quite good. This estimate actually represents only the volume of an embedded solid hyperbolic cylinder about the degree p singular set and the examples of §4 show that this bound is sharp for each $p \geq 7$ in this regard.

2. Preliminary results

The main tool that we shall use in the proof of the results stated above is the sharp collaring theorem for elliptic elements of a Kleinian group together with the fact that the extremal situation occurs for $(2, 3, p)$-triangle groups. This result was established in our earlier work with F. W. Gehring [5]. We recall some terminology in order to state the result we need.

Other than the identity the elements of a Kleinian group Γ can be classified into three distinct types: If $g \in \Gamma$, then g is either

- *elliptic*: conjugate to a rotation $z \to cz$, $z \in \mathbb{C}$ and $|c| = 1$. The order of g is the period of the rotation.
- *parabolic*: conjugate to the translation $z \to z + 1$.
- *loxodromic*: conjugate to a dilation $z \to cz$, $z \in \mathbb{C}$ and $|c| \neq 1$.

Elliptic and loxodromic transformations g have two fixed points in the Riemann sphere $\bar{\mathbb{C}}$ and we call the hyperbolic line joining these fixed points the *axis* of g, denoted *axis*(g). Let f and g be elliptic or loxodromic Möbius transformations. We say that the axes of f and g are *parallel* if they lie in a common hyperbolic plane and do not meet. We say that the axes are *perpendicular* if one of the axes lies in a hyperbolic plane which meets the other axis at a right angle.

We define the numbers $\delta(p, q)$ for integers p and q, $2 \leq p, q < \infty$, $\max\{p, q\} \geq 3$, as follows:

$$\delta(p, q) = \inf_{f,g} \rho(axis(f), axis(g)) \tag{2.1}$$

where the infimum is taken over all pairs of Möbius transformations f and g such that f is elliptic of order p, g is elliptic of order q and $\langle f, q\rangle$ is a Kleinian group. The restriction $\max\{p, q\} \geq 3$ is necessary as two elements of order two can never generate a Kleinian group. Basic compactness theorems imply the infimum is actually a minimum [11] (see for instance the proof of our Theorem 3.2 here).

The main tool we shall use in this paper can be found in [5], Theorem 6.19, and also [6].

THEOREM 2.2. *For all $p \geq 7$,*

- $\delta(2, p) = \operatorname{arccosh}(1/(2 \sin \pi/p))$.
  Let $\langle f, g\rangle$ be a Kleinian group with f and g elliptic of order 2 and p respectively. Then $\rho(axis(f), axis(g)) = \delta(2, p)$ only if the axes of f and g are parallel or perpendicular and $\langle f, g\rangle$ is the $(2, 3, p)$-triangle group or a $\mathbb{Z}_2$-extension of this group.
- $\delta(3, p) = \operatorname{arccosh}(\cot(\pi/p)/\sqrt{3})$.
  Let $\langle f, g\rangle$ be a Kleinian group with f and g elliptic of order 3 and p respectively. Then $\rho(axis(f), axis(g)) = \delta(3, p)$ only if the axes of f and g are parallel and $\langle f, g\rangle$ is the $(2, 3, p)$-triangle group.
- $\delta(p, p) = 2\delta(2, p)$.
  Let $\langle f, g\rangle$ be a Kleinian group with f and g elliptic of order p. Then $\rho(axis(f), axis(g)) = \delta(p, p)$ only if the axes of f and g are parallel and $\langle f, g\rangle$ is the $(2, 3, p)$-triangle group. □

In [5] [6] we have identified many other values for $\delta(p, q)$. It turns out that the set of possible values for $\rho(axis(f), axis(g))$ for f of order p and g of order q and $\langle f, g\rangle$ Kleinian, initially comes in a discrete part and then a continuous part (roughly the continuous part comes from the groups that are free on the two elliptic generators). The value $\delta(p, q)$ is the smallest possible nonzero value. While other triangle groups are not necessarily extremal for the distance between axes of elliptic elements of other orders, all that is really necessary for our proof is that the set of values is discrete and that we can identify all those axial distances which are less than the values for the triangle groups in question. This is the basis for our claim in the introduction that our results also hold for other infinite families of triangle groups. For instance $\delta(4, p)$ is not attained in the $(2, 4, p)$-triangle group, however it very nearly is. A similar situation arises in the case of $\delta(5, p)$ and the $(2, 5, p)$-triangle groups. However these are the only other infinite families for which we know our results are valid.

We shall also need the following results [5], Theorem 6.19.

THEOREM 2.3. *For $n \geq m \geq 7$ we have the estimate*

$$\delta(m, n) \geq \operatorname{arcsinh}\left(\frac{\sqrt{2\cos(\pi/n) - 1}}{2\sin(\pi/n)\sin(\pi/m)}\right). \tag{2.4}$$

We now give a few results which are used in the proof of Theorems 1.7 and 1.9. The first lemma we need is a simple consequence of convexity and some elementary hyperbolic trigonometry.

LEMMA 2.5. *Let T be a hyperbolic triangle with vertex angles $\pi/2$, $\pi/3$ and π/p, $p \geq 7$. If z, $w \in T$, then*

$$\rho(z, w) \leq \delta(3, p)$$

with equality and only if z and w lie on the vertices with angles $\pi/3$ and π/p. □

The next lemma will give an estimate of the distance between a line lying in a hyperbolic plane stabilized by a $(2, 3, p)$-triangle group to an elliptic axis of order 2 or 3.

LEMMA 2.6. *Let Δ be a $(2, 3, p)$-triangle subgroup of a Kleinian group Γ stabilizing the hyperbolic plane Π_Δ. If l is a hyperbolic line in Π_Δ, then there is a point $v_1 \in \Pi_\delta$ stabilized by an element of order 3 in Δ such that*

$$\rho(v_1, l) \leq \operatorname{arccosh}(2\cos(\pi/p)/\sqrt{3}). \tag{2.7}$$

Moreover there is a point $v_2 \in \Pi_\Delta$ stabilized by an element of order 2 or 3 in Δ such that

$$\rho(v_2, l) \leq \frac{1}{2}\operatorname{arccosh}(2\cos(\pi/p)/\sqrt{3}). \tag{2.8}$$

Proof. We view Π_Δ as the unit disk model of hyperbolic 2-space $\mathbb{H}^2$ and Δ as a triangle group acting on it. The hyperbolic plane $\mathbb{H}^2$ is tesselated by hyperbolic triangles whose vertices meet the axes of elliptics in Δ. The line l lies in Π_Δ and therefore meets some hyperbolic triangle T of this tesselation. Conjugate by a Möbius transformation preserving Π_Δ, so that T is a hyperbolic triangle in the tesselation such that the vertex with angle π/p is at the origin. Now the link of this vertex in the triangulation of $\mathbb{H}^2$ obtained from this tesselation is a regular hyperbolic p-gon all of whose internal angles are $2\pi/3$. The line l meets this p-gon and therefore meets a side. Each side of this p-gon has endpoints which meet the axes of elliptics of order 3 and the axis of an elliptic of order 2 in the middle. he side length is $2\operatorname{arccosh}(2\cos(\pi/p)/\sqrt{3})$. The result is now easily seen to follow. □

THEOREM 2.9. *Let Δ be a $(2,3,p)$-triangle subgroup of a Kleinian group Γ stabilizing the hyperbolic plane Π_Δ. If $g \in \Gamma$ is elliptic of order* 2, 3 *or $q \geq p$, then*

$$axis(g) \cap \Pi_\Delta = \varnothing \tag{2.10}$$

unless $q \in \{2,3,p\}$ and $g \in \Delta$, or g is elliptic of order 2 *and $axis(g) \subset \Pi_\Delta$.*

Proof. Again we identify Π_Δ and $\mathbb{H}^2$. The triangle group Δ then tesselates Π_Δ and the vertices of the tesselating triangles are stabilised by elliptic elements of Δ.

Case 1. $q = 3$ *or* $q \geq p$.

Suppose first that g is elliptic of order q and $q = 3$ or $q \geq p$. It is immediate from Lemma 2.5 that the axis of g cannot meet the interior of any triangle of the tesselation, or at any point which is not a vertex of a triangle in the tesselation. This is since the function $\delta(3,p)$, as a function of p, is strictly increasing and the vertices with angles $\pi/3$ and π/p of each triangle are exactly the distance $\delta(3,p)$ apart. Now suppose that the axis of g meets at a vertex of some triangle in the tesselation. Since $q \geq p \geq 7$ or $q = 3$, if the vertex is stabilized by an elliptic of order p we must have $p = q$, by the classification of the elementary discrete groups [2], and this axis must coincide with that of the elliptic of order p in Δ. Thus $g \in \Delta$. If the axis of g meets a triangle of the tesselation in a vertex stabilised by an elliptic of order 2 we get a contradiction because the two vertices with angles $\pi/2$ and π/p are at a distance $\delta(2,p)$ which is smaller than $\delta(p,q)$ for both $q = 3$ and $q \geq p$. Finally in this case if the axis of g meets a triangle of the tesselation in a vertex stabilized by an elliptic of order 3, then the classification of the elementary groups implies that g has order 3. By the second part of Theorem 2.2 (since g has order 3 and its axis is at most the distance $\delta(3,p)$ from an order p axis in Δ) the axis of g is parallel to the elliptic axis of order p in Δ. It again follows that $g \in \Delta$. Thus Theorem 2.9 has been verified if g has order 3 or $q \geq p$.

Case 2. $q = 2$.

There remains the possibility that g has order 2 and its axis meets Π_Δ. It is possible that the axis lies in Π_Δ, and we have allowed for this in our conclusion. We may therefore assume that $axis(g)$ meets Π_Δ at some unique point, say w. Let θ be the angle of intersection between the axis of g and the hyperbolic plane Π_Δ. If $\theta = \pi/2$, then $\langle g, \Delta\rangle$ is a discrete Fuchsian group. (It is discrete since it is a subgroup of Γ and it is Fuchsian since both g and elements of Δ fix Π_Δ and are orientation preserving when restricted to this hyperbolic plane). Since Δ is a $(2,3,p)$-triangle group it follows that $g \in \Delta$. Thus we assume $0 < \theta < \pi/2$. Let $\Pi'_\Delta = g(\Pi_\Delta)$. The dihedral angle of intersection between Π_Δ and Π'_Δ is either 2θ or $\pi - 2\theta$ whichever is the smaller.

By virtue of the fact that these two hyperbolic planes meet there is a vertex $v_p \in \Pi_\Delta$ of the tesselation lying on an elliptic axis l of order p for which

$$\rho(v_p, \Pi'_\Delta) < \cosh(\delta(3, p)).$$

Let w be the closest point of Π'_Δ to v_p. From what we have proved in Case 1, $w \neq v_p$. Next there is a point $v'_p \in \Pi'_\Delta$ such that

$$\rho(v'_p, w) < \cosh(\delta(3, p)).$$

Since the line from v_p to w is perpendicular to Π'_Δ we see that the angle at w formed by the two line segments from w to v_p and from w to v'_p is $\pi/2$.

Therefore

$$\cosh(\rho(v_p, v'_p)) = \cosh(\rho(w, v_p)) \cosh(\rho(w, v'_p)) \tag{2.11}$$

$$= \cosh^2(\delta(3, p)). \tag{2.12}$$

However we must also have

$$\cosh(\rho(v_p, v'_p)) \geq \cosh(2\delta(2, p))$$

since the two vertices lie on different elliptic axes of order p. Thus

$$\cosh(2\delta(2, p)) \leq \cosh^2(\delta(3, p)). \tag{2.13}$$

Substituting in the values for $\delta(2, p)$ and $\delta(3, p)$ from Theorem 2.2 we obtain

$$\frac{1}{2\sin^2(\pi/p)} - 1 \leq \frac{\cot^2(\pi/p)}{3} \tag{2.14}$$

or

$$1 \leq 4\sin^2(\pi/p). \tag{2.15}$$

Equation (2.15) implies that $p \leq 4$. This contradiction proves Theorem 2.9. □

We want to use Theorem 2.9 to show that the hyperbolic planes stabilized by different triangle groups do not meet. We begin with an auxiliary lemma which bounds the angle of intersection if they do in fact meet.

LEMMA 2.16. *Let $\Delta(p)$ and $\Delta'(q)$ by $(2, 3, p)$ and $(2, 3, q)$-triangle subgroups of a Kleinian group Γ stabilizing different hyperbolic planes Π_p and Π'_q respectively. Suppose that $p \leq q$ and that the hyperbolic planes Π_p and Π'_q intersect at the dihedral angle θ. Then*

$$\sin(\theta) \geq (1/2 + \cos(\pi/p)/\sqrt{3})^{-1/2} \geq 0.95. \tag{2.17}$$

Proof. To see this we argue in a similar manner as in the proof of Theorem 2.9. By Lemma 2.6, if η is the line of intersection of the two hyperbolic planes, there is an elliptic element $f \in \Gamma$ of order 2 or 3 whose axis $l = axis(f)$ is perpendicular to Π_p and lies at a distance at most

$$r = \operatorname{arccosh}(2\cos(\pi/p)/\sqrt{3})/2$$

from η. Theorem 2.9 implies that l cannot meet the hyperbolic plane Π'_q. Again consider the hyperbolic plane Π perpendicular to η and containing l, and the two lines formed by the intersections with Π of the hyperbolic planes Π_p and Π'_q. The dihedral angle θ of the intersection of Π_p and Π'_q is the angle between these two lines in Π. Elementary hyperbolic trigonometry implies the estimate

$$\cosh(r) \geq \frac{1}{\sin(\theta)}$$

which yields the desired result. □

The preceding result implies that if the two hyperbolic planes Π_p and Π'_q do intersect, then they do so nearly perpendicularly. We will use this fact to show that they don't intersect at all.

THEOREM 2.18. *Let $\Delta(p)$ and $\Delta'(q)$ be $(2, 3, p)$ and $(2, 3, q)$ triangle subgroups of a Kleinian group Γ stabilizing hyperbolic planes Π_p and Π'_q respectively. Then*

$$\Pi_p \cap \Pi'_q = \varnothing$$

unless $p = q$, $\Pi_p = \Pi'_q$ and $\Delta(p) = \Delta'(q)$.

Proof. It is clear that the hyperbolic planes cannot coincide unless all the attributed equalities are valid. We therefore suppose that the hyperbolic planes do meet with dihedral angle $\theta > \theta$ along the line η. Let $w \in \eta$. Then w lies in a triangle of the tesselation of Π_p and also a triangle of the tesselation of Π'_q. Therefore Lemma 2.5 asserts that there are points $v_p \in \Pi_p$ and $v_q \in \Pi'_q$ stabilized by the elliptics of order p and q respectively and such that $\rho(w, v_p) \le \delta_p = \delta(3, p)$ and $\rho(w, v_q) \le \delta_q = \delta(3, q)$. Next from the law of hyperbolic cosines we have

$$\cosh(\rho(v_p, v_q)) \le \cosh(\delta_p)\cosh(\delta_q) - \sinh(\delta_p)\sinh(\delta_q)\cos(\theta). \tag{2.19}$$

Now substituting in the values of δ_p and δ_q given in Theorem 2.2 and our estimate for the dihedral angle of intersection given in Lemma 2.16 we obtain the following inequality.

$$\cosh(\rho(v_p, v_q)) \le \frac{1}{3}\left(\cot(\pi/p)\cot(\pi/q) - .96\sqrt{\cot^2(\pi/p) - 3}\sqrt{\cot^2(\pi/q) - 3}\right).$$

However we also have the estimate of Theorem 2.3:

$$\cosh(\rho(v_p, v_q)) \ge \cosh(\delta(p, q)) = \sqrt{1 + \frac{2\cos(\pi/q) - 1}{4\sin^2(\pi/p)\sin^2(\pi/q)}}.$$

Using elementary calculus one can show that these two estimates are incompatible for $7 \le p \le q$. □

Theorem 2.18 implies that the hyperbolic planes Π_p and Π'_q cannot meet at any finite point of $\mathbb{H}^3$. We next want to show that they do not meet at infinity either.

THEOREM 2.20. *Let $\Delta(p)$ and $\Delta'(q)$ be distinct $(2, 3, p)$ and $(2, 3, q)$ triangle subgroups of a Kleinian group Γ stabilizing hyperbolic plane Π_p and Π'_q respectively. Then*

$$\rho(\Pi_p, \Pi'_q) > 0.$$

Proof. Suppose that $\rho(\Pi_p, \Pi'_q) = 0$. Then, since the two hyperbolic planes cannot meet at any finite point of $\mathbb{H}^3$ by Theorem 2.18, there is a point of tangency of the two hyperbolic planes at some point of $\bar{\mathbb{C}}$ which we may as well assume to be ∞. In this situation the two hyperbolic planes are parallel vertical euclidean planes. Consider the axes of elliptics of order 3 (for instance) in $\Delta(p)$. They are hyperbolic lines which are perpendicular to Π_p and do not meet Π'_q by Theorem 2.9. However they occur at greater and greater heights and it is geometrically evident that they must therefore eventually meet Π'_q. This contradiction establishes Theorem 2.20. □

3. Invariant tubular neighbourhoods

Theorem 2.20 implies that the hyperbolic plane stabilized by a $(2, 3, p)$-triangle group is precisely invariant (see (1.4)). In this section we shall give estimates, some of which are sharp, on the distance between the invariant hyperbolic planes stabilized by different $(2, 3, p)$-triangle groups. We begin by defining certain numbers $\alpha(p, q)$ which bound from below the distance between any pair of hyperbolic planes stabilized by $(2, 3, p)$ and $(2, 3, q)$-triangle groups.

For integers p and q with $7 \leq p, q < \infty$ let $\alpha(p, q)$ be the largest real number with the following property: If Γ is a Kleinian group and $\Delta(p)$ and $\Delta'(q)$ are $(2, 3, p)$ and $(2, 3, q)$-triangle subgroups respectively stabilizing hyperbolic planes Π_p and Π'_q, then

$$\rho(\Pi_p, \Pi'_q) \geq \alpha(p, q). \tag{3.1}$$

THEOREM 3.2. *For all* $7 \leq p, q < \infty$,

$$0 < \alpha(p, q) < \infty.$$

Moreover for each such p and q there is a Kleinian group $\Gamma_{p,q}$ *with subgroups* $\Delta(p)$ *and* $\Delta'(q)$, *which are* $(2, 3, p)$ *and* $(2, 3, q)$*-triangle groups respectively, stabilizing hyperbolic planes* Π_p *and* Π'_q *with*

$$\rho(\Pi_p, \Pi'_q) = \alpha(p, q).$$

Proof. Let p and q be given as in the statement of Theorem 3.2 and set $\alpha = \alpha(p, q)$. We first use the combination theorems to provide examples to show that $\alpha < \infty$. Let $\Delta(p)$ and $\Delta'(q)$ be triangle groups. Then both $\Delta(p)$ and $\Delta'(q)$ act discontinuously in $\bar{\mathbb{C}} \backslash \{C_p\}$ and $\bar{\mathbb{C}} \backslash \{C_q\}$ respectively, where C_p, C_q are circles or lines in $\bar{\mathbb{C}}$ (the limit sets of the groups). Let F_p and F_q be open subsets of fundamental domains for the respective actions. Choose a Möbius transformation g of $\bar{\mathbb{C}}$ such that

$$g(F_q) \supset \bar{\mathbb{C}} \backslash F_p.$$

Then the Klein-Maskit combination theorems [12] imply that the group

$$\langle \Delta(p), g\Delta'(q)g^{-1} \rangle$$

is a discrete nonelementary, hence Kleinian, group which is algebraically isomorphic to the free product group $\Delta(p) * \Delta'(q)$. Since this Kleinian group contains a $(2, 3, p)$ and a $(2, 3, q)$ triangle groups as subgroups it follows that $\alpha < \infty$.

Next we show that $\alpha > 0$. Let Γ_n be a sequence of Kleinian groups with triangle subgroups $\Delta_n(p)$ and $\Delta'_n(q)$ stabilizing hyperbolic planes $\Pi_{n,p}$ and $\Pi'_{n,q}$ such that

$$\rho(\Pi_{n,p}, \Pi'_{n,q}) \to \alpha.$$

Consider the sequence of subgroups of Γ_n defined by

$$\Gamma_n(p, q) = \langle \Delta_n(p), \Delta'_n(q) \rangle.$$

After conjugating by a suitable hyperbolic isometry of $\mathbb{H}^3$ we may assume that the common perpendicular between $\Pi_{n,p}$ and $\Pi'_{n,q}$ is bisected by the point $(0, 0, 1) \in \mathbb{H}^3$ and lies in the line $\{(0, 0, t) \in \mathbb{H}^3\}$. Choose generators $\{g_{n,1}, f_{n,1}\}$ and $\{g_{n,2}, f_{n,2}\}$ for $\Delta_n(p)$ and $\Delta'_n(q)$ respectively such that $g_{n,i}$ are elliptic of order 2 and $f_{n,i}$ are elliptic of order 3 and the axes of $f_{n,i}$ and $g_{n,i}$ are at a distance no more than $\max\{\delta(3, p), \delta(3, q)\}$ from the line $\{(0, 0, t) \in \mathbb{H}^3\}$. This is possible since the common perpendicular meets the tesselation of both hyperbolic planes. It now follows from the local compactness of the Möbius group that for $i = 1, 2$

$$g_{n,i} \to g_i \quad \text{and} \quad f_{n,i} \to g_i \qquad \text{as } n \to \infty$$

By the theorem of Jørgensen [11] on the algebraic convergence of sequences of discrete nonelementary Kleinian groups, the limit group

$$\Gamma(p, q) = \langle g_1, f_1, g_2, f_2 \rangle$$

is discrete and nonelementary. Also, the algebraic convergence theorem is easily seem to imply that $\Delta_\infty(p) = \langle g_1, f_1 \rangle$ is a Fuchsian group which is the homomorphic image of a $(2, 3, p)$-triangle group. It follows tht $\Delta_\infty(p)$ is a $(2, 3, p)$-triangle group. Similarly for the group $\Delta'_\infty(q) = \langle g_2, f_2 \rangle$. There are of course invariant hyperbolic planes $\Pi_{\infty,p}$ and $\Pi'_{\infty,q}$ and it remains only to observe that

$$\rho(\Pi_{\infty,p}, \Pi'_{\infty,q}) = \alpha,$$

and that $\alpha > 0$ by Lemma 2.16. □

We now give two different ways of estimating the number $\alpha(p, q)$ which will in turn give bounds for it. We begin by defining a sequence of numbers $a_p(m)$ as follows:

Let $a_p(m)$ be the largest number with the following property: If Γ is a Kleinian group, $\Delta(p)$ a $(2, 3, p)$-triangle subgroup stabilizing a hyperbolic plane Π, and $f \in \Gamma$

is an elliptic of order m with $axis(f) \cap \Pi = \emptyset$, then

$$\cosh(\rho(\Pi, axis(f))) \geq a_p(m). \tag{3.3}$$

It is not difficult to show using a compactness argument, much the same as we used in the proof of Theorem 3.2, that $0 < a_p(m) < \infty$ and that the value $a_p(m)$ is attained in an example.

Let us now give an estimate on the value of $a_p(m)$ in terms of the value $\alpha(p, p)$.

THEOREM 3.4. *For all $m \geq 2$ and $p \geq 7$*

$$\cosh(a_p(m)) \geq \frac{\cosh(\alpha(p,p)/2)}{\sin(\pi/m)}. \tag{3.5}$$

Proof. Let Γ be a Kleinian group and $\Delta(p)$ a $(2, 3, p)$-triangle subgroup stabilizing a hyperbolic plane Π_Δ. Let f be an elliptic of order m in Γ such that $axis(f) \cap \Pi_\Delta = \emptyset$. We may conjugate the group Γ so that $axis(f)$ is the line $\{(0, 0, t) \in \mathbb{H}^3\}$. The collection of hyperbolic planes $f^k(\Pi_\Delta), k = 1, 2, \ldots, m$ now form m congruent euclidean hemispheres. The hyperbolic distance between any pair of them is at least $\alpha(p, p)$ since each hyperbolic plane $f^k(\Pi_\Delta)$ is stabilized by the $(2, 3, p)$-triangle group $f^k\Delta(p)f^{-k}$.

Let Π be the hyperbolic plane which is perpendicular to $axis(f)$ and Π_Δ. Then Π meets the other m-hyperbolic planes perpendicularly as well, moreover the common perpendicular between $axis(f)$ and any of the other hyperbolic plane $f^k(\Pi_\Delta)$ lies in Π as does the common perpendicular between any adjacent pair of hyperbolic planes, $f^k(\Pi_\Delta)$ and $f^{k+1}(\Pi_\Delta)$. We now view Π as the disk model of hyperbolic 2-space. In Π we see m complete hyperbolic lines, each congruent by a euclidean rotation (induced by f), the distance between any pair of lines is at least $\alpha(p, p)$. The distance $a_p(m)$ we are trying to estimate is, in the worst case, the distance between any one of these lines and the point 0. Choose one of these lines l and construct a Lambert quadrilateral as follows. Let l_1 be the geodesic line segment from 0 to l, l_2 be the initial half (starting from l) of the geodesic segment which is perpendicular to l and either adjacent geodesic line. Let l_3 be the geodesic segment from the endpoint of the segment l_2, not in l, to 0. Finally l_4 is the portion of l connecting l_1 to l_2. All angles of this quadrilateral are $\pi/2$ except for the angle at 0 which is π/m by symmetry. We seek the length of l_1, and know the length of l_2 is at least $\alpha(p, p)/2$. Then by formulas of hyperbolic trigonometry, [2] Theorem 7.17.1 we find

$$\cosh(l_1) \geq \cosh(l_2)/\sin(\pi/m).$$

From which the desired result follows. □

Suppose that $\Delta = \Delta(p)$ and $\Delta' = \Delta'(q)$ are $(2, 3, p)$ and $(2, 3, q)$-triangle subgroups stabilizing hyperbolic planes Π_p and Π'_q whose common perpendicular is l and that $\langle \Delta, \Delta' \rangle$ is discrete. Then for $m \in \{2, 3, p\}$ we set

$$c_m(\Delta, \Delta') = \min\{\rho(l, axis(g)) : g \in \Delta \text{ has order } m \text{ and } axis(g) \neq l\}. \tag{3.6}$$

Thus $c_m(\Delta, \Delta')$ is the distance from the common perpendicular of Π_p and Π'_q to the closest elliptic axis of order m which is perpendicular to Π_p and not coincident with the common perpendicular. Notice that it follows from Lemma 2.5 that $0 < c_m(\Delta, \Delta') \leq \delta(p, p)$.

The following theorem is used to give an estimate on the value of $\alpha(p, q)$ in terms of the values $a_p(m)$ and $c_m(\Delta, \Delta')$.

THEOREM 3.7. *Let Γ be a Kleinian group and let p and q be integers with $7 \leq p, q < \infty$. Suppose that $\Delta = \Delta(p)$ and $\Delta' = \Delta'(q)$ are $(2, 3, p)$ and $(2, 3, q)$-triangle subgroups stabilizing hyperbolic planes Π_p and Π'_q respectively. Then*

$$\sinh(\rho(\Pi_p, \Pi'_q)) \geq \frac{\cosh(a_q(m))}{\sinh(c_m(\Delta, \Delta'))} \tag{3.8}$$

for all $m \in (2, 3, p)$.

Proof. Again we conjugate the group Γ so that the common perpendicular between the two hyperbolic planes is a subset of the line $\{(0, 0, t) \in \mathbb{H}^3\}$. Let $m \in \{2, 3, p\}$ and let l be an axis of an elliptic of order m perpendicular to Π_p closest to the intersection of, but not coincident with, the common perpendicular. The line l is at least the distance $a_q(m)$ from the other hyperbolic plane Π'_q. The reason for this is simply that l and Π'_q are disjoint (if they were to meet they would meet perpendicularly since l is the axis of an elliptic element of Γ. Hence l would be the common perpendicular between Π_p and Π'_q which it is not).

We now construct a regular hyperbolic pentagon all of whose angles are $\pi/2$ as follows. Let l_1 be the geodesic line segment which is the common perpendicular between l and Π'_q. l_2 is the portion of l joining l_1 to Π_p. l_3 is the common perpendicular between Π_p and Π'_q. Complete the pentagon in the obvious manner, denoting that segment in Π_p by l_4. We seek the length of l_3, we have a lower bound on the length of l_1 given by $a_q(m)$. The length of l_4 is equal to $c_m(\Delta, \Delta')$. Again by the formulas of hyperbolic trigonometry ([2] Theorem 7.18.1) we have

$$\sinh(l_4)\sinh(l_3) = \cosh(l_1). \tag{3.9}$$

Thus $\sinh(\rho(\Pi_p, \Pi'_q)) \geq \cosh(a_q(m))/\sinh(c_m(\Delta, \Delta'))$ as desired. □

We now put together the two estimates of Theorem 3.4 and Theorem 3.7 to obtain the following corollary.

COROLLARY 3.10. *If Γ is a Kleinian group and $\Delta(p)$ and $\Delta'(q)$ are $(2, 3, p)$ and $(2, 3, q)$-triangle subgroups stabilizing hyperbolic planes Π_p and Π'_q, then*

$$\sinh(\rho(\Pi_p, \Pi'_q)) \geq \frac{\cosh(\alpha(q, q)/2)}{\sinh(c_m(\Delta, \Delta'))\sin(\pi/m)} \tag{3.11}$$

for all $m \in (2, 3, p)$.

The point to the corollary is that with $p = q$ in the extremal configuration the number $\alpha(p, p)$ will occur on both sides of equation (3.11) and can be bounded purely in terms of the number $c_m(\Delta, \Delta')$ for which we already have estimates.

THEOREM 3.12.

$$\cosh(\alpha(p, p)) = 1 + \frac{2\sin^2(\pi/p)}{1 - 4\sin^2(\pi/p)}. \tag{3.13}$$

Proof. Theorem 3.2 asserts the existence of an extremal Kleinian group Γ with two $(2, 3, p)$-triangle groups stabilizing hyperbolic planes Π_1 and Π_2 with

$$\rho(\Pi_1, \Pi_2) = \alpha(p, p) = \alpha. \tag{3.14}$$

Squaring both sides of the inequality (3.11) of Corollary 3.10 and using the obvious trigonometric identities we find that

$$\cosh^2(\alpha) \geq a\cosh(\alpha) + 1 + a \tag{3.15}$$

where $a = 1/(2\sin^2(\pi/m)\sinh^2(c_m(\Delta, \Delta'))) > 0$. Thus for all $m \in \{2, 3, p\}$ we have

$$\cosh(\alpha) \geq 1 + \frac{1}{2\sin^2(\pi/m)\sinh^2(c_m(\Delta, \Delta'))}. \tag{3.16}$$

We now set $m = p$. By Lemma 2.5 we have $c_p(\Delta, \Delta') \leq \delta(3, p) < \delta(p, p)$ unless the common perpendicular between Π_1 and Π_2 is the axis of an elliptic of order p and $c_p(\Delta, \Delta') = \delta(p, p)$. Substituting the value of $\delta(p, p)$ from Theorem 2.2 into equation (3.18) yields

$$\cosh(\alpha) \geq 1 + \frac{2\sin^2(\pi/p)}{1 - 4\sin^2(\pi/p)}.$$

The converse inequality will be shown by example in the next section. □

COROLLARY 3.17. *For every $p, q \geq 7$ with $p \neq q$*

$$\sinh^2(\alpha(p, q)) \geq \frac{3}{1-4\sin^2(\pi/p)} \frac{1-3\sin^2(\pi/q)}{1-4\sin^2(\pi/q)}. \tag{3.18}$$

Proof. Theorem 3.2 guarantees the existence of the extremal group Γ containing the two triangle subgroups stabilizing hyperbolic planes a distance $\alpha(p, q)$ apart. If $p \neq q$, then the common perpendicular cannot coincide with an elliptic axis of order p and q and hence $c_p(\Delta, \Delta') \leq \delta(3, p)$ by Lemma 2.5. Corollary 3.10 implies

$$\sinh(\alpha(p, q)) \geq \frac{\cosh(\alpha(q, q)/2)}{\sinh(\delta(3, p))\sin(\pi/p)}. \tag{3.19}$$

From Theorem 3.12

$$\cosh(\alpha(q, q)/2) = \sqrt{1 + \frac{\sin^2(\pi/q)}{1-4\sin^2(\pi/q)}},$$

which together with (3.19) and Theorem 2.2 gives the desired result. □

A point to notice is that for all p and q the left hand side of equation (3.18) in Corollary 3.17 is bounded below by 3.

COROLLARY 3.20. *For every $p, q \geq 7$ with $p \neq q$*

$$\alpha(p, q) \geq \operatorname{arcsinh}(\sqrt{3}) \approx 1.3170\ldots.$$

Of course one can use the ideas here to obtain estimates under other assumptions. For instance one might assume the common perpendicular is an axis of order 2 or 3. We leave it to the reader to obtain these estimates.

We also wish to record the following result which gives estimates when $p = q$ and the common perpendicular is not an elliptic axis of order p. The proof is similar to that of Corollary 3.17.

THEOREM 3.21. *Suppose that Δ_1 and Δ_2 are two $(2, 3, p)$-triangle subgroups of a Kleinian group Γ which stabilize distinct hyperbolic planes Π_1 and Π_2 and that the common perpendicular is not coincident with an elliptic axis of order p. Then*

$$\begin{aligned}\sinh^2(\rho(\Pi_1, \Pi_2)) &\geq \frac{3-9\sin^2(\pi/p)}{(1-4\sin^2(\pi/p))^2}\\ &\geq 3.\end{aligned}$$

□

Our results now give estimates for the numbers $a_p(m)$ via Theorem 3.4.

THEOREM 3.22. *For $p \geq 7$,*

$$\cosh(a_p(m)) \geq \frac{\sqrt{1+\sin^2(\pi/p)/(1-4\sin^2(\pi/p))}}{\sin \pi/m}$$
$$\geq \frac{1}{\sin(\pi/m)}.$$

Finally, let us say a few words about the algebraic intersection theorem mentioned in the introduction. It is easily seen that the intersection of a $(2,3,p)$-triangle group and a $(2,3,q)$-triangle group cannot contain any element of infinite order as each such is a hyperbolic transformation whose axis must lie in both invariant hypberbolic planes. But these hyperbolic planes do not meet. Therefore the intersection, if nontrivial, must be generated by an elliptic element of order 2, 3, p or q. If $p \neq q$ it is clear that the intersection cannot contain either the elliptic of order p or q simply because the associated triangle groups cannot contain these extra elliptics and remain discrete.

4. Examples

The examples that exhibit the sharpness of our Theorem 3.12 were explicitly constructed in an earlier joint work with M. Conder [3]. The groups in question are subgroups of groups generated by reflections in the faces of a hyperbolic pentahedra. These groups were constructed by opening up a cusp of a certain tetrahedral orbifold group. More precisely the group generated by reflections in the sides of a hyperbolic tetrahedron with Coxeter diagram 3-3-6, whose index two orientation preserving subgroup is the minimal co-volume cusped (finite volume noncompact) orbifold. Interestingly (and not unrelatedly) this Kleinian group has minimal co-volume among all Kleinian groups with an element of order 6 [8]. We continuously decreased the dihedral angle of the edge whose stabilizer was the elliptic of order 6 while keeping all the other dihedral angles fixed. When the angle is decreased until it has the form π/p, $p \geq 7$, the group is discrete. However it has infinite co-volume. The tetrahedron has opened up and subtends a $(2,3,p)$ triangle on the sphere at infinity. We show how, in these circumstances, to construct a hyperbolic plane Π perpendicular to the three faces of the unbounded tetrahedron which subtend the triangle. The three faces then subtend a $(2,3,p)$-triangle in Π and the group generated by these three reflections is a Z_2-extension of the $(2,3,p)$-triangle group which stabilizes Π. If we adjoin to this group the reflection in the hyperbolic plane Π we find from the Poincaré Polyhedron Theorem [12] that the group obtained is discrete. The index two orientation preserving subgroup is the

Kleinian group we want. Call it $\Gamma^0(p)$. It contains a $(2, 3, p)$-triangle subgroup stabilizing the hyperbolic plane Π and has finite co-volume (actually, it contains a Z_2-extension of this triangle group which is orientation preserving but when restricted to Π is orientation reversing as there are elliptic axes of order two lying in Π). It is extremal with respect to the collaring theorems because it contains a $(2, 3, p)$-triangle subgroup. Because of the explicit construction we can compute the length of the primitive loxodromic element sharing an axis with the elliptic of order p. The calculations occur in §3 of [3] and the value we want is twice the value identified in equation (3.5) in that paper, since this is the distance between perpendicular elliptic axes of order two and hence half the translation length of the primitive loxodromic. The minimal distance between $(2, 3, p)$-triangle subgroups is then bounded above the translation length of this loxodromic. We record this discussion in the following theorem.

THEOREM 4.1. *For each $p \geq 7$ there is a Kleinian group $\Gamma^0(p)$ containing two distinct $(2, 3, p)$-triangle subgroups stabilizing hyperbolic planes Π_1 and Π_2 with*

$$\cosh(\rho(\Pi_1, \Pi_2)) = 1 + \frac{2\sin^2(\pi/p)}{1 - 4\sin^2(\pi/p)}.$$

Proof. Equation (3.5) of [3] gives the value l for the distance between perpendicular elliptic axes of order two along the axis of the elliptic of order p in the group $\Gamma^0(p)$, where

$$\sinh^2(l) = \frac{1}{\csc^2(\pi/p) - 4}. \tag{4.2}$$

The translation length of the primitive loxodromic element is then $2l$. But then the above equation gives

$$\begin{aligned}\cosh(2l) &= 1 + \frac{2}{\csc^2(\pi/p) - 4}\\ &= 1 + \frac{2\sin^2(\pi/p)}{1 - 4\sin^2(\pi/p)}.\end{aligned}$$ □

Actually, the construction of these groups by continuous variation of a single dihedral angle makes the volume easy to compute as well, and this is related in [3]. We also should point out that the co-volumes of the groups $\Gamma^0(p)$ have a finite limit which is the smallest possible value for the limit of a sequence of finite co-volumes of *any* sequence of Kleinian groups [1]. Thus these groups are extremal in many different ways. A short list of the volumes of these groups is given in the next section where we compare them with general lower bounds.

5. Lower bounds on the co-volume

We use two different methods to obtain estimates on the co-volumes of Kleinian groups with $(2, 3, p)$-triangle subgroups as follows. First we use the action of the triangle group on a precisely invariant neighbourhood of the stable hyperbolic plane; second we use the collaring theorem about the elliptic axis of order p together with a bound obtained on the translation length of any loxodromic element whose axis coincides with the elliptic axis of order p. This latter bound is given by the size of the precisely invariant neighbourhood of the hyperbolic plane. Let us state the following simple corollary of the results of the previous section.

LEMMA 5.1. *Let Γ be a Kleinian group and $\Delta(p)$ a $(2, 3, p)$-triangle subgroup. Let Π denote the invariant hyperbolic plane stabilized by $\Delta(p)$. Then Π has a precisely invariant tubular neighbourhood of radius $\alpha(p, p)/2$.*

Proof. Let $g \in \Gamma$ and N the tubular neighbourhood of Π of radius $\alpha(p, p)/2$. Suppose that $g(N) \cap N \neq \emptyset$. Then $\rho(\Pi, g(\Pi)) < \alpha(p, p)$. As both Π and $g(\Pi)$ are stabilized by $(2, 3, p)$-triangle groups we must have $g(\Pi) = \Pi$ by the definition of the numbers $\alpha(p, p)$. It follows that $g(N) = N$, so that N is precisely invariant. □

Suppose now that Γ is a Kleinian group with a $(2, 3, p)$-triangle subgroup, $\Delta(p)$ stabilizing a hyperbolic plane Π. Let ε be the radius of a precisely invariant tubular neighbourhood of Π. It follows that the volume of $\mathbb{H}^3/\Gamma$ is bounded below by the volume of $N_\varepsilon(\Pi)/\Gamma_\Pi$. Here Γ_Π is the stabilizer of Π. Because there may be an involution stabilizing Π whose axis lies in Π we see that the triangle group $\Delta(p)$ has index at most two in the stabilizer of Π and therefore has index of at most two in the stabilizer of $N_\varepsilon(\Pi)$. It is at most two since anything stabilizing $N_\varepsilon(\Pi)$ stabilizes Π and therefore has a Fuchsian subgroup of index two. This Fuchsian subgroup contains the $(2, 3, p)$-triangle group, and therefore is the $(2, 3, p)$-triangle group. (We note that the index is indeed two in the conjectured extremals.)

It is easy to see that a fundamental domain for the action of the triangle group $\Delta(p)$ on $N_\varepsilon(\Pi)$ is the set $\Omega = T \times [-\varepsilon/2, \varepsilon/2]$ where T is a fundamental domain for the action of $\Delta(p)$ on Π. Here the interval $[-\varepsilon, \varepsilon]$ is identified with a geodesic line segment perpendicular to T whose bisector 0 lies in T. Therefore in order to get a lower bound on the co-volume of the Kleinian group Γ, we need to compute the volume of Ω, with $\varepsilon = \alpha(p, p)/2$, and then divide by 2. Here is how to compute the volume of Ω.

LEMMA 5.2. *Let Π be a hyperbolic plane in $\mathbb{H}^3$ and A a measurable subset of Π. Let $\Omega_A(\varepsilon)$ be the union of all geodesic line segments of length 2ε which are perpendicular to A and whose bisector lies in A. Then*

$$Vol_{hyp}(\Omega_A(\varepsilon)) = \frac{1}{2} Area_{hyp}(A)(\sinh(2\varepsilon) + 2\varepsilon). \tag{5.3}$$

Proof. We use the coordinates (τ, ρ) where $\tau \in \Pi$ and ρ measures the oriented hyperbolic distance (that is choose a + and a − direction) from Π. In these coordinates the volume element can be computed as

$$dVol_{hyp} = \cosh^2(\rho)\, dA(\tau)\, d\rho \tag{5.4}$$

where $dA(\tau)$ is the hyperbolic area measure in Π (see [7] for this calculation in all dimensions). The result now follows. □

The area of a fundamental domain $\mathscr{F}$ for the $(2, 3, p)$-triangle group is

$$Area_{hyp}(\mathscr{F}) = 2\pi\left(\frac{1}{6} - \frac{1}{p}\right).$$

Lemma 5.2 now yields the following volume estimates in view of our previous discussion.

THEOREM 5.5. *Let Γ be a Kleinian group with a $(2, 3, p)$-triangle subgroup. Then*

$$Vol_{hyp}(\mathbb{H}^3/\Gamma) \geq \frac{\pi}{2}\left(\frac{1}{6} - \frac{1}{p}\right)(\sinh(\alpha(p, p)) + \alpha(p, p)). \tag{5.6}$$

As we mentioned above we can also use the collaring theorems, Theorem 2.2, to give the volume estimate. This is because the elliptic of order p has a precisely invariant tubular neighbourhood of radius $\delta(p, p)/2 = \delta(2, p)$. The stabilizer of this solid hyperbolic cylinder is generated by the elliptic of order p, a primitive loxodromic element (whose translation length we have bounded below by $\alpha(p, p)$) and possibly an involution of order two whose axis is perpendicular to the elliptic axis of order p. The hyperbolic volume of a solid cylinder of length τ and radius r is $\pi\tau \sinh^2(r)$. We therefore have the following theorem.

THEOREM 5.7. *Let Γ be a Kleinian group with a $(2, 3, p)$-triangle subgroup. Then*

$$Vol_{hyp}(\mathbb{H}^3/\Gamma) = \frac{\pi}{2}\,\alpha(p,p)\,\sinh^2(\delta(2,p))$$
$$= \frac{\pi}{2}\left(\frac{1-4\sin^2(\pi/p)}{4\sin^2(\pi/p)}\right)\operatorname{arccosh}\left(\frac{1-2\sin^2(\pi/p)}{1-4\sin^2(\pi/p)}\right).$$

In fact we have shown that under the circumstances of the theorem that there is an embedded solid hyperbolic cylinder whose volume is given by Theorem 5.7. The examples of Section 4 show that this estimate is sharp for every $p \geq 7$. Notice that as $p \to \infty$

$$\frac{\pi}{2}\left(\frac{1-4\sin^2(\pi/p)}{4\sin^2(\pi/p)}\right)\operatorname{arccosh}\left(\frac{1-2\sin^2(\pi/p)}{1-4\sin^2(\pi/p)}\right) \to \frac{1}{4}. \tag{5.8}$$

The estimate given by Theorem 5.5 is better than that of Theorem 5.7 only when $p \in \{7, 8\}$. Here is a table comparing our estimates and the conjectured best bound.

p	lower bound on co-volume	conjectured sharp bound
7	0.145686...	0.17712...
8	0.164406...	0.21442...
9	0.179486...	0.2365...
10	0.194991...	0.25106...
∞	0.25	0.30532...

Acknowledgment

This work is largely motivated by continuing joint work with F. W. Gehring on universal constraints for the geometry of Kleinian groups and can be viewed as part of that general program. Many of the ideas and basic results used here can be found in our earlier works. Also thanks are due to the referee who contributed significantly to the clarity and exposition of this work.

REFERENCES

[1] C. C. ADAMS, *Limit volumes of hyperbolic 3-orbifolds*, J. Diff. Geom. *34* (1991) 115–141.

[2] A. BEARDON, *The geometry of discrete groups*, Springer-Verlag, 1983.

[3] M. D. E. CONDER and G. J. MARTIN, *Cusps, triangle groups and hyperbolic 3-folds*, J. Australian Math. Soc. *55* (1993) 149–182.

[4] F. W. GEHRING and G. J. MARTIN, *Inequalities for Möbius transformations and discrete groups*, J. reine angew. Math. *418* (1991) 31–76.

[5] F. W. GEHRING and G. J. MARTIN, *Commutators, collars and the geometry of Möbius groups*, J. d'Analyse Math *63* (1994) 175–219.

[6] F. W. GEHRING and G. J. MARTIN, *Kleinian groups with an elliptic generator*, to appear.
[7] F. W. GEHRING and G. J. MARTIN, *n-dimensional collaring theorems and the volume of symmetric hyperbolic n-manifolds*, to appear.
[8] F. W. GEHRING and G. J. MARTIN, 6-*Torsion and hyperbolic volume*, Proc. Amer. Math. Soc. *117* (1993) 727–735.
[9] F. W. GEHRING and G. J. MARTIN, *On the hyperbolic* 3-*fold of minimal co-volume* (with F. W. Gehring), Mathematical Research Letters *1* (1994) 107–115.
[10] F. W. GEHRING, C. MACLACHLAN, G. J. MARTIN and A. REID, *Arithmeticity, Discreteness and Volume*, Trans. Amer. Math. Soc.
[11] T. JØRGENSEN, *On discrete groups of Möbius transformations*, Amer. J. Math. *98* (1976) 739–749.
[12] B. MASKIT, *Kleinian groups*, Springer-Verlag, 1987.
[13] T. MARSHALL, *Hyperbolic Geometry and Reflection Groups*, PhD Thesis, The University of Auckland, 1994.
[14] W. P. THURSTON, *The Geometry and Topology of* 3-*manifolds*, Princeton University Lecture Notes, 1976.

Gaven J. Martin
Centre for Mathematics & Applications
Australian National University
Canberra ACT
Australia

Department of Mathematics
The University of Auckland
Auckland
New Zealand
e-mail address: martin@math.auckland.ac.nz

Received September 23, 1993; December 12, 1994 and October 30, 1995