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Local structure of the moduli space of vector bundles over curves

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0. Introduction

Let X be a smooth, projective and connected curve (over an algebraically closed field of characteristic zero) of genus $g(\mathrm{X}) \geq 2$. Let x be a (closed) point of X and $\mathrm{SU_X}(r, d)$ the moduli space of semi-stable vector bundles on X of rank $r \geq 2$ and determinant $\mathcal{O}(dx)$. As usual, the geometric points of $\mathrm{SU_X}(r, d)$ correspond to *polystable* bundles, namely direct sums $\mathrm{E} = \oplus \mathrm{E}_i$ where E_i is stable of slope $\mu(\mathrm{E}_i) = d/r$ (and $\otimes_i \det(\mathrm{E}_i) = \mathcal{O}(dx)$).

DEFINITION. The number of stable summands in the preceding sum is called the length of the polystable bundle E.

The singular locus of $\mathrm{SU_X}(r, d)$ consists exactly of the non stable points (except if $r = g(\mathrm{X}) = 2$ and d even). In this case, $\mathrm{SU_X}(r, d) = \mathbf{P}^3$ [N-R1]). In particular, except in the exceptional case above, $\mathrm{SU_X}(r, d)$ is smooth if and only if r and d are relatively prime. General facts about the action of reductive groups ensure that $\mathrm{SU_X}(r, d)$ is Cohen-Macaulay [E-H], normal and that the singularities are rational [B]. The principal aim of this paper is to give additional information about the singularities, essentially the description of the completion of the local ring at a singular point of $\mathrm{SU_X}(r, d)$ and to compute the multiplicity and the tangent cones at those singular points E which are not too bad, i.e. $l(\mathrm{E}) = 2$ (or equivalently $\mathrm{Aut}(\mathrm{E}) = \mathrm{G}_m \times \mathrm{G}_m$). Further, we give a complete description in the rank 2 case (corollaries III.2 and III.3 and theorem IV.4).

As recalled with some details (following a request of the referee), Narasimhan and Ramanan [N-R2] have a long time ago discovered the link between the rank 2 vector bundles on a canonical curve X of genus 3 and the geometry of the Kummer variety $\kappa(\mathrm{X})$ of JX. The crucial point is that $\mathrm{SU_X}(2)$ is in this case canonically isomorphic to the so called *Coble quartic*. We get the local form of this

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quartic and prove that the $\kappa(X)$ is *schematically* defined by 8 cubics, the partials derivatives of the Coble quartic (theorem IV.6), although the corresponding homogeneous ideal is not generated by these cubics.

One could also give partial information at least if Aut(E) is a torus, or by using results of [P], if $\mathrm{Aut(E)} = \mathrm{Gl}_r(k)$ (the latter case meaning that E is a twist of the trivial bundle). But it seems to be difficult (and somewhat messy) to calculate for instance the multiplicity. In the last part of the paper, we compute the multiplicity of a generalized theta divisor of $\mathrm{SU_X}(2, \mathcal{O})$ at a point $[\mathrm{L} \oplus \mathrm{L}^\vee]$, where $\mathrm{L}^2 \neq \mathcal{O}$. In fact, this computation could be done with only minor changes for a point E of any rank with $\det(\mathrm{E}) = \mathcal{O}$ and $\mathrm{Aut(E)} = \mathrm{G}_m \times \mathrm{G}_m$.

Let us also mention that similar results could be obtained exactly in the same way for certain surfaces. But, all the future applications that we have in mind as well as the applications that we have in our hand are for curves. Therefore, we have restricted ourselves to the case of curves.

Notations and conventions

All the schemes are of finite type over an algebraically closed field k of characteristic $p \neq 2$. Except in the first section, it will be assumed of characteristic zero. By point we mean closed point. The scheme $\mathbf{P}(\mathrm{V})$ is the projective space $\mathrm{Proj}(\mathrm{Sym}\, \mathrm{V}^\vee)$ of lines of V and V will also denote the pointed affine space $\mathrm{Spec}(\mathrm{Sym}\, \mathrm{V}^\vee) = \mathrm{Spec}\, k[\mathrm{V}]$ (notice that $k[\mathrm{V}]$ is the coordinate ring of V).

The notations X (resp. x) will denote a smooth projective connected curve over k of genus $g \geq 2$ (resp. a closed point x of X). If $\mathscr{L}$ is a line bundle on a scheme, the linear system $\mathrm{PH}^0(\mathscr{L})$ is denoted by $|\mathscr{L}|$ and the dual projective space by $|\mathscr{L}|^\vee$. Fix also two integers $r \geq 2$ and d. Finally, E will always denote a rank r polystable bundle on X of determinant dx and $\mathrm{SU_X}(r, d)$ is the moduli space of rank r semi-stable vector bundles of determinant $\mathcal{O}(dx)$ on X.

I. Around the Coble quartic

In this section, we first recall for the non-expert reader some facts about the geometry of the Kummer variety (essentially due to Mumford, Kempf and Khaled). We explain then the link between the Coble quartic and the Kummer of the jacobian JX of a canonical genus 3 curve X on one hand, and, in the other hand the identification of the Coble quartic and $\mathrm{SU_X}(2, 0)$. Finally, we state our result about the Coble quartic.

Generalities on linear system on abelian varieties

Let A be an abelian variety and L an ample line bundle on it. It is well-known that L^n is very ample if $n \geq 3$. A more subtle result due to Kempf (see [Kh1]) says that the products

$$H^0(A, L^n) \otimes H^0(A, L^m) \to H^0(A, L^{n+m})$$

are surjective if $n \geq 2$ and $m \geq 3$. In particular, the complete linear system $A \to |L^n|^\vee$ is projectively normal. Let I_n be the corresponding homogeneous ideal

$$I_n = \mathrm{Ker}\left(\bigoplus_{k \geq 0}^{\infty} \mathrm{Sym}^k H^0(A, L^n) \twoheadrightarrow \bigoplus_{k \geq 0}^{\infty} H^0(A, L^{nk})\right).$$

Kempf and Khaled (cf. [Kh1]) have also shown that I_n is generated by quadrics and cubics if $n \geq 3$ and by quadrics if $n \geq 4$.

The situation is very different for $n = 2$. In this case, the linear system $|L^2|$ is in general only base point free.

From now on, A is a principally polarized irreducible abelian variety

Let Θ be a symmetric divisor representing the polarization. Then the line bundle $\mathscr{L} = \mathcal{O}(2\Theta)$ does not depend on the particular choice of such a Θ. Then, the canonical morphism

$$\kappa_A : A \to |\mathscr{L}|^\vee$$

induces an embedding of $A/\pm 1$ in $|\mathscr{L}|^\vee$. The image $\kappa(A)$ of κ_A is the Kummer variety of A. The line bundle $\mathscr{L}$ is canonically $\{\pm 1\}$-linearized (as the pull-back of $\kappa_A^* \mathcal{O}(1)$ and one has

$$H^0(\kappa(A), \mathcal{O}(n)) = H^0(A, \mathscr{L}^n)^+$$

where $H^0(A, \mathscr{L}^n)^+$ is the invariant part of $H^0(A, \mathscr{L}^n)$ under -1.

Remark. If $k = \mathbf{C}$, the link with the classical geometry of theta functions can be explained as follows. The Chern class $c_1(\mathscr{L}) \in H^2(A, \mathbf{Z})$ give a symplectic non degenerate $\mathbf{Z}$-bilinear form on $H_1(A, \mathbf{Z})$ which depends only on the polarization $[\mathscr{L}]$. The abelian variety A is the quotient of $T_0(A)$ by the lattice $\Gamma = h_1(A, \mathbf{Z})$.

Choose a symplectic basis $(\gamma_1, \ldots, \gamma_{2g})$ of Γ. Then $(\gamma_1, \ldots, \gamma_g)$ is a basis of $T_0(A)$ which therefore will be identified to $\mathbf{C}^g$. The matrix τ of coordinates of $(\gamma_{g+1}, \ldots, \gamma_{2g})$ in $\mathbf{C}^g$ is called the period matrix and one has by construction $A = \mathbf{C}^g/(\mathbf{Z}^g \oplus \tau \mathbf{Z}^g)$. With this description of A, the space $H^0(A, \mathscr{L}^n)$ becomes the space of order $2n$ theta functions (with respect of τ). The sections of $H^0(\kappa(A), \mathcal{O}(n))$ are the even theta functions of order $2n$.

Notice the formula $\dim H^0(\kappa(A), \mathcal{O}(n)) = 2^{g-1}(n^g + 1)$ (see [C], page 56 in the case $k = \mathbf{C}$ and formula $(+)$ below in general).

The theorem of Mumford-Koizumi [Ko] says that the Kummer variety is projectively normal if and only if A has no vanishing theta-nulls. Geometrically, this condition means that among the 2^{2g} symmetric theta divisors Θ representing the principal polarization, there is no Θ vanishing at the origin with even multiplicity. In the case where A is the jacobian of a smooth curve X, this condition simply means that there is no effective even theta-characteristic on X.

Suppose now that A has no one vanishing theta-null

The number $h(n, g)$ of independent degree n hypersurfaces containing $\kappa(A)$ is

$$h(n, g) = \binom{2^g + n - 1}{n} - 2^{g-1}(n^g + 1).$$

In particular $\kappa(A)$ is never contained in a quadric. If $g = 2$, one has $h(3, g) = 0$ and $h(4, g) = 1$: the Kummer surface is a quartic surface with $2^{2g} = 16$ nodes. Suppose now $g \geq 3$; then $h(3, g) > 0$ and $\kappa(A)$ is always contained in cubics. Let I_A (resp. $\mathscr{I}_A$) be the homogeneous ideal (resp. the sheaf of ideal) of $\kappa(A)$; by definition, one has

$$I_A = \operatorname{Ker}\left(\bigoplus_{k \geq 0}^{\infty} \operatorname{Sym}^k H^0(A, \mathscr{L}) \twoheadrightarrow \bigoplus_{k \geq 0}^{\infty} H^0(A, \mathscr{L}^k)^+ \right)$$

and the dimension of the degree δ component $I_A[d] = \dim H^0(\kappa_A, \mathcal{O}(\delta))$ of I_A is $h(g, \delta)$. One has the

THEOREM 1 ([Kh 2]). *With the notations above,* I_A *is generated by* **quartics** *in degree* ≥ 4.

In particular, the ideal $\mathscr{I}_A$ is itself generated by quartics.

Remark. When $g = 3$ or A is generic, this theorem was known from Wirtinger [Wi].

If $g = 3$: the Kummer variety is embedded in $\mathbf{P}^7 = |\mathscr{L}|^{\vee}$. In this case, $\dim \mathrm{I}_{\mathrm{A}}[3] = 8$ and $\dim \mathrm{I}_{\mathrm{A}}[4] = 70 > 64 = \dim \mathrm{I}_{\mathrm{A}}[3]. \dim \mathrm{H}^0(\mathbf{P}^7, \mathcal{O}(1))$ which shows that $\mathrm{I}_{\mathrm{A}}[4]$ **is not generated by cubics.**

(2) One can ask the following natural questions:

(2.1) Is the Kummer $\kappa(\mathrm{A})$ variety *set-theoretically* defined by the cubics of $\mathrm{I}_{\mathrm{A}}[3]$?

(2.2) Is the Kummer $\kappa(\mathrm{A})$ variety *schematically* defined by the cubics of $\mathrm{I}_{\mathrm{A}}[3]$? In other words, is the canonical map $\mathrm{I}_{\mathrm{A}}[3] \otimes \mathcal{O}(-3) \to \mathscr{I}_{\mathrm{A}}$ surjective?

The dimension 3 *case*: *the Coble quartic*

Let us first recall general facts from [M] about the Mumford group $\mathrm{G}(\mathscr{L})$, the group of pairs (α, a) where α is an isomorphism of $\tau_a^*(\mathscr{L})$ on $\mathscr{L}$ (τ_a is the translation by a). The morphism $(\alpha, a) \mapsto a$ surjects onto A_2 and one has a canonical exact sequence

$$1 \to k^* \to \mathrm{G}(\mathscr{L}) \to \mathrm{A}_2 \to 0$$

with k^* central. Let V be a $\mathrm{G}(\mathscr{L})$-module. We say that V is of level n if $0 < \dim(\mathrm{V}) < \infty$ and k^* acts on V by $t \mapsto t^n$. If V is irreducible, there exist an unique integer n such that V is of level n. The basic fact about the representation theory of $\mathrm{G}(\mathscr{L})$ is the following classical observation:

LEMMA 3. *Let w be an odd number. There exists only one* (*up to isomorphism*) *irreducible* $\mathrm{G}(\mathscr{L})$*-module* V_w *of level w. Moreover, if* V *is any* $\mathrm{G}(\mathscr{L})$*-module of level w, then* V *is a sum of copies of* $\mathrm{V}(w)$.

Let us explicit the representation $\mathrm{V}(w)$. Let $\mathrm{G}(2)$ be the group which is set theoretically the product $k^* \times \mathbf{F}_2^g \times \widehat{\mathbf{F}_2^g}$ with the group law

$$(\lambda, \varepsilon, \hat{\varepsilon}) \cdot (\mu, \eta, \hat{\eta}) = ((-1)^{\langle \hat{\eta}, \varepsilon \rangle} \lambda\mu, \varepsilon + \eta, \hat{\varepsilon} + \hat{\eta}).$$

Because the characteristic is $\neq 2$, the group $\mathrm{G}(\mathscr{L})$ is (non canonically) isomorphic to $\mathrm{G}(2)$: an isomorphism is called a level 2 theta structure on A.

Remark. If $k = \mathbf{C}$, the polarization can be seen as a symplectic non degenerate form on $H_1(A, \mathbf{Z})$ (for instance, it is the Chern class of a symmetric divisor Θ as above). The choice of a symplectic basis $\mathscr{B}$ of $H_1(A, \mathbf{Z})$ defines the period matrix τ of A and identifies A and $\mathbf{C}^g/(\mathbf{Z}^g + \tau\mathbf{Z}^g)$. Moreover $\mathscr{B}$ defines canonically a theta structure on $\mathscr{L}$.

From now on, A is endowed with a level 2 structure

Then $V(w)$ can be viewed as $\mathrm{Map}(\mathbf{F}_2^g, k)$ with the action given by the formula

$$(\lambda, \varepsilon, \hat{\varepsilon}) \cdot f(v) = (-1)^{\langle \hat{\varepsilon}, v+\varepsilon \rangle} \lambda^w f(v+\varepsilon), \qquad v \in \mathbf{F}_2^g, \qquad f \in \mathrm{Map}(\mathbf{F}_2^g, k).$$

Of course, $H^0(A, \mathscr{L})$ is a level 1 module of dimension $2^g = \dim \mathrm{Map}(\mathbf{F}_2^g, k)$. Therefore, there exists a $G(\mathscr{L})$ isomorphism $V(1) \tilde{\to} H^0(A, \mathscr{L})$ well defined up to a non zero scalar. The action on ± 1 is given by the *inverse formula* (p. 331 of [M]): this is just the multiplication by -1 on $\mathbf{F}_2^g$. More generally, using a level $2n$ theta structure, $H^0(A, \mathscr{L}^n)^+$ can be identified with the **even** functions on $(\mathbf{Z}/2n\mathbf{Z})^g$ and we get the formula

$$\dim H^0(A, \mathscr{L}^n)^+ = 2^{g-1}(n^g + 1). \tag{+}$$

For $v \in \mathbf{F}_2^g$, let $Z_v \in V(w)$ be the characteristic function of $\{v\}$. We have then

$$(\lambda, \varepsilon, \hat{\varepsilon}) \cdot Z_v = (-1)^{\langle \hat{\varepsilon}, \varepsilon+v \rangle} \lambda^w Z_{v+\varepsilon}.$$

Remark. If $k = \mathbf{C}$ and $A = \mathbf{C}^g/(\mathbf{Z}^g \oplus \tau\mathbf{Z}^g)$ as above, $H^0(A, \mathscr{L})$ is canonically identified with the space of level 2 theta function with respect of τ and Z_v corresponds to the theta function with characteristic $(v, 0)$

$$\theta\begin{bmatrix} v \\ 0 \end{bmatrix}(2z, 2\tau).$$

We now focus our attention to the 3-dimensional case. Because A is assumed to be irreducible with no vanishing theta-nulls, A is the jacobian JX of canonical curve X of genus 3.

Then $I_{JX}[3]$ is a level 3 $G(\mathscr{L})$-module of dimension 8 and therefore isomorphic to $V(3)$. Let $C_v \in I_{JX}[3]$ be the cubic corresponding to Z_v. The cubic C_0 is invariant under the subgroup $I = \{1\} \times \{0\} \times \widehat{\mathbf{F}_2^g}$ of $G(2) = G(\mathscr{L})$. Notice that a monomial $Z_u Z_v Z_w$ is invariant under I if and only if $w = u + v$. We think to $\mathbf{F}_2^3$ as $\mathbf{P}^2(\mathbf{F}_2) \cup 0$. It follows that C_0 is of the form

$$\sum_{\alpha_v \in \mathbf{F}_2^3} Z_0 Z_v^2 + \sum_{\underline{d} \in \mathbf{P}^2 \mathbf{F}_2^\vee} \alpha_{\underline{d}} Z^{\underline{d}}$$

where $Z^{\underline{d}} = \Pi_{0 \neq v \in \underline{d}} Z_v$. The group I acts on C_v by $(1, 0, \hat{\varepsilon}) \mapsto (-1)^{\langle \hat{\varepsilon}, v \rangle}$ and therefore $Z_v \cdot C_v$ is also invariant under I and can be written using the coefficients α_v and $\alpha_{\underline{d}}$ of C_0. Let C be the quartic

$$\sum_{v \in \mathbf{F}_2^g} Z_v \cdot C_v.$$

By direct computation, one observes $\partial C / \partial Z_0 = 4C_0$.
By construction, C is invariant under $\{1\} \times \mathbf{F}_2^g \times \{0\}$ which gives $\partial C / \partial Z_v = 4C_v$

for all $c \in \mathbf{F}_2^g$. In particular, there exists a quartic containing $\kappa(JX)$ in its singular locus. Notice that C being invariant under I as $Z_v \cdot C_v$ is, the quartic div(C) of $|\mathscr{L}|^\vee$ is invariant under the whole $G(\mathscr{L})$. Let us prove the

LEMMA 4. *The Coble quartic is the unique* $G(\mathscr{L})$*-invariant quartic in* $|\mathscr{L}|^\vee$ *containing* $\kappa(JX)$ *in his singular locus.*

Remark. This is probably the meaning of the assertion (6) page 106 of [C].
Proof of the lemma: let F be an equation of a $G(\mathscr{L})$-invariant quartic. There exists a character χ of $G(\mathscr{L})$ such that (with the notations above)

$$F((\lambda, \varepsilon, \hat{\varepsilon}) \cdot Z_v) = \chi((\lambda, \varepsilon, \hat{\varepsilon})) \cdot F(Z_v). \tag{1}$$

Let $p \in \mathbf{F}_2^g$ such that $\chi(1, 0, \hat{\varepsilon}) = (-1)^{{}^t p \cdot \hat{\varepsilon}}$. Differentiating (1) with respect of Z_v gives that there exists a scalar λ_v such that

$$\partial F / \partial Z_v = \lambda_v \, \partial C / \partial Z_{v+p}. \tag{2}$$

Because $p \neq 2$, the endomorphism given by $Z_v \to Z_{v+p}$ is semi-simple and the system (2) becomes

$$dF = dC \circ M \tag{3}$$

where M is a semi-simple matrix. After eventually replacing F and C by projectively equivalent hypersurfaces, one can assume that M is *diagonal.* In this case, (3) gives that F and some multiple of C differs by a p^{th}-power of some form G. By degree consideration (using $p \neq 2$), one obtains $G = 0$. □

To go further, one has to construct geometrically the Coble quartic: this has been done in [N-R2].

Rank 2-vector bundles and the Coble quartic

Let X be a canonical curve of genus 3 (recall that $p = \mathrm{char}(k) \neq 2$). Let A be the jacobian JX. Let Θ be the (canonical) theta divisor on $\mathrm{Pic}^{g-1}(X)$. If $[E] \in SU_X(2, 0)$, the locus

$$\Theta_E = \{L \in \mathrm{Pic}^{g-1}(X) \mid H^0(X, E \otimes L) \neq 0\}$$

is a divisor in $\mathbf{P}H^0(\mathrm{Pic}^{g-1}(X), \mathcal{O}(2\Theta))$ and the $G(\mathcal{O}(2\Theta))$-equivariant morphism $\tilde{\varphi}_X$

$$\tilde{\varphi}_X : \begin{cases} SU_X(2, 0) & \to \quad \mathbf{P}H^0(\mathrm{Pic}^{g-1}(X), \mathcal{O}(2\Theta)) \\ E & \mapsto \quad \Theta_E \end{cases}$$

is an embedding onto a quartic C(X) by [N&R]. The Riemann's bilinear relations furnish a canonical identification $|\mathscr{L}|^\vee = \mathbf{P}H^0(\mathrm{Pic}^{g-1}(X), \mathcal{O}(2\Theta))$ and the corresponding $G(\mathscr{L})$-equivariant morphism $\varphi_X : SU_X(2, 0) \hookrightarrow |\mathscr{L}|^\vee$ restricts on $\mathrm{Sing}(SU_X(2, 0)) = A/\pm 1$ to the Kummer morphism κ_{JX}.

Exercise. Suppose that k is of characteristic 0. Using the Verlinde formula, prove directly that φ_X is an embedding. Notice that this formula was unknown when Narasimhan and Ramanan stated and proved their theorem

Notice that $\varphi_X[\mathrm{Sing}(SU_X(2, 0))] = \kappa(JX)$. In particular C(X) is a quartic singular exactly along $\kappa(JX)$. The $G(\mathscr{L})$-invariance and the lemma above shows then that **C(X) is the Coble quartic**. In fact, at least if k is of characteristic zero, a stronger statement due to Beauville than the lemma 4 is true.

PROPOSITION 5 (Beauville). *Assume that* $p = \mathrm{char}(k)$ *is zero. The Coble quartic is the unique quartic in* $|\mathscr{L}|^\vee$ *containing* $\kappa(JX)$ *in its singular locus.*

Sketch of proof. let F be a quartic singular along $\kappa(\mathrm{JX})$. Notice that C is not a cone. The condition to be a cone being algebraic, the generic member of the pencil $(\mathrm{C}+t\mathrm{F})_{t\in k}$ is not a cone. In particular, one can assume that F is not a cone. The partial derivatives of C are then a basis of the cubics containing the Kummer. This fact together with the fact that F is not a cone proves that F and C have the same jacobian ideal. A result of [Do] ensures that F and C are related by an invertible linear transformation T. Because C is singular exactly along $\kappa(\mathrm{JX})$, the linear morphism T leaves $\kappa(\mathrm{X})$ invariant. An easy adaptation of the Torelli theorem proves that T is induced by an element of Aut(X) which leaves obviously $\mathrm{div}(\mathrm{C}) = \varphi_{\mathrm{X}}(\mathrm{SU}_{\mathrm{X}}(2, 0))$ invariant. □

The beautiful corollary of the equality $\mathrm{Sing}(\mathrm{C}(\mathrm{X})) = \kappa(\mathrm{JX})$ is the

THEOREM 6 [N-R2]. *The Kummer variety of the jacobian of a genus* 3 *canonical curve is set theoretically defined by cubics.*

This gives a positive answer to the question (2.1).

More surprisingly (recall that $\mathrm{I}_{\mathrm{JX}}[3]$ does not generate $\mathrm{I}_{\mathrm{JX}}[4]$), the main result of the present paper is

THEOREM IV.6. *Assume that* $p = \mathrm{char}(k)$ *is zero. The scheme defined by the eight partial derivatives of the Coble quartic is reduced.*

The proposition and the fact that these eight cubics defines set-theoretically the Kummer variety of $\kappa(\mathrm{JX})$ proves the

COROLLARY. *Assume that* $p = \mathrm{char}(k)$ *is zero. Let* X *be a canonical genus* 3 *curve over. The sheaf of ideals* $\mathscr{I}_{\mathrm{JX}}$ *of the Kummer variety* $\kappa(\mathrm{JX})$ *is generated by cubics.*

These facts will be proved in section IV and give a positive answer to the question (2.2).

II. Local structure of $\mathrm{SU}_{\mathrm{X}}(r, d)$ and classical invariant theory

From now, k is of characteristic zero. If $(\mathrm{X}_i)_{1\leq \mathrm{N}}$ (resp. $(n_i)_{1\leq \mathrm{N}}$) are indeterminates (resp. non negative integers), let me denote by $\underline{\mathrm{X}}$, $\underline{n}$ and $\mathrm{X}^{\underline{n}}$ the N-tuple $\underline{\mathrm{X}} = (\mathrm{X}_i)$, the multi-index $\underline{n} = (n_i)$ and the product $\mathrm{X}^{\underline{n}} = \Pi_{i=1}^{\mathrm{N}} x_i^{n_i}$ respectively. For

V a finite dimensional vector space with dual $V^\vee$, the ring $k[V]$ (resp. $k[[V]]$) is the polynomial ring $\mathrm{Sym}\, V^\vee$ (resp. its completion at the origin).

It is well known (see [S] for instance) that the key ingredient to analyse the local structure of $SU_X(r, d)$ is the étale slice theorem of Luna. Let us recall this analysis.

Take n big enough such that $E(nx)$ is globally generated and has no H^1 for every semi-stable rank r vector bundle E of degree d (every n such that $rn + d > r(2g - 1)$ has this property). Let $\chi = \chi(E(nx))$ be the corresponding Euler-characteristic. In Grothendieck's scheme $\mathcal{Q}uot$ which parametrizes quotients

$$\mathcal{O}(-nx)^{\oplus\chi} \twoheadrightarrow E,$$

let $\mathcal{Q}$ be the open set whose closed points correspond to such quotients with the following properties:

(i) E is semi-stable of rank r and degree d.
(ii) $H^1(X, E(nx)) = 0$ and the natural map $H^0(X, \mathcal{O}^{\oplus\chi}) \to H^0(X, E(nx))$ is onto.

Let $\mathscr{E}$ be the universal quotient bundle on $\mathcal{Q}$. The scheme $\mathcal{Q}$ is smooth and the semi-simple group $G = PGL_\chi$ acts on it. The moduli space $SU_X(r, d)$ is the GIT quotient of $\mathcal{Q}/G$.

For (E_i) be a set of a vector bundles over X, the kernel of the trace map

$$\mathrm{Ker}(\oplus \mathrm{Ext}^1(E_i, E_i) \xrightarrow{\oplus \mathrm{Tr}_i} H^1(X, \mathcal{O}))$$

will be denoted by $(\oplus \mathrm{Ext}^1(E_i, E_i))_0$.

Let $q = [\mathcal{O}(-nx)^{\oplus\chi} \twoheadrightarrow E]$ a point of $\mathcal{Q}$ and gr the corresponding graded object. By the very definition of semi-stability, there exists a G-stable open affine neighborhood Ω and the fibre of $\Omega \to \Omega/G$ at [E] contains a unique closed orbit $G(q)$. This orbit is either characterized as being of minimal dimension, or as having an isotropy group $G_q = \mathrm{Aut}(E)$ of maximal dimension: one checks that this exactly means that E is *polystable*. The corresponding orbit is closed. The closedness of the orbit and the smoothness of Ω corresponding to E allows us to use the Luna étale slice theorem (more precisely the remarque page 97 of [Lu]) which gives precisely the:

THEOREM 1 (Luna). *Let* E *be a polystable of rank* r *and determinant* $\mathcal{O}(dx)$. *There exists an étale neighborhood of* $(\mathrm{Ext}_0^1(E, E)/\mathrm{Aut}(E), 0)$ *(resp.* $(SU_X(r, d), E)$*) which are isomorphic.*

Remarks 2. (a) The group Aut(E) acts by functoriality on both arguments of $\mathrm{Ext}^1_0(\mathrm{E}, \mathrm{E})$. Notice that the scalars acts trivially and therefore

$$\mathrm{Ext}^1_0(\mathrm{E}, \mathrm{E})/\mathrm{Aut}(\mathrm{E}) = \mathrm{Ext}^1_0(\mathrm{E}, \mathrm{E})/\mathrm{G}_\mathrm{E}$$

where

$$\mathrm{G}_\mathrm{E} = \mathrm{Ker}\{\mathrm{Aut}(\mathrm{E}) \xrightarrow{\det} \mathrm{G}_m\}.$$

(b) We'll say for short that $\mathrm{SU}_\mathrm{X}(r, d)$ is ètale locally isomorphic to $\mathrm{Ext}^1_0(\mathrm{E}, \mathrm{E})/\mathrm{G}_\mathrm{E}$ at E.

(c) By definition, one has

$$\mathrm{Ext}^1_0(\mathrm{E}, \mathrm{E})/\mathrm{G}_\mathrm{E} = \mathrm{Spec}(\mathrm{A}_\mathrm{E})$$

where A_E is the ring of polynomial maps on $\mathrm{Ext}^1_0(\mathrm{E}, \mathrm{E})$ invariant under G_E.

Unless otherwise stated, all bundles will be polystable

COROLLARY. *The local ring* $\hat{\mathcal{O}}_{\mathrm{SU}_\mathrm{X}(r,d),[\mathrm{E}]}$ *depends only on the numerical invariants of* X *and* E.

Suppose once for all that E is non stable ($l(\mathrm{E}) > 1$). One has of course the inequalities

$$1 \leq \dim \mathrm{G}_\mathrm{E} \leq r^2 - 1 \tag{1}$$

with equality on the left hand side (resp. right hand side) of (1) if $\mathrm{G}_\mathrm{E} = \mathrm{G}_m$ (**Case 1**) (resp. $\mathrm{G}_\mathrm{E} = \mathrm{Sl}_r$ (**Case 2**)). Let's examine these 2 cases.

III. Case 1: $\mathrm{G}_\mathrm{E} = \mathrm{G}_m$

In this case, E is a direct sum $\mathrm{E} = \mathrm{E}_1 \oplus \mathrm{E}_2$ where E_i is stable of slope d/r, rank $r_i \neq 0$ and $\mathrm{E}_1 \not\simeq \mathrm{E}_2$. Each element $(\alpha_1, \alpha_2) \in \mathrm{G}_\mathrm{E}(k)$ acts by multiplication by $\alpha_j \cdot \alpha_i^{-1}$ on each factor $\mathrm{Ext}^1(\mathrm{E}_i, \mathrm{E}_j)$ of

$$\mathrm{Ext}^1(\mathrm{E}, \mathrm{E}) = \oplus \mathrm{Ext}^1(\mathrm{E}_i, \mathrm{E}_j).$$

Let

$$d_{i,j} = \dim \operatorname{Ext}^1(E_i, E_j) = \begin{cases} r_i r_j (g-1) & \text{if } i \neq j \\ r_i r_j (g-1) + 1 & \text{if } i = j \end{cases}$$

and $X_{i,j}^k$, $k = 1, \dots, d_{i,j}$ a basis of $\operatorname{Ext}_0^1(E_i, E_j)^\vee$.

The ring

$$A_E \subset k[\operatorname{Ext}^1(E, E)] = k[X_{i,j}^k, 1 \leq i, j \leq n, 1 \leq k \leq d_{i,j}]$$

is the ring generated by $(\operatorname{Ext}^1(E_1, E_1) \oplus \operatorname{Ext}^1(E_2, E_2))_0^\vee$ and the products

$$\langle X_{1,2}^k \cdot X_{2,1}^l, 1 \leq k, l \leq d_{i,j} \rangle.$$

Let $\mathcal{S}$ be the affine cone of the Segre variety

$$\mathbf{P}(\operatorname{Ext}^1(E_1, E_2)) \times \mathbf{P}(\operatorname{Ext}^1(E_2, E_1)) \subset \mathbf{P}(\operatorname{Ext}^1(E_1, E_2) \otimes_k \operatorname{Ext}^1(E_2, E_1)).$$

PROPOSITION 1. *There is an isomorphism*

$$\operatorname{Spec}(A_E) \xrightarrow{\sim} (\operatorname{Ext}^1(E_1, E_1) \oplus \operatorname{Ext}^1(E_2, E_2))_0 \times \mathcal{S}.$$

Note that $\operatorname{Spec}(A_E)$ is a cone.

COROLLARY 2. *With the previous notations,* $SU_X(r, d)$ *is étale locally isomorphic to* $(\operatorname{Ext}^1(E_1, E_1) \oplus \operatorname{Ext}^1(E_2, E_2))_0 \times \mathcal{S}$ *at* $E = E_1 \oplus E_2$.

Using furthermore that the multiplicity at the origin of the affine cone of a projective variety is just its degree, one gets the

COROLLARY 3. *The Zariski tangent space is*

$$(\operatorname{Ext}^1(E_1, E_1) \oplus \operatorname{Ext}^1(E_2, E_2))_0 \oplus (\operatorname{Ext}^1(E_1, E_2) \otimes_k \operatorname{Ext}^1(E_2, E_1)).$$

Moreover the multiplicity of $SU_X(r, d)$ *at* [E] *is*

$$\operatorname{mult}_{[E]}(SU_X(r, d)) = \binom{2 \cdot d^{1,2} - 2}{d_{1,2} - 1}$$

(*recall that* $d_{1,2} = r_1 \cdot r_2(g-1)$).

Remark 4. Note that $\binom{2 \cdot d_{1,2} - 2}{d_{1,2} - 1} = 1$ if and only if $g = 2$ and $r_1 = r_2 = 1$. Using that the singular locus of $\mathrm{Sing}\,\mathrm{SU}_X(r, d)$ is closed in $\mathrm{SU}_X(r, d)$, one obtains easily in this way another proof of the fact that $\mathrm{Sing}\,\mathrm{SU}_X(r, d)$ is the non stable locus, except if $g = r = 2$ and $d = 0$.

IV. Case 2: $G_E = Sl_r$

In this case, $E = L^{\oplus r}$ where $L^{\otimes r} = \mathcal{O}$. Using a translation by L^{-1} which induces an automorphism of $\mathrm{SU}_X(r, 0)$, one may assume $L = \mathcal{O}$. The ring A_E is the ring of polynomial maps on $M_0(r)^g$ invariant under Sl_r. This group Sl_r acts diagonally by conjugation on each factor $M_0(r)$, which is the space of traceless matrices of size r. For general r, Procesi [P] and Rasmyslev [Ras] have obtained the following description of the first 2 syzigies of A_E:

- Generators: for every sequence $\underline{i} = (i_1, \ldots, i_{N(\underline{i})})$ of integers of $[1, \ldots, g]$, let $t_{\underline{i}}$ be the invariant polynomial map

$$t_{\underline{i}}: \begin{cases} M_0(r)^g & \rightarrow \quad k \\ (X_1, \ldots, X_g) & \mapsto \quad Tr(X_{i_1}, \ldots, X_{i_N}) \end{cases}.$$

  Then the $t_{\underline{i}}$ with $N(\underline{i}) \leq 2^g - 1$ form a system of generators.
- Relations between all the $t_{\underline{i}}'s$: Let P_X be the characteristic polynomial of the general matrix X. The homogeneous polynomial $X \mapsto Tr(X \cdot P_X(X))$ gives by polarization (namely by taking the total differential of order $g + 1$) a multilinear map $F(H_1, \ldots, H_{g+1})$ where the H_i's runs in the set of all possible monomials in the X_i.

Although this description is quite explicit, it looks difficult to obtain a complete **finite** set of relations between the (finite) set of generators constructed above.

As far as I know, the only case where such a finite description is available is for $r = 2$(*).

(*) According to some experts of invariant theory, it is more or less hopeless to obtain such a finite description of A_E in the general case.

Assume moreover that $r = 2$

In this case, A_E can be described by using classical results of the geometric invariant theory of $SO_3(k)$. Following [LeB], I.4, let me briefly explain this description.

For $X \in M_0(2)$, let $u(X) = (u_1(X), u_2(X), u_3(X)) \in k^3$ be defined by the equality

$$X = \begin{pmatrix} u_1(X) & u_2(X) - \sqrt{-1}u_3(X) \\ u_2 + \sqrt{-1}u_3(X) & -u_1(X) \end{pmatrix}.$$

By theorem 4.1 of [LeB] the isomorphism

$$\begin{cases} M_0(2) & \to & k^3 \\ X & \mapsto & u(X) \end{cases}$$

induces an identification of A_E with the polynomial maps of $(k^3)^{\oplus g}$ invariant under the canonical diagonal action of $SO_3(k)$.

Let $T_{i,j}$ be the invariant function corresponding to $(u_1, \ldots, u_n) \mapsto (u_i \cdot u_j)$ (scalar product), namely $T_{i,j}(X_1, \ldots, X_g) = \frac{1}{2}Tr(X_iX_j)$.

Let $T_{i,j,k}$ be the invariant function corresponding to $(u_1, \ldots, u_n) \mapsto u_i \wedge u_j \wedge u_k$ (the wedge product lives in $\wedge^3 k^3 \overset{\text{can}}{=} k$), namely $T_{i,j,k}(X_1, \ldots, X_g) = Tr(X_iX_jX_k)$. With some abuse of notation, one can now use the results of H. Weyl [W], theorem (2.9 A) and (2.17 B) and it's sequel on page 77 which says the following:

- Generators for the invariants maps under $O_3(k)$: the set

  $$\langle T_{i,j} \rangle$$

- Relations between the generators: the 4-minors of the $g \times g$-symmetric matrix

  $$\begin{pmatrix} & \vdots & \\ \cdots & T_{i,j} & \cdots \\ & \vdots & \end{pmatrix}$$

  and the relations $T_{i,j} = T_{j,i}$. One recognizes the coordinate ring of the (affine) cone C of symmetric matrices of rank ≤ 3. This scheme is well understood: it is integral and normal [easy], Cohen-Macaulay [H-R], it multiplicity at the origin (or the degree of the projectivization PC) is known [H-T]...

- **Generators** for the $SO_3(k)$-invariants maps are: the $T_{i,j}$'s and the $T_{i,j,k}$'s.
- **Relations**: the previous 4-minors and:

$$T_{i_1,i_2,i_3} \cdot T_{j_1,j_2,j_3} = \det([T_{i_n j_m}]_{1 \leq n,m \leq 3}) \tag{1}$$

$$T_{i_0,i_4}T_{i_1,i_2,i_3} - T_{i_1,i_4}T_{i_0,i_2,i_3} + T_{i_2,i_4}T_{i_0,i_1,i_3} - T_{i_3,i_4}T_{i_0,i_1,i_2} \tag{2}$$

and the relations given by the symmetry of $T_{i,j}$ and the skew symmetry of $T_{i,j,k}$ in the indices.

Let $\bar{C}$ be the tangent cone of $\mathrm{Spec}(A_E)$ at the origin. It is the subscheme of

$C \times \mathrm{Spec}\, k[T_{i,j,k}]$

whose ideal is generated by

$$T_{i_1,i_2,i_3} \cdot T_{j_1,j_2,j_3} \tag{3}$$

and

$$T_{i_0,i_4}T_{i_1,i_2,i_3} - T_{i_1,i_4}T_{i_0,i_2,i_3} + T_{i_2,i_4}T_{i_0,i_1,i_3} - T_{i_3,i_4}T_{i_0,i_1,i_2} \tag{4}$$

and relations given by the skew symmetry of $T_{i,j,k}$ in the indices. The ideal described above is the ideal of initial forms of the ideals given by (1) and (2).

Let $k(C)$ be the function field of C and K its algebraic closure. Note that, according to (3), the ideal $I_{C/\bar{C}}$ of C in $\bar{C}$ is nilpotent. This implies by [F] example 4.3.4, the formula

$$\mathrm{mult}_0\, \bar{C} = \mathrm{length}\,\mathcal{O}_{C,\bar{C}} \cdot \mathrm{mult}_0(C). \tag{5}$$

The next formula is clear

$$\mathrm{length}\,\mathcal{O}_{C,\bar{C}} = 1 + \dim_{k(C)} I_{C/\bar{C}} \otimes_{\mathcal{O}_{\bar{C}}} k(C) = 1 + \dim_K I_{C/\bar{C}} \otimes_{\mathcal{O}_{\bar{C}}} K. \tag{6}$$

One therefore has to compute the dimension over K of the sub-vector space V_T of the dual space of $W = \oplus K \cdot T_{i,j,k}$ of equations given by (4) and the skew symmetry condition for the $T_{i,j,k}$. This vector space is isomorphic to $I_{C/\bar{C}} \otimes_{\mathcal{O}_{\bar{C}}} K$.

LEMMA 2. *The dimension of* V_T *depends only on the conjugation class of* T.

Proof. The symmetric matrix

$$T = [T_{i,j}] \in M_g(K)$$

acts on the dual vector space V of K^g. Let $(e_i)_{1 \leq i \leq g}$ be the canonical basis and $(e_i^\vee)_{1 \leq i \leq g}$ its dual basis. The map

$$\begin{cases} W & \rightarrow & \wedge^3 V \\ T_{i,j,k} & \mapsto & e_i \wedge e_j \wedge e_k \end{cases}$$

identifies W and $\wedge^3 V^\vee$. With this identification, the relations (4) become

$$-T(e_{i_4}^\vee) \lrcorner e_{i_0} \vee e_{i_1} \vee e_{i_2} \vee e_{i_3}$$

and $\dim V_T$ is the corank of

$$\begin{cases} V^\vee \otimes \wedge^4 V & \rightarrow & \wedge^3 V \\ x^\vee \otimes y & \mapsto & T(x^\vee) \lrcorner y \end{cases}$$

This map depends only on the conjugation class of T. □

One can therefore assume that T is diagonal of eigenvalues λ_i with $\lambda_i = 0$ for $3 < i \leq g$ and $\lambda_i \neq 0$ if $i \leq 3$.

Let us prove this simple lemma

LEMMA 3. *The dimension of* V_T *is* $[g(g-1)(g-2)/6] - [(g-3)(g-4)(g-5)/6]$ *if* $g \geq 3$ *and* 0 *if* $g \leq 2$.

Proof. If $g \leq 2$, the vector space $\wedge^3 V$ is zero and so is V_T. Suppose $g \geq 3$. Consider an equation

$$T_{i_0,i_4} T_{i_1,i_2,i_3} - T_{i_1,i_4} T_{i_0,i_2,i_3} + T_{i_2,i_4} T_{i_0,i_1,i_3} - T_{i_3,i_4} T_{i_0,i_1,i_2} \tag{4}$$

defining V_T. If $i_4 \notin \{1, 2, 3\}$ or $i_4 \notin \{i_0, i_1, i_2, i_3\}$ then the equation (4) is trivial. Let me suppose that $i_4 \in \{1, 2, 3\}$ and for instance that $i_4 = i_0$. If $i_4 \in \{i_1, i_2, i_3\}$, the equation is just a consequence of the skew symmetry of T_{i_1,i_2,i_3}. To get a new relation, one has therefore further to suppose further that $i_4 \notin \{i_1, i_2, i_3\}$ and the equation becomes

$$\lambda_{i_0} \cdot \mathrm{T}_{i_1,i_2,i_3} = 0.$$

Of course, the other cases are obtained by symmetry. One has proved the following: the equations (4) are non trivial if and only if

$$\{i_0, i_1, i_2, i_3\} = \{i, j, k\} \cup \{i_4\} \quad \text{and} \quad i_4 \notin \{i, j, k\}.$$

In this case the relation (4) becomes

$$\mathrm{T}_{i,j,k} = 0,$$

or equivalently

$$\mathrm{T}_{i,j,k} = 0 \quad \text{if} \quad \{i, j, k\} \cap \{1, 2, 3\} \neq \emptyset.$$

In particular this corank is

$$\dim \wedge^3 \mathrm{V} - \mathrm{A}^3_{g-3} = \frac{g(g-1)(g-2)}{6} - \frac{(g-3)(g-4)(g-5)}{6}. \qquad \square$$

The degree d^r_g of the locus (*) of $g \times g$-symmetric matrices of corank $\geq r$ is computed in [H–T], proposition 12.b:

$$d^r_g = \deg \mathbf{PC} = \prod_{\alpha=0}^{r-1} \frac{\binom{g+\alpha}{r-\alpha}}{\binom{2\alpha+1}{\alpha}}.$$

Using the formulas (5) and (6) and the lemma 3, one obtains the

(*) In [H-T], this locus is endowed with the reduced scheme structure. But it is known in full generality that the natural scheme structure given by the vanishing of the $(g-r+1)$-minors is Cohen-Macaulay [J] and generically reduced [easy] and therefore reduced. In our case ($r = g-3$), this reduceness is obvious, because C is a ring of invariants.

THEOREM 4. *The multiplicity of* $[\mathcal{O} \oplus \mathcal{O}]$ *in* $SU_X(2, 0)$ *is*

$$\left(1 + \frac{g(g-1)(g-2)}{6} - \frac{(g-3)(g-4)(g-5)}{6}\right) \cdot d_g^{g-3}$$

if $g \geq 3$ *and* 1 *if* $g = 2$.

Remarks 5.
1. The preceding discussion gives a precise description of the tangent cone $\bar{C}$ of $SU_X(2, 0)$. In particular, the Zariski tangent space at the trivial bundle is

   $$T_{[\mathcal{O} \oplus \mathcal{O}]} SU_X(2, 0) = \mathrm{Sym}^2 V \oplus \wedge^3 V.$$

2. One recovers the smoothness of $SU_X(2, 0)$ if $g = 2$.

In the case of a non hyperelliptic genus 3 curve, the generalized Θ divisor embeds $SU_X(2, 0)$ as the Coble quartic $SU_X(2, 0)$ in $\mathbf{P}H^0(J^2, 2 \cdot \Theta_{J^2})$ (see [N-R2] and [D-O] pages 184–185).

THEOREM 6. *The scheme defined by the eight partial derivatives of the Coble quartic* C(X) *is reduced.*

This theorem is the consequence of the

THEOREM 6 bis. (i) *The (étale) local equation of the Coble quartic at the trivial bundle is*

$$T^2 = \det([T_{i,j}]_{1 \leq i,j \leq 3})$$

in the affine space $\mathbf{A}^7$ *with coordinates* T, $T_{i,j}$ *with* $T_{i,j} = T_{j,i}$.

(ii) *The (étale) local equation of the Coble quartic at* $E = E_1 \oplus E_2$ *with* $E_1 \neq E_2$ *is a rank* 4 *quadric in* $\mathbf{A}^7$.

(iii) *The ideal generated by the* 8 *cubic equations which are the partials derivatives of the Coble quartic is prime.*

Proof. The first 2 points are clear from proposition II.2 and (1), (2). Let me prove (iii). Let $\tilde{K}$ the scheme defined by the partial derivatives of an equation of $\mathcal{C}$(X). The Kummer variety K(X) is the reduced scheme of $\tilde{K}$. It is therefore enough to prove that the completion of $\tilde{K}$ at each non stable point E of $K(X) \subset \tilde{K}$ is *reduced.* Because of the invariance of the Coble quartic under the

Theta group of $2 \cdot \Theta_{J2}$, one has 2 cases to examine: either E is trivial, or $E = E_1 \oplus E_2$ with $E_1 \neq E_2$. In the first case, by (i), the equations in $k[[T, T_{i,j}]]$ of the completion of $\tilde{K}$ at $[\mathcal{O} \oplus \mathcal{O}]$ are T and the 2×2-minors of $[T_{i,j}]_{1 \leq i,j \leq 3}$. It is precisely the (completion at the origin) of the cone over the Veronese surface in $\mathbf{P}^5$ (with homogeneous coordinates $T_{i,j}$) and K is therefore reduced. The second case is even simpler, $\tilde{K}$ being (the completion of) a 3-plane in $\mathbf{A}^7$ (the tangent of K(X)). □

V. The case $SU_X(3, 0)$ for of a genus 2 curve

Suppose in this section that X has genus 2 and let $\mathcal{M} = SU_X(3, 0)$ be the moduli space of rank 3 semi-stable vector bundles on X with trivial determinant. Consider a non stable point E of $\mathcal{M}$.

The case $G_E = G_m$ has been treated in section II: in this case, the completion of $\mathcal{M}$ at E is the completion at the origin of a rank 4 quadric in $\mathbf{A}^9$.

Suppose now that $G_E = G_m \times G_m$ which means

$$E = E_1 \oplus E_2 \oplus E_3 \qquad \text{with } E_i \neq E_j \text{ for } i \neq j \text{ and } \deg(E_i) = 0.$$

Let $X_{i,j}$ be a basis of $\mathrm{Ext}^1(E_i, E_j)^\vee$ for $i \neq j$. Let $\mathcal{A}_E = k[X_{i,j}]^{G_E}$ be the ring of invariants of $k[X_{i,j}]$ under G_E with the action defined by the following rule

$$\underline{\alpha} \cdot X^{\underline{n}} = \prod \left(\frac{\alpha_i}{\alpha_j}\right)^{n_{i,j}} X^{\underline{n}}$$

for $(\alpha_1, \sigma_2, \alpha_3) \in G_E(k) = (k^*)^3/k^*$. The following equality is easy

$$A_E = k[(\oplus \mathrm{Ext}^1(E_i, E_i))_0] \otimes \mathcal{A}_E.$$

A polynomial $\Sigma p_{\underline{n}} X^{\underline{n}}$ is in $\mathcal{A}_E$ if and only if

$$\sum_j n_{j,i} = \sum_j n_{i,j} \qquad \text{for } i = 1, 2, 3 \tag{1}$$

if $p_{\underline{n}} \neq 0$. Therefore, $\mathcal{A}_E$ is generated by the monomials

$X^{\underline{n}}$ such that $\underline{n}$ satisfies (1).

LEMMA 1. *The ring* $\mathscr{A}_{\mathrm{E}}$ *is generated by*

$$X_{3,2}X_{2,1}X_{1,3}, \qquad X_{1,2}X_{2,3}X_{3,1} \quad \text{and} \quad X_{i,j}X_{j,i}, \quad i<j.$$

Proof. put $\delta_{i,j} = n_{i,j} - n_{j,i}$ and let $\underline{n}$ be a multi-index satisfying (1). The relations (1) become

$$\delta_{1,2} + \delta_{1,3} = 0, \qquad \delta_{1,2} = \delta_{2,3}, \qquad \delta_{1,3} + \delta_{2,3} = 0.$$

If $\delta^+ = \delta_{1,2} \geq 0$, we write $\underline{n} = (n_{2,1} + \delta^+, n_{2,1}, n_{1,3}, n_{1,3} + \delta^+, n_{3,2} + \delta^+, n_{3,2})$ and use the monomial $X_{1,2}X_{2,3}X_{3,1}$ corresponding to $\underline{n}_0 = (1, 0, 0, 1, 1, 0)$ to write $\underline{n} = \underline{m} + \delta^+ \cdot \underline{n}_0$. This allows us to write

$$X^{\underline{n}} = (X_{1,2}X_{2,3}X_{3,1})^{\delta+} \prod_{i<j} (X_{i,j} \cdot X_{j,i})^{m_{i,j}}$$

with $m_{i,j} \geq 0$. In the same way, when $\delta^- = -\delta_{1,2} > 0$, one has an equality

$$X^{\underline{n}} = (X_{3,2}X_{2,1}X_{1,3})^{\delta-} \prod_{i<j} (X_{i,j} \cdot X_{j,i})^{m_{i,j}}$$

with $m_{i,j} \geq 0$. □

For $i \in \mathrm{I} = \{1, 2, 3\}$, put

$$x_i = X_{j,k}, \qquad x_{i+3} = X_{k,j}, \qquad \zeta_i = x_i x_{i+3}$$

with $\mathrm{I} = \{i, j, k\}$ and $j < k$. Let $\zeta_4 = X_{3,2}X_{2,1}X_{1,3}$ and $\zeta_5 = X_{1,2}X_{2,3}X_{3,1}$. There is an equality

$$\zeta_4\zeta_5 = \zeta_1\zeta_2\zeta_3.$$

PROPOSITION 2. *The natural morphism*

$$f: \begin{cases} k[X_i] & \to \quad \mathscr{A}_{\mathrm{E}} \\ X_i & \mapsto \quad \zeta_i \end{cases}$$

gives an isomorphism

$$k[X_i]/(X_4X_5 - X_1X_2X_3) \xrightarrow{\sim} \mathscr{A}_{\mathrm{E}}.$$

Proof. Let $\mathfrak{p}$ be the (prime) ideal generated by $X_4X_5 - X_1X_2X_3$. Let

$$P = \sum \alpha_{\underline{n}} X^{\underline{n}}$$

be an element of $\mathrm{Ker}(f)$. Then, one finds by simple expansion

$$f(P) = \sum \alpha_{\underline{n}} \prod x^{\phi(\underline{n})} = \sum_{\underline{m}} x^{\underline{m}} \sum_{\phi(\underline{n}) = \underline{m}} \alpha_{\underline{n}}$$

with

$$\phi(\underline{n}) = (n_1 + n_5, n_2 + n_4, n_3 + n_5, n_1 + n_4, n_2 + n_5, n_3 + n_5)$$

which implies

$$\sum_{\phi(\underline{n}) = \underline{m}} \alpha_{\underline{n}} = 0. \tag{2}$$

(Here $X^{\underline{n}} = \Pi_i X_i^{n_i}$ and $x^{\underline{m}} = \Pi_i x_i^{m_i}$ $\underline{n} = (n_i)_{1 \leq i \leq 5}$ and $\underline{m} = (m_i)_{1 \leq i \leq 6}$ are multi-indices).

The kernel of ϕ is generated by $(1, 1, 1, -1, -1)$: if $\phi(\underline{m}') = \phi(\underline{m})$, there exists $\alpha \in \mathbf{Z}_+$ such that

$$\pm\alpha(1, 1, 1, 0, 0) + \underline{m}' = \pm\alpha(0, 0, 0, 1, 1) + \underline{m}.$$

In particular, one has the congruence

$$(X_4 \cdot X_5)^{\alpha} \cdot X^{\underline{m}'} \equiv (X_4 \cdot X_5)^{\alpha} \cdot X^{\underline{m}} \bmod \mathfrak{p}. \tag{3}$$

According to (2) and (3), we get the existence of a positive integer a such that

$$(X_4 \cdot X_5)^a \cdot P \equiv 0 \bmod \mathfrak{p}.$$

Since the ideal $\mathfrak{p}$ is prime and $(X_4 \cdot X_5) \notin \mathfrak{p}$ and therefore $P \in \mathfrak{p}$. □

COROLLARY 3. *Let* E *a point of* $\mathcal{M}$ *satisfying* $G_E = G_m \times G_m$. *Then* $\mathcal{M}$ *is étale locally isomorphic at* E *to*

$$(\oplus \mathrm{Ext}^1(E_i, E_i))_0 \times \mathrm{Spec}(k[X_i]/(X_4X_5 - X_1X_2X_3)).$$

Its tangent cone is a rank 2 *quadric in the Zariski tangent space* $T_{[E]}\mathcal{M} = \mathbf{A}^9$.

In particular, there exists a family **E** of semi-stable bundles of trivial determinant over a germ of curve such that:

(i) The group G_{E_η} of the generic bundle $\mathbf{E}_\eta$ is $G_m \otimes_k k(\eta)$.
(ii) The group of G_{E_s} the special bundle $\mathbf{E}_s$ is $G_m \times G_m$.
(iii) The multiplicity of $\mathcal{M}$ at $[\mathbf{E}_\eta]$ and $[\mathbf{E}_s]$ are the same.

This shows that 2 points of $\mathcal{M}$ can have the same multiplicity without having the same group of automorphisms.

When the E has 3 summands for which at least 2 are isomorphic, or equivalent if G_E is not a torus, the calculations are very intricate (but seem to be possible). In fact, one can in spite of this obtain the following

PROPOSITION 4. *The tangent cone at each non stable point* E *such that* G_E *is not a torus is a quadric in* $\mathbf{A}^9$ *of rank* ≤ 2.

Proof. let E be a semi-stable vector bundle of rank 3 and determinant $\mathcal{O}$ and Θ_{J^1} the canonical theta divisor on $\mathrm{Pic}^1(X)$. Using the corollary 1.7.4 of [Ray], the determinantal locus Θ_E in $\mathrm{Pic}^1(X)$

$$\Theta_E = \{L \in \mathrm{Pic}^1(X) \text{ such that } H^0(X, E \otimes L) \neq 0\}$$

is a *divisor* in $\mathbf{P}H^0(J^1, \mathcal{O}(3\Theta_{J^1}))$. The Picard group of $\mathcal{M}$ is cyclic with ample generator $\mathcal{O}(\Theta)$ [D-N]. By [B-N-R], the inverse image of $\mathcal{O}(1)$ by the morphism

$$\pi : \begin{cases} \mathcal{M} & \to \quad \mathbf{P}H^0(J^1, \mathcal{O}(3\Theta_{J^1})) \\ [E] & \mapsto \quad \Theta_E \end{cases}$$

is $\mathcal{O}(\Theta)$ and π^* is a (canonical) isomorphism

$$|\Theta|^\vee \xrightarrow{\sim} \mathbf{P}H^0(J^1, \mathcal{O}(3\Theta_{J^1})).$$

Using this isomorphism, π becomes the morphism given by the complete linear system $|\Theta|$ (in particular, $|\Theta|$ has no base point in this case!). There are various ways to prove this simple lemma, but the following one can be generalized for the higher rank case.

LEMMA 5. *The morphism π is finite of degree 2 over $\mathbf{P}^8$.*

Proof of the lemma. Using the isomorphism $\pi^*\mathcal{O}(1) \xrightarrow{\sim} \mathcal{O}(\Theta)$, we get that π is finite of degree $c_1(\Theta)^8$ onto $\mathbf{P}^8$. One therefore has to compute the degree of π. Although there exists a general beautiful formula due to Witten to evaluate the volume

$$\frac{c_1(\Theta)^{\dim \mathrm{SU}_X(r0)}}{(\dim \mathrm{SU}_X(r, 0))!},$$

we give a simplest (and elementary) method to get this volume for $\mathcal{M}$. One has to prove that the leading term of the Hilbert polynomial $n \mapsto \mathrm{P}(n) = \chi(\mathrm{X}, \Theta^n)$ is $2/8!$. The canonical divisor of $\mathcal{M}$ is Θ^{-6} [D-N]. Serre duality implies therefore the symmetry

$$\mathrm{P}(n) = \mathrm{P}(-6-n). \tag{4}$$

The Grauert-Reimenschneider vanishing theorem (recall that $\mathcal{M}$ has rational singularities) gives the equality

$$\mathrm{P}(n) = \dim \mathrm{H}^0(\mathcal{M}, \Theta^n) \qquad \text{for } n \geq -5.$$

One therefore obtains the values

$$\mathrm{P}(n) = 0 \qquad \text{for } n = -5, \ldots, -1,\ \mathrm{P}(0) = 1 \text{ and } \mathrm{P}(1) = 9. \tag{5}$$

By (4) and (5), one obtains

$$\mathrm{P}(\mathrm{X}) = \lambda(\mathrm{X}+5)(\mathrm{X}+2)(\mathrm{X}+3)^2(\mathrm{X}+2)(\mathrm{X}+1)(\mathrm{X}-\alpha)(\mathrm{X}+6+\alpha).$$

The equalities $\mathrm{P}(0) = 1$ and $\mathrm{P}(1) = 9$ imply

$$\alpha = -3 \pm \sqrt{-47} \quad \text{and} \quad \lambda = \frac{2}{8!}. \qquad \square$$

One has proved that the morphism π is finite of degree 2 onto $|\Theta|^\vee = \mathbf{P}^8$. Since $\mathcal{M}$ is Cohen-Macaulay and $\mathbf{P}^8$ smooth, this double covering is flat ([EGA IV]15.4.2) and is given locally by an equation

$$t^2 = f(\underline{x})$$

where $\underline{x}$ are local coordinates on $\mathbf{P}^8$. This implies that the multiplicity of each point of $\mathcal{M}$ is ≤ 2.

Take a point $E \in \mathcal{M}$ such that G_E is not a torus: it is a non smooth point of $\mathcal{M}$, therefore the tangent cone is a quartic (the initial term of f is not linear). But, such a point is a specialization of a point E_η such that $G_{E_\eta} = G_m \times G_m$: by the (obvious) semi-continuity of the rank of the quartic cone of E, the inequality rank ≤ 2 follows from the corollary 3. □

Remark 6 (*Dolgacev*). The morphism π is ramified along a sextic S(X) in $\mathbf{P}H^0(J^1, \mathcal{O}(3\Theta_{J^1}))$. Consider the embedding

$$J^1 \hookrightarrow |3\Theta_{J^1}|^\vee = \mathbf{P}^8.$$

The variety J^1 is contained in 9 quadrics. Using the action of the Mumford group as in I, it can be shown that there exists a unique cubic $\mathscr{C}(X)$ of $|3\Theta_{J^1}|^\vee$ which is both invariant under the Mumford group and singular along J^1.

QUESTION (Dolgacev). *Are the sextic* S(X) *and the cubic* $\mathscr{C}(X)$ *dual to each other*?

VI. Multiplicity of the theta divisor (rank 2 case)

Recall that there exists a (Cartier) divisor Θ on $SU(2, \omega_X)$ which is characterized by the following universal property [D-N]: let S be a k-scheme and **E** a family of semi-stable vector bundles over $X_S = X \times_k S$ of determinant ω_X. Let $\pi: S \to SU_X(2, \omega_X)$ be the classifying map corresponding to **E**. Then, one has the equality

$$\pi^*(\Theta) = \operatorname{div}(\det Rp_*\mathbf{E})^\vee.$$

By construction, the geometric points of Θ are the classes $[E] \in SU(2, \omega_X)$ such that $H^0(X, E) \neq 0$.

Remark. The vanishing of $H^0(X, E)$ depends on the S-equivalence class [E] of E.

Let E be a semi-stable vector bundle of determinant ω_X. Recall ([La], theorem III.3) that for E stable

$$\mathrm{mult}_{[E]}\,\Theta = \dim H^0(X, E)$$

and that the tangent cone is defined in $\mathrm{Ext}^1_0(E, E)$ by the ideal of the determinant of linear forms defined by the cup-product

$$H^0(X, E) \otimes \mathrm{Ext}^1_0(E, E) \to H^1(X, E).$$

These facts can be generalized formally (using the universal property of Θ) as follows. Let $[E] \in \mathrm{Su}(2, \omega_X)$ a non stable point of Θ of graded object $E = E_1 \oplus E_2$ and let

$$h = \tfrac{1}{2} \dim H^0(X, E) = \dim H^0(X, E_1) = \dim H^0(X, E_2)$$

(note that $E_1 \otimes E_2 = \omega_X$ which implies by Serre duality and Riemann-Roch the equality

$$\dim H^0(X, E_2) = \dim H^1(X, E_2) = \dim H^0(X, E_1)). \tag{1}$$

With the notations of I, let $V = \mathrm{Spec}\, k[[\mathrm{Ext}^1_0(E, E)]]$ be a (formal) étale slice of $\mathcal{Q}$ at E and

$$\pi\colon V \to V/\mathrm{Aut}(E) \xrightarrow{\text{étale}} \mathrm{SU}(2, \omega_X)$$

the canonical morphism. Then, the induced map

$$\pi^*(\Theta)/\mathrm{Aut}(E) \to \Theta$$

is étale. The tangent cone of $\pi^*(\Theta)$ is given by the determinant

$$d_E \in \mathrm{Sym}^{2h}\, \mathrm{Ext}^1_0(E, E)^\vee$$

defined (up to a non zero scalar) by the cup-product

$$H^0(X, E) \otimes \mathrm{Ext}^1_0(E, E) \to H^1(X, E)$$

In particular, a point $e \in \mathrm{Ext}^1_0(E, E)$ is in the tangent cone of $\pi^*(\Theta)$ at [E] if and only if the cup-product

$$\cup e\colon \mathrm{H}^0(\mathrm{X}, \mathrm{E}) \to \mathrm{H}^1(\mathrm{X}, \mathrm{E})$$

is not onto.

PROPOSITION 1. *Assume that* $\mathrm{E}_1 \neq \mathrm{E}_2$. *Then, the multiplicity of* Θ *at* [E] *is*

$$\mathrm{mult}_{[\mathrm{E}]}\, \Theta = \tfrac{1}{2} \dim \mathrm{H}^0(\mathrm{X}, \mathrm{E}) \cdot \mathrm{mult}_{[\mathrm{E}]}\, \mathrm{SU}_{\mathrm{X}}(2, 0).$$

Proof. With the notation of the second section, the completion of $\mathrm{SU}(2, \omega_{\mathrm{X}})$ at [E] is the completion at the origin of

$$(\mathrm{Ext}^1(\mathrm{E}_1, \mathrm{E}_1) \oplus \mathrm{Ext}^1(\mathrm{E}_2, \mathrm{E}_2))_0 \times \mathscr{S}$$

where $\mathscr{S}$ is the cone over the Segre variety

$$\mathbf{P}(\mathrm{Ext}^1(\mathrm{E}_1, \mathrm{E}_2)) \times \mathbf{P}(\mathrm{Ext}^1(\mathrm{E}_2, \mathrm{E}_1)) \subset \mathbf{P}(\mathrm{Ext}^1(\mathrm{E}_1, \mathrm{E}_2)) \otimes_k \mathrm{Ext}^1(\mathrm{E}_2, \mathrm{E}_1)).$$

Fix coordinates

$$\underline{\mathrm{X}} = (\mathrm{X}^k_{i,j}) \text{ on } \mathrm{Ext}^1(\mathrm{E}_i, \mathrm{E}_j) \qquad \text{for } i \neq j \text{ and } (\underline{\mathrm{Y}}) = \mathrm{Y}^k_i \text{ on } \mathrm{Ext}^1(\mathrm{E}_i, \mathrm{E}_i).$$

The equation F of $\pi^*(\Theta)$ is of the form

$$\mathrm{F} = d_{\mathrm{E}} + \mathrm{G}_{2h+1}$$

where G_{2h+1} vanishes at the origin with order $\geq 2h+1$. The polynomials d_{E} and G_{2h+1} are G_{E} invariant and therefore (see section 2) can be written in terms of $\underline{\mathrm{Y}}$ and $z^{k,l} = \mathrm{X}^k_{1,2} \cdot \mathrm{X}^l_{2,1}$. Let me decompose d_{E} as

$$d_{\mathrm{E}} = \sum_{i=0}^{2h} \mathrm{Q}_i(\underline{\mathrm{X}})\mathrm{P}_{2h-i}(\underline{\mathrm{Y}})$$

where the degree of Q_i (resp. P_{2h-i}) is i (resp. $2h - i$) and $\mathrm{P}_0 = 1$. Using the degrees, one finds the following properties:

- If i is odd, then $\mathrm{Q}_i(\underline{\mathrm{X}})\mathrm{P}_{2h-i}(\underline{\mathrm{Y}}) = 0$.
- If $\mathrm{P}_{2h-2i} \neq 0$, then Q_{2i} is invariant and therefore

$$\mathrm{Q}_{2i}(\underline{\mathrm{X}}) = \mathrm{R}_i(\underline{z})$$

with R_i is a polynomial in $\underline{z}$ of degree i which is defined up to the ideal of the Segre cone $\mathscr{S}$.

It follows that the equation of $\pi^*(\Theta)$ can therefore be written as

$$R_h(\underline{z}) + S(\underline{z}, \underline{Y}) \tag{2}$$

where S vanishes at the origin with order $\geq h+1$ at the origin.

LEMMA 2. *The polynomial* $Q_{2h}(\underline{X})$ *is non zero.*

Proof of the lemma. According to the previous discussion, one just has to prove the existence of $e \in \mathrm{Ext}^1(E_1, E_2) \oplus \mathrm{Ext}^1(E_2, E_1) \subset \mathrm{Ext}^1_0(E, E)$ such that the cup product $\cup e\colon H^0(X, E) \to H^1(X, E)$ is onto. By symmetry, one only has to prove the existence of $e_1 \in \mathrm{Ext}^1(E_1, E_2)$ such that the cup product $\cup e_1\colon H^0(X, E_1) \to H^1(X, E_2)$ is onto. This is classical (see [La], lemma II.8): let Γ be the variety

$$\Gamma = \{(k \cdot s, k \cdot e) \in \mathbf{P}H^0(X, E_1) \times \mathbf{P}\,\mathrm{Ext}^1(E_1, E_2) \text{ such that } s \cup e = 0\}$$

and p (resp. q) the first (resp. second) projection. Let $0 \neq s \in H^0(X, E_1)$ and $D = \mathrm{div}(s)$ its zero divisor. The canonical surjection $\cup s\colon \mathscr{H}om(E_1, E_2) \twoheadrightarrow E_2(-D)$ gives a surjection

$$\cup s\colon \mathrm{Ext}^1(E_1, E_2) \twoheadrightarrow H^1(X, E_2). \tag{3}$$

By (3) the dimension of $p^{-1}(k \cdot s)$ is

$$\dim p^{-1}(k \cdot s) = \dim \mathbf{P}\,\mathrm{Ext}^1(E_1, E_2) - \mathbf{P} \dim H^0(X, E_1) - 1$$

Therefore $\dim \Gamma = \dim \mathbf{P}\,\mathrm{Ext}^1(E_1, E_2) - 1$ and $q(\Gamma) \neq \Gamma$. □

The polynomial R_h can be thought of as an element of

$$H^0(\mathbf{P}\,\mathrm{Ext}^1(E_1, E_2) \times \mathbf{P}\,\mathrm{Ext}^1(E_2, E_1), \mathcal{O}(h, h))$$

which by the lemma 2 is non zero. According to (2), the completion of the tangent cone is therefore the hypersurface of $(\mathrm{Ext}^1(E_1, E_1) \oplus \mathrm{Ext}^1(E_2, E_2))_0 \times \mathscr{S}$ given by R_h. The proposition follows. □

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REFERENCES

[B-N-R] A. Beauville, M. S. Narasimhan and M. S. Ramanan, *Spectral curves and the generalized theta divisor*, J. reine angew. Math. *398* (1989) 169–179.

[B] J. F. Boutot, *Singularites rationelles et quotients par les groupes reductifs*, Inv. Math. *88* (1987) 65–68.

[C] A. B. Coble, *Algebraic geometry and Theta functions*, Amer. Math. Soc. (1929).

[D-N] J. M. Drezet and M. S. Narasimhan, *Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques*, Invent. Math. *97* (1989) 53–94.

[D-O] I. Dolgacev and D. Ortland, *Point sets in projective spaces and theta functions*, Astérisque, *165* (1988).

[EGA IV] A. Grothendieck, *Elements de géométrie algébrique*, avec la collaboration de J. Dieudonne, Publ. Math. I.H.E.S. *24* (1965).

[F] W. Fulton, *Intersection theory*, Ergeb. der Math. und ihrer Grenzgebiete, 3. Folge Band 2, Springer-Verlag, Berlin Heidelberg New York Tokyo (1984).

[H-R] M. Hochster, J. Roberts, *Rings of invariants of reductive groups acting on regular local rings are Cohen-Macaulay*, Adv. in Math. *13* (1974) 115–175.

[H-T] J. Harris and L. W. Tu, *On symmetric and skew symmetric determinantal varieties*, Topology *23* (1984) 71–84.

[Ha] R. Hartshorne, *Algebraic Geometry*, GTM 52, Springer Verlag, Berlin-Heidelberg-New York (1977).

[J] T. Jozefiak, *Ideals generated by minors of a symmetric matrix*, Comment. Math. Helvetici *53* (1978) 595–607.

[Kh1] A. Khaled, *Projective normality and equations of Kummer varieties*, J. reine angew. Math. *465* (1995), 197–217.

[Kh2] A. Khaled, *Equations des variétés de Kummer*, Math. Annal. *295* (1993) 685–701.

[Ko] S. Koizumi, *Theta relations and projective normality of Abelian varieties*, Amer. Journ. of Math. *98* (1976) 865–889.

[La] Y. Laszlo, *Un théorème de Riemann pour les diviseurs Thêta généralisés sur les espaces de modules de fibrés stables sur une courbe*, Duke Math. Journ. *64* (1991) 333–347.

[LeB] L. Le Bruyn, *Trace rings of generic 2 by 2 matrices*, Mem. A.M.S. *66* (1987).

[Lu] D. Luna, *Slices étales*, Mém. Soc. Math. France *33* (1973) 81–105.

[M] D. Mumford, *On equations defining Abelian varieties*, Invent. Math. *1* (1966) 287–354.

[N-R1] M. S. Narasimhan and M. S. Ramanan, *Moduli of vector bundles on a compact Riemann surface*, Ann. of Math. *89* (1969) 14–51.

[N-R2] M. S. Narasimhan and M. S. Ramanan, *2θ-linear system on abelian varieties*, in *Vector bundles and algebraic varieties*, Oxford University Press, (1987) 415–427.

[P] C. Procesi, *The invariant theory of n by n matrices*, Adv. in Math. *19*, 306–381.

[Ras] J. Rasmyslev, *Trace identities of full matrix algebras over a field of characteristic zero*, Izv. Akad. Nauk. U.S.S.R. *8* (1974) 727–760.

[Ray] M. Raynaud, *Sections des fibŕs vectoriels sur une courbe*, Bull. Soc. Math. France *110* (1982) 103–125.

[S] C. S. Seshadri, *Fibrés vectoriels sur les courbes algébriques* (rédigé par J. M. Drezet), Astérisque *96* (1982).

[W] H. WEYL, *The classical groups*, Princeton Univ. Press, Princeton, N.J. (1946).
[W] H. WIRTINGER, *Untersuchung über thetafunctionen*, Teubner, Leipzig (1935).

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