Projective and Hilbert modules over group algebras, and finitely dominated spaces.

Autor(en): **Eckmann, Beno**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **71 (1996)**

PDF erstellt am: **01.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-53853

Comment. Math. Helvetici 71 (1996) 453–462

0010-2571/96/030453-10\$1.50 + 0.20/0

Projective and Hilbert modules over group algebras, and finitely dominated spaces

BENO ECKMANN

0. Introduction

The objective of this note is to consider ℓ_2-Betti numbers for FP-complexes and finitely dominated spaces, and to compare the corresponding Euler characteristic $\bar{\chi}$ with the ordinary χ. The results depend on the validity of the Bass conjectures for the group G involved. We first outline the contents.

0.1. Let G be a countable group, P a finitely generated projective $\mathbb{Z}G$-module, and $\ell_2 G$ the Hilbert space of square-integrable real functions on G with left and right G-action by translation. Then $P^{(2)} = \ell_2 G \otimes_G P$ is a Hilbert-G-module. We write rkP for the "naive" rank $dim_{\mathbb{R}} \mathbb{R} \otimes_G P$ of P. We will show (Theorem 1 in Section 3) that $P^{(2)}$ is a "free" Hilbert-G-module, i.e., a direct sum of copies of $\ell_2 G$ their number being $= rkP$, provided $(*)$ G fulfills the strong Bass conjecture, or G is residually finite. The Bass conjectures are explained in Section 2 (for more details we refer to Bass' fundamental paper [B]).

0.2. A complex P_* is said of type FP, or an FP-complex, if it is of finite length $n > 0$ ($P_i = 0$ for $i > n$ and $i < 0$) and if all P_i are finitely generated projective $\mathbb{Z}G$-modules; it is said of type FF if the P_i are free. G is always assumed to be finitely generated. In the FF-case $\ell_2 G \otimes_G P_*$ is a complex of free Hilbert-G-modules. Under condition $(*)$ this is also the case for an FP-complex, and the rank of $P_i^{(2)}$ is $= rk\, P_i$; one has (reduced) homology groups, ℓ_2-Betti numbers $\bar{\beta}_i$, and an ℓ_2-Euler characteristic $\bar{\chi}$ which is equal to the ordinary homological Euler characteristic χ (Section 1, and Theorem 2 in Section 3), all depending only on the chain homotopy type of P_*.

More generally, $w(P_*) = \Sigma_0^n (-1)^i [P_i] \in K_0(\mathbb{Z}G)$ is the *Wall obstruction* of P_*, where $[P_i]$ is the class of $P_i \in K_0(\mathbb{Z}G)$. Then $rk\ w(P_*)$ is the Euler characteristic, and $\tilde{w}(P_*)$ corresponding to $w(P_*)$ in $\tilde{K}_0(\mathbb{Z}G)$ is the finiteness (or rather freeness) obstruction. Thus our result tells that $\ell_2 G \otimes_G$- annihilates $\tilde{w}$ and leaves χ unchanged.

0.3. In topology *FP*-complexes occur in connection with a finitely dominated connected space X. The chain complex of the universal cover $\tilde{X}$ is chain homotopy equivalent to an *FP*-complex P_* over $\mathbb{Z}G$ where G is the fundamental group of X; it is finitely presented. One can write $w(X) = w(P_*)$; $rk\, w(X)$ is the Euler characteristic of X and $\tilde{w}(X)$ the Wall finiteness obstruction of X. Under condition $(*)$ on G one has $\bar{\chi}(X) = \chi(X)$, and the ℓ_2-finiteness obstruction of X is 0.

0.4. If a complex P_* fulfills Poincaré duality of dimension n (PD^n) then it is an *FP*-complex, and its ℓ_2-Betti numbers also fulfill PD^n (Section 4). This applies to finitely dominated spaces, but also to PD^n-groups (which are finitely generated, but need not be finitely presented).

1. *FP*-complexes and ℓ_2-Betti numbers

1.1. We recall that a space X is said to be finitely dominated, cf. [M], if it is a retract, in the homotopy category, of a finite *CW*-complex. Such a space is homotopy equivalent to a (finite dimensional, in general infinite) *CW*-complex. In the following "space" will always mean connected *CW*-complex.

Let G be the fundamental group of the space X, and $\tilde{X}$ the universal cover. The cellular chain complex $C_*\tilde{X}$ is a complex of free $\mathbb{Z}G$-modules. If X is finitely dominated then $C_*\tilde{X}$ is $\mathbb{Z}G$-chain homotopy equivalent to a complex P_* of type *FP*; i.e., P_* has finite length n ($P_i = 0$ for $i < 0$ and $i > n$) and all P_i are finitely generated projective $\mathbb{Z}G$-modules. Moreover G is finitely presentable (and for such G the converse holds: if $C_*\tilde{X}$ is of type *FP* then X is finitely dominated).

Most of our arguments deal in fact just with *FP-complexes* P_* *over a group ring* $\mathbb{Z}G$. Finite presentability of G is in general irrelevant.

1.2. Let $\beta_i X = \dim_{\mathbb{R}} H_i(X; \mathbb{R})$ be the i-th Betti number of X. If X is finitely dominated the $\beta_i X$ are finite and the homological Euler characteristic χX is defined as $\Sigma_0^n (-1)^i \beta_i X$. Homology can be computed from any P_* equivalent to $C_*\tilde{X}$ through $\mathbb{R} \otimes_G P_*$. We write $rk\, P_i = \dim_{\mathbb{R}} \mathbb{R} \otimes_G P_i$. The usual Euler–Poincaré argument then yields

$$\chi X = \sum_0^n (-1)^i rk\, P_i.$$

If X itself is a *finite CW*-complex then we can take $P_* = C_*\tilde{X}$ (all P_i are finitely generated free over $\mathbb{Z}G$, of rank $= rk\, P_i$; such a P_* is said to be of type *FF*). We recall that, in that case, $rk\, P_i$ is equal to the number α_i of i-cells of X. Thus

$\chi X = \Sigma_0^n(-1)^i \alpha_i$ becomes the ordinary "combinatorial" Euler characteristic of the finite cell-complex X.

1.3. For a *finite CW-complex X* (or an *FF*-complex P_*) it is well-known that χX can be expressed by ℓ_2-Betti numbers $\bar{\beta}_i X$ as $\chi X = \bar{\chi} X = \Sigma_0^n(-1)^i \bar{\beta}_i X$. For ℓ_2-homology and ℓ_2-Betti numbers we refer, e.g., to [Lü] and references given there. Here we just recall the main facts.

Let $\ell_2 G$ be as in 0.1. The complex $\ell_2 G \otimes_G P_*$ consists of "free" Hilbert-G-modules $\ell_2 G \otimes_G P_i = \ell_2 G \oplus \cdots \oplus \ell_2 G$, α_i terms, also written $\ell_2 G^{\alpha_i}$, with the induced boundary operators ∂. The *reduced* homology group $\bar{H}_i \tilde{X} = \bar{H}_i P_* = \{$cycles in $\ell_2 G \otimes_G P_i\}$/closure of $\partial(\ell_2 G \otimes_G P_{i+1})$ is a Hilbert-G-module, since it can be imbedded in $\ell_2 G^{\alpha_i}$ (as orthogonal complement of the above closure in the cycle space). Its von Neumann dimension $\dim_G \bar{H}_i \tilde{X}$ is $\bar{\beta}_i X$; it is a homotopy invariant of X. Since $\dim_G \ell_2 G \otimes_G P_i = \alpha_i$, and since the von Neumann dimension behaves as a rank (although it is a real number ≥ 0) the Euler–Poincaré argument applied to $\ell_2 G \otimes_G P_*$ yields

$$\bar{\chi} X = \sum_0^n (-1)^i \bar{\beta}_i X = \sum_0^n (-1)^i \alpha_i$$

whence $\bar{\chi} X = \chi X$.

1.4. In order to apply the same procedure to the general case of a finitely dominated space X (an *FP*-complex P_*) we look closer at $\ell_2 G \otimes_G P$ where P is a finitely generated projective $\mathbb{Z}G$-module. It is clearly a Hilbert-G-module since it imbeds in a free one, namely in $\ell_2 G \otimes_G F$ where F is a finitely generated free $\mathbb{Z}G$-module containing P as a direct summand.

It turns out (Theorem 1 in Section 3) that *if G fulfills the Bass conjecture or if G is residually finite then $\ell_2 G \otimes_G P$ is isometrically G-isomorphic to $\ell_2 G^k$ where* $k = rk\, P = \dim_{\mathbb{R}} \mathbb{R} \otimes_G P$. (The residually finite case has already been treated by Lück [Lü]; his approach is slightly different from ours where the residually finite case is a corollary of the strong Bass case.)

Now the arguments of 1.3 apply to $\ell_2 G \otimes_G P_*$ and to its ℓ_2-Betti number $\bar{\beta}_i X$ except that α_i is replaced by $rk\, P_i$

$$\bar{\chi} X = \sum_0^n (-1)^i \bar{\beta}_i X = \sum_0^n (-1)^i\, rk\, P_i$$

whence, according to 1.1, $= \chi X$. Similarly for an arbitrary *FP*-complex.

1.5. The advantage resulting from $\bar{\chi}X$ instead of the classical χX is due to the fact that for the $\bar{\beta}_i X$ certain vanishing theorems are known. They imply properties of χX or of the $\beta_i X$. Some immediate applications are given in Section 4: To Poincaré duality spaces, to groups of type *FP*, to groups fulfilling PD^2 (Poincaré duality of formal dimension 2).

2. Rank concepts for projective modules

2.1. In this section we recall the various rank concepts and their relations (Bass conjectures) since we will make essential use of them. P will always denote a finitely generated projective $\mathbb{Z}G$-module, G an arbitrary group.

(a) The "homological" or "naive" rank $rk\, P = \dim_{\mathbb{R}} \mathbb{R} \otimes_G P$ is being used for the definition of homology, Betti numbers, and Euler characteristic. We note that it is not clear a priori whether $rk\, P = 0$ implies $P = 0$.

(b) The "Kaplansky" rank κP. It is defined as follows: Let F be a finitely generated free $\mathbb{Z}G$-module containing P as a direct summand. In a basis of F, consisting of m elements, the endomorphism f of $\mathbb{Z}G^m$ which projects F onto P is given by an $m \times m$ matrix M_P with $\mathbb{Z}G$-entries. κP is the coefficient of $1 \in G$ of trace M_P; it is easily seen to be independent of the choice of F and of the basis.

Another way of describing f is by a matrix (f_{ij}), f_{ij} being the $\mathbb{Z}G$-endomorphism of $\mathbb{Z}G$ determined by $f_{ij}(1) = a_{ij} \in \mathbb{Z}G$.

(c) The "Hattori–Stallings" rank r_P. The trace $\in \mathbb{Z}G$ of the matrix M_P above depends on the choices. However, if one passes from $\mathbb{Z}G$ to $\overline{\mathbb{Z}G} = \mathbb{Z}G/\{xy - yx\}$ where $\{xy - yx\}$ is the submodule of $\mathbb{Z}G$ generated by all additive commutators (it suffices to take $xy - yx$ for all $x, y \in G$) then the value of trace M_P in $\overline{\mathbb{Z}G}$, denoted by r_P, is independent of the choices. $\overline{\mathbb{Z}G}$ is obtained from $\mathbb{Z}G$ by identifying conjugate elements of G; it is thus the free Abelian group generated by the conjugacy classes $[x]$ of G. One can write r_P as

$$r_P = \sum r_P(x)[x]$$

where x is an arbitrary representative of $[x]$.

2.2. One immediately notes that $r_P(1) = \kappa P$. Moreover, $\Sigma_x\, r_P(x) = rk\, P$. Indeed $\Sigma\, r_P(x)$ is the sum of all coefficients in trace $M_P \in \overline{\mathbb{Z}G}$, i.e., it is induced by the augmentation $\varepsilon: \mathbb{Z}G \to \mathbb{Z}$; and ε turns P into the free Abelian group $\mathbb{Z} \otimes_G P$. By naturality $\varepsilon_* r_P$ is the rank, with respect to $\mathbb{Z}$, of $\mathbb{Z} \otimes_G P$ which is $= rk\, P$.

The *Strong Bass Conjecture* (SB) claims that all $r_P(x)$, $x \neq 1$, are 0. It implies that $\kappa P = rk\, P$. That equality is the *Weak Bass Conjecture* (WB). It has been proved quite generally by Linnell [Li, p. 96] that $r_P(x)$ vanishes on all elements $x \neq 1$ of finite order.

(SB) has been established for various classes of groups. We mention the following:

1) Finite groups (Swan's Theorem [S])
2) Linear groups ([B], [E])
3) Negatively curved groups
4) Solvable groups [E]
5) Groups of cohomology dimension ≤ 2 over $\mathbb{R}$ ([E])

As for 3), this does not seem to be explicitly in the literature. It follows from the method used in [E] and the fact that in such a group the centralizer of an element of infinite order is virtually infinite cyclic.

(WB) holds for all residually finite groups [B], but it is not known whether these fulfill (SB).

2.3. We introduce an adhoc *notation* $c_x(\alpha)$ = coefficient of $x \in G$ in $\alpha \in \mathbb{Z}G$ (or $\mathbb{R}G$, or $\ell_2 G$) to be used in this and the next sections.

As a consequence of the definitions one has

PROPOSITION 1. *The von Neumann dimension* $\dim_G(\ell_2 G \otimes_G P)$ *is equal to* κP.

Proof. We recall that the von Neumann dimension of a Hilbert-G-module $H \subset \ell_2 G^m$ is defined as "trace" of the projection operator φ of $\ell_2 G^m$ having H as image. φ is given by an $m \times m$ matrix (φ_{ij}) with $\varphi_{ij} \in NG$, the ring of bounded G-equivariant operators in $\ell_2 G$ (the von Neumann algebra of G); "trace" is to be understood as $c_1 \Sigma_{i=1}^m \varphi_{ii}(1)$. If $H = \ell_2 G \otimes_G P$ then (φ_{ij}) is the same as the matrix $M_P = (f_{ij}), f_{ij}(1) = a_{ij} \in \mathbb{Z}G$; indeed the endomorphism ring of $\mathbb{Z}G$ is naturally imbedded in NG. Thus the two "traces" $c_1(\Sigma f_{ii}(1)) = c_1(\Sigma\, \varphi_{ii}(1))$ are the same.

3. Structure of $\ell_2 G \otimes_G P$

3.1. If G fulfills (*SB*) or is residually finite it follows from Proposition 1 that $\dim_G \ell_2 G \otimes_G P = rk\ P$. In this section we will show that $\ell_2 G \otimes_G P$ and $\ell_2 G^{rk\,P}$ are even isometrically and G-equivariantly isomorphic. We need some preparations.

3.2. Let CG be the center of $\mathbb{R}G$; it has a basis consisting of all sums $\sigma_x = \Sigma_{y \in [x]}\, y$ for the *finite* conjugacy classes in G.

PROPOSITION 2. *Let P be a finitely generated projective $\mathbb{Z}G$-module, ζ an element of CG. If the Hattori–Stallings ranks r_P vanishes on finite conjugacy classes $[x]$, $x \neq 1$ then*

$$c_1(\zeta \text{ trace } M_p) = c_1(\zeta)\kappa P.$$

Proof. It suffices to prove that $c_1(\sigma_x \text{ trace } M_P) = 0$ for $x \neq 1$. Now

$$\begin{aligned} c_1(\sigma_x \text{ trace } M_P) &= \sum_{y \in [x]} c_1(y \text{ trace } M_P) \\ &= \sum_{y \in [x]} c_{y^{-1}}(\text{trace } M_P) \\ &= \sum_{y \in [x^{-1}]} c_y(\text{trace } M_P) = r_P(x^{-1}) = 0. \end{aligned}$$

The assumption of Proposition 2 holds if G fulfills (SB), and then $\kappa P = rk\, P$. The result also holds if G is *residually finite*: We choose a normal subgroup N of finite index in G which does not contain the elements of σ_x, of trace M_P, nor of their product. Passing from $\mathbb{Z}G$ to $\mathbb{Z}(G/N)$ we get a projective $\mathbb{Z}(G/N)$-module $P' = \mathbb{Z}(G/N) \otimes_G P$ and an element $\sigma'_x \in C(G/N)$. The coefficient c_1 of σ_x, of trace M_P, and of the product remains unchanged. By Swan's theorem [S], G/N fulfills (SB); therefore

$$r_P(x^{-1}) = c_1(\sigma_x \text{ trace } M_P) = c_1(\sigma'_x \text{ trace } M_{P'}) = r_{P'}(x^{-1}) = 0$$

if $x \neq 1$. Moreover $\kappa P = \kappa P' = rk\, P'$ and

$$\begin{aligned} rk\, P' &= \dim_{\mathbb{R}}(\mathbb{R} \otimes_{G/N} P') = \dim_{\mathbb{R}}(\mathbb{R} \otimes_{G/N} \mathbb{Z}G/N \otimes_G P) \\ &= \dim_{\mathbb{R}}(\mathbb{R} \otimes_G P) = rk\, P. \end{aligned}$$

COROLLARY 2'. *If G is residually finite then*

$$c_1(\zeta \text{ trace } M_P) = c_1(\zeta) rk\, P.$$

3.3. We assume that r_P vanishes on finite conjugacy classes $[x]$, $x \neq 1$ and show that for the two Hilbert-G-modules $\ell_2 G \otimes_G P$ and to $\ell_2 G^{\kappa P}$ the *center-valued trace* (cf. [KR]), *ctr* $\in$ center of NG, applied to the respective $\Sigma\, \varphi_{ii}$ is the same. This implies that they are isometrically G-isomorphic.

For $\ell_2 G \otimes_G P$ one has $ctr(\Sigma\, \varphi_{ii}) = ctr(\text{trace } M_P)$ which lies in CG since $ctr\, x = (1/card[x])\, \Sigma_{y \in [x]}\, y$ if $[x]$ is finite, and 0 otherwise. Of course for $\ell_2 G^{\kappa P}$, the

ctr of the matrix is $\kappa P \cdot E$, $E = \text{identity} \in NG$. Thus

$$\xi = ctr\left(\sum \varphi_{ii} - \kappa P \cdot E\right)$$

is an element of CG, and so is its adjoint ξ^* in NG.

Since the 1-coefficient of the central-valued trace fulfills quite generally $c_1(ctr\ \varphi) = c_1\varphi(1)$, $\varphi \in NG$, we have

$$c_1(\xi^*\xi) = c_1(\xi^*(\text{trace } M_P - \kappa P));$$

by Proposition 2 this is equal to $c_1(\xi^*)c_1(\text{trace } M_P - \kappa P) = 0$. Since $c_1(\xi^*\xi) = \langle \xi^*\xi, 1\rangle = \langle \xi, \xi\rangle$ in $\ell_2 G$, it follows that $\xi = 0$:

PROPOSITION 3. *If r_P vanishes on finite conjugacy classes $[x]$, $x \neq 1$ then $\ell_2 G \otimes_G P$ and $\ell_2 G^{\kappa P}$ are isometrically G-isomorphic.*

By Corollary 2′ the same arguments work in the residually finite case. We summarize the important cases as follows.

THEOREM 1. *If G fulfills (SB), or if G is residually finite, then $\ell_2 G \otimes_G P$ is isometrically G-isomorphic to $\ell_2 G^{rk\, P}$.*

3.4. As stated in Section 1.4 this implies

THEOREM 2. *If G fulfills (SB), or if G is residually finite, then for any finitely dominated space X with fundamental group G, or for any FP-complex P_* over $\mathbb{Z}G$, the ℓ_2-Euler characteristic $\bar{\chi}$ and the ordinary Euler characteristic χ coincide.*

3.5. An *FP*-complex P_* over $\mathbb{Z}G$ is always chain homotopy equivalent to a complex where the P_i are all *free* except (possibly) for the *top module* P_n. Then for $\ell_2 G \otimes_G P_i$ to be free of rank $= rk\ P_i$, $i < n$, it is not necessary to apply Theorem 1, i.e., to assume (*SB*) or G residually finite.

We recall that for P_* chosen as above the class of P_n in $\tilde{K}_o(\mathbb{Z}G)$ is, up to sign, the "Wall obstruction"; it is 0 if and only if P_* is equivalent to an *FF*-complex. In that sense Theorem 1 tells that, under the appropriate assumption the *ℓ_2-Wall obstruction always vanishes.*

Remark. We have worked throughout with homology. Everything could also be carried through in cohomology, based on $Hom_G(P_*, \ell_2 G) = P_*^{\text{dual}} \otimes_G \ell_2 G$; $P_i^{\text{dual}} = Hom_G(P_i, \mathbb{Z}G)$ is again finitely generated projective. We note that in case P_i is free,

P_i^{dual} can be identified with P_i, so that $\bar{H}^i = \bar{H}_i$. But this also holds in the general projective case. Indeed the projection matrix for P_i^{dual} can be taken to be the transposed of M_{P_i}; thus the trace and the various ranks are the same.

4. Applications

4.1. *Groups of type FP*

These are groups G for which there exists a resolution $P_* \twoheadrightarrow \mathbb{Z}$ over $\mathbb{Z}G$ with P_* of type *FP*. We assume that the cohomology dimension cdG, the minimal length n of P_*, is ≥ 2 (if $cdG = 1$ then G is finitely generated free). It is not known whether such a group is necessarily of type *FF*; cf. [Br] for a discussion of that problem.

G being infinite one notes that, without further assumptions on G, $H^o(G; \ell_2 G) = \ell_2 G^G = 0$ (an element of $\ell_2 G$ cannot be invariant unless it is 0). Then $\bar{H}^0 G = 0$ and $\bar{H}_0 G = 0$.

If G fulfills (*SB*) or is residually finite then $\bar{\chi} G = \chi G$, hence $\Sigma_0^n (-1)^i \beta_i G = \Sigma_1^n (-1)^i \bar{\beta}_i G$. If, moreover, G is amenable then all $\bar{\beta}_i G$ are 0 (see [CG] or [E2]), and thus $\chi G = 0$.

4.2. *Spaces with Poincaré duality*

If the space X fulfills (ordinary) oriented Poincaré duality of formal dimension n

$$(*) \quad H^i(X; A) \cong H_{n-i}(X; A)$$

for all $i \in \mathbb{Z}$ and all $\mathbb{Z}G$-modules A, $G = \pi_1 X$, then $C_* \tilde{X}$ is equivalent to an *FP*-complex P_* over $\mathbb{Z}G$ of length n. This follows from the fact that homology, and therefore cohomology, commutes with direct limits in A (the isomorphisms $(*)$ are assumed to be natural in A); and from the finiteness criterion of Bieri–Eckmann–Brown, see [Br]. If moreover G is finitely presented then X is finitely dominated, but this is not of importance here.

We further assume that $(*)$ is given by the cap-product $e \cap -$ where e is a generator of $H_n(G; \mathbb{Z}) = H^0(G; \mathbb{Z}) = \mathbb{Z}$. Then $e \cap -$ maps $Hom_G(P_*, \mathbb{Z}G) = P_*^{\text{dual}}$ to $P_* = \mathbb{Z}G \otimes_G P_*$. We reverse the numbering of P_*^{dual} ($i \sim n - i$) to make this a map of degree 0; it induces homology isomorphism since $H_i(P_*^{\text{dual}}) = H^{n-i}(P_*; \mathbb{Z}G) = H_i(P_*)$ and is therefore a chain homotopy equivalence. Tensoring with $\ell_2 G$ and passing to reduced homology yields

$$\bar{H}^{n-i}P_* \cong \bar{H}_i P_*$$

for all i, whence $\bar{\beta}_{n-i}X = \bar{\beta}_i X$. If G is infinite $\bar{\beta}_0 X = 0$ as in 4.1 and then also $\bar{\beta}_n X = 0$, without the special assumptions on G.

PROPOSITION 4. *If X fulfills Poincaré duality of formal dimension $n = 2k$ (in short PD^n) then*

$$\chi X = 2 - 2\beta_1 X + \cdots + (-1)^k \beta_k X = -2\bar{\beta}_1 X + \cdots + (-1)^k \bar{\beta}_k X,$$

provided *(SB) holds for G, or G is residually finite.*

4.3. *Example.* $n = 4$ of Proposition 4. In that case

$$\chi X = 2 - 2\beta_1 X + \beta_2 X = -2\bar{\beta}_1 X + \bar{\beta}_2 X$$

$\bar{\beta}_1 X$ depends on G only (one can obtain a $K(G, 1)$ by adding cells of dimension ≥ 3), it can be written $\bar{\beta}_1 G$.

PROPOSITION 5. *If X fulfills PD^4, and if a) G fulfills (SB) or is residually finite, and b) $\bar{\beta}_1 G = 0$ then $\chi X \geq 0$.*

For groups with $\bar{\beta}_1 G = 0$ see [BV].

4.4. *Example.* $n = 2$ of Proposition 4. In that case we first note that if G is *infinite* then $H_2\tilde{X} = \pi_2 X = 0$. Indeed, $H_2\tilde{X} = H_2(X; \mathbb{Z}G) = H^0(X; \mathbb{Z}G) = 0$. Thus X is aspherical, i.e. a $K(G, 1)$, and G is a PD^2-group. All groups of cohomology dimension 2 (over $\mathbb{R}$) fulfill (*SB*), see [E]. We thus get

$$\chi X = \chi G = 2 - \beta_1 G = -\bar{\beta}_1 G$$

whence $\beta_1 G \geq 2$.

This may seem trivial since it is known that the orientable PD^2-groups are just the fundamental groups of closed orientable surfaces of genus $g \geq 1$, so $\beta_1 G = 2g$. However, $\beta_1 G \geq 2$ was an important ingredient in the proof of that result (see [EL]).

REFERENCES

[B] HYMAN BASS, *Euler characteristics and characters of discrete groups*, Inventiones Math. *35* (1976) 155–196.

[Br] KENNETH S. BROWN, *Cohomology of Groups*, Springer, New York, 1982.
[BV] M. E. B. BEKKA and ALAIN VAETTE, *Group cohomology, harmonic functions and the first L^2-Betti number*, preprint.
[CG] JEFF CHEEGER and MIKHAEL GROMOV, *L_2-cohomology and group cohomology*, Topology *25* (1986) 189–215.
[E] BENO ECKMANN, *Cyclic homology of groups and the Bass conjecture*, Comment. Math. Helv. *61* (1986) 193–202.
[E2] BENO ECKMANN, *Amenable groups and Euler characteristic*, Comment. Math. Helv. *67* (1992) 383–393.
[EL] BENO ECKMANN and PETER LINNELL, *Poincaré duality groups of dimension 2, II*, Comment. Math. Helv. *58* (1983) 111–114.
[KR] R. V. KADISON and J. R. RINGROSE, *Fundamentals of the theory of operator algebras*, Volume II, Academic Press, 1986.
[Li] P. A. LINNELL, *Decomposition of augmentation ideals and relation modules*, Proc. London Math. Soc. *47* (1983) 83–127.
[Lü] WOLFGANG LÜCK, *Approximating L^2-invariants by their finite-dimensional analogues*, GAFA *4* (1994) 455–481.
[M] GUIDO MISLIN, *Wall's finiteness obstruction*. In *Handbook of Algebraic Topology*, Elsevier Science, 1995.
[S] R. C. SWAN, *Induced representations and projective modules*, Ann. of Math. *71* (1960) 552–578.

Beno Eckmann
Mathematik
ETH-Zentrum
CH-8092 Zürich
Switzerland

Received September 21, 1995