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Fenchel type theorems for submanifolds of S^n

REMI LANGEVIN and HAROLD ROSENBERG

We dedicate this paper to the memory of Nicolaas Kuiper

The total curvature of compact hypersurfaces M of $\mathbf{R}^n$ ($\int_M |K|$) is related to the topology of M and to the manner in which M is embedded in $\mathbf{R}^n$. K is the Gauss-Kronecker curvature of M, i.e., the determinant of the second fundamental form.

For curves C in $\mathbf{R}^3$, the theorems of Fenchel and Fary-Milnor, state the total curvature of C is at least 2π (with equality precisely for convex planar curves) and if C is knotted in $\mathbf{R}^3$ then $\int_C |k| > 4\pi$, [Fe], [Fa], [M_1], [M_2].

Chern and Lashof observed the total curvature of $M^k \subset \mathbf{R}^n$ is

$$c \int_{P^{n-1}} |\mu|(M, l),$$

where c is a constant depending only on n and k, P^{n-1} is the projective space of lines l through the origin in $\mathbf{R}^n$ and $|\mu|(M, l)$ is the number of critical points of the projection of M to l. Since this projection is a Morse function for almost all l, they obtained $c\beta$ as a minoration of the total curvature, β the sum of the betti numbers of M[C-L].

In particular for surfaces in $\mathbf{R}^3$ one has

$$\int_M |K| \geq 2\pi(2g+2),$$

g the genus of M. If a torus is knotted in $\mathbf{R}^3$, then the total curvature is at least twice as large, i.e., 16π [L-R]. Results of this type for knotted surfaces of higher genus in $\mathbf{R}^3$ have been obtained by Kuiper and Meeks [K-M].

In this paper we establish results of this nature for submanifolds of S^n. For surfaces in S^3, it is not sufficient to consider $\int_M |K|$, where K is the extrinsic curvature of M (consider the boundary of a small tubular neighborhood of a geodesic. Any two points of M differ by an isometry of S^3 so the intrinsic curvature of M is constant; it is zero by Gauss-Bonnet. So $|K| = 1$ and $\int_M |K|$ is the area of M). In fact, for curves C in S^2, it's easy to see that $\int_C (|k_g|+1) \geq 2\pi$, and equality holds precisely when C is a geodesic; k_g the geodesic curvature of C. However for surfaces M in S^3, it is still not enough to consider $\int_M (|K|+1)$. One must add to $|K|+1$, a function $h_1(x)=$ the average of the absolute values of the normal curvatures to M at x. Then one has the desired results:

$$C(M)=\int_M (c_2|K|+c_1h_1(x)+c_0) \geq 2\pi(2g+2),$$

for certain constants c_0, c_1, c_2, and g the genus of M. Moreover, if M is knotted in S^3, then $C(M) \geq 2\pi(2g+4)$.

The function $\int_M h_1$ has an interesting geometric interpretation. It is the total number of folds of M. We call this the 1-length of M. It is a one dimensional measure of M; for M in $\mathbf{R}^3$ and tM the homothety of M by t, one has $L_1(tM)=tL_1(M)$. In general, for M a p dimensional submanifold of $\mathbf{R}^n$ or S^n, we introduce i-length of M for every $i \leq p$. We then study the behaviour of i-length through projections and intersections obtaining local and cinematic-type formulae.

Notice that $h_1(x)$ is not (except if M is convex) the first symmetric function of curvature σ_1 of M at x. Chern and Slavsky have studied $\int_M \sigma_1$, for M in $\mathbf{R}^n$ and proved cinematic formulae for these functions [Ch], [Sl].

The 2-length of $M \subset S^3$, $L_2(M)$, is the area of M, $L_0(M)$ is the total curvature of M. We define $L_1(M)$ as follows. Let Σ be a geodesic 2-sphere of S^3 with x a conjugate point of Σ (i.e., $\mathrm{dist}(x, \Sigma)=\pi/2$). Let $p: S^3-\{x, -x\} \to \Sigma$ be the projection along the geodesics starting at x. Denote by γ_Σ the critical values of p/M. Define

$$L_1(M)=\frac{1}{\pi^2}\int_{G(4,3)} |\gamma_\Sigma| \, d\Sigma,$$

where $G(4, 3)$ is the Grassmann manifold of 3-planes through the origin of $\mathbf{R}^4$, identified with the space of geodesic 2-spheres of S^3.

We prove $L_1(M)=\pi^2 \int_M h_1$. Also we establish

$$L_0(M)=\frac{1}{2\mathrm{Vol}\, G(4,2)}\int_{G(4,2)} |\gamma_l| \, dl,$$

where $l \in G(4, 2)$ is a geodesic of S^3, and $|\gamma_l|$ is the number of critical points of the projection of M to l (along the geodesic spheres orthogonal to l).

Now one uses the cinematic formulae to relate $L_0(M)+L_1(M)+L_2(M)$ to the critical points of a Morse function on M. For this, we construct an "adapted" singular foliation of S^3.

The theory is much simpler for curves on S^2; we indicate the argument here.

Let $l \in G(3,2)$ denote a geodesic of S^2 and for each $y \in P^2$ ($y = a$ pair of antipodal points of S^2), let $\mathscr{F}(y)$ be the foliation of S^2 (singular at y) by geodesics passing through y.

We have

$$\int_C |k_g| = \frac{1}{2}\int_{P^2} |\mu|(C, \mathscr{F}(y))\, dy,$$

where $|\mu|(C, \mathscr{F}(y))$ denotes the number of contact points of C and $\mathscr{F}(y)$. Also

$$|C| = \frac{1}{2}\int_{l\in G(3,2)} \#(C\cap l)\, dl = \frac{1}{2\pi}\int_y \left(\int_{l\in\mathscr{F}(y)} \#(C\cap l)\right) dy,$$

where $|C|$ denotes the length of C. Hence

$$\int_C (|k_g|+1) = \frac{1}{2}\int_y \left[|\mu|(C, \mathscr{F}(y)) + \frac{1}{\pi}\int_{l\in\mathscr{F}(y)} \#(C\cap l)\, dl\right] dy.$$

Now for $y \in P^2$, if C intersects every $l \in \mathscr{F}(y)$, then C intersects every such l in at least two points and

$$\int_{l\in\mathscr{F}(y)} \#(C\cap l) \geq 2\pi$$

If C is disjoint from $l \in \mathscr{F}(y)$, then a moments thought shows there are at least two points of contact of C with $\mathscr{F}(y)$. Thus $|\mu|(C, \mathscr{F}(y)) \geq 2$; so $\int_C (|k_g|+1) \geq 2\pi$. This illustrates the integral geometric technique but for curves the result is not interesting since the last inequality is just an application of Fenchel's theorem for curves in $\mathbb{R}^3 = (k = \sqrt{k_g^2+1}$ is the curvature of C in $\mathbf{R}^3$).

For surfaces in S^3 the argument requires the introduction of a foliation adapted to a flag of geodesic spheres.

We remark that this notion of length has been applied in oceanography [J-L].

I. The length functions for submanifolds of $\mathbf{R}^n$ and their cinematic formulae

Let M be a p-dimensional submanifold of $\mathbf{R}^n$ and let h be a $i+1$ dimensional linear subspace of $\mathbf{R}^n$ (we will denote by $G(n, i+1)$ the Grassmann manifold of all such h). The critical points of the orthogonal projection p_h of M to h will be denoted by $\Gamma_h(M)$ (or Γ_h if there is no ambiguity) and we denote the set of critical values of p_h by γ_h, or $\gamma(M, h)$.

When $p \geq i$, for almost every $h \in G(n, i+1)$, Γ_h is almost everywhere an i-dimensional submanifold of M and for almost every $x \in \Gamma_h$, $T_x(\Gamma_h)$ and $h^\perp$ are transverse in $T_x(M)$, so γ_h is a hypersurface of h in a neighborhood of $p_h(x)$.

We define the i-length functional as:

$$L_i(M) = c \int_{G(n,i+1)} |\gamma_h| \, dh,$$

where $|\gamma_h|$ denotes the volume of γ_h (when $i=0$, γ_h is a finite set and $|\gamma_h|$ is the number of points in γ_h), and the constant c is chosen so that if M is the boundary of an ε-tubular neighborhood of an i-dimensional submanifold C of an affine $p+1$ dimensional subspace of $\mathbf{R}^n$, then $\lim_{\varepsilon \to 0} L_i(M) = |C|$.

If tM denotes a homothety of M by $t > 0$, then clearly

$$L_i(tM) = t^i L_i(M).$$

The constant c occurring in the definition of L_0 is $1/2|\mathbf{P}_{n-1}|$, since a sphere of any dimension ≥ 1 satisfies $|\gamma_l| = 2$ for every line $l \in G(n, 1)$. We will see shortly that $L_0(M)$ is the total curvature of M.

Here are some examples of 1-lengths of surfaces in $\mathbf{R}^3$:

$$L_1(M) = \frac{1}{\pi^2} \int_{G(3,2)} |\gamma_h| \, dh.$$

If M is a round cylinder of height λ, then γ_h is (for almost all h) two parallel segments of length $\lambda |\cos \theta|$ where θ is the angle between the axis of M and the plane h. Hence $L_1(M) = \lambda$. If M is a sphere of radius R, γ_h is a circle of radius R and $L_1(M) = 4R$.

I.1. The local formulae

We define extrinsic curvature functions h_i on $M^p \subset \mathbf{R}^n$, and we prove $L_i(M) = c \int_M h_i(x)\, dx$, where $c = c(n, p, i)$.

Let us begin by L_0 and L_1 of a surface M in $\mathbf{R}^3$. We know that

$$L_0(M) = \frac{1}{4\pi} \int_{\mathbf{P}_2} |\gamma_l| \, dl,$$

where $|\gamma_l|$ is the number of critical points of the projection of M to l.

Let $\phi\colon M \to E$ be the map $\phi(x) = (l(x), p_{l(x)}(x))$, where $l(x)$ is the line through the origin parallel to the normal line to M at x, $p_{l(x)}(x)$ is the orthogonal projection of x to $l(x)$, and E is the tautological line bundle over P_2. Let $N = \phi(M)$ and H be the horizontal plane field of the Riemannian fibration $\pi\colon E \to P_2$.

Clearly $\pi\phi$ is the Gauss map of M with $|\mathrm{Jac}(\pi\phi)| = |K(x)|$, K the Gauss curvature of M at x; so

$$|K(x)| = |\mathrm{Jac}\,\phi(x)|\,|\mathrm{Jac}\,p_{H(x)}|,$$

where $\mathrm{Jac}\,p_{H(x)}$ is the Jacobian of the orthogonal projection (in E) of $T_{\phi(x)}N$ to $H_{\phi(x)} = H(x)$.

Hence

$$\int_{\mathbf{P}_2} |\gamma_l|\,dl = \int_N |\mathrm{Jac}(p_H)| = \int_M |\mathrm{Jac}(\phi)|\,|\mathrm{Jac}\,p_H|\,dx = \int_M |K(x)|\,dx.$$

The first equality is a special case of the coarea formula and the second is a change of variables. Hence

$$L_0(M) = \frac{1}{4\pi}\int_M |K(x)|\,dx.$$

This formula for the total curvature of M is the basis of the Chern-Lashof theorem and easily generalises to $\mathbf{R}^n$ [C-L].

For future calculations it is useful to introduce the following notation. Let $p\colon E \to B$ be a Riemannian fibration and $N \subset E$ a submanifold transverse to the fibers $F(y) = p^{-1}(y)$, $y \in B$. Let H be the horizontal plane field of the fibration. At $x \in N$, $T_x(N)$ is the orthogonal sum $T_x(N \cap F_x) + V(x)$ where $V(x)$ is a subspace transverse to the fibers of dimension that of $H(x)$. Denote by $\mathrm{Jac}\,p_{H(x)}$ the Jacobian of the orthogonal projection of $V(x)$ to $H(x)$. Then the coarea formula yields:

$$\int_N |\mathrm{Jac}\,p_{H(x)}|\,dx = \int_B |F(y) \cap N|\,dy,$$

and more generally, if $\phi\colon M \to E$ is an immersion transverse to the fibers, $N = \phi(M)$, then

$$\int_M |\mathrm{Jac}\,\phi|\,|\mathrm{Jac}\,p_{H(x)}| = \int_N |\mathrm{Jac}\,p_{H(x)}|\,dx = \int_B |F(y) \cap N|\,dy.$$

Now we derive the local formula for a surface M in $\mathbf{R}^3$. Let l be a line in the tangent space to $x \in M$, and let $|k(x, l)|$ be the module of the normal curvature of M at x in the direction l; i.e., $k(x, l)$ is the curvature of the plane curve $M \cap (v_x \oplus l)$, v_x the normal line to M at x.

We define

$$h_1(x) = \frac{1}{\operatorname{Vol}(\mathbf{P}_1)} \int_{\mathbf{P}_1(T_x(M))} |k(x, l)| \, dl.$$

When M is convex at x, $h_1(x)$ is the mean curvature of M at x.

PROPOSITION I.2. *For M a surface in $\mathbf{R}^3$,*

$$L_1(M) = \frac{1}{\pi} \int_M h_1(x) \, dx.$$

Proof. Let $\pi: E = E(3, 2) \to G(3, 2) = G$ be the tautological line bundle, $E = \{h \in G, x \in h\}$.

Let $\phi: P_1(M) \to E$ be the map

$$\phi(x, l) = (h = l^{\perp}, p_h(x)),$$

and let $\phi(P_1(M)) = N$. We know that

$$\int_G |\gamma_h| \, dh = \int_{P_1(M)} |\operatorname{Jac} \phi| \, |\operatorname{Jac} p_H|,$$

so we compute the Jacobians.

Let l be a line through x in $T_x(M)$, v_x denote the line normal to M at x, $h = l^{\perp}$ the subspace of $\mathbf{R}^3$ orthogonal to l and W the orthogonal to v_x in h; cf. Figure 1.

We choose a basis of $T_{(x,l)}(P_1(M))$ as follows:

- U_f is a unit vector tangent to the circle fiber of $\mathbf{P}_1(M)$ at x,
- U_Γ is a horizontal lift of a unit vector tangent to Γ_h at x,
- U_l is a horizontal lift of a unit vector tangent to $(l \oplus v_x) \cap M$ at x.

Also, let U_γ be a horizontal lift (in E) of a unit vector tangent to γ_h at y.

The volume of the parallelepiped generated by the first three vectors is $|\cos \theta|$ where θ is the angle between $T_x\Gamma_h$ and h.

The image $d\phi(U_\Gamma)$ is the vector $\pm\cos(\theta) U_\gamma$. The vector $d\phi(U_f)$ and $d\phi(U_l)$ are projected by the differential $d\pi$ of the projection $\pi: E(3, 2) \to G(3, 2)$ on two orthogonal vectors of $T_{\pi\phi(x)} G(3, 2)$; the first unitary and the second of norm $|k(x, l)|$.

Hence

$$|\text{Jac}\,\phi(x)|\,|\text{Jac}\,p_H| = |k(x, l)|,$$

and I.2 follows by integrating over the fibers of $\mathbf{P}_1(M)$.

Remark. A different proof of this can be found in [L-S] based on a Meusnier formula.

Now we define the functions $h_i(x)$ when $M \subset \mathbf{R}^n$ is a hypersurface. Let $l = l^i$ be an i-dimensional subspace of $T_x(M)$, and let $\nu(x)$ be the normal line to M at x. Denote by $|K|(x, l)$ the absolute value of the Gauss-Kronecker curvature at x of the hypersurface $M \cap (l \oplus \nu(x))$ of $l \oplus \nu(x)$. Then we define

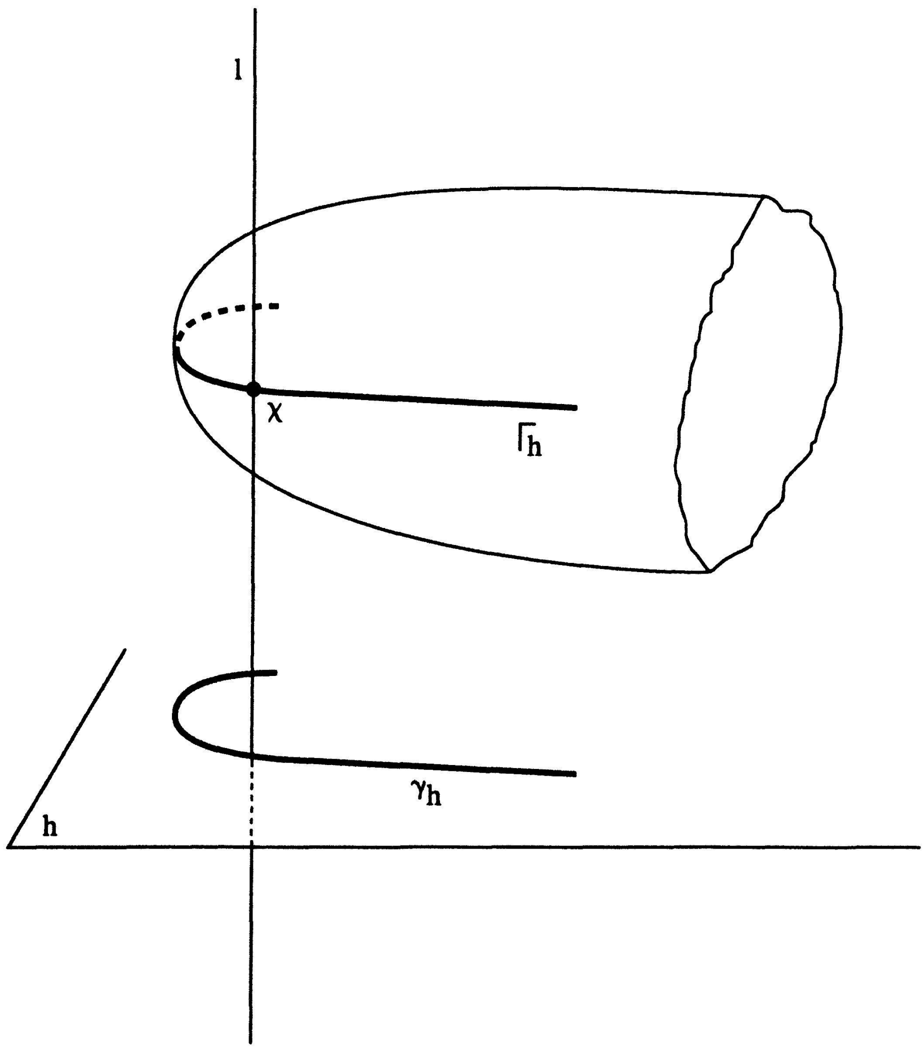

Figure 1

$$h_i(x) = \frac{1}{\operatorname{Vol} G(n-1, i)} \int_{G(T_x M, i)} |K|(x, l)\, dl,$$

where $G(T_x M, i)$ is the i-dimensional subspaces of $T_x(M)$.

Now I.2 generalizes to $\mathbf{R}^n$.

PROPOSITION I.3. *The functions $h_{n-i}(x)$ localize the functions $L_i(M)$; more precisely,*

$$\int_M h_{n-i}(x) = cL_i(M),$$

where the constant c depends only on the dimensions.

Proof. Let G be the bundle over M whose fibers are the spaces $G(T_x M, l)$, l an $n-1-i$ dimensional subspace of $T_x M$, and let $E = E(n, i+1) \to G(n, i+1)$ be the tautological bundle.

Define $\phi\colon G \to E$ by

$$\phi(x, l) = (h = l^{\perp}, p_h(x)).$$

Notice that the dimension of $G(M, n-1-i)$ is equal to the dimension of $N = \bigcup_{h \in G(n, i+1)} \gamma_h$, which is $in + n + i^2 - i - 1$.

Now the proof proceeds as in I.2; we leave the details to the reader.

I.4. The cinematic formulae

We will show that the p-length of a submanifold $M \subset \mathbf{R}^n$ is equal to the $(p-i)$-length of the sections of M by affine subspaces of codimension i (up to a constant only depending on dimensions; we will denote such constants by c here).

The idea is to use the Cauchy formula and a projection in cascade.

Let D denote the flag of all pairs (h, L) where $g \in G_{n,p+1}$ and L is an affine subspace of h of codimension i.

When L is transverse to γ_h, the points of $\gamma_h \cap L$ are the critical points of the projection of $M \cap (L \oplus h^{\perp})$ to the vector subspace l determined by L. Let $H = L \oplus h^{\perp}$; H is an affine subspace of codimension i in $\mathbf{R}^n$.

Since $\gamma(M \cap H, l) = \gamma_h \cap L$, we have

$$|\gamma_h| = c \int_{L \in A(h, p+1-i)} |\gamma(M \cap H, l)|.$$

Hence

$$L_p(M) = c\int_{G(n,p+1)}\left(\int_{A(h,p+1-i)} |\gamma(M\cap H, l)|\right).$$

Notice that D can be thought of as $\{H \in A(n, n-i), l \in G(H, p+1-i)\}$, hence D is a Riemannian fibration over $A(n, n-i)$ with fiber $G(H, p+1-i)$.

Now

$$c \cdot L_{p-i}(M\cap H) = \int_{G(H,p+1-i)} |\gamma(M\cap H, l)|,$$

hence one has the cinematic formula:

$$L_p(M) = c\int_{A(n,n-i)} L_{p-i}(M\cap H).$$

II. Surfaces in S^3

In this section we will define the length functionals of surfaces in S^3 and establish the local and cinematic-type formulae. There are technical difficulties that arise here (in contrast to $\mathbf{R}^3$) due to the fact that the distortion of the projection in S^3 to a geodesic sphere depends on the point.

We begin with $L_2(M)$ (=the area of M) and the spherical Cauchy-Crofton formula [Sa].

THEOREM II.1. *For M a compact surface in S^3,*

$$L_2(M) = \frac{1}{\pi}\int_{G(4,2)} |M\cap l|\, dl,$$

where l is a great circle of S^3 (which we can think of as a 2-plane through the origin of $\mathbf{R}^4$), $|M\cap l|$ is the number of points of $M\cap l$.

Proof. Consider the map $\phi\colon P(TS^3/M)\to G(4,2)$, $\phi(x, L) = l$ where l is the great circle whose tangent at x is L

Write the tangent space to $G(4, 2)$ at l_0 as an orthogonal sum:

$$T_{l_0}G(4,2) = T_{l_0}\{l/x\in l\} \oplus T_{l_0}\{l\perp\Sigma_{l_0,x}\},$$

where $\Sigma_{l,x}$ is the geodesic 2-sphere at x orthogonal to l.

Write $T_{(x,L)}(PTS^3/M) = V \oplus H$ where V is the tangent space to the fiber and $H = V^{\perp}$. Then

$$d\phi = \begin{pmatrix} Id & * \\ \bigcirc & p_{L^{\perp}} \end{pmatrix},$$

where $p_{L^{\perp}}$ is the orthogonal projection of T_xM to $T_x(\Sigma_{l,x}) = L^{\perp}$. Then

$$\int_{L \in P_x(TS^3/M)} |\mathrm{Jac}\, d\phi| = \int_{P_2} |\cos \measuredangle (L^{\perp}, T_xM)| = \pi.$$

Since

$$\int_{G(4,2)} |\phi^{-1}(l)| = \int_{G(4,2)} |l \cap M|,$$

we have

$$\int_{G(4,2)} |l \cap M| = \pi |M|.$$

Now we discuss $L_1(M)$. Let $a = (x, -x) \in G(4, 1)$, be a pair of antipodal points of S^3 which are not on M. This point a determines a projection $p_{\Sigma}: M \to \Sigma$ where Σ is the geodesic 2-sphere of S^3 conjugate to a (i.e. dist $(x, \Sigma) = \pi/2$). By definition $p_{\Sigma}(y)$ is the point of Σ which is the intersection with Σ of the geodesic of S^3 through a and y. Let Γ_{Σ} be the critical points of p_{Σ} and γ_{Σ} the critical values.

DEFINITION. $L_1(M) = (1/2\pi^2) \int_{G(4,3)} |\gamma_{\Sigma}| \, d\Sigma$.

The constant is chosen so that the 1-length of an ε tubular neighborhood of a curve C tends to the length of C as $\varepsilon \to 0$. This choice will be justified once we have established the cinematic formulae for L_1.

Now just as in $\mathbf{R}^3$ we define an extrinsic function h_1 on M. Let $k(x, l)$ be the geodesic curvature at x of the curve $\Sigma_l \cap M$ in Σ_l, where Σ_l is the geodesic 2-sphere at x tangent to l and $v_x = T_x(M)^{\perp}$. Then define

$$h_1(x) = \frac{1}{\pi} \int_{P_1(T_xM)} |k(x, l)| \, dl.$$

THEOREM II.2. *For M a compact surface in S^3,*

$$L_1(M)=\frac{1}{\pi}\int_M h_1.$$

Proof. For $x\in M$, let Σ_x be the geodesic 2-sphere tangent to M at x. Let P be the bundle over M with fiber the projective space P_2:

$$P=\{(x,a)/a=(y,-y),\, y\in\Sigma_x\}.$$

Denote by Σ_a^* the geodesic 2-sphere conjugate to the pair $a=(y,-y)$, and let $E=E(4,3)\to G(4,3)=G$ be the tautological bundle:

$$E=\{(\Sigma,y)/\Sigma \text{ a geodesic 2-sphere}, y\in\Sigma\}.$$

Then define $\phi\colon P\to E$ by:

$$\phi(x,a)=(\Sigma_a^*, p_{\Sigma_a^*}(x)).$$

By construction $N=\phi(P)$ is the union of the critical values γ_Σ; $N=\bigcup_\Sigma \gamma_\Sigma$ (cf. Figure 2; the polar curve Γ_Σ is the set of critical points of the orthogonal projection on Σ, and the critical values Γ_Σ is in $p_\Sigma(\Gamma_\Sigma)$).

Then

$$\int_{G_{4,3}}|\gamma_\Sigma|\,d\Sigma=\int_P|\text{Jac}\,\phi|\,|\text{Jac}\,p_H|,$$

so we must calculate the Jacobians.

To do this we decompose $T_{(x,a)}P$ and TN.

As y varies on Σ, Σ_y^* spans a sphere $S(\Sigma)$ contained in G.

Let F be the 3-dimensional orthogonal complement of $T\gamma_\Sigma$ in TN, at the point $u=(\Sigma_a^*, p_{\Sigma_a^*(x)})$. Write $F=F_1\oplus F_2$ (at x), where F_1 is the lift of $T_{\Sigma_a^*}(S(\Sigma))$ to F and F_2 is the orthogonal complement of F_1 in F. So $TN=F_1\oplus F_2\oplus T\gamma_\Sigma$, at x. Let H_1 be the horizontal lift to $H(E)$ of $T_{\Sigma_a^*}(S(\Sigma))$, and let H_2 be $H_1^\perp$ in $H(E)$.

Now define a splitting of $T_{(x,y)}P$, non orthogonal in general, as follows. Write $T_xM=T_x\Gamma_{\Sigma_y^*}+L$, where L is the line tangent to the circle l joining x to y (this is not orthogonal in general). Let h_1 and h_2 be the horizontal lifts to P of $T_x\Gamma_{\Sigma_y^*}$ and L respectively.

We shall see that the matrix of $p_H\circ d\phi$ is then:

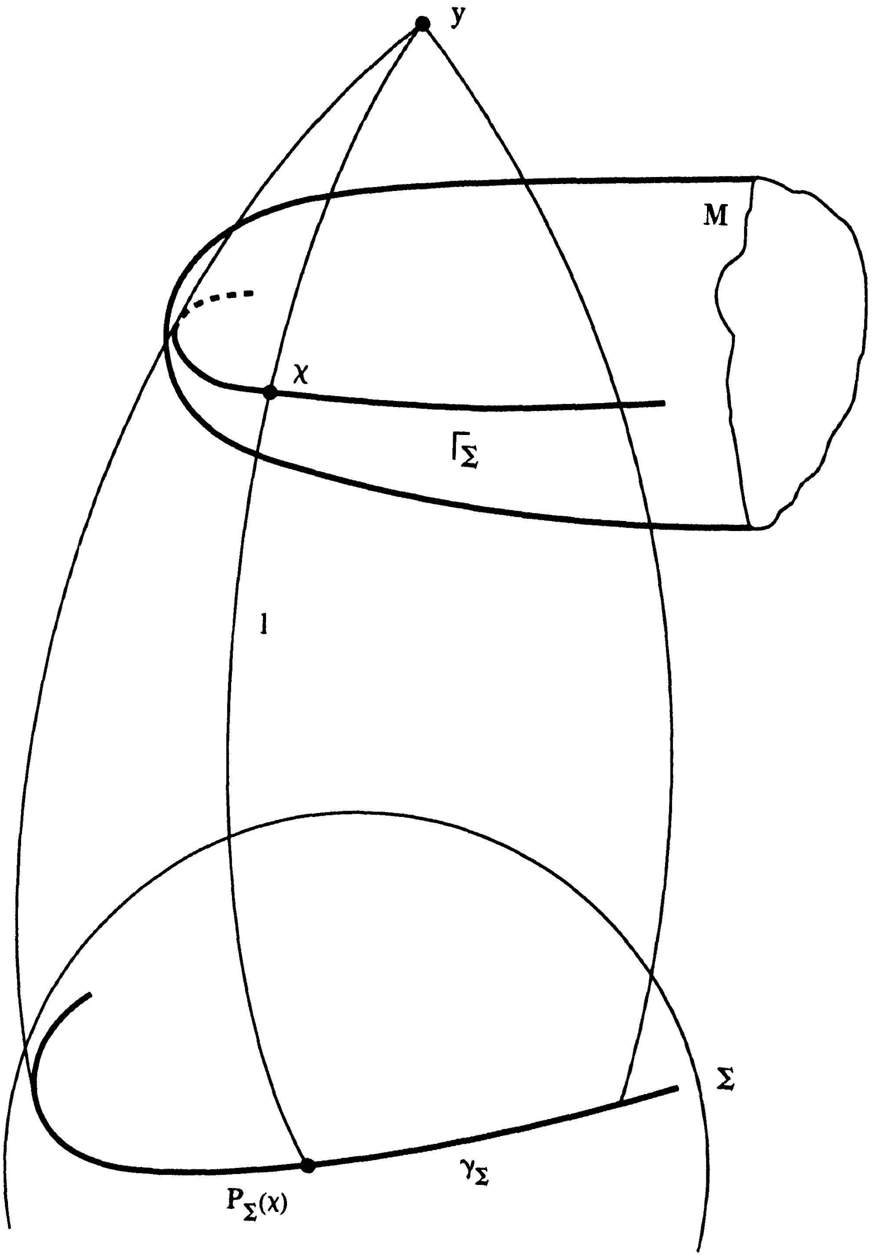

Figure 2

$$\begin{pmatrix} \alpha & * & * \\ 0 & Id & * \\ 0 & 0 & k(x, L)|\sin\theta| \end{pmatrix}$$

here α is the Jacobian of the projection of Γ_Σ on γ_Σ and θ is the arclength on l between x and y. This matrix is computed with respect to the basis vectors $\{h_1, T_{(x,y)}\Sigma_x, h_2\}$ of the domain and the basis vectors $\{T\gamma_\Sigma, H_1, H_2\}$ of the range. We calculate the matrix of $p_H \circ d\phi$ on $H_1 \oplus H_2$; identifying $H_1 \oplus H_2$ with TG.

By definition of Γ_Σ, $d\phi(h_1) \subset T\gamma_\Sigma$.

The coefficient α satisfies: $\alpha|\sin\theta| = \alpha_0$, where α_0 is the Jacobian of the projection of Γ_{Σ_0} on γ_{Σ_0}, when the geodesic sphere Σ_0 is orthogonal to l at x. This follows from lemma II.3, which we prove shortly.

By definition of $T_{\Sigma_a^*}(S(\Sigma))$, $d\phi(T_{(x,y)}\Sigma_x)$ is of the form:

$$\begin{pmatrix} * \\ Id \\ 0 \end{pmatrix}.$$

It remains to determine the component of $d(p \circ \phi)(h_2)$ on H_2. For that, we follow a point on the circle tangent at ξ, where ξ is a point moving on the curve C of intersection of M with the geodesic sphere at x containing l and the normal geodesic circle to M at x (cf. Figure 3). Figure 3 shows the analogous map for a curve on S^2: the length of the arc of the evolute (image of the arc dl between x and

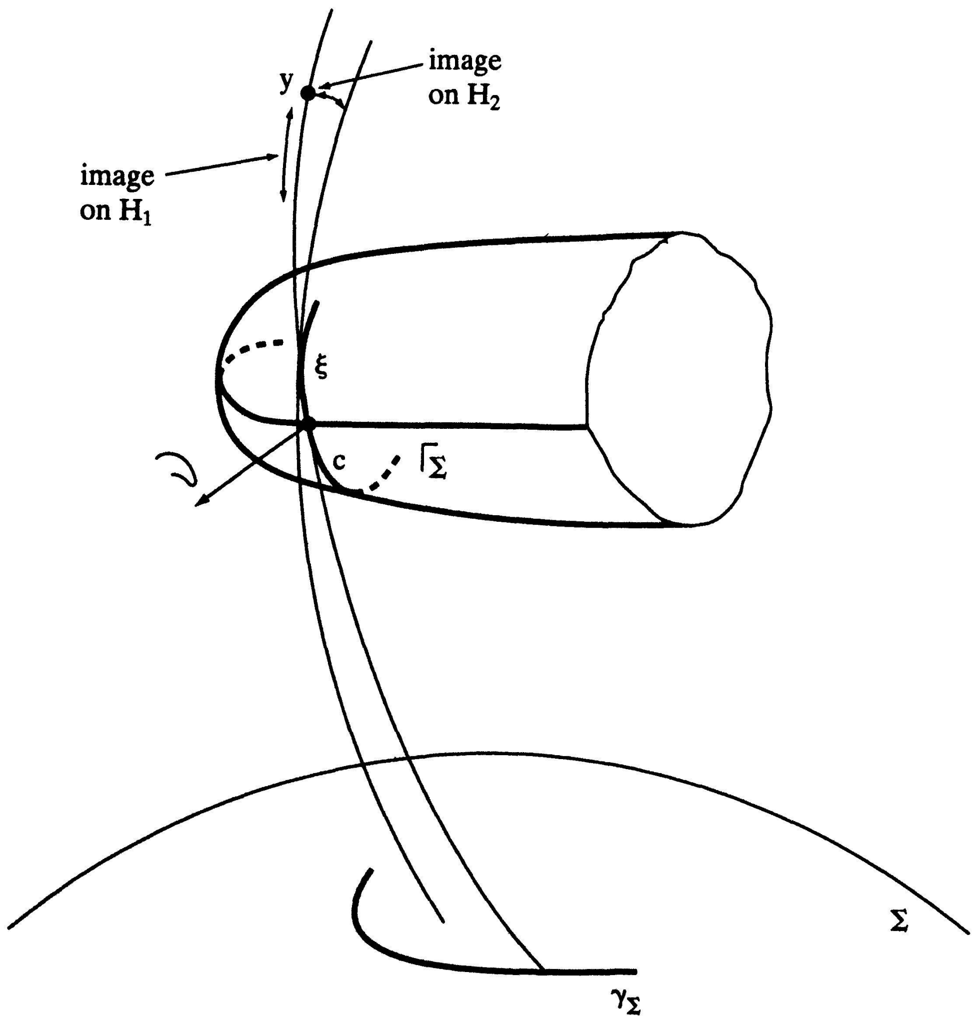

Figure 3

ξ) is $k(x)|\sin\theta|$, up to first order, where θ is the arc length along l between x and y (since $k(x)=d\varphi/ds$).

The same analysis applies in S^3; one gets $k(x,l)|\sin\theta|$.

The decomposition of TP is not orthogonal; the volume of the parallelepiped generated by h_1, $T_{(x,y)}\Sigma_x$ and h_2 is α_0.

The volume density on $P(\Sigma_x)$ is $|\sin\theta\,d\theta\wedge d\varphi|$ where (θ,φ) are polar coordinates at x on the space $P(\Sigma_x)$ of pairs of antipodal points on Σ_x.

Hence

$$\int_P |\text{Jac}\,\phi|\,|\text{Jac}\,p_H| = \int_M\int_{P(\Sigma_x)} \frac{\alpha_0|k(x,l)|\,|\sin\theta|\,|d\theta\wedge d\varphi|}{\alpha_0}$$

$$= 2\pi\int_M h_1(x)\,dx.$$

To complete the proof of theorem II.2 we now prove Lemma II.3.

LEMMA II.3. *Let $C(t)$ be a curve on a surface M embedded in $\mathbf{R}^3$. Assume $\dot{C}(t)$ is not in the kernel of γ at $C(t)$, γ the Gauss map of M. Then the characteristic line of the envelope of the family of tangent planes to M along $C(t)$ is $d\gamma(\dot{C})^\perp$.*

Proof. The equations of the envelope are:

$$\langle X-x, \gamma(x)\rangle = 0,$$

$$\langle X-x, d\gamma(\dot{C})\rangle = 0.$$

As an immediate corollary of this lemma we have: if $K(x)\neq 0$ (so $d\gamma(x)$ is non singular), all the curves C through x (C on M), such that the characteristic line through x of the envelope of the family of planes $T_{C(t)}M$ is a given line D, are tangent at x to the line Δ such that $d\gamma(\Delta)=D$.

The analogous result in S^3, using envelopes of geodesic spheres tangent to M along a curve, follows from the following remark concerning cones in $\mathbf{R}^4$, over $M\subset S^3$ and $C(t)$ a curve on M. Then the envelope of the family $T_{C(t)}(Z)$, contains the 2-plane $(d\gamma(\dot{C}(t)))^\perp$, (orthogonal in $T_{C(t)}Z$ to $d\gamma(\dot{C}(t))$) whenever $\dot{C}(t)$ is not contained in $Ker\,d\varphi$. This remark is clear since the equations of the 2-plane are as before:

$$\langle X-C(t), \gamma(C(t))\rangle = 0$$

$$\langle X-C(t), d\gamma(\dot{C}(t))\rangle = 0.$$

We finish this section with a discussion of $L_0(M)$. By definition:

$$L_0(M) = \frac{1}{2\,\mathrm{Vol}(G(4,2))} \int_{G(4,2)} |\gamma_l|\, dl,$$

where $|\gamma_l|$ is the number of critical points of the projection of M to the geodesic l; the projection along the (singular) foliation $\mathscr{F}(l)$ of geodesic 2-spheres orthogonal to l. Notice that $|\gamma_l|$ is the number of points of contact of M and $\mathscr{F}(l)$, for almost all l. The constant is chosen so that $L_0(\partial B(x, \varepsilon)) = 1$, for $\varepsilon \to 0$.

THEOREM II.4. *Let M be a surface in S^3 and $K(x)$ be the extrinsic Gauss curvature of M at x. Then*

$$L_0(M) = \frac{1}{4\pi} \int_M |K(x)|.$$

Proof. Let $E = E(4, 2) \to G(4, 2) = G$ be the tautological fibration and let $P(M)$ be the bundle over M of the geodesic 2-spheres tangent to M. Define $\phi: P \to E$ by:

$$\phi(x, y) = (y, l \text{ is orthogonal to } \Sigma_x \text{ at } y).$$

Here Σ_x is the geodesic sphere tangent to M at x. Let $N = \phi(P)$ and H be the horizontal field of the bundle $E \to G$.

Take a basis of $T_{(x,y)}P$ composed of a unitary frame tangent to Σ_x at y and two horizontal unit vectors that project to two unitary vectors tangent to the principal directions to M at x. Then it is clear that the proof of II.4 follows from Lemma II.5 below.

First we define the 0-length of a curve C on S^2:

$$L_0(C) = \frac{1}{4\pi} \int_{G(3,2)} |\gamma_l|\, dl.$$

Then we have:

LEMMA II.5. *Let k_g be the geodesic curvature of a curve $C \subset S^2$. Then*

$$L_0(C) = \frac{1}{2\pi} \int_C |k_g|.$$

Proof. Let $E = E(3, 2) \to G(3, 2) = G$ be the tautological fibration and $P(C)$ the bundle over C with fibers the geodesic circles of S^2 tangent to C. Define $\phi: P(C) \to E$ by

$$\phi(x, y) = (y, l \text{ is orthogonal to } \Sigma_x \text{ at } y).$$

Here Σ_x is the geodesic circle tangent to C at x. We have

$$|\operatorname{Jac} p_H| = |\cos d(x, y)|\, |k_g|,$$

so integrating on the fibers of $P(C)$ we have

$$\int_C |k_g| = C_0 \cdot L_0(C).$$

Since

$$\lim_{\varepsilon \to 0} \int_{\partial B(x,y)} |k_g| = 2\pi,$$

we see that $C_0 = 2\pi$.

Now we derive a cinematic-type formula satisfied by $L_1(M)$.

THEOREM II.6. *Let M be a surface in S^3. Then*

$$L_1(M) = \frac{1}{\pi} \int_{G(4,3)} L_0(M \cap \Sigma).$$

The constant is obtained by considering small spheres S_t. Then $L_1(S_t) \sim 4t$ and $\int_{G(4,2)} L_0(S_t \cap \Sigma) \sim 4\pi t$.

Proof. By definition,

$$L_1(M) = \frac{1}{2\pi^2} \int_{G(4,3)} |\gamma_\Sigma|.$$

The Cauchy-Crofton formula in S^2 says:

$$|\gamma_\Sigma| = \frac{1}{2} \int_{G(3,2)} |\gamma_\Sigma \cap l|.$$

The inverse image of the orthogonal projection onto Σ of the great circle l is a sphere Σ_l. The points of $\gamma_\Sigma \cap l$ are the critical points of the orthogonal projection of $\Sigma_l \cap M$ onto l. Hence

$$L_1(M) = \frac{1}{4\pi^2} \int_{G(4,3)} \int_{G(3,2)} |\gamma_\Sigma \cap l| = \frac{1}{4\pi^2} \int_{D(4,3,2)} |\mu|(\Sigma_l \cap M, P_l),$$

where P_l is the (singular) foliation of Σ_l by geodesics orthogonal to l. Here $D = D(4, 3, 2)$ is the space of flags (Σ, l), $\Sigma \supset l$. The map $D \mapsto D$, $(\Sigma \supset l) \mapsto (l \subset \Sigma)$, is an isometry of D. Hence

$$L_1(M) = \frac{1}{4\pi^2} \int_{G(4,3)} 4\pi L_0(\Sigma \cap M) = \frac{1}{\pi} \int_{G(4,3)} L_0(\Sigma \cap M),$$

which completes the proof of II.6.

III. The Fenchel theorem for surfaces in S^3

Let $D = D(4, 3, 2, 1)$ be the space of flags $\Delta = (y \subset l \subset \Sigma)$ where y is a pair of antipodal points of a geodesic l contained in a geodesic sphere Σ of S^3. Given Δ, let $\mathcal{F}(y)$ be the foliation (singular) of Σ by the geodesics of Σ passing through y and let $\mathcal{F}(l)$ be the foliation of S^3 by the geodesic spheres of S^3 containing l.

For M a compact surface in S^3 we define the geometry of M with respect to Δ, by

$$\text{Geom}(M, \Delta) = \#(l \cap M) + |\mu|(M \cap \Sigma, \mathcal{F}(y)) + |\mu|(M, \mathcal{F}(l)),$$

where $|\mu|(M \cap \Sigma, \mathcal{F}(y))$ is the number of points of contact of $M \cap \Sigma$ and $\mathcal{F}(y)$, and $|\mu|(M, \mathcal{F}(l))$ the number of contact points of M and $\mathcal{F}(l)$. If M is transvere to Δ (i.e. $y \notin M$ and l and Σ are transverse to M) and if $M \cap \Sigma$ is in general position with respect to $\mathcal{F}(y)$, M in general position with respect to Δ, then $\text{Geom}(M, \Delta)$ is well defined. This holds for almost every $\Delta \in D$.

Hence we can define the geometry of M:

$$\text{Geom}(M) = \frac{1}{\text{Vol}(D)} \int_D \text{Geom}(M, \Delta).$$

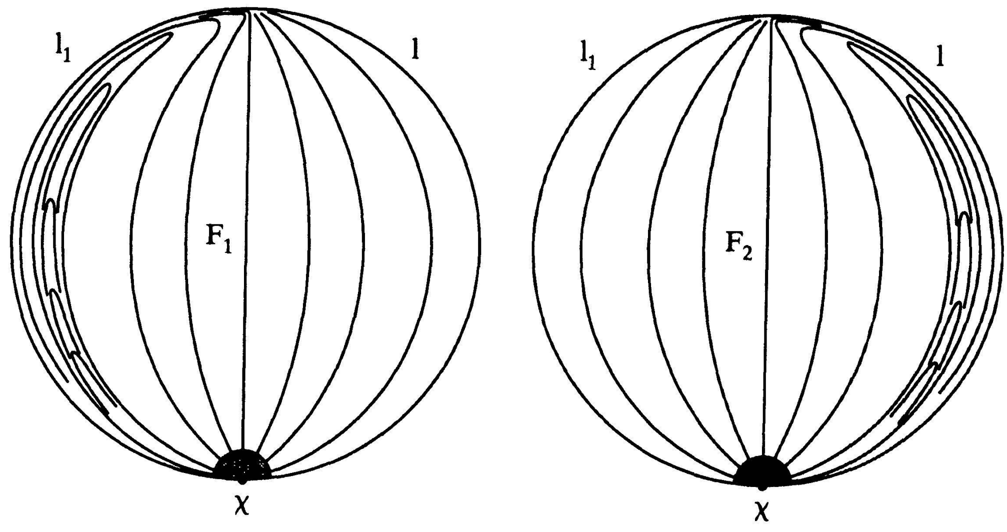

Figure 4

THEOREM III.1. Geom$(M) \geq 2g+2$, *g the genus of M, and if M is knotted in* S^3 Geom$(M) \geq 2g+4$. (*M oriented*).

Proof. It suffices to prove the inequalities for Geom(M, Δ) whenever M is transverse to Δ and in general position with respect to $\mathcal{F}(y)$ and $\mathcal{F}(l)$. To do this we shall construct a foliation $\mathcal{F} = \mathcal{F}(t)$ of $S^3 - B(x, t)$ for $t > 0$ small, $x \in y$, $B(x, t)$ the t-ball of S^3 centered at x, satisfying:

- Geom$(M, \Delta) = |\mu|(M, \mathcal{F})$
- $\mathcal{F}$ is smoothly equivalent to a foliation of $\mathbf{R}^3$ by parallel planes,
- M is in general position with respect to $\mathcal{F}$.

Then the standard Morse theory applies and the theorem follows.

Let $t > 0$ be chosen so that $B(x, t)$ is disjoint from M. Let Σ_1 be one of the hemispheres of Σ bounded by l, $\Sigma = \Sigma_1 \cup \Sigma_2$, $\Sigma_1 \cap \Sigma_2 = l$. Let $\mathcal{F}_1$ be a one-dimensional foliation of $\Sigma_1 - B(x, t)$ as in Figure 4). Notice that l is a leaf of $\mathcal{F}_1$ (actually $l - B(x, t)$). We require the leaves of $\mathcal{F}_1$ to be geodesics of Σ_1 through y, outisde of a small tubular neighborhood of l in Σ_1.

This foliation of Σ_1 has a "Reeb-type" component near an arc $x = l_1$ of l going from $-x$ to $\partial B(x, t)$ (the left side of l in Figure 4). Notice that if C is a curve on Σ, transverse to l_1, then the foliation $\mathcal{F}_1$ can be constructed so that $\#(C \cap l_1) =$ the number of contact points of C and the Reeb-type component of $\mathcal{F}_1$. It suffices to construct $\mathcal{F}_1$ so the Reeb-type component is close enough to l_1.

Similarly, define a foliation $\mathcal{F}_2$ of $\Sigma_2 - B(x, t)$, with the Reeb type component of $\mathcal{F}_2$ close to the other arc of l, i.e. $l - l_1$; cf. Figure 4.

Now define $\mathcal{F}(\varepsilon)$; the trace of $\mathcal{F}(\varepsilon)$ on Σ will be $\mathcal{F}_1 \cup \mathcal{F}_2$; $\varepsilon = t$.

Each leaf α of $\mathcal{F}_1$ bounds a 2-disk in Σ_1 (more precisely, each leaf of $\mathcal{F}_1$, together with an arc on $B(x, \varepsilon) \cap \Sigma_1$ joining the extremities of α, bounds a disk in Σ_1). Let

α_1 be a leaf of $\mathcal{F}_1$ as indicated in Figure 4, and consider the leaves of α of $\mathcal{F}_1$ inside the disk of Σ_1 bounded by α_1. Let $D(\alpha)$ be the disk of Σ_1 bounded by α. Let $F(\alpha)$ be a 2-disk in S^3 which is a thickened $D(\alpha)$; imagine $F(\alpha)$ as a thin pancake over $D(\alpha)$. $F(\alpha)$ is orthogonal to Σ_1 and $F(\alpha)\cap\Sigma_1=\alpha$. In S^3, Σ separates S^3 into two balls B_1 and B_2, and $F(\alpha)$ intersects each ball in a 2-disk close to $D(\alpha)$.

Choose the $D(\alpha)$, α inside $D(\alpha_1)$, so that the $\bigcup_\alpha F(\alpha)$ foliate a part of S^3, and all the $F(\alpha)$ are sufficiently flat so the foliated set is close to $D(\alpha)$. (One can do this by pushing one's thumb into $S^3-B(x,\varepsilon)$, starting at $a\in\partial B(x,\varepsilon)$ to create the Reeb component. One keeps on pushing almost until x. The thumb starts out as a very thin thumb and then spreads out as a thin pancake till α_1.)

Let $\Sigma(l)$ be the geodesic 2-sphere of S^3 containing l, which is orthogonal to Σ along l (in the ball B_1 for example, if one imagines Σ_1 as the upper hemisphere, then $\Sigma(l)\cap B_1$ is the equatorial plane). Now foliate the region of $S^3-B(x,\varepsilon)$ between $F(\alpha_1)$ and $\Sigma(l)-B(x,l)$ by "blowing out" $F(\alpha_1)$ to $\Sigma(l)$. More precisely, the region in question is topologically $F(\alpha_1)\times[0,1]$. One puts the product foliation in the region. However one does this so all the leaves outside a small tubular neighborhood of Σ, are leaves of $\mathcal{F}(l)$, i.e. they coincide with geodesic spheres containing l, outside of a tubular neighborhood of Σ.

This defines $\mathcal{F}(\varepsilon)$ on half of $S^3-B(x,\varepsilon)$. To extend to the other half, one does the same thing we just did, blowing down to the foliation by thin pancakes close to the foliation $\mathcal{F}_2$ of Σ_2. In fact, if β is the geodesic of S^3 through y and orthogonal to Σ, then one extends $\mathcal{F}(\varepsilon)$ by rotating $\mathcal{F}(\varepsilon)$ by π around β.

By construction, all the leaves of $\mathcal{F}(\varepsilon)$, outside a tubular neighborhood of Σ, are parts of the geodesic spheres of $\mathcal{F}(l)$. Now if M is a surface in S^3, transverse to Σ, $y\notin M$ (i.e. $x\notin M$ and $-x\notin M$) and M in general position with respect to $\mathcal{F}(y)$ and $\mathcal{F}(l)$, then constructing $\mathcal{F}(\varepsilon)$ so that the tubular neighborhoods of l (to define $\mathcal{F}_1$) and of Σ, are small, one sees that $\mathrm{Geom}(M,\Delta)=|\mu|(M,\mathcal{F}(\varepsilon))$. A moments inspection shows $\mathcal{F}(\varepsilon)$ is equivalent to a parallel foliation of $\mathbf{R}^3$. This completes the proof of Theorem III.1.

THEOREM III.2. *Let M be a compact surface in S^3. Then* $\mathrm{Geom}(M)$ *is a linear combination of $L_0(M)$, $L_1(M)$ and $L_2(M)$*:

$$\mathrm{Geom}(M)=\pi^3L_2(M)+4\pi^3L_1(M)+2\pi^2\,\mathrm{Vol}\,G(4,2)L_0(M).$$

Proof. We have

$$\int_D |l\cap M|=\pi^2\int_{G(4,2)}|l\cap M|=\pi^3L_2(M)\qquad\text{by II.1.}$$

Also

$$\int_D |\mu|(M \cap \Sigma, \mathscr{F}(y)) = \pi \int_{D(4,3,1)} |\mu|(M \cap \Sigma, \mathscr{F}(y))$$

$$= \pi \int_{G(4,3)} 4\pi L_0(M \cap \Sigma) = 4\pi^3 L_1(M) \qquad \text{by II.6.}$$

Finally

$$\int_D |\mu|(M, \mathscr{F}(l)) = \pi^2 \int_{G(4,2)} |\mu|(M, \mathscr{F}(l))$$

$$= 2\pi^2 \operatorname{Vol}(G(4,2)) L_0(M) \qquad \text{by definition of } L_0(M).$$

COROLLARY III.3.

$$\operatorname{Geom}(M) = \int_M \pi^3 + 2\pi h_1(x) + \frac{\pi}{2} \operatorname{Vol} G(4,2)|K(x)|.$$

Proof. This follows immediately from Theorem III.2 and the local formulae.

IV. Geometry of $M^{n-1} \subset S^n$

Let $D = D(n, n-1, \ldots, 1)$ be the space of flags $\Delta = (\Sigma^0 \subset \Sigma^1 \subset \cdots \subset \Sigma^n = S^n)$ each Σ^i and i-dimensional geodesic sphere of S^n. Define $\mathscr{F}(i, i+2)$ to be the (singular) foliation of Σ^{i+2} by geodesic $i+1$ spheres that contain Σ^i. Denote $M \cap \Sigma^{i+2}$ by M_i when M is in general position with respect to Δ (we subsequently assume this).

We define the geometry of M with respect to Δ.

$$\operatorname{Geom}(M, \Delta) = |M \cap \Sigma^1| + \sum_{i=2}^{n} |\mu|(M_i, \mathscr{F}(i-2, i)).$$

As in the proof of III.1 one has:

THEOREM IV.1. *Let $M^{n-1} \subset S^n$ be in general position with respect to the flag Δ. Then there is an $\varepsilon > 0$ and foliation $\mathscr{F} = \mathscr{F}(\Delta)$ of $S^n - B(x, \varepsilon)$, $x \in \Sigma^0$, satisfying:*

– $\mathrm{Geom}(M, \Delta) = |\mu|(M, \mathscr{F})$, *and*
– $\mathscr{F}$ *is smoothly equivalent to a foliation of* $\mathbf{R}^n$ *by parallel hyperplanes.*

THEOREM IV.2. $\mathrm{Geom}(M)$ *is a linear combination of* $L_0(M)$, $L_1(M), \ldots, L_{n-1}(M)$;

$$\mathrm{Geom}(M) = \int_D \mathrm{Geom}(M, \Delta) = \sum_{i=0}^{n-1} c_i L_i(M),$$

where $c_0, \ldots, c_{n-1}$ *are dimension constants.*

COROLLARY IV.3. *For* $M^{n-1} \subset S^n$, *one has*

$$\sum_{i=0}^{n-1} c_i L_i(M) \geq \beta(M),$$

$\beta(M)$ *the sum of the Betti numbers of* M.

V. The geometry of submanifolds $M \subset S^n$ of arbitrary codimension

Similar results can be obtained in higher codimension. The construction of the foliation associated to a complete flag is unchanged. Therefore we can extend the results obtained in $\mathbf{R}^n$ (see [C-L], [Fe], [L-R]).

THEOREM V.1. *Let* V *be a compact manifold immersed in* S^n. *Then*

$$\mathrm{Geom}(V) \geq \sum \beta_i,$$

where the β_i *are the Betti numbers of* V.
If V *is the sphere* S^p *and is embedded, the condition*

$$\mathrm{Geom}(V) < 4$$

implies that V *is an unknotted sphere* (*topologically and differentiably for* $p = 1$, *all* n; $p = 2$ $n = 4$; $p \geq 5$, $n = p + 2$).

The integral geometric construction requires one more step. For example, in the codimension 2 case ($V^{n-3} \subset S^{n-1}$), we need to consider the "quasi flag space" $D(n, n-2, n-1, n-2)$ of

$$\{h \subset k \supset l, \dim(h) = n-2, \dim(k) = n-1, \dim(l) = n-2\}.$$

Notice that the dimension of the fiber bundle $\mathfrak{D}$ on V

$$\mathfrak{D} = \{x \in V, h_x \subset k \supset l, \dim(k) = n-1, \dim(l) = n-2\},$$

where h_x is the vector space spanned by the geodesic sphere tangent at x to V, is $2(n-2)$, the same as that of the Grassmann manifold $G(n, n-2)$.

THEOREM V.2. *A curve C embedded in S^3 satisfies*

$$\int_C |k_g| + 1 \geq 2\pi$$

$$\int_C |k_g| + 1 \geq 4\pi$$

if C is knotted, and more precisely

$$\int_C |k_g| + 1 \geq 2\pi \cdot (\text{bridge number of } C).$$

The first result was already proved by Banchoff [Ba]; the two others extend results of Fenchel, Fary and Milnor [Fe], [Fa], [M_1], [M_2]; and Sunday [Su].

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