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Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **71 (1996)**

PDF erstellt am: **01.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-53862

Comment. Math. Helvetici 71 (1996) 628–663 0010-2571/96/040628-36$1.50 + 0.20/0

Halves of a real Enriques surface

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Abstract. The real part $E_{\mathbb{R}}$ of a real Enriques surface E admits a natural decomposition in two halves, $E_{\mathbb{R}} = E_{\mathbb{R}}^{(1)} \cup E_{\mathbb{R}}^{(2)}$, each half being a union of components of $E_{\mathbb{R}}$. We classify the triads $(E_{\mathbb{R}}; E_{\mathbb{R}}^{(1)}, E_{\mathbb{R}}^{(2)})$ up to homeomorphism. Most results extend to surfaces of more general nature than Enriques surfaces. We use and study in details the properties of Kalinin's filtration in the homology of the fixed point set of an involution, which is a convenient tool not widely known in topology of real algebraic varieties.

Introduction

A *real Enriques surface* is a complex Enriques surface equipped with an anti-holomorphic involution, called *complex conjugation*; its fixed point set is called the *real part* of the surface. This involution lifts to an involution of the covering $K3$-surface (Lemma 1.3.1). Thus the study of real Enriques surfaces is equivalent to the study of real $K3$-surfaces equipped with a holomorphic fixed point free involution which commutes with the real structure.

A systematic study of the topological properties of real Enriques surfaces was started by V. Nikulin. It is his preprint [N2] that stimulated our investigation. In our preceding paper [DK1] we have completed the classification of real Enriques surfaces by the topological types of their real part.

This classification has a natural refinement (also first studied by V. Nikulin): the real part $E_{\mathbb{R}}$ of a real Enriques surface admits a natural decomposition in two halves $E_{\mathbb{R}} = E_{\mathbb{R}}^{(1)} \cup E_{\mathbb{R}}^{(2)}$, each half being a union of components of $E_{\mathbb{R}}$. This splitting is due to the fact that the real structure lifts to the covering K3 surface in two different ways: each half is covered by the fixed point set of one of the two liftings (see 1.3). This gives rise to the following problem: to classify the triads $(E_{\mathbb{R}}; E_{\mathbb{R}}^{(1)}, E_{\mathbb{R}}^{(2)})$ up to homeomorphism.

For a large number of topological types an arbitrary splitting is realizable. For some other types the splittings are determined by the only restriction: the orientation double covering of a half must either consist of two topological tori or have at

1991 Mathematics Subject Classification. 14J28, 14P25, and 57S25.
Key words and phrases. Enriques surface, real algebraic surface, involution on manifold.

most one nonspherical component. The surfaces constructed in [DK1] show the existence of such splittings in many cases. On the other hand, as it was discovered by Nikulin, there are topological types whose distributions must satisfy to certain restrictions.

It is the distribution of the components between the two halves that is the principal subject of the present paper. Similar to what happened during the investigation of other special classes of surfaces, the present study is stipulated by and based on the discovery of some new prohibitions. These prohibitions (see 2.1) apply not only to Enriques surfaces but as well to other classes of surfaces with non simply connected complexification. More precisely, in this paper we treat what we call *generalized Enriques surfaces*: quotients of a nonsingular compact complex surface X with $H_1(X; \mathbb{Z}/2) = 0$ and $w_2(X) = 0$ by a fixed point free holomorphic involution (see 1.2 and Appendix B).

Note that there are quite 'classical' examples of generalized Enriques surfaces: in Horikawa's construction (see Section 8.1) bi-degree $(4, 4)$ can be replaced with $(4k, 4k)$, $k \in \mathbb{Z}_+$ (and even with $(4k + 2, 4k + 2)$, $k \in \mathbb{Z}_+$; this leads to Spin-surfaces, see Appendix B). Thus, our results also provide some prohibitions on the topology of symmetric real curves on real quadrics.

The prohibitions obtained (see 2.1 and Appendix B) are a combination of the inequality-type and congruence-type prohibitions. To an extent they may be regarded as some kind of refinement of the Smith-Thom inequality and extension of the Arnold-Rokhlin congruences to non simply connected surfaces. (Additional prohibitions of this kind, which also have no precise analogues in the simply connected case and whose proofs are based on similar techniques, can be found in [DK3].)

We apply these results to the classical Enriques surfaces and complete the classification of the distributions of their components (see 2.2.2).

Another by-product are new proofs which clarify the nature of the prohibitions obtained in our previous paper, devoted to the topological classification of real Enriques surfaces (see 2.2 and [DK1, 3.7–3.10]).

The key rôle in our present study is played by so called Kalinin's spectral sequence and Viro homomorphisms, used in combination with more traditional tools of topology of real algebraic varieties. The spectral sequence in question is derived from the Borel-Serre spectral sequence: it is some sort of its stabilization with only one grading. It converges to the homology of the fixed point set, and the corresponding filtration and identification with the limit term are given by the Viro homomorphisms, which have an explicit geometrical definition (see Section 5 for the details).

The paper consists of eight sections and two appendices. In Section 1 we introduce the main objects, such as a generalized $K3$-surface (which, from our point of view, is just a Spin-surface X with $H_1(X; \mathbb{Z}/2) = 0$) and a generalized Enriques surface, give some definitions and fix the principal notation. In Section 2 we formulate the main results and apply them to the classical Enriques surfaces. In Section 3 we expose some auxiliary results on the arithmetic of involutions. Section 4 is devoted to the study of the basic topological properties of generalized Enriques surfaces. In Section 5 we introduce Kalinin's homology spectral sequence and Viro homomorphisms and examine their general properties which we need in subsequent proofs; these results are then applied to generalized Enriques surfaces in Section 6. Finally, in Section 7 we prove the main results announced in Section 2, and in Section 8 we construct some surfaces to extend the list of distributions found in [DK1] and thus complete the classification for the case of classical Enriques surfaces.

In Appendix A we study the multiplicative structure in Kalinin's spectral sequence and prove Theorem 5.2.3, which in the case of an involution on a closed manifold relates the intersection pairings on the manifold and on the fixed point set.

In Appendix B we introduce Spin generalized Enriques surfaces and extend to them the main results of Section 2. (The proofs are found in [DK2], along with the necessary information on the Steenrod operations in Kalinin's spectral sequence.)

1. Preliminary definitions and notation

1.1. *Notation*

We agree that, unless specified explicitly, the coefficients of all the homology and cohomology groups are $\mathbb{Z}/2$. Both the cohomology characteristic classes of a closed smooth manifold and their dual homology classes are denoted by w_i. Throughout the paper we use the following notation:

- b_r and β_r stand for the Betti numbers with the integral and $\mathbb{Z}/2$-coefficients respectively: $b_r(\cdot) = \operatorname{rk} H_r(\cdot\,; \mathbb{Z})$ and $\beta_r(\cdot) = \dim H_r(\cdot)$;
- β_* is the total Betti number: $\beta_*(\cdot) = \Sigma_{r \geq 0}\, \beta_r(\cdot)$;
- $\chi(X)$ is the Euler characteristic of a topological space X;
- $\sigma(M)$ is the signature of an oriented manifold M;
- $\operatorname{Tors}_2 G$ is the 2-primary component of an abelian group G.

1.2. *Generalized Enriques surfaces*

A nonsingular compact complex surface X will be called a *generalized $K3$-surface* if $H_1(X; \mathbb{Z}/2) = 0$ and $w_2(X) = 0$. A *generalized Enriques surface* is a complex surface E which (1) has $w_2(E) \neq 0$, and (2) can be obtained as the orbit space X/τ of a generalized $K3$-surface by a fixed point free holomorphic involution $\tau\colon X \to X$; the latter is called the *Enriques involution.*

As it follows, e.g., from the Ghysin exact sequence, $H_1(E; \mathbb{Z}/2) = \mathbb{Z}/2$ (cf. 4.2.1). Thus, X is the only double covering space of E, and τ is its deck translation. Hence, they are both determined by E.

Remark. Orbit spaces of generalized $K3$-surfaces with $w_2(E) = 0$ are considered in Appendix B.

1.3. *Decomposition of the real part*

As usually, by a *real structure* on a nonsingular complex surface we mean an anti-holomorphic involution. When not empty, the fixed point set of such an involution is a real 2-manifold.

Let E be a generalized Enriques surface, and let conj: $E \to E$ be the real structure on E. Denote by $E_{\mathbb{R}}$ the real part, $E_{\mathbb{R}} = \operatorname{Fix}\operatorname{conj}$.

1.3.1. LEMMA. *There are exactly two liftings $t^{(1)}, t^{(2)}\colon X \to X$ of* conj *to X. They are both anti-holomorphic involutions, commute with each other, and their composition is τ. Both the real parts $X_{\mathbb{R}}^{(i)} = \operatorname{Fix} t^{(i)}$, $i = 1, 2$, and their images $E_{\mathbb{R}}^{(i)}$ in E are disjoint, and $E_{\mathbb{R}}^{(1)} \cup E_{\mathbb{R}}^{(2)} = E_{\mathbb{R}}$.*

Proof. The case $E_{\mathbb{R}} = \varnothing$ is considered in [Ht]. If $E_{\mathbb{R}} \neq \varnothing$, the proof is obvious as soon as the points of X are represented by homotopy classes of paths in E starting at a point of $E_{\mathbb{R}}$: two paths define the same point in X iff they differ by a loop homologous to zero in $H_1(E; \mathbb{Z}/2)$. □

Due to this lemma, $E_{\mathbb{R}}$ canonically splits into two disjoint parts, which we will refer to as the *halves* of $E_{\mathbb{R}}$. Both $E_{\mathbb{R}}^{(1)}$ and $E_{\mathbb{R}}^{(2)}$ consist of whole components of $E_{\mathbb{R}}$, and $X_{\mathbb{R}}^{(i)}$ is an unramified double covering of $E_{\mathbb{R}}^{(i)}$, $i = 1, 2$. In most cases these coverings are determined by $E_{\mathbb{R}}$ intrinsically:

1.3.2. LEMMA. *The real parts $X_{\mathbb{R}} = X_{\mathbb{R}}^{(1)} \cup X_{\mathbb{R}}^{(2)}$ are orientable. The restriction of the projection $X \to E$ to $X_{\mathbb{R}} \to E_{\mathbb{R}}$ is the orientation double covering unless $\sigma(X) \equiv$* (mod 32), *one of the halves of $E_{\mathbb{R}}$ is empty, and the nonempty half is orientable.*

The orientability is well known, see [E], [S], or [K]. The rest follows from the fact that the canonical orientations of $X_{\mathbb{R}}$ are reversed by τ. For classical Enriques surfaces these orientations are given by an exterior holomorphic 2-form ω which is nowhere zero, τ-skew-invariant and becomes $t^{(i)}$-real (i.e., satisfying $\bar{\omega} = t^{(i)}\omega$) after multiplication by a proper constant a_i. In the general case the construction is slightly different. In the proof below we use the Spin-structures as in [DK1, Theorem A.2].

Proof of 1.3.2. Since $H^1(X) = 0$, on X there is a unique Spin-structure ψ. In particular, ψ is equivariant in respect to any involution, i.e., it takes equal values on symmetric framed loops. Let $X_{\mathbb{R}}^{(1)}$ be a nonempty half. In order to compare local orientations of $X_{\mathbb{R}}^{(1)}$ at two points $x, y \in X_{\mathbb{R}}^{(1)}$, represent them by 2-frames and complete these frames to positive 4-frames of X by some pairs of $t^{(1)}$-skew-invariant vectors. Then pick a path γ connecting x and y, extend the 4-frames to a field $\Xi = (\xi_1, \xi_2, \xi_3, \xi_4)$ on γ, and evaluate ψ on the loop $\gamma * t^{(1)}\gamma^{-1}$ framed with $\Xi * \Xi'$, where $\Xi' = (dt^{(1)}\xi_1, dt^{(1)}\xi_2, -dt^{(1)}\xi_3, -dt^{(1)}\xi_4)$. (The latter framed loop is called a *test loop*.) The two orientations are considered coherent iff the value obtained is 0. This construction is consistent since ψ is equivariant; thus, it gives a canonical pair of opposite orientations of $X_{\mathbb{R}}^{(1)}$, and it remains to check that τ reverses them.

For any orientation preserving free involution $c: X \to X$ with X/c not Spin (in particular, for $c = \tau$) the value of ψ on a c-symmetric loop with a 4-frame field $\Xi = (\xi_1, \xi_2, \xi_3, \xi_4)$ is 1 if Ξ is c-invariant and 0 if Ξ is c-skew-invariant, i.e., $dc(\xi_1, \xi_2, \xi_3, \xi_4) = (\xi_1, \xi_2, -\xi_3, -\xi_4)$. Thus, it suffices to construct a τ-invariant test loop. If $X_{\mathbb{R}}^{(2)} \neq \varnothing$, pick $x \in X_{\mathbb{R}}^{(1)}$ and $a \in X_{\mathbb{R}}^2$, join them by an arc (xa), and let γ be the loop formed by (xa), $t^{(1)}(xa)$, $\tau(xa)$, and $t^{(2)}(xa)$. Pick a $t^{(1)}$-invariant frame at x and a $t^{(2)}$-invariant frame at a, complete them by pairs of $t^{(1)}$-skew-invariant (respectively, $t^{(2)}$-skew-invariant) vectors to positive 4-frames, and extend these 4-frames to a 4-frame field over (xa). Reflection gives a τ-invariant continuous 4-framing over γ.

Let now $X_{\mathbb{R}}^{(2)} = \varnothing$ and $\sigma(X) \not\equiv 0 \pmod{32}$. Then $X/t^{(2)}$ is not Spin, since $\sigma(X/t^{(2)}) = \frac{1}{2}\sigma(X) \not\equiv 0 \pmod{16}$. Pick a point $a \in X$ whose orbit a, $t^{(1)}a$, τa, $t^{(2)}a$ consists of four elements and form a loop from the same four arcs as above, an arc δ connecting a and $t^{(2)}a$, and $t^{(1)}\delta$. The test loop constructed as before is the sum of a τ-invariant loop (obtained by replacing $t^{(1)}\delta$ with $\tau\delta$) and a $t^{(2)}$-skew-invariant one, and ψ equals 1 on the former portion and 0 on the latter one (as $t^{(2)}$ is also free now), which totals to 1 on γ.

Finally, if $X_{\mathbb{R}}^{(1)}$ is nonorientable, the result follows from the obvious fact that, since ψ is τ-equivariant, τ either preserves or reverses the canonical orientation of all the components of $X_{\mathbb{R}}^{(1)}$ simultaneously. □

Since E is a compact surface, each component C of $E_{\mathbb{R}}$ is a closed manifold. By the first part of 1.3.2, C may be of one of the following three types:

S_g – a trivially covered orientable surface of genus $g \geq 0$;
V_g – a nonorientable surface of genus $g > 0$, $V_g \cong \#_g \mathbb{R}\mathrm{p}^2$, covered by an orientable component $S_{g-1} \subset X_{\mathbb{R}}$;
T_g – a nontrivially covered orientable surface of genus $g > 0$.

In our notation we use any of $S = S_0 = V_0$ for S^2. To describe the decomposition of $E_{\mathbb{R}}$ into the two halves, we write $E_{\mathbb{R}} = \{\text{half } E_{\mathbb{R}}^{(1)}\} \sqcup \{\text{half } E_{\mathbb{R}}^{(2)}\}$.

Remark. According to 1.3.2, the type T_g is very special: $E_{\mathbb{R}}$ may have such a component only if $\sigma(X) \equiv 0 \pmod{32}$ (or, equivalently, $\sigma(E) \equiv 0 \pmod{16}$), one of the halves of $E_{\mathbb{R}}$ is empty, and the other one is orientable. In particular, this type never occurs in the case of the classical Enriques surfaces.

Remark. Lemma 1.3.2 gives rise to the following problem: Let X be a closed complex surface with $H_1(X) = 0$ and $w_2(X) = 0$, and let τ and conj be two commuting fixed point free involutions on X, holomorphic and antiholomorphic respectively. If X/τ is not Spin, can X/conj be Spin?

1.4. *Types of the real part*

Given a nonsingular compact complex surface Y with real structure, its real part $Y_{\mathbb{R}}$ has a well defined $\mathbb{Z}/2$-homology fundamental class $[Y_{\mathbb{R}}]$. We say that $Y_{\mathbb{R}}$ and Y are of type I_0 (respectively, I_w) if $Y_{\mathbb{R}}$ is homologous to zero (respectively, $w_2(Y)$) in $H_2(Y)$. The surface is said to be of type I if it is of type I_0 or I_w; otherwise it is said to be of type II.

In the case of a generalized Enriques surface E and its double covering X the notion of type obviously extends to the halves $E_{\mathbb{R}}^{(i)}$ and $X_{\mathbb{R}}^{(i)}$. For the covering and its halves the types I_0 and I_w coincide.

1.5. *$(M-d)$-surfaces*

According to the Smith-Thom inequality, for any complex surface Y with real structure one has $\beta_*(Y_{\mathbb{R}}) \leq \beta_*(Y)$, and the difference $\beta_*(Y) - \beta_*(Y_{\mathbb{R}})$ is even. By definition, Y is called an *$(M-d)$-surface* if the above difference is $2d$.

2. Main results

From now on we fix a generalized real Enriques surface E with $E_{\mathbb{R}} \neq \varnothing$ and follow the notation of Section 1: conj: $E \to E$ is the real structure on E, X is the double covering of E with Enriques involution $\tau: X \to X$, and $t^{(1)}$, $t^{(2)}$ are the two real structures on X determined by conj (see 1.3.1).

2.1. *General prohibitions*

2.1.1. THEOREM. *Let $X_{\mathbb{R}}^{(1)}$ be of type* I *and both the halves nonempty. Then*
(1) *$E_{\mathbb{R}}$ has no nonorientable components of odd genus (i.e., V_{2g+1});*
(2) *at least one of the two halves $E_{\mathbb{R}}^{(1)}$, $E_{\mathbb{R}}^{(2)}$ is orientable.*

2.1.2. THEOREM. *Suppose that $E_{\mathbb{R}}$ is orientable. Then E is an $(M-d)$-surface with $d \geq 2$, and*
(1) *if $d=2$, then $\chi(E_{\mathbb{R}}) \equiv \sigma(E) \pmod{16}$ and $E_{\mathbb{R}}$ is of type* I;
(2) *if $d=3$, then $\chi(E_{\mathbb{R}}) \equiv \sigma(E) \pm 2 \pmod{16}$;*
(3) *if $d=4$ and $\chi(E_{\mathbb{R}}) \equiv \sigma(E)+8 \pmod{16}$, then $E_{\mathbb{R}}$ is of type* I.
If, in addition, all the components of $E_{\mathbb{R}}$ are spheres, then $d \geq 3$.

Remark. The last assertion of Theorem 2.1.2 follows from Comessatti-Severi inequality $\chi(E_{\mathbb{R}}) \leq h^{1,1}(E)$ (see [Co]), which transforms into $d \geq 3+h^{2,0}(E)$ for a generalized Enriques $(M-d)$-surface with only spherical components. Thus, such a surface may exist only if $d \geq 3$, and if $d=3$, the lattice $H_2(E;\mathbb{Z})$ must be hyperbolic (as this is the case, e.g., for classical Enriques surfaces).

2.1.3. THEOREM. *Suppose that $E_{\mathbb{R}}$ consists of a single half and does not have nonorientable components of odd genus (i.e., V_{2g+1}). Then E is an $(M-d)$-surface with $d \geq 2$, and*
(1) *if $d=2$, then $\chi(E_{\mathbb{R}}) \equiv \sigma(E) \pmod{16}$ and $E_{\mathbb{R}}$ is of type* I;
(2) *if $d=3$, then $\chi(E_{\mathbb{R}}) \equiv \sigma(E) \pm 2 \pmod{16}$;*
(3) *if $d=4$ and $\chi(E_{\mathbb{R}}) \equiv \sigma(E)+8 \pmod{16}$, then $E_{\mathbb{R}}$ is of type* I.

2.1.4. THEOREM. *Let E be an $(M-3)$-surface with $E_{\mathbb{R}} = kS$. Then $E_{\mathbb{R}} = \{4pS\} \sqcup \{(4q+1)S\}$, both the halves being nonempty unless $k \equiv 1 \pmod 8$.*

2.1.5. THEOREM. *Let $E_{\mathbb{R}} = V_{2g} \sqcup kS$, $g>0$. Suppose that E is an $(M-d)$-surface and $\chi(E_{\mathbb{R}}) \equiv \sigma(E)+2\delta \pmod{16}$. Then for the values of (d, δ) listed in Table*

Table 1

d	δ	$k^{(2)}$ (mod 4)
0	0	0
1	1	0, 1
	-1	0, 3
2	0	0, 2 (if $E_{\mathbb{R}}$ is of type I) 0, 1, 3 (if $E_{\mathbb{R}}$ is of type II)
	2	0, 1, 2
	-2	0, 2, 3
	4	0, 2
3	± 3	0, 1, 2, 3

1 *one has* $E_{\mathbb{R}} = \{V_{2g} \sqcup k^{(1)}S\} \sqcup \{k^{(2)}S\}$, *where* $k^{(2)}$ (mod 4) *takes one of the values given in the table; furthermore,* $k^{(2)} \neq 0$ *with the possible exception of the case* $d = 2$, $\delta = 0$, $E_{\mathbb{R}}$ *is of type* I. *Besides, there are the following additional prohibitions*:

(1) *if* $d = 0$, *then* $E_{\mathbb{R}}^{(1)}$ *is of type* I_0 *and* $E_{\mathbb{R}}^{(2)}$ *is of type* I_w;

(2) *if* $d = 0$, *then* $k^{(1)} \neq 0$ *unless* $k \equiv 0$ (mod 8);

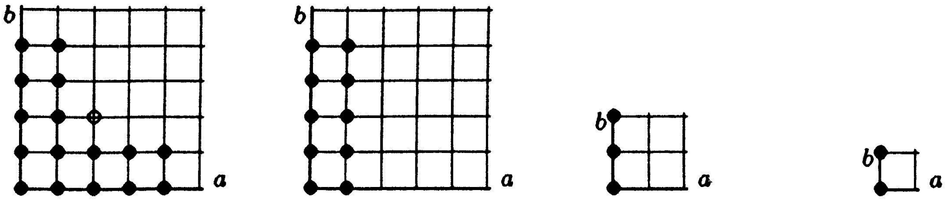
(3) *if* $d = 1$ *and* $k^{(1)} = 0$, *then either* $k \equiv \delta$ (mod 8), *or* $k \equiv 0$ (mod 4) *and* $E_{\mathbb{R}}^{(2)}$ *is of type* I_w.

Remark. Note that in the case $d = 3$ the last theorem only states that, if $\chi(E_{\mathbb{R}}) \equiv \sigma(E) \pm 6$ (mod 16), then both the halves are not empty. This follows also from Theorem 2.1.3.

2.2. *Classical Enriques surfaces*

The topological types realizable by the real part of a classical Enriques surface were enumerated in [DK1], where we treated separately the types $6S$, $S_1 \sqcup 5S$, $3V_2$ and series $S_1 \sqcup V_1 \sqcup \cdots$ not prohibited by the standard inequalities and congruences known in topology of real algebraic varieties. The prohibition of these types is now an immediate consequence of the results of Section 2.1: the first two are prohibited by Theorem 2.1.2, the others – by Theorem 2.1.1. To apply Theorem 2.1.1 one should note that, if the real part of a real $K3$-surface X contains two components S_1, then X is of type I and $X_{\mathbb{R}}$ has no other components, see [Kh1].

Consider now the decomposition $E_{\mathbb{R}} = E_{\mathbb{R}}^{(1)} \cup E_{\mathbb{R}}^{(2)}$. The following obvious observation can be found, e.g., in [DK1]:



$\{aS\}\sqcup\{bS\}$, $\{V_2\sqcup aS\}\sqcup\{bS\}$ $\{V_4\sqcup aS\}\sqcup\{bS\}$ $\{V_6\sqcup aS\}\sqcup\{bS\}$ $\{V_8\sqcup aS\}\sqcup\{bS\}$, $\{V_{10}\sqcup aS\}\sqcup\{bS\}$

Figure 1. Exceptional topological types.

2.2.1. *Each half of a classical real Enriques surface may only be either*
(1) $\alpha V_g \sqcup aV_1 \sqcup bS$ *with* $g > 1$, $a \geq 0$, $b \geq 0$, $\alpha = 0, 1$, *or*
(2) $2V_2$, *or*
(3) S_1.

In [DK1] and in Section 8 we construct a number of realizations of Enriques surfaces sufficient to show that, with few exceptions, any distribution satisfying 2.2.1 is realizable. The exceptional topological types are listed in Figure 1: the distributions marked by the black nodes are realized, e.g., in [DK1]; the white node represents the distributions $\{2S\} \sqcup \{2S\}$ and $\{V_2 \sqcup 2S\} \sqcup \{2S\}$ constructed in [N2]. Theorems 2.1.4 and 2.1.5 imply that this list is complete.

2.2.2. THEOREM. *With the exception of the types kS and $V_{2r} \sqcup kS$ any distribution of components of a real Enriques surface satisfying* 2.2.1 *is realizable. The exceptional types admit only the distributions listed in Figure* 1.

Remark. The distributions $\{2S\} \sqcup \{2S\}$, $\{V_2 \sqcup 2S\} \sqcup \{2S\}$, $\{V_2 \sqcup 2S\} \sqcup \{V_2 \sqcup 2S\}$, and $\{V_2 \sqcup 4S\} \sqcup \{V_2\}$ are not constructed in [DK1] or Section 8; their existence is announced in [N2]. The first two of them *cannot* be obtained by our construction, i.e., the covering $K3$-surface is not a double of a symmetric quadric. (Proof will be published elsewhere.)*

3. Involutions on modules

In this section we expose some elementary facts on the Galois cohomology of modules with involution and on the discriminant forms of integral lattices with involution. Most results appear, explicitly or implicitly, in [N1]. We give proofs when it is easier than to find a precise reference or when the direct proof is simpler.

**Added in proof.* Now we can prove the existence of these 4 distributions.

3.1. *Galois cohomology of $\mathbb{Z}/2$-vector spaces with involution*

The zero-dimensional cohomology group of a $\mathbb{Z}/2$-vector space V with involution c is $H^0(V) = \mathrm{Ker}(1+c)$. All the other cohomology groups are isomorphic to $\mathrm{Ker}(1+c)/\mathrm{Im}(1+c)$; to be short and in accordance with the notation commonly used in the literature we denote them by $\hat{H}^0(V)$.

3.1.1. LEMMA. *Let V and V' be finite dimensional vector spaces over $\mathbb{Z}/2$ with involution. If they are connected by one of the following two short exact sequences of spaces with involution*

$$0 \to \mathbb{Z}/2 \to V \to V' \to 0 \quad \textit{or} \quad 0 \to V' \to V \to \mathbb{Z}/2 \to 0,$$

then $\dim \hat{H}^0(V) - \dim \hat{H}^0(V') = \pm 1$. *In the former case the difference is -1 if and only if the generator of the subgroup $\mathbb{Z}/2$ vanishes in $\hat{H}^0(V)$. In the latter case it is -1 if and only if the generator of the quotient group $\mathbb{Z}/2$ does not lift to $\hat{H}^0(V)$, i.e., does not belong to the image of* $\mathrm{Ker}(1+c) \subset V$.

Proof. Denote by c, c', and c_0 the involutions on V, V', and $\mathbb{Z}/2$ respectively. Then $\mathrm{Ker}(1+c_0) = \mathrm{Coker}(1+c_0) = \mathbb{Z}/2$, and the result follows immediately from the additivity of dimension and the Ker-Coker exact sequences (see, e.g., [CE, Lemma V.10.1])

$$0 \to \mathrm{Ker}(1+c_0) \to \mathrm{Ker}(1+c_0) \to \mathrm{Ker}(1+c') \to \mathrm{Coker}(1+c_0) \to \mathrm{Coker}(1+c)$$

and

$$\mathrm{Ker}(1+c) \to \mathrm{Ker}(1+c_0) \to \mathrm{Coker}(1+c') \to \mathrm{Coker}(1+c) \to \mathrm{Coker}(1+c_0) \to 0.$$

□

Suppose now that V is equipped with a c-equivariant symmetric bilinear form $\circ\colon V \otimes V \to \mathbb{Z}/2$. Then $\circ$ induces, in a natural way, a symmetric bilinear form on $\hat{H}^0(V)$.

3.1.2. LEMMA. *If $\circ\colon V \otimes V \to \mathbb{Z}/2$ is nondegenerate, then so is the induced form $\circ\colon \hat{H}^0(V) \otimes \hat{H}^0(V) \to \mathbb{Z}/2$.*

Proof. Since $\hat{H}^0(V) = \mathrm{Ker}(1+c)/\mathrm{Im}(1+c)$, the result follows from the additivity of dimension and the existence of the induced form. □

3.2. *Free abelian groups with involution*

Let L be a finitely generated free abelian group with involution c. Let $L^{\pm} = \{x \in L \mid cx = \pm x\}$ be its eigensubgroups and $\hat{H}(L) = \hat{H}^0(L/2L)$ the cohomology group of the associated $\mathbb{Z}/2$-vector space $L/2L = L \otimes \mathbb{Z}/2$. Obviously, both $L^{\pm}$ are primitive in L (i.e., the quotients $L/L^{\pm}$ are torsion free), and $L^+ \cap L^- = 0$.

3.2.1. LEMMA. *One has*

$$\operatorname{Ker}[(1+c)\colon L/2L \to L/2L] = (L^+/2L) + (L^-/2L),$$

$$\operatorname{Im}[(1+c)\colon L/2L \to L/2L] = (L^+/2L) \cap (L^-/2L),$$

$$\dim \hat{H}(L) = \dim L - 2\dim[(L^+/2L) \cap (L^-/2L)].$$

Proof. In $L \otimes \mathbb{Q}$ each element x is represented as $x = x^+ + x^-$, where $x^+ = \frac{1}{2}(x + cx)$ and $x^- = \frac{1}{2}(x - cx)$. The first statement follows from the fact that, given an $x \in L$, the elements $\frac{1}{2}(x + cx)$ and $\frac{1}{2}(x - cx)$ belong to L if and only if $x \equiv cx \pmod{2L}$. To prove the second statement just notice that $(1+c)y \equiv (1-c)y \pmod{2L}$ for any $y \in L$, and that whenever $x^+ \in L^+$ and $x^- \in L^-$ are such that $x^+ \equiv x^- \pmod{2L}$, one has $x^+ = y + cy$, where $y = \frac{1}{2}(x^+ + x^-) \in L$.

The last statement is an immediate consequence of the first two. □

3.3. *Integral lattices*

Suppose now that L is a unimodular integral even lattice, i.e., L is supplied with a symmetric bilinear pairing $\circ\colon L \otimes L \to \mathbb{Z}$ so that (1) the *correlation* $\varphi\colon L \to L^* = \operatorname{Hom}(L, \mathbb{Z})$, $\varphi x(y) = x \circ y$, is an isomorphism (L is unimodular), and (2) $x \circ x \in 2\mathbb{Z}$ for any $x \in L$ (L is even). Assume also that L is supplied with an involution $c\colon L \to L$ which is a lattice morphism, i.e., $cx \circ cy = x \circ y$ for any $x, y \in L$. Under these assumptions each of the sublattices $L^{\pm}$ is the orthogonal complement of the other one, and they are both nondegenerate, i.e., their correlations are injective.

Recall that, given a nondegenerate even lattice M, one can define a quadratic space $\operatorname{discr} M$, called the *discriminant space*, in the following way: the underlying finite group, called the *discriminant group*, is $\operatorname{discr} M = M^*/M$, where M^* is considered, via the correlation, as an extension of M in $M \otimes \mathbb{Q}$. The quadratic function $q\colon \operatorname{discr} M \to \mathbb{Q}/2\mathbb{Z}$ is induced from $\circ$ extended to $M \otimes \mathbb{Q}$: given $x \in M^* \subset M \otimes \mathbb{Q}$, define $q(x) = x \circ x \pmod 2$.

Let $(\mathcal{D}^{\pm}, q)$, or briefly $\mathcal{D}^{\pm}$, be the discriminant spaces $\operatorname{discr} L^{\pm}$.

3.3.1. LEMMA (see [N1]). *Spaces* $(\mathscr{D}^{\pm}, q)$ *are anti-isometric, i.e., there exists a group isomorphism* $\alpha\colon \mathscr{D}^{+} \to \mathscr{D}^{-}$ *such that* $q(\alpha x) = -q(x)$ *for any* $x \in \mathscr{D}^{+}$.

At the group level this statement has the following consequence:

3.3.2. LEMMA. $2(L^{\pm})^{*} \subset L$ *and the quotient* $\alpha^{\pm}\colon \mathscr{D}^{\pm} = (L^{\pm})^{*}/L^{\pm} \to L/2L$ *of the multiplication by* 2 *is an isomorphism* $\mathscr{D}^{\pm} \to (L^{+}/2L) \cap (L^{-}/2L) \subset L/2L$. *In particular,* $\mathscr{D}^{\pm}$ *are 2-periodic groups and* $\dim \hat{H}(L) = \operatorname{rk} L - 2\dim \mathscr{D}^{\pm}$.

Proof. Let $x \in (L^{+})^{*}$, i.e., let $x \in L^{+} \otimes \mathbb{Q}$ be an element such that $x \circ L^{+} \in \mathbb{Z}$. Then for any $y \in L$ one has $2x \circ y = 2x \circ (y^{+} + y^{-}) = 2x \circ y^{+} = x \circ (y + cy) \in \mathbb{Z}$. Hence, $2x \in L^{*} = L$ and $2(L^{+})^{*} \subset L$. Since $2L^{+} \subset 2L$, the multiplication by 2 has a well defined quotient $\alpha^{+}\colon \mathscr{D}^{+} = (L^{+})^{*}/L^{+} \to L/2L$.

Let $x \in \operatorname{Ker} \alpha^{+}$, i.e., $2x \in 2L$. Then $x \in L \cap (L^{+} \otimes \mathbb{Q}) = L^{+}$, i.e., $x = 0$ in $\mathscr{D}^{+}$. Thus, $\operatorname{Ker} \alpha^{+} = 0$ and $\mathscr{D}^{+}$ is a 2-periodic group.

Given $2x = (1 + c)y \in (L^{+}/2L) \cap (L^{-}/2L)$ (see Lemma 3.2.1), for any $z \in L^{+}$ one has $x \circ z = \frac{1}{2}(y \circ z + cy \circ cz) \in \mathbb{Z}$, i.e., $x \in (L^{+})^{*}$. This proves that $\operatorname{Im} \alpha^{+} \supset (L^{+}/2L) \cap (L^{-}/2L)$.

Since $\mathscr{D}^{+}$ is a 2-periodic group, $2x \in L^{+}$ for any $x \in (L^{+})^{*}$. Hence $\operatorname{Im} \alpha^{+} \subset L^{+}/2L$. Since L^{+} is primitive in the unimodular lattice L, the map $L = L^{*} \to (L^{+})^{*}$ induced by the inclusion $L^{+} \subset L$ is onto, and, given $x \in (L^{+})^{*}$, there is some $y \in L$ so that $(x - y) \circ L^{+} = 0$. Then $z = 2x - 2y \in L^{-} = (L^{+})^{\perp}$ and $2x \equiv z \pmod{2L}$. Hence $\operatorname{Im} \alpha^{+} \subset L^{-}/2L$. This completes the proof for α^{+}; the other isomorphism is constructed similarly. □

3.3.3. COROLLARY. *An* $x \in L^{+}$ *vanishes in* $\hat{H}(L)$ *if and only if* $x \circ L^{+} \in 2\mathbb{Z}$.

Proof. According to Lemmas 3.2.1 and 3.3.2, x vanishes in $\hat{H}(L)$ if and only if $x \bmod 2L \in \operatorname{Im} \alpha^{+}$, i.e., $\frac{1}{2}x \in (L^{+})^{*}$. □

3.3.4. To formulate the next statement, remind that, given a (not necessary unimodular) nondegenerate lattice M and nondegenerate primitive sublattice $M' \subset M$, one can define subgroups $\Gamma' \subset \operatorname{discr} M'$ and $\Gamma'' \subset \operatorname{discr} M'^{\perp}$ and an anti-isometry $\alpha\colon \Gamma' \to \Gamma''$ so that M is the pull back of the graph Γ of α under the projection $(M')^{*} \oplus (M'^{\perp})^{*} \to \operatorname{discr} M' \oplus \operatorname{discr} M'^{\perp}$ and $\operatorname{discr} M = \Gamma^{\perp}/\Gamma$. (Details can be found in [N1].)

3.3.5. LEMMA. *Suppose that* M' *is a primitive nondegenerate sublattice of* L^{+} *and* M *is the primitive hull of* $M' \oplus L^{-}$ *in* L. *Let* $x \in M' \subset L^{+}$ *be an element with* $x \circ M' \in 2\mathbb{Z}$, *so that* $\frac{1}{2}x$ *defines an element in* $\operatorname{discr} M'$. *If this element belongs to the subgroup* Γ' *defined above, then* x *vanishes in* $\hat{H}(L)$.

Proof. According to Nikulin's construction, if the element defined by $\frac{1}{2}x$ in $\operatorname{discr} M'$ belongs to Γ', there are some $y \in L^-$ and $z \in M$ such that $z = \frac{1}{2}x + \frac{1}{2}y$. Then $x = 2z - y$ and $x \circ L^+ \in 2\mathbb{Z}$ (since $y \circ L^+ = 0$). The statement follows now from Corollary 3.3.3. □

4. Basic topological properties of generalized Enriques surfaces

4.1. *General facts*

First, consider an arbitrary complex algebraic surface Y equipped with a real structure $\operatorname{conj}\colon Y \to Y$. Let $L = H_2(Y; \mathbb{Z})/\operatorname{Tors}$, $\mathcal{D}^\pm = \operatorname{discr} L^\pm$, where $L^\pm$ are the subgroups of conj_*-invariant and conj_*-skew-invariant elements of L, and $\operatorname{Br} \mathcal{D}^\pm$ the Brown invariant of $\mathcal{D}^\pm$.

4.1.1. LEMMA. *The fundamental class $[Y_\mathbb{R}] \in H_2(Y)$ and the Stiefel-Whitney class $w_2(Y)$ are integral, i.e., belong to the image of $H_2(Y; \mathbb{Z})$ in $H_2(Y)$.*

Proof. As it is known (see [HH]), $w_2(Y)$ is integral for any closed orientable 4-dimensional manifold.[1] According to [Ar], Lemma 3[2], $[Y_\mathbb{R}]$ is the characteristic class of the *twisted intersection form* $(x, y) \mapsto x \circ \operatorname{conj}_* y$. In particular, it is orthogonal to the image of $\operatorname{Tors} H_2(Y; \mathbb{Z})$ in $H_2(Y)$, which, by Poincaré duality, is the orthogonal complement of the image of $H_2(Y; \mathbb{Z})$. □

Thus, the projections of $[Y_\mathbb{R}]$ and $w_2(Y)$ to $L/2L$ are well defined, and since both these classes are conj_*-invariant, they further descend to $\hat{H}(L)$.

4.1.2. LEMMA. *The projections of $[Y_\mathbb{R}]$ and $w_2(Y)$ in $\hat{H}(L)$ coincide.*

Proof. Since $\hat{H}(L)$ consists of only conj_*-invariant classes, the twisted and the standard intersection forms on it coincide, and so do their characteristic classes (Lemma 3.1.2). On the other hand, $[Y_\mathbb{R}]$ is the characteristic class of the twisted intersection form (Arnol'd Lemma, *loc. cit.*), and $w_2(Y)$ is the characteristic class of the standard intersection form. □

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[1]For complex manifolds this assertion is completely obvious as $w_2(Y) = c_1(Y) \bmod 2$.

[2]Arnol'd formulates and proves this assertion only for orientable $Y_\mathbb{R}$; the proof in the general case is literally the same.

4.1.3. LEMMA. *If Y is an $(M-d)$-surface, then*
(1) $\chi(Y_{\mathbb{R}}) \equiv \sigma(Y) + 2\ \mathrm{Br}\,\mathscr{D}^-$ (mod 16);
(2) $\dim \mathscr{D}^- \equiv d$ (mod 2);

Proof. Hirzebruch's signature theorem gives $\nu(Y_{\mathbb{R}}) = \sigma(L^+) - \sigma(L^-)$. The left hand side here equals $-\chi(Y_{\mathbb{R}})$ as the normal Euler number of $Y_{\mathbb{R}}$ in Y; the right hand side is $-\sigma(Y) + 2\sigma(L^+) \equiv -\sigma(Y) - 2\ \mathrm{Br}\,\mathscr{D}^-$ (mod 16), since due to Lemma 3.3.1. one has $\mathrm{Br}\,\mathscr{D}^- = -\mathrm{Br}\,\mathscr{D}^+ \equiv -\sigma(L^+)$ (mod 8). This proves (1).

Since Y is an algebraic surface, $\sigma(Y) \equiv -\chi(Y) \equiv -\beta_*(Y)$ (mod 4). By definition, $\beta_*(Y) = \beta_*(Y_{\mathbb{R}}) + 2d$. Substitution to (1) and replacing $\chi(Y_{\mathbb{R}})$ with $-\beta_*(Y_{\mathbb{R}}) \equiv \chi(Y_{\mathbb{R}})$ (mod 4) and $\mathrm{Br}\,\mathscr{D}^-$ with $\dim \mathscr{D}^- \equiv \mathrm{Br}\,\mathscr{D}^-$ (mod 2) gives (2). □

4.1.4. LEMMA. *The quadratic space $\mathscr{D}^-$ is even (i.e., $q(\hat{x}) \in \mathbb{Z}/2\mathbb{Z}$ for any $\hat{x} \in \mathscr{D}^-$) iff $[Y_{\mathbb{R}}] - w_2(Y)$ belongs to the image of* Tors $H_2(Y; \mathbb{Z})$ *in $H_2(Y)$.*

Proof. $[Y_{\mathbb{R}}]$ and $w_2(Y)$ are the characteristic classes of the (respectively, twisted and standard) intersection forms. In particular, they are both orthogonal to the image of Tors $H_2(Y; \mathbb{Z})$ in $H_2(Y)$. In addition, they are both integral (see Lemma 4.1.1). Thus, the condition that $[Y_{\mathbb{R}}] - w_2(Y)$ belongs to the image of Tors $H_2(Y; \mathbb{Z})$ in $H_2(Y)$ is equivalent to the condition that this difference annihilates all the integral classes, which, in turn, is equivalent to the congruence $x^2 \equiv x \circ \mathrm{conj}_* x$ (mod 2) for any $x \in L$.

Let $x^{\pm} = \frac{1}{2}(x \pm \mathrm{conj}_* x) \in L^{\pm} \otimes \mathbb{Q}$. Then $x = x^+ + x^-$ and $x^2 - x \circ \mathrm{conj}_* x \equiv 2(x^-)^2$ (mod $2\mathbb{Z}$). Since $x^- \circ L^- = x \circ L^-$ takes integral values, x^- belongs to $(L^-)^*$ and, hence, represents an element in $\mathscr{D}^-$. Moreover, each element in $\mathscr{D}^-$ admits such a representative. Thus, $(x^-)^2 \in \mathbb{Z}$ for any $x \in L$ if and only if $\mathscr{D}^-$ is even. □

4.1.5. COROLLARY. *Suppose that the 2-primary component* $\mathrm{Tors}_2\, H_2(Y; \mathbb{Z})$ *is generated by $w_2(Y)$. (This is the case for generalized Enriques surfaces; see Lemma 4.2.3 below.) Then $Y_{\mathbb{R}}$ is of type* I *if and only if $\mathscr{D}^-$ is even.*

All the statements above except Lemma 4.1.3[3] extend literally to any (not necessary anti-holomorphic) orientation preserving involution conj on any (not necessary complex) oriented 4-manifold Y. Lemma 4.1.4 has then the following corollary:

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[3]4.1.3 extends to any anti-holomorphic involution on any quasi-complex variety, cf. [Wi].

4.1.6. COROLLARY. *Let* conj *be a fixed point free orientation preserving involution on an oriented* 4*-manifold* Y. *Then the quadratic spaces* $\mathscr{D}^{\pm}$ *are even if and only if so is* $H_2(Y; \mathbb{Z})/\mathrm{Tors}$.

4.2. *Homology of a generalized Enriques surface*

We now consider a generalized Enriques surface E covered by a generalized $K3$-surface X with Enriques involution τ. We denote by pr: $X \to E$ the projection and by tr: $H_*(E; R) \to H_*(X; R)$ the transfer (with coefficients in a group R).

Note that $H_1(X) = 0$ implies $\mathrm{Tors}_2\, H_2(X; \mathbb{Z}) = 0$.

4.2.1. LEMMA. *There are isomorphisms* $\mathrm{Tors}_2\, H_1(E; \mathbb{Z}) = H_1(E) = \mathbb{Z}/2$ *and an exact sequence*

$$0 \to \mathrm{Tors}_2\, H_2(E; \mathbb{Z}) \to H_2(E) \xrightarrow{\mathrm{tr}} H_2(X),$$

where $\mathrm{Tors}_2\, H_2(E; \mathbb{Z}) = \mathbb{Z}/2$ *is generated by* $w_2(E)$.

Proof. From the Smith-Ghysin exact sequence it follows that $H_1(E) = \mathbb{Z}/2$ and $\mathrm{Ker}[\mathrm{tr}_2\colon H_2(E) \to H_2(X)] = \mathbb{Z}/2$. As $\mathrm{tr}\, w_2(E) = w_2(X) = 0$ and $w_2(E) \neq 0$, the only nontrivial element of $\mathrm{Ker}\,\mathrm{tr}_2$ is $w_2(E)$. By the Poincaré duality and universal coefficient formula, from $H_1(E) = \mathbb{Z}/2$ it follows that $\mathrm{Tors}_2\, H_2(E; \mathbb{Z}) = \mathrm{Tors}_2\, H_1(E; \mathbb{Z})$ is a cyclic group. It cannot be larger than $\mathbb{Z}/2$ since otherwise X would have a nontrivial double covering. □

4.2.2. LEMMA. *For any* $p = 1, 2, 3$ *there is a short exact sequence*

$$0 \to \mathrm{Tors}_2\, H_p(E; \mathbb{Z}) \to H_p(E; \mathbb{Z}) \xrightarrow{\mathrm{tr}_p} H_p^{+\tau}(X; \mathbb{Z}) \to 0,$$

where $H_p^{+\tau}(X; \mathbb{Z})$ *denotes the subgroup of* τ_**-invariant elements.*

4.2.3. LEMMA. *Let* $\bar{L} = H_2(X; \mathbb{Z})/\mathrm{Tors}$ *and let* $\bar{L}^{\pm\tau}$ *be the sublattices of* τ_**-invariant and* τ_**-skew-invariant elements of* $\bar{L}$. *Then* $H_2(E; \mathbb{Z})/\mathrm{Tors}$ *is an even lattice isometric via* tr *to* $\bar{L}^{+\tau}(\frac{1}{2})$, *which is* $\bar{L}^{+\tau}$ *with modified pairing* $(x, y) \mapsto \frac{1}{2}(x \circ y)$.

Proof of Lemmas 4.2.2 *and* 4.2.3. The transfer $H_*(E; R) \to H_*^{+\tau}(X; R)$ for $R = \mathbb{Q}$ and $R = \mathbb{Z}/q$, q odd, is an isomorphism (see, e.g., [B]). Thus, in the integral homology $\mathrm{Ker}\,\mathrm{tr}_p = \mathrm{Tors}_2\, H_p(E; \mathbb{Z})$, and to prove 4.2.2 it remains to show that tr_2 reduced modulo torsion maps $H_2(E; \mathbb{Z})/\mathrm{Tors}$ onto $\bar{L}^{+\tau}$.

Let $L = H_2(E; \mathbb{Z})/\mathrm{Tors}$ and $L' = \overline{\mathrm{tr}}\, L \subset \bar{L}$, where $\overline{\mathrm{tr}}$ is the integral transfer modulo torsion. Then $L' \subset \bar{L}^{+\tau}$ is a subgroup of finite index. The identity $\mathrm{tr}\, x \circ \mathrm{tr}\, y = 2(x \circ y)$ implies that $L = L'(\frac{1}{2})$ as a lattice and, since L is unimodular, $\operatorname{discr} L'$ is a 2-periodic group of dimension $\operatorname{rk} L = \operatorname{rk} L'$. Since, due to Lemma 4.2.1, the index of L' in $\bar{L}^{+\tau}$ is odd ($\overline{\mathrm{tr}} \otimes \mathbb{Z}/2$ is a monomorphism) and $\operatorname{discr} \bar{L}^{+\tau}$ is also 2-periodic (Lemma 3.3.2), these two subgroups coincide.

Thus $\overline{\mathrm{tr}}_2$ provides an isometry between the lattices $H_2(E; \mathbb{Z})/\mathrm{Tors}$ and $\bar{L}^{+\tau}(\frac{1}{2})$ and an isomorphism between the groups $H_2(E; \mathbb{Z})/\mathrm{Tors}$ and $\bar{L}^{+\tau}$. The lattice $\bar{L}^{+\tau}(\frac{1}{2})$ is even due to Corollary 4.1.6. □

4.3. *Eigenspaces of conj$_*$*

Let now E be a generalized Enriques surface with real structure conj: $E \to E$. The following fact is well known and follows from the Lefschetz fixed point theorem (part (1)) and Hirzebruch signature theorem (part (2)). Note that (2) applies, in fact, to any real algebraic surface, and (1) applies to any surface E with $H_1(E; \mathbb{Q}) = 0$.

4.3.1. LEMMA. *Let* $L = H_2(E; \mathbb{Z})/\mathrm{Tors}$ *and let* $L^{\pm}$ *be the subgroups of* conj_*-*invariant and* conj_*-*skew-invariant elements of* L. *Then*

(1) $\operatorname{rk} L^+ = \frac{1}{2}(b_2(E) + \chi(E_{\mathbb{R}})) - 1$, $\operatorname{rk} L^- = \frac{1}{2}(b_2(E) - \chi(E_{\mathbb{R}})) + 1$;

(2) $\sigma(L^+) = \frac{1}{2}(\sigma(E) - \chi(E_{\mathbb{R}}))$, $\sigma(L^-) = \frac{1}{2}(\sigma(E) + \chi(E_{\mathbb{R}}))$.

5. Kalinin's spectral sequence and Viro homomorphisms

In this section we summarize some auxiliary results from algebraic topology of involutions. The constructions, which we present in their homology form, require, in principle, a cautious choice of the homology theory, as well as certain appropriate conditions on the underlying topological spaces. One possibility is to use the sheaf theories and suppose that the topological spaces are locally compact and finite dimensional. However, as we apply the results to the best topological spaces one can possibly expect – smooth compact manifolds – we do not need any definite choice and can use any theory.

Throughout this section Y is a good (see the paragraph above) topological space with involution c: $Y \to Y$.

5.1. *Kalinin's homology spectral sequence*

5.1.1. *There exist a filtration*

$$0=\mathcal{F}^{n+1}\subset\mathcal{F}^{n}\subset\cdots\subset\mathcal{F}^{0}=H_*(\operatorname{Fix}c),$$

a $\mathbb{Z}$-graded spectral sequence (H^r_*, d^r_*), where

$$d^r_q\colon H^r_q\to H^r_{q+r-1},\qquad d^r_{q+r-1}\circ d^r_q=0,$$

(H^0_*, d^0_*) is the chain complex of Y, and $H^{r+1}_q=\operatorname{Ker}d^r_q/\operatorname{Im}d^r_{q-r+1}$,

and homomorphisms $\mathrm{bv}_r\colon\mathcal{F}^r\to H^\infty_r$ *such that*

(1) $H^1_*=H_*(Y)$ *and* $d^1_*=1+c_*$;

(2) *a cycle $x_p\in H^0_p$ survives to H^r_p if and only if there are some chains $y_p=x_p, y_{p+1},\dots,y_{p+r-1}$ in Y so that $\partial y_{i+1}=(1+c_*)y_i$. In this case* $d^r_px_p=(1+c_*)y_{p+r-1}$;

(3) bv_q *annihilates $\mathcal{F}^{q+1}$ and maps $\mathcal{F}^q/\mathcal{F}^{q+1}$ isomorphically onto H^∞_q*;

(4) *the filtration, spectral sequence, and homomorphisms are all natural with respect to equivariant mappings.*

When necessary, we will use the notation $H^r_q=H^r_q(Y)$ and $\mathcal{F}^q=\mathcal{F}^q(Y)$ to indicate the original space Y.

The original construction of this spectral sequence is due to I. Kalinin [Ka], who derived it from the Borel-Serre spectral sequence and related results by Borel (see [Bo]). This construction is briefly outlined in Appendix A. Property (2) is proved in [D]. An alternative description of Kalinin's spectral sequence, based upon the Smith exact sequence, can be found in [DK2].

The following results are straightforward consequences of 5.1.1.

5.1.2. COROLLARY. *If Y is connected and $\operatorname{Fix}c\neq\varnothing$, then $H_0(Y)=H^2_0(Y)=H^\infty_0(Y)=\mathbb{Z}/2$ and each nonzero element of $H^2_1(Y)$ which survives to $H^\infty_1(Y)$ is nonzero in $H^\infty_1(Y)$.*

5.1.3. COROLLARY–DEFINITION. *If a cycle admits a representation by an equivariant chain, it survives to $H^\infty_*(Y)$. Thus, in particular, there are tautological homomorphisms $H_p(\operatorname{Fix}c)\to H^\infty_p(Y)$; with certain abuse of terminology we will call them the* inclusion homomorphisms.

5.1.4. COROLLARY. *One has* $H_2^2(Y) = \hat{H}^0(H_2(Y))$.

The homomorphisms bv_* were first discovered, in an equivalent form, by O. Viro. That is why we call them *Viro homomorphisms*. The following geometrical description, close to the original one (cf. [VZ]), is found in [D].

5.1.5. *Suppose that* $\mathrm{Fix}\, c \neq \varnothing$. *Then*

(1) $\mathrm{bv}_0\colon H_*(\mathrm{Fix}\, c) \to H_0^\infty(Y)$ *is zero on* $H_{\geq 1}(\mathrm{Fix}\, c)$; *its restriction to* $H_0(\mathrm{Fix}\, c) \to H_0^\infty(Y) = H_0(Y)$ *coincides with the inclusion homomorphism* (*cf.* 5.1.2 *and* 5.1.3);

(2) *a* (*nonhomogeneous*) *element* $x \in H_*(\mathrm{Fix}\, c)$ *represented by a cycle* $\Sigma\, x_i$ *belongs to* $\mathscr{F}_p = \mathrm{Ker}\, \mathrm{bv}_{p-1}$ (*see* 5.1.1) *if and only if there exist some chains* y_i, $1 \leq i \leq p$, *so that* $\partial y_1 = x_0$ *and* $\partial y_{i+1} = x_i + (1 + c_*)y_i$ *for* $i \geq 1$; *the class of* $x_p + (1 + c_*)y_p$ *in* $H_p^\infty(Y)$ *represents then* $\mathrm{bv}_p\, x$.

5.1.6. EVIDENT COROLLARY. *For any p the Viro homomorphism* bv_p *is zero on* $H_{>p}\mathrm{Fix}\, c)$ *and coincides with the inclusion homomorphism* (*see* 5.1.3) *when restricted to* $H_p(\mathrm{Fix}\, c) \to H_p^\infty(Y)$.

5.2. *Kalinin's intersection pairing*

The original construction presented in [Ka] gives a cohomology spectral sequence (H_r^*, d_r^*) starting at $H_1^q = H^q(Y)$ and converging to $H^*(\mathrm{Fix}\, c)$. We denote by $\mathscr{F}_q$ the corresponding filtration on $H^*(\mathrm{Fix}\, c)$ and by $\mathrm{bv}^q\colon H_\infty^q \rightharpoonup H^*(\mathrm{Fix}\, c)$ the cohomology Viro homomorphisms. This spectral sequence is dual to its homology counter-part 5.1.1; the cup-product in $H^*(Y)$ converts H_r^* to a spectral sequence of $\mathbb{Z}$-graded algebras, and 5.1.1 is a spectral sequence of graded H_r^*-moduli. The following result, which, to our knowledge, is stated explicitly only in [Ka], is proved in [DK2]:

5.2.1. PROPOSITION. *If Y is a closed n-dimensional manifold and* $\mathrm{Fix}\, c \neq \varnothing$, *then for any r*, $1 \leq r \leq +\infty$, *one has* $H_r^n \cong \mathbb{Z}/2$, *and the product map* $H_r^p \otimes H_r^{n-p} \to H_r^n$ *is a nondegenerate pairing.*

5.2.2. COROLLARY (the dual version of 5.2.1). *If Y is a closed n-dimensional manifold and* $\mathrm{Fix}\, c \neq \varnothing$, *then the intersection pairing in* $H_*(Y)$ *descends to a nondegenerate pairing* $H_p^\infty \otimes H_{n-p}^\infty \to \mathbb{Z}/2$.

Corollary 5.2.2 is a paraphrase of 5.2.1 using the Poincaré duality. The pairing $H_p^\infty \otimes H_{n-p}^\infty \to \mathbb{Z}/2$ is called *Kalinin's intersection form*. Its relation to the standard intersection form in $H_*(\operatorname{Fix} c)$ is given by the following theorem, which we prove in Appendix A.

5.2.3. THEOREM. *Let Y be a smooth closed N-dimensional manifold with smooth involution $c\colon Y \to Y$ and $F = \operatorname{Fix} c$ the fixed point set of c. Then for any two classes $a \in \mathscr{F}^p$ and $b \in \mathscr{F}^q$ one has $w(\nu) \cap (a \circ b) \in \mathscr{F}^{p+q-N}$ and $\operatorname{bv}_p a \circ \operatorname{bv}_q b = \operatorname{bv}_{p+q-N}[w(\nu) \cap (a \circ b)]$, where $w(\nu)$ is the total Stiefel-Whitney class of the normal bundle ν of F in Y.*

5.3. *Application to a real structure of a complex surface*

Let Y be a compact nonsingular complex surface with real structure $c\colon Y \to Y$. Then the $\mathbb{Z}/2$-homology fundamental class $[Y_\mathbb{R}]$ of $Y_\mathbb{R} = \operatorname{Fix} c$ is well defined.

5.3.1. LEMMA. *The Stiefel-Whitney class $w_2(Y)$ survives to $H_2^\infty(Y)$. The projection of $w_2(Y)$ in $H_2^\infty(Y)$ coincides with $\operatorname{bv}_2[Y_\mathbb{R}]$.*

Proof. As any Chern or Stiefel-Whitney class, $w_2(Y)$ is realized by the fundamental class of a c-invariant divisor. (The earliest reference which we could find in the literature is [BH]; the statement is based on the simple observation that Schubert cycles are defined over $\mathbb{R}$ and even over $\mathbb{Z}$.) Thus, w_2 survives to $H_2^\infty(Y)$. The other part of the lemma follows from 5.2.2, 5.1.4, and the fact that the image of $[Y_\mathbb{R}]$ in $H_2(Y)$ coincides with the characteristic class of the twisted intersection form (cf. the proof of Lemmas 4.1.1 and 4.1.2). □

Denote by $\langle C_i \rangle \in H_0(\operatorname{Fix} c)$ and $[C_i] \in H_2(\operatorname{Fix} c)$ the classes represented by a component C_i of $Y_\mathbb{R}$. It is clear that $H_{\le 2}^\infty$ is spanned by the following values of Viro homomorphisms: (we abbreviate $\langle C_i - C_j \rangle = \langle C_i \rangle - \langle C_j \rangle$)

- $\operatorname{bv}_0 \langle C_i \rangle$ in $H_0^\infty(Y)$;
- $\operatorname{bv}_1 \alpha$ and $\operatorname{bv}_1 \langle C_i - C_j \rangle$ in $H_1^\infty(Y)$, where $\alpha \in H_1(Y_\mathbb{R})$;
- $\operatorname{bv}_2[C_i]$, $\operatorname{bv}_2 \alpha$, $\operatorname{bv}_2 \langle C_i - C_j \rangle$, and $\operatorname{bv}_2(\alpha + \langle C_i - C_j \rangle)$ in $H_2^\infty(Y)$.

From 5.1.5 (which also gives an explicit geometric description of the corresponding chains) and 5.1.6 it immediately follows that:

- all the above classes but the last three are always well defined;
- $\operatorname{bv}_2 \alpha$ is defined if and only if $\operatorname{bv}_1 \alpha = 0$;

Table 2

	$\mathrm{bv}_2\langle C_i - C_j\rangle$	$\mathrm{bv}_2\,\alpha$	$\mathrm{bv}_2[C_i]$
$\mathrm{bv}_2\langle C_k - C_l\rangle$	0	0	$\delta_{ik} + \delta_{il}$
$\mathrm{bv}_2\,\beta$	0	$(\alpha \circ \beta)[Y_{\mathbb{R}}]$	$(\beta \circ \beta)[C_i]$
$\mathrm{bv}_2[C_k]$	$\delta_{ik} + \delta_{jk}$	$(\alpha \circ \alpha)[C_k]$	$\delta_{ik}\chi(C_i)$

- $\mathrm{bv}_2\langle C_i - C_j\rangle$ is defined if and only if $\mathrm{bv}_1\langle C_i - C_j\rangle = 0$;
- $\mathrm{bv}_2(\alpha + \langle C_i - C_j\rangle)$ is defined if and only if $\mathrm{bv}_1\,\alpha = \mathrm{bv}_1\langle C_i - C_j\rangle$.

Theorem 5.2.3 gives the following values for the intersection numbers:

5.3.2. INTERSECTION MATRIX. *The intersection form on* $H_2^{\infty}(Y) = \operatorname{Im} \mathrm{bv}_2$ *is that defined by Table* 2, *where* $C_i, \dots, C_l$ *are some connected components of* $Y_{\mathbb{R}}$, *and* α, β *are some* 1-*dimensional homology classes in* $Y_{\mathbb{R}}$. *The intersection* $\alpha \circ \beta$ *is regarded as an element of* $H_0(Y_{\mathbb{R}})$, *and* $(\alpha \circ \beta)[Y_{\mathbb{R}}]$ *and* $(\alpha \circ \beta)[C_i]$ *are, respectively, the total intersection number and its part which falls into* C_i. δ_{ij} *stands for the Kronecker symbol*: $\delta_{ii} = 1$ *and* $\delta_{ij} = 0$ *if* $i \neq j$. *The intersection form extends linearly to the classes of the form* $\mathrm{bv}_2(\alpha + \langle C_i - C_j\rangle)$, *as if* $\mathrm{bv}_2\,\alpha$ *and* $\mathrm{bv}_2\langle C_i - C_j\rangle$ *were well defined.*

Remark. Note that in this dimension one can avoid reference to 5.2.3 and use the standard geometric techniques: represent classes by chains given by 5.1.5, smoothen them, bring to general position, and count the intersection points. Since the intersection numbers are considered modulo 2, the imaginary intersection points, which appear in pairs, can be ignored (cf., e.g., [Kh2, Lemma 2.3]).

6. Viro homomorphisms in generalized Enriques surfaces

Recall that we denote by E a generalized real Enriques surface. We assume that $E_{\mathbb{R}} \neq \varnothing$. The main goal of this section is to prove Propositions 6.1 and 6.2 below. We use the homology spectral sequence H^r_* and denote $\beta^r_p = \dim H^r_p$.

6.1. DIMENSION OF THE DISCRIMINANT SPACE. *Let* E *be an* $(M - d)$-*surface, and let* $\mathcal{D}^-$ *be the discriminant space of the sublattice of* conj-*skew-invariant vectors in* $H_2(E; \mathbb{Z})/\mathrm{Tors}$. *Then*:

$d - \dim \mathcal{D}^- = 0$ *if either*

(1) $E_{\mathbb{R}}$ *has a component* V_{2g+1} (*i.e.*, $w_2(E_{\mathbb{R}}) \neq 0$), *or*
(2) $E_{\mathbb{R}}$ *is nonorientable and both the halves are nonempty*;

$d - \dim \mathcal{D}^- = 2$ *if either*

(1) $E_{\mathbb{R}}$ *is nonorientable*, $w_2(E_{\mathbb{R}}) = 0$, *and one of the halves is empty, or*
(2) $E_{\mathbb{R}}$ *is orientable and both the halves are nonempty*;

$d - \dim \mathcal{D}^-$ *may be* 2 *or* 4 *if* $E_{\mathbb{R}}$ *is orientable and one of the halves is empty.*

6.2. RELATIONS BETWEEN REAL COMPONENTS. *There is at least one and at most two relations between the elements of* $H_2^{\infty}(E)/w_2(E)$ *realized by the fundamental classes of the components of* $E_{\mathbb{R}}$. *One relation is* $\mathrm{bv}_2[E_{\mathbb{R}}] = w_2(E)$; *the only other possible relation is* $\mathrm{bv}_2[E_{\mathbb{R}}^{(1)}] \equiv \mathrm{bv}_2[E_{\mathbb{R}}^{(2)}] \equiv 0 \pmod{w_2(E)}$.

6.3. *Proof of Proposition* 6.1

6.3.1. LEMMA. *Let* C_1, C_2 *be two components of* $E_{\mathbb{R}}$. *Then* $\mathrm{bv}_1\langle C_1 - C_2\rangle = 0$ *if and only if these two components belong to the same half of* $E_{\mathbb{R}}$.

Proof. Pick two points $c_i \in C_i$ and connect them with a path γ in E. By 5.1.2, $\mathrm{bv}_1\langle C_1 - C_2\rangle = 0$ if and only if the loop $\delta = (\mathrm{conj}\,\gamma)^{-1} \cdot \gamma$ is homologous to zero in $H_1(E)$. Thus $\mathrm{bv}_1\langle C_1 - C_2\rangle = 0$ if and only if δ lifts to a loop in X. Suppose that $C_1 \in E_{\mathbb{R}}^{(1)}$ and lift γ to a path $\tilde{\gamma}$ with the endpoints $\tilde{c}_1, \tilde{c}_2$. Then $\tilde{\delta} = \tilde{\gamma} \cdot (t^{(1)}\tilde{\gamma})^{-1}$ is a lift of δ which connects $t^{(1)}\tilde{c}_2$ and $\tilde{c}_2$. It is a loop if and only if $t^{(1)}\tilde{c}_2 = \tilde{c}_2$, i.e., $c_2 \in E_{\mathbb{R}}^{(1)}$. □

6.3.2. LEMMA. *Let* α *be an element of* $H_1(E_{\mathbb{R}})$. *Then* $\mathrm{bv}_1\,\alpha \neq 0$ *if and only if* $\omega \circ \alpha = 1$, *where* $\omega \in H_1(E_{\mathbb{R}})$ *is the characteristic element of the covering* $X_{\mathbb{R}} \to E_{\mathbb{R}}$. *Moreover*, $\mathrm{bv}_1\,\alpha \neq 0$ *whenever* $\alpha^2 = 1$.

Proof. Since $H_1(E) = \mathbb{Z}/2$, from 5.1.2 it follows that $\mathrm{bv}_1\,\alpha = 0$ if and only if $\mathrm{in}_*\,\alpha \in H_1(E)$ is zero, or equivalently, if $\omega \circ \alpha = 0$. The last assertion follows from Lemma 1.3.2: if $w_1(E_{\mathbb{R}}) \neq 0$, then $\omega = w_1(E_{\mathbb{R}})$. □

6.3.3. LEMMA. *The Stiefel-Whitney class $w_2(E)$ (which, due to 5.3.1, always survives to $H_2^\infty(E)$) represents a nonzero element in $H_2^\infty(E)$ if and only if either*

(1) *$E_\mathbb{R}$ has a component V_{2g+1} (i.e., $w_2(E_\mathbb{R}) \neq 0$),* or

(2) *$E_\mathbb{R}$ is nonorientable and both the halves are nonempty.*

Proof. By 5.2.2 and since $w_2(E)$ is a characteristic element of the intersection form, $w_2(E) \neq 0$ in $H_2^\infty(E)$ if and only if there is an element $x \in H_*(E_\mathbb{R})$ with $(\mathrm{bv}_2\, x)^2 \neq 0$. According to 5.3.2 such an x can be found in one of the following three forms: (i) $x = [C_1]$, where $C_1 \subset E_\mathbb{R}$ is a component of odd Euler characteristic; (ii) $x = \alpha + \langle C_1 - C_2 \rangle$, where $\alpha \in H_1(E_\mathbb{R})$ is an element with $\alpha^2 = 1$ and $\mathrm{bv}_1\, \alpha \neq 0$; (iii) $x = \alpha \in H_1(E_\mathbb{R})$ with $\alpha^2 = 1$ and $\mathrm{bv}_1\, \alpha = 0$. In (i) we have case (1) of the lemma. In (ii), according to 6.3.1, we have case (2). Finally, (iii) contradicts to 6.3.2. □

6.3.4. LEMMA. *$H_1^\infty(E) \neq 0$ if and only if either*

(1) *$E_\mathbb{R}$ is nonorientable, or*

(2) *$E_\mathbb{R}$ has a component T_g, or*

(3) *both the halves of $E_\mathbb{R}$ are nonempty.*

If $H_1^\infty(E) \neq 0$, then the spectral sequence collapses at H_*^2; in particular, $\beta_2^2 - \beta_2^\infty = 0$. If $H_1^\infty(E) = 0$, then $\beta_2^2 - \beta_2^\infty = 0$ or 2 and $\beta_1^\infty = \beta_3^\infty = 0$.

Proof. By 5.1.5, $H_1^\infty(E) = \mathrm{bv}_1\, H_{\leq 1}(\mathrm{Fix}\, c)$. According to 6.3.1 and 6.3.2, a homogeneous element $x \in H_*(E_\mathbb{R})$ with $\mathrm{bv}_1\, x \neq 0$ is either $\alpha \in H_1(E_\mathbb{R})$ with $\omega \circ \alpha = 1$ (cases (1) and (2) of the lemma, see 1.3.2) or $\langle C_1 - C_2 \rangle$, where $C_i \subset E_\mathbb{R}^{(i)}$ are two components from different halves of $E_\mathbb{R}$ (case 3)).

The last statement is a straightforward consequence of the relations $\beta_0^2 = \beta_0^\infty = 1$ and $\beta_1^2 = 1 \geq \beta_1^\infty$ and the existence of a nondegenerate pairing in the spectral sequence. When $H_1^\infty = 0$ one has $\beta_2^2 - \beta_2^\infty = 0$ if $H_1^2(E)$ is killed by d^3 and $\beta_2^2 - \beta_2^\infty = 2$ if it is killed by d^2. □

6.3.5. *End of the proof*

By definition, $2d = \beta_*(E) - \beta_*^\infty$. According to Lemma 4.3.1, we have $2 \dim \mathscr{D}^- = b_2(E) - b_2^2$, where $b_2^2 = \dim \hat{H}(\mathrm{conj}_*, H_2(E; \mathbb{Z})/\mathrm{Tors})$. Therefore,

$$2(d - \dim \mathscr{D}^-) = [(2 - \beta_1^\infty - \beta_3^\infty) + (\beta_2^2 - \beta_2^\infty)] + [2 - (\beta_2^2 - b_2^2)].$$

The first term of this expression is zero if $H_1^\infty(E) \neq 0$ and 2 or 4 otherwise, see 6.3.4. Applying Lemma 3.1.1 to the exact sequences

$$0 \to \operatorname{Tors}_2 H_2(E; \mathbb{Z}) \to H_2(E; \mathbb{Z}) \otimes \mathbb{Z}/2 \to (H_2(E; \mathbb{Z})/\operatorname{Tors}) \otimes \mathbb{Z}/2 \to 0,$$

$$0 \to H_2(E; \mathbb{Z}) \otimes \mathbb{Z}/2 \to H_2(E) \to \mathbb{Z}/2 \to 0$$

gives that $\beta_2^2 - b_2^2$ is equal to 2 if $w_2(E) \neq 0$ in $H_2^2(E)$, and it is equal to 0 or -2 otherwise. The combination $\beta_2^2 - b_2^2 = 0$ and $w_2(E) \neq 0$ in $H_2^2(E)$ is excluded by an additional argument: the intersection form on $H_2^2(E)$ is nondegenerate, hence, $w_2(E)$, which generates $\operatorname{Tors}_2 H_2(E; \mathbb{Z}/2) \subset H_2(E)$, and an arbitrary element, which generates the quotient $H_2(E)/(H_2(E; \mathbb{Z}) \otimes \mathbb{Z}/2)$ and thus has a nonzero intersection with $w_2(E)$, must either both survive to $H_2^2(E)$ or both disappear. Now the lemma follows from Lemmas 6.3.3 and 6.3.4 and the (mod 2)-congruence 4.1.3(2). □

6.4. *Proof of Proposition* 6.2

The relation $\mathrm{bv}_2[E_{\mathbb{R}}] = w_2(E)$ is given by Lemma 5.3.1.

Suppose that $\mathrm{bv}_2([C_1] + \cdots + [C_r]) = kw_2(E)$, $k \in \mathbb{Z}/2$, is a relation other than $\mathrm{bv}_2[E_{\mathbb{R}}^{(1)}] \equiv 0 \pmod{w_2(E)}$ or $\mathrm{bv}_2[E_{\mathbb{R}}^{(2)}] \equiv 0 \pmod{w_2(E)}$. This means that one of the components C_i involved in the relation, say C_1, belongs to $E_{\mathbb{R}}^{(1)}$, and there is another component of $E_{\mathbb{R}}^{(1)}$, say D, which does not belong to the relation. Then $\mathrm{bv}_2\langle C_1 - D\rangle$ is well defined, and, according to 5.3.2, $\mathrm{bv}_2\langle C_1 - D\rangle \circ \mathrm{bv}_2([C_1] + \cdots + [C_r]) = 1$ and $(\mathrm{bv}_2\langle C_1 - D\rangle)^2 = 0$. On the other hand, $w_2(E)$ survives to $H_2^\infty(E)$, and, since $w_2(E)$ is the characteristic class, one has $\mathrm{bv}_2\langle C_1 - D\rangle \circ w_2(E) = (\mathrm{bv}_2\langle C_1 - D\rangle)^2 = 0$. This contradicts to $\mathrm{bv}_2([C_1] + \cdots + [C_r]) = kw_2(E)$ and $\mathrm{bv}_2\langle C_1 - D\rangle \circ \mathrm{bv}_2([C_1] + \cdots + [C_r]) = 1$.

7. Proof of the main results

Below, as in Section 2, E is a generalized real Enriques surface with nonempty real part, conj: $E \to E$ is the real structure on E, and X is the double covering of E with Enriques involution $\tau: X \to X$ and two real structures $t^{(1)}$, $t^{(2)}$ determined by conj.

7.1. *Proof of Theorem* 2.1.1. By the hypothesis, the fundamental class of $X_{\mathbb{R}}^{(1)}$ vanishes in $H_2(X)$. On the other hand, it is equal to the image of the fundamental class of $E_{\mathbb{R}}^{(1)}$ under the transfer tr: $H_2(E) \to H_2(X)$, whose kernel is generated by $w_2(E)$ (see Lemma 4.2.1). Thus, the half $E_{\mathbb{R}}^{(1)}$ realizes either 0 or $w_2(E)$ in $H_2(E)$. Since, according to Lemma 5.3.1, the union $E_{\mathbb{R}}^{(1)} \cup E_{\mathbb{R}}^{(2)}$ realizes $w_2(E)$ in $H_2^\infty(E)$, the half $E_{\mathbb{R}}^{(2)}$ realizes either $w_2(E)$ or 0. In any case at least one of the two halves realizes zero in $H_2^\infty(E)$.

Suppose that there is a component $C_1 \subset E_{\mathbb{R}}^{(1)}$ of type V_{2k+1}. Then, according to 5.3.2, $\mathrm{bv}_2[E_{\mathbb{R}}^{(1)}] \circ \mathrm{bv}_2[C_1] = w_1^2(C_1) = 1$, i.e., $\mathrm{bv}_2[E_{\mathbb{R}}^{(1)}] \neq 0$. Furthermore, by assumption there also is a component $C_2 \subset E_{\mathbb{R}}^{(2)}$. Then $x = \mathrm{bv}_2(w_1(C_1) + \langle C_1 - C_2 \rangle)$ is well defined (see 5.3), and, due to 5.3.2, $\mathrm{bv}_2[E_{\mathbb{R}}^{(2)}] \circ x = 1$, i.e., also $\mathrm{bv}_2[E_{\mathbb{R}}^{(2)}] \neq 0$. This contradiction to the previous paragraph proves the first assertion.

Let now each of the halves contain a nonorientable component $C_i \subset E_{\mathbb{R}}^{(i)}$ (which, due to the first statement, are of even genus). Pick some classes $\alpha_i \in H_1(C_i)$ with $\mathrm{bv}_1\, \alpha_i \neq 0$. Then for both $(i, j) = (1, 2)$ and $(i, j) = (2, 1)$ one has $\mathrm{bv}_2(\alpha_j + \langle C_1 - C_2 \rangle) \circ \mathrm{bv}_2[E_{\mathbb{R}}^{(i)}] = 1$, which is also a contradiction. □

7.2. *Proof of Theorems* 2.1.2 *and* 2.1.3. Let $\mathcal{D}^-$ be the discriminant form of the sublattice of conj_*-skew-invariant vectors in $H_2(E; \mathbb{Z})/\mathrm{Tors}$. From Lemma 6.1 it follows that, under the hypotheses, $d - \dim \mathcal{D}^- = 2$ or 4. Since the dimension is nonnegative, $d \geq 2$.

All the congruences are derived from $\chi(E_{\mathbb{R}}) \equiv \sigma(E) + 2 \operatorname{Br} \mathcal{D}^- \pmod{16}$ given by Lemma 4.1.3(1) (just like the other congruences known in topology of real algebraic manifolds, cf. [Kh3], [M], and [N1]).

If $d = 2$, then $\mathcal{D}^- = 0$ and $\operatorname{Br} \mathcal{D}^- = 0$. This gives the congruence. The fact that $E_{\mathbb{R}}$ is of type I follows from Corollary 4.1.5.

If $d = 3$, then $\dim \mathcal{D}^- = 1$. Hence $\mathcal{D}^- = \langle \pm\frac{1}{2} \rangle$ and $\operatorname{Br} \mathcal{D}^- = \pm 1$.

If $d = 4$ and $\chi(E_{\mathbb{R}}) \equiv \sigma(E) + 8 \pmod{16}$, then $\operatorname{Br} \mathcal{D}^- = 4$ and $\dim \mathcal{D}^- = 2$. The only such form is the one given by the (2×2)-matrix $\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$ (see Table 3); it is even and Corollary 4.1.5 applies to prove that $E_{\mathbb{R}}$ is of type I. □

7.3. *Proof of Theorems* 2.1.4 *and* 2.1.5. In addition to the lattice $L = H_2(E; \mathbb{Z})/\mathrm{Tors}$ with involution conj_*, the eigenlattices $L^\pm$ of conj_*, and their discriminant forms $\mathcal{D}^\pm$, let us consider the sublattice M' of L^+ generated by the classes $s_1, \dots, s_k \in L$ realized by the spherical components of $E_{\mathbb{R}}$ (with some

Table 3. Discriminant forms of even rank ≤ 2

Odd forms		Even forms	
$\mathcal{D}^-$	$\operatorname{Br} \mathcal{D}^-$	$\mathcal{D}^-$	$\operatorname{Br} \mathcal{D}^-$
$\langle \frac{1}{2} \rangle \oplus \langle \frac{1}{2} \rangle$	2	0	0
$\langle \frac{1}{2} \rangle \oplus \langle -\frac{1}{2} \rangle$	0	$\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$	0
$\langle -\frac{1}{2} \rangle \oplus \langle -\frac{1}{2} \rangle$	-2	$\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$	4

orientations), and denote by N the orthogonal complement of M' in L^+. Recall that L and all its sublattices are even, see 4.2.3.

7.3.1. LEMMA. *If M' is not primitive in L^+, then either $E_{\mathbb{R}}$ has a half $\{lS\}$ of type* I *with $l \equiv 0 \pmod 4$, or $E_{\mathbb{R}} = kS$, it is of type* I, *and $k \equiv 0 \pmod 4$. If all the k spherical components constitute one half of $E_{\mathbb{R}}$ and, besides, $\mathcal{D}^- = 0$ and* $\operatorname{rk} N = k - 2$, *then* $k \equiv 0 \pmod 8$.

Proof. Since $s_i \circ s_j = -2\delta_{ij}$, nonprimitiveness of M' means that there is an $x \in L$ such that $2x = s_1 + \cdots + s_l$, $l > 0$. (We simplify the notation and assume that the relation involves the first l components.) Pick such a relation with the smallest possible number l of components. Then, due to 6.2 and 6.3.4, either the first l spherical components form a half $\{lS\}$ of $E_{\mathbb{R}}$ of type I, or $lS = E_{\mathbb{R}}$ and $E_{\mathbb{R}}$ is of type I. Since $l = -2x^2$, the first part of the lemma follows from the fact that L^+ is an even lattice.[4]

Suppose that all the spherical components form together one half of $E_{\mathbb{R}}$. As it follows from the first part of the proof, no partial sum of $s_1, \dots, s_k$ is divisible by 2 (as otherwise the corresponding components would form a half), and the primitive hull M'' of M' in L^+ is generated by M' and an $x \in L$ such that $2x = s_1 + \cdots + s_k$. Thus, the discriminant form of M'' is the nondegenerate part of the restriction of $-\frac{1}{2}(\theta_1^2 + \cdots + \theta_k^2)$, $\theta_j \in \mathbb{Z}/2$, to $\theta_1 + \cdots + \theta_k = 0$. In particular, $\dim \operatorname{discr} M'' = k - 2$ and $\operatorname{discr} M''$ is an even form. If $\mathcal{D}^- = 0$, then $\mathcal{D}^+ = 0$ and L^+ is unimodular. If, in addition, $\operatorname{rk} N = k - 2$, then, since $\dim \operatorname{discr} N = \dim \operatorname{discr} M'' = k - 2$, the lattice $\frac{1}{2}N$ is integral and unimodular. Besides, it is even, since so are $\operatorname{discr} M''$ and L^+. Hence, $k = -\sigma(M') = \sigma(\frac{1}{2}N) - \sigma(L^+) \equiv 0 \pmod 8$. □

7.3.2. LEMMA. *If M' is primitive in L^+ and* $\dim \operatorname{discr} M' + \dim \mathcal{D}^- > \dim \operatorname{discr} N$, *then either $E_{\mathbb{R}}$ has a half $\{lS\}$, or $E_{\mathbb{R}} = lS$, where $l \neq 0$ and $l \equiv 2q(y)$* $\pmod 4$ *for some non trivial element $y \in \mathcal{D}^-$. If, in addition, $l = k$, $\dim \mathcal{D}^- = 1$, and* $\operatorname{rk} N = k - 1$, *then* $k \equiv \operatorname{Br} \mathcal{D}^- \pmod 8$.

Remark. If $\dim \mathcal{D}^- = 1$, then $\mathcal{D}^-$ contains only one nontrivial element, and $2q(y) = \operatorname{Br} \mathcal{D}^- \pmod 8$. In all cases $y = \frac{1}{2}y_- \pmod{L^-}$ for some element $y_- \in L^-$, and $2q(y) \equiv \frac{1}{2}y_-^2 \pmod 4$.

Proof. Denote by M the primitive hull of $L^- \oplus M'$ in L. Since M and N are the orthogonal complements of each other in the unimodular even lattice L, their

[4]Since the Chern classes have equivariant representatives (cf. 5.3.1), L^+ is even for any compact complex (and even quasicomplex) surface with real structure.

discriminant forms are anti-isometric. On the other hand, $\dim \operatorname{discr} M' + \dim \mathscr{D}^- > \dim \operatorname{discr} N = \dim \operatorname{discr} M$ by the hypotheses, and, hence, $L^- \oplus M'$ is not primitive in L and the subgroup $\Gamma' \subset \operatorname{discr} M'$ (see 3.3.4) is nontrivial: for some $l > 0$ there exists an element $y_- \in L^-$ which represents a nonzero element $y \in \operatorname{discr} M'$ so that the class $s = \frac{1}{2}(y_- + s_1 + \cdots + s_l)$ belongs to L. Then $s_1 + \cdots + s_l = s + \operatorname{conj}_* s$. Thus $s_1 + \cdots + s_l$ vanishes in $\hat{H}(L)$ and therefore the element realized by the corresponding l spherical components of $E_{\mathbb{R}}$ in $\hat{H}(H_2(E, \mathbb{Z}))$ is either 0 or w_2.

Due to 6.2 and 6.3.4, either these components form a half of $E_{\mathbb{R}}$, or $E_{\mathbb{R}} = lS$ and $l = k$. Furthermore, $2q(y) \equiv \frac{1}{2}y_-^2 \equiv \frac{1}{2}(s_1 + \cdots + s_l)^2 \equiv l \pmod 2$.

If the additional assumptions hold, then $\operatorname{discr} M$ is an even discriminant form of dimension $(k-1)$. Therefore, as in 7.3.1, $\frac{1}{2}N$ is an integral even unimodular lattice and $k - \operatorname{Br} \mathscr{D}^- \equiv \sigma(\frac{1}{2}N) \equiv 0 \pmod 8$. □

In order to complete the proof, consider separately the different cases.

7.3.3. *The case* $E_{\mathbb{R}} = kS$ (*Theorem* 2.1.4). Comessatti-Severi inequality $\chi(E_{\mathbb{R}}) \le h^{1,1}(E)$ gives $d \ge 3 + h^{2,0}(E)$. Hence $d \ge 3$ and, if $d = 3$, then $\sigma(E) = 2 - b_2(E)$. In the latter case a calculation using 4.3.1 shows that L^- is a positive definite lattice of rank 1 and L^+ is a negative definite lattice of rank $2k - 1$. Hence, $\dim \mathscr{D}^- = 1$ and $\operatorname{Br} \mathscr{D}^- = 1$. By 4.1.3, this implies that $k \equiv 1 \pmod 4$. This congruence excludes, in particular, the second choice $E_{\mathbb{R}} = kS$, $k \equiv 0 \pmod 4$ in Lemma 7.3.1. The theorem follows now from 7.3.1 and 7.3.2, which cover the two possibilities for M' and both give the same decomposition $\{4pS\} \sqcup \{(4q+1)S\}$ (with $l = 4q + 1$ in the latter case). □

7.3.4. *The case* $E_{\mathbb{R}} = V_{2g} \sqcup kS$ (*Theorem* 2.1.5). From Lemma 4.3.1(1) it follows that $\operatorname{rk} L^+ = 2k + d - 2$ and, hence, $\dim \operatorname{discr} N \le \operatorname{rk} N = k + d - 2$. If $d = 0$, then L^+ is a unimodular lattice and $\dim \operatorname{discr} M' > \dim \operatorname{discr} N$. Hence M' cannot be primitive and 7.3.1 applies. Corollary 4.1.5 gives the missing information: $E_{\mathbb{R}}$ is of type I. If $d = 1$, then $\dim \mathscr{D}^- = 1$ and $\dim \operatorname{discr} N \le k - 1$, and the statement follows from 7.3.1 and 7.3.2. The possibility "$k \equiv 0 \pmod 4$, $E_{\mathbb{R}}^{(2)}$ is of type I" for $k^{(1)} = 0$ arises from the case when M' is not primitive: then $k = k^{(2)}$ must be divisible by 4. If $d = 2$, then $\mathscr{D}^-$ is one of the forms given in Table 3. $\mathscr{D}^- = 0$ is the exceptional case of Theorem 2.1.5 when $k^{(2)}$ may be trivial. (In fact, $k^{(2)}$ is trivial in this case since $\dim \mathscr{D}^- = d - 2$ and, according to Lemma 6.1, $E_{\mathbb{R}}$ must consist of a single half.) In all the other cases 7.3.1 and 7.3.2 give all the values of $k^{(2)} \pmod 4$ listed in Table 1.

The remaining case $d = 3$, $\delta = \pm 3$ follows from Theorem 2.1.3, though, due to 6.6 and 4.1.3, in this case $\dim \mathscr{D}^- = 3$, and one can also apply 7.3.2.

Finally, to distinguish between types I_0 and I_w in (1) and (3) it suffices to notice that, under the hypotheses, $w_2(E) \neq 0$ in $H_2^\infty(E)$ (see Lemma 6.3.3) and, hence, a half is of type I_0 if and only if its fundamental class vanishes in $H_2^\infty(E)$, i.e., belongs to the kernel of the intersection form. Using 5.3.2 one can see that the spherical half realizes $w_2(E)$; hence, it is of type I_w. $\square$

8. Construction

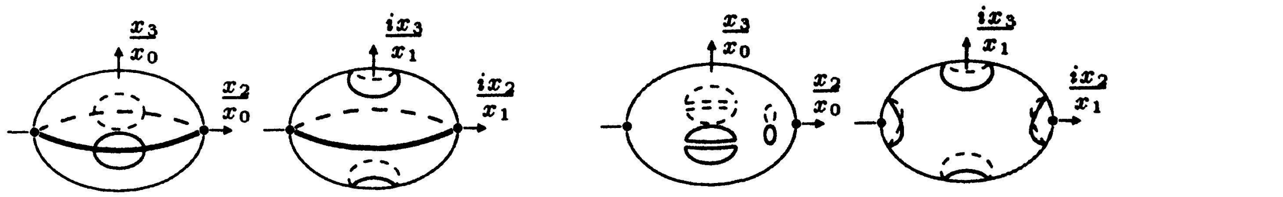
8.1. *General idea* (see [DK1] for details)

Let X be the $K3$-surface obtained as the double covering of $Y = \mathbb{C}p^1 \times \mathbb{C}p^1$ branched over a non-singular bi-degree $(4, 4)$ curve $C \subset Y$. Let $s\colon Y \to Y$ be the Cartesian product of the nontrivial involutions $(u\colon v) \mapsto (-u; v)$ of the factors. If C is s-symmetric, s lifts to two different involutions on X, commuting with the deck translation d of $X \to Y$. If, besides, C contains no fixed points of s, then exactly one of these two involutions, which we denote by τ, is fixed point free (see, e.g., [H] or [BPV]), and, hence, the orbit space $E = X/\tau$ is an Enriques surface.

Suppose that Y is equipped with a real structure conj_Y commuting with s and C is a real curve. Then $s \circ \operatorname{conj}_Y$ is another real structure on Y and C. We denote the real point sets of these structures by $Y_{\mathbb{R}}^{(i)}$ and $C_{\mathbb{R}}^{(i)}$, $i = 1, 2$ ($i = 1$ corresponding to conj_Y) and call them the *halves* of Y and C. The involutions conj_Y and $s \circ \operatorname{conj}_Y$ lift to four different commuting real structures ($t^{(1)}$, $t^{(2)} = \tau \circ t^{(1)}$, $d \circ t^{(1)}$, and $d \circ t^{(2)}$) on X, which, in turn, descend to two real structures on E, called the *expositions* of E. A choice of an exposition is determined by a choice of one of the two liftings $t^{(1)}$, $t^{(2)}$ of conj_Y to X.

We use for Y a quadric in $\mathbb{C}p^3$ real in respect to the standard complex conjugation involution and invariant in respect to the real symmetry $s\colon \mathbb{C}p^3 \to \mathbb{C}p^3$, $(x_0\colon x_1\colon x_2\colon x_3) \mapsto (x_0\colon x_1\colon -x_2\colon -x_3)$. Since the bi-degree of C is even, $C_{\mathbb{R}}^{(i)}$ separates $Y_{\mathbb{R}}^{(i)}$ into two parts with common boundary $C_{\mathbb{R}}^{(i)}$; at least one of them is non-empty unless $Y_{\mathbb{R}}^{(i)}$ is empty. The fixed point set $X_{\mathbb{R}}^{(i)}$ of $t^{(i)}$ is the pull-back of one of the parts. Thus, a choice of $t^{(1)}$ is equivalent to a choice of one of the two parts of $Y_{\mathbb{R}}^{(1)}$, and, since $t^{(2)} = \tau \circ t^{(1)}$, the latter determines as well the part of $Y_{\mathbb{R}}^{(2)}$ whose pull-back is $\operatorname{Fix} t^{(2)}$: as $X_{\mathbb{R}}^{(1)}$ and $X_{\mathbb{R}}^{(2)}$ are disjoint, the pull-back of a point of $Y_{\mathbb{R}}^{(1)} \cap Y_{\mathbb{R}}^{(2)}$ is contained in exactly one of $X_{\mathbb{R}}^{(1)}$, $X_{\mathbb{R}}^{(2)}$. (Note that in all the examples we use here $Y_{\mathbb{R}}^{(1)} \cap Y_{\mathbb{R}}^{(2)} \neq \varnothing$.)

The branch curve $C \in Y$ is constructed by perturbing the equation $f = 0$ of a singular s-symmetric curve $\tilde{C} \in Y$ to $f + \varepsilon h = 0$; here f and h are homogeneous real s-symmetric polynomials of bi-degree $(4, 4)$ and ε is a small real parameter. All the



(a) (b)

Figure 2

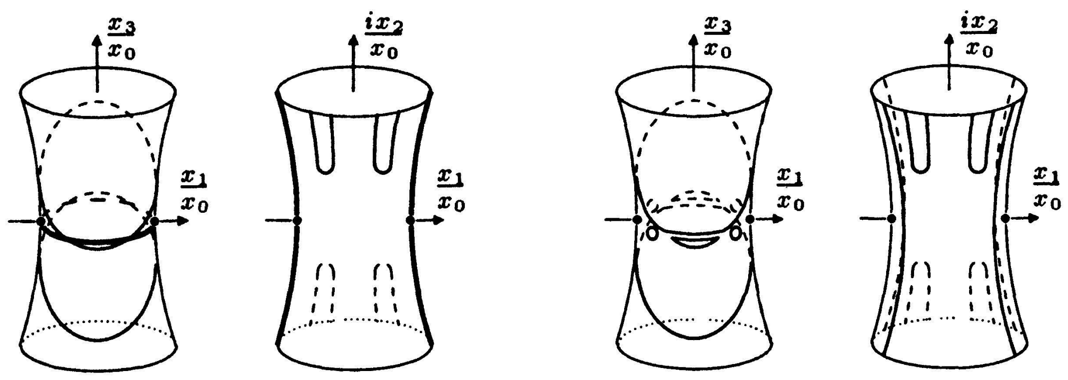
facts necessary to construct a perturbation and to control its topology can be found in [DK1, Sect. 4].

8.2. *The distributions of* $2V_1 \sqcup kS$

It suffices to construct the distributions $\{aS\} \sqcup \{2V_1 \sqcup bS\}$ and $\{V_1 \sqcup aS\} \sqcup \{V_1 \sqcup bS\}$ with $(a, b) = (1, 3)$, $(2, 2)$ or $(3, 1)$; the rest is constructed in [DK1]. Let Y be the ellipsoid given by $x_0^2 = x_1^2 + x_2^2 + x_3^2$ and $\tilde{C} = \tilde{C}_1 \cup \tilde{C}_2$, where $\tilde{C}_1$ and $\tilde{C}_2$ are cut on Y by $x_3^2 = 0$ and $2(x_2^2 - x_3^2) = x_0^2$ respectively (see Figure 2(a), which represents the two halves of $Y_{\mathbb{R}}$ and $\tilde{C}$. The two black dots in each figure are the fixed points of the restriction of s to the corresponding half.) To perturb $\tilde{C}$ take for h the equation of a bi-degree $(4, 4)$ s-symmetric real curve which intersects the two real halves of $\tilde{C}_1$ at eight points (the *ramification points*); all these points must be outside of the ovals of $\tilde{C}_2$ and different from the fixed points of s. Then, under a proper choice of the sign of ε, the portions of the real part of $\tilde{C}_1$ which are either inside the ovals of $\tilde{C}_2$ or between pairs of the ramification points double, and the rest of $\tilde{C}_1$ disappears (see, e.g., Figure 2(b), corresponding to $\{3S\} \sqcup \{2V_1 \sqcup S\}$; to obtain the other distributions note that one or both the ovals surrounding the fixed points can be moved to the 'left hand' half, and the pair of small ovals can be moved to the 'right hand' half). If the exposition is chosen so that $X_{\mathbb{R}}^{(2)}$ covers the interior of the two ovals surrounding the fixed points of s, then these two ovals produce the V_1 components of $E_{\mathbb{R}}$; the other pairs of symmetric ovals produce spheres.

8.3. *The distributions of* $2V_2 \sqcup kS$

The distributions constructed here are $\{V_2 \sqcup aS\} \sqcup \{V_2 \sqcup bS\}$ for all (a, b) except $(0, 0)$, $(4, 0)$, $(2, 2)$, and $(0, 4)$. (The first exception is found in [DK1], the others, in [N2], see the remark at the end of 2.2.) Let Y be the hyperboloid $x_0^2 = x_1^2 + x_2^2 - x_3^2$ and $\tilde{C} = \tilde{C}_1 \cup \tilde{C}_2$, where $\tilde{C}_1$ and $\tilde{C}_2$ are given, respectively, by



(c) (b)

Figure 3

$x_3^2 = 0$ and $(2x_3 - x_2)^2 = \varepsilon(x_0^2 + x_3^2)$ for some small real $\varepsilon > 0$ (see Figure 3(a)). The perturbative term h (see 8.1) is chosen so that its zero set does not intersect the right half of $\tilde{C}_1$ and intersects its left half at $4(a - 1)$ points, $a = 1, 2, 3$, close to the fixed points of s. Under a proper choice of the sign of the perturbation, the right half of $\tilde{C}_1$ doubles and the ramification points generate $2(a - 1)$ ovals which do not contain the fixed points of s (Figure 3(b)). The exposition is chosen so that the two strips containing the fixed points of s in $Y_{\mathbb{R}}^{(2)}$ are covered by $X_{\mathbb{R}}^{(2)}$; they produce the components V_2 of $E_{\mathbb{R}}$. Thus we obtain the distributions $\{V_2 \sqcup aS\} \sqcup \{V_2 \sqcup bS\}$ with $a = 1, 2, 3$ and $b = 1$. To construct surfaces with $b = 0$, we replace $\tilde{C}_2$ with the curve given by $(2x_3 - x_2)^2 = \varepsilon(x_0^2 - x_3^2)$; its right hand half is empty.

8.4. *The distributions of* $V_3 \sqcup V_1 \sqcup kS$

We construct the distributions $\{V_3 \sqcup V_1 \sqcup aS\} \sqcup \{bS\}$ and $\{V_3 \sqcup aS\} \sqcup \{V_1 \sqcup bS\}$ with $1 \le a + b \le 4$ and $a \ge 1$; the rest is found in [DK1]. Start with a quartic $Q \subset \mathbb{R}\mathrm{p}^2$ with $(k + 1)$ real components, $1 \le k \le 3$, obtained by perturbing the union of two conics (see Figure 4, where $k = 3$). Pick an oval O (the lowest one in Figure 4) and denote by L the double tangent to O and by L_b, $0 \le b \le k$, another tangent, which together with L separates O in $\mathbb{R}\mathrm{p}^2$ from b other ovals of Q.

We use the following technical result, proved at the end of this section.

8.4.1. LEMMA. *The union $L \cup L_b$ can be perturbed to an irreducible conic K which is still tangent to O at three points, has no other real intersection points with Q, and such that O is in the outer part of the oval of K.*

Let K be the conic given by the lemma. Consider the double cover $\bar{Y}$ of the projective plane branched over K. Denote by $\bar{s}$ the deck translation involution, by

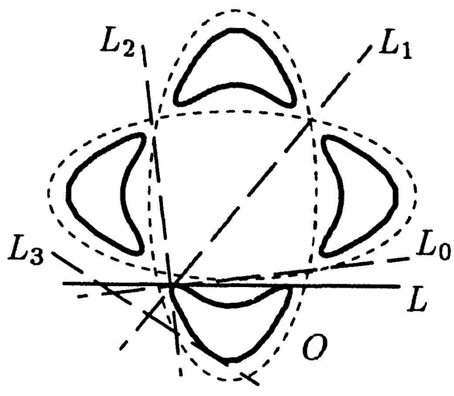


Figure 4

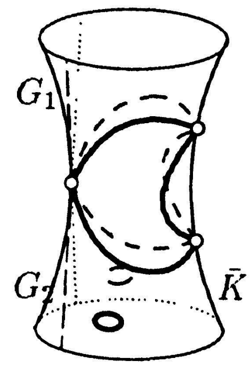


Figure 5

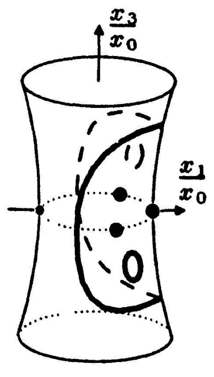


Figure 6

$\bar{K}$ its fixed point set (which projects to K), and by $\bar{Q}$ the pull-back of Q. Due to $\bar{s}$ (cf. 8.1), each of $\bar{Y}$, $\bar{Q}$ and $\bar{K}$ has two real halves. $\bar{Y}^{(1)}_{\mathbb{R}}$ is the hyperboloid shown in Figure 5: $\bar{Q}^{(1)}_{\mathbb{R}}$ has a component $\bar{O}$ (the pull-back of O) with three nondegenerate double points in $\bar{K}^{(1)}_{\mathbb{R}}$ and $(k-b)$ pairs of symmetric ovals. The other half $\bar{Y}^{(2)}_{\mathbb{R}}$ is an ellipsoid in which $\bar{Q}^{(2)}_{\mathbb{R}}$ has b pairs of ovals disjoint from $\bar{K}^{(2)}_{\mathbb{R}}$. Now (Y, s) is obtained from $(\bar{Y}, \bar{s})$ by the following real $\bar{s}$-symmetric birational transformation: blow up the singular points of $\bar{Q}$ and then blow down the transforms of $\bar{K}$ and the two generatrices G_1, G_2 of $\bar{Y}$ through the singular point of $\bar{Q}$ whose image in $\mathbb{R}p^2$ is close to the tangency point of L_b and O. Let $\tilde{C}$ be the transform of $\bar{Q}$ (Figure 6). Clearly, $\tilde{C}^{(1)}_{\mathbb{R}}$ consists of a large oval $\tilde{O}$ (the transform of $\bar{O}$) surrounding $(k-b)$ pairs of symmetric ovals and three isolated double points: the image of $\bar{K}$, fixed under s, and the image of G_1, G_2, symmetric to each other. The other half consists of b pairs of ovals and an isolated double point, the image of $\bar{K}$. All the ovals but $\tilde{O}$ are not nested and do not surround the singular points of $\tilde{C}$. Finally, perturb $\tilde{C}$ to a nonsingular symmetric curve C (see 4.3.1 in [DK1]); the fixed double point, which produces the V_1 component of $E_{\mathbb{R}}$, can be made to pop up in either side, and the two symmetric double points may either form a pair of symmetric ovals or disappear. Thus, we obtain $\{V_3 \sqcup V_1 \sqcup (k-b+\delta)S\} \sqcup \{bS\}$ and $\{V_3 \sqcup (k-b+\delta)S\} \sqcup \{V_1 \sqcup bS\}$ with $\delta = 0, 1$.

Proof of Lemma 8.4.1. Given an imaginary point $u \in Q$, define an involution ρ_u of a Zariski open subset of the symmetric power S^3Q in the following way: for a generic triple $(x_1, x_2, x_3) \in S^3Q$ there is a unique conic through $u, \bar{u}, x_1, x_2, x_3$; it intersects Q at three more points y_1, y_2, y_3, and we let $\rho_u(x_1, x_2, x_3) = (y_1, y_2, y_3)$. Clearly, the above conic is tangent to Q at x_1, x_2, x_3 if and only if (x_1, x_2, x_3) is a fixed point of ρ_u.

Denote by a_1, a_2, a_3 the three tangency points of $L \cup L_b$ and Q, and by v one of the two imaginary intersection points of L_b and Q. Then the graph Γ_v of ρ_v intersects the diagonal $\Delta \subset S^3Q \times S^3Q$ at $a = (a_1, a_2, a_3) \times (a_1, a_2, a_3)$ transversally. (Note that S^3Q is smooth at this point.) Indeed, let p_1, p_2 be the two projections $S^3Q \times S^3Q \to S^3Q$, and let $\mathbf{e}_i$ be some real generators of the tangent spaces $T_{a_i}Q$, which we regard as basis vectors of $T_{(a_1,a_2,a_3)}S^3Q$. Then $T_a\Delta$ is spanned by $p_1^*\mathbf{e}_i + p_2^*\mathbf{e}_i$, $i = 1, 2, 3$, and $T_a\Gamma_v$ is spanned by $p_1^*\mathbf{e}_i + \alpha_i p_2^*\mathbf{e}_i$, $i = 1, 2, 3$, with some real $\alpha_i < 0$. (To see that, one can move one point at a time; then the conic is still reducible, and it is easy to estimate the tangent vectors.) Thus, for any other point v' close to v the graph of $\rho_{v'}$ also has a unique (and hence real) intersection point with Δ close to a, i.e., there is a real conic K through v' tangent to Q at three real points close to a_1, a_2, a_3. If the line $(v'\bar{v}')$ is not tangent to Q, this conic is irreducible. Finally, to control the topology (actually, to choose one of the two possible real directions of the perturbation), just note that K has no real intersection points with $(v'\bar{v}')$; hence, this line lies outside of the oval of K, and if v' is chosen so that $(v'\bar{v}')$ intersects O at two real points, then O is also outside. □

Remark. The involution ρ_u is similar to that in [GH, Sect. 7], where it is used for a similar purpose. It also seems possible to apply Shustin's approach [Sh].

Appendix A. Kalinin's intersection form

A.1. *The local case*

Kalinin's spectral sequence and, in particular, Viro homomorphisms admit an obvious relative version. We make use of such a version to do some calculations in a neighborhood of the fixed point set. Then, in the next subsection, we apply the result obtained to prove Theorem 5.2.3.

A.1.1. LEMMA. *Let v be an m-dimensional vector bundle over a finite cell complex F, and let T and ∂T be the associated disk and sphere bundles, respectively, supplied with the antipodal involution. Then the homology filtration $\mathscr{F}^*$ associated with Kalinin's spectral sequence of $(T, \partial T)$ is given by $\mathscr{F}^{m+p} = w(v)^{-1} \cap H_{\geq p}(F)$, where $w(v) = 1 + w_1(v) + w_2(v) + \cdots$ is the total Stiefel-Whitney class of v.*

Proof. Given a topological space Y with involution $c\colon Y \to Y$ and an integer k, $0 \leq k \leq \infty$, denote by Y_k the twisted product

(A.1.2) $Y_k = Y \times S^k/\{(y, s) \sim (cy, gs)\}$,

where $g\colon S^k \to S^k$ is the antipodal involution on the standard sphere S^k. It is clear (see, e.g., [D]) that T_k and $(\partial T)_k$ are, respectively, the disk and the sphere bundles associated with $v \otimes \eta$ over $F_k = F \times \mathbb{R}\mathrm{p}^k$, where η is the tautological linear bundle over $\mathbb{R}\mathrm{p}^k$. Let $h_i \in H_i(\mathbb{R}\mathrm{p}^k)$ be the generators. (We let $h_i = 0$ for $i < 0$ or $i > k$.) According to [D], a sufficient condition for a class Σx_i, $x_i \in H_i(F)$, to belong to $\mathscr{F}_q$ is that the image of $\Sigma x_i \otimes h_{q-1-i}$ in $H_{q-1}(T_q, \partial T_q)$ under the inclusion map $H_*(F_q) \to H_*(T_q, \partial T_q)$ should vanish. (In [D] the absolute case is considered, but the proof transfers literally to the relative case.) The inclusion map $H_*(F_q) \to H_*(T_q, \partial T_q)$ is equal to the composition of the multiplication by $w_m(v \otimes \eta) = \Sigma w_i(v) \otimes h^{m-i}$ and Thom isomorphism, and spelling out the product $w_m(v \otimes \eta) \cap \Sigma x_i \otimes h_{q-1-i}$ and taking into account the coefficients of those of h_j which are not identically zero in $H_*(\mathbb{R}\mathrm{p}^q)$ shows that the above sufficient condition is equivalent to $w(v) \cap \Sigma x_i \in H_{\geq q-m}(F)$, i.e., $\Sigma x_i \in w(v)^{-1} \cap H_{\geq q-m}(F)$. *A priori*, the subgroup obtained is only a portion of $\mathscr{F}^q$, but comparing the dimensions shows that, in fact, these two subgroups coincide. □

A.1.3. COROLLARY. *Let F, v, T, and ∂T be as in Lemma A.*1.1, *and let* th: $H_{q+m}(T, \partial T) \to H_q(F)$ *be the Thom isomorphism. Then for any class $a \in H_q(F)$ one has* $\mathrm{bv}_{q+m}(w^{-1}(v) \cap a) = \mathrm{th}^{-1}\, a$.

Proof. The result is actually proved for the case when F is a q-dimensional polyhedron with $H_q(F) = \mathbb{Z}/2$, and a is the generator of $H_q(F)$: in this case $w^{-1}(v) \cap a$ is the only nontrivial element in $\mathscr{F}^{q+m}$, $\mathrm{th}^{-1}\, a$ is the only nontrivial element in $H_{q+m}(T, \partial T)$, and $\mathrm{bv}_{q+m}\colon \mathscr{F}^{q+m} \to H_{q+m}(T, \partial T)$ is an isomorphism. In general, one can find a singular q-dimensional polyhedron $f\colon P \to F$ with $H_q(P)$ generated by a single element $[P]$ so that $a = f_*[P]$. The result follows then from the naturality of bv_* and th. □

A.2. Proof of Theorem 5.2.3

A.2.1. LEMMA. *Let Y, c, and F be as above. Denote by $D_Y\colon H^*(Y) \to H_*(Y)$ and $D_F\colon H^*(F) \to H_*(F)$ the Poincaré duality maps in Y and F respectively, and by $D_c\colon H^*(F) \to H_*(F)$ the map $\alpha \mapsto \alpha \cap (w^{-1}(v) \cap [F])$. Then*:

(1) *D_c induces isomorphisms $\mathscr{F}_{N-p} \to \mathscr{F}^p$*;

(2) *given $x \in \mathscr{F}_p$, one has* $\mathrm{bv}^{N-p}(D_Y^{-1}\, \mathrm{bv}_p\, x) \equiv D_c^{-1} x \bmod \mathscr{F}_{N-p-1}$.

Proof. From the naturality of Kalinin's spectral sequence and Corollary A.1.3 it follows that the only nontrivial element of $\mathscr{F}^N$ is $w^{-1}(v) \cap [F]$ and, hence, $[Y] = \mathrm{bv}_N(w^{-1}(v) \cap [F])$. Thus, D_c is the multiplication by the generator of $\mathscr{F}^N$; hence, it maps $\mathscr{F}_{N-p}$ to $\mathscr{F}^p$. Furthermore, D_c is an isomorphism (as composition of Poincaré duality and multiplication by an invertible element), and comparing the dimensions shows that so is its restriction to $\mathscr{F}_{N-p} \to \mathscr{F}^p$. (Recall that $\dim \mathscr{F}_{N-p} = \dim \mathscr{F}^p$ due to 5.2.2 and duality between H^*_∞ and H^∞_*.)

It follows that $D_c \operatorname{bv}^{N-q}(D_Y^{-1} \operatorname{bv}_q x) \in \mathscr{F}_p$, and one has:

$$\operatorname{bv}_p(D_c \operatorname{bv}^{N-p}(D_Y^{-1} \operatorname{bv}_p x)) = D_Y^{-1} \operatorname{bv}_p x \cap [Y] = \operatorname{bv}_p x;$$

since $\operatorname{Ker} \operatorname{bv}_p = \mathscr{F}^{p+1}$, this gives $D_c \operatorname{bv}^{N-p}(D_Y^{-1} \operatorname{bv}_p x) \equiv x \bmod \mathscr{F}^{p+1}$. □

Proof of Theorem 5.2.3. By the definition, $w(v) \cap (a \circ b) = D_c^{-1} a \cap b \in \mathscr{F}_{N-p} \cap \mathscr{F}^q \subset \mathscr{F}^{p+q-N}$, and a direct calculation using Lemma A.2.1(2) shows that $\operatorname{bv}_{p+q-N}(D_c^{-1} a \cap b) = D_Y^{-1} \operatorname{bv}_p a \cap \operatorname{bv}_q b = \operatorname{bv}_p a \circ \operatorname{bv}_q b$. □

Mention also the following immediate consequence of A.1.1 and A.1.3:

A.2.2. PROPOSITION. *Let Y, c, F, and v be as in Theorem 5.2.3. Pick a component $F_i \subset F$ of dimension $(N - m)$, and denote by $\operatorname{in}_i\colon F_i \to Y$ the inclusion. Then $\mathscr{F}^q \cap H_*(F_i) \subset w^{-1}(v) \cap H_{\geq q-m}(F_i)$, and for any class $a \in \mathscr{F}^q$ one has $\operatorname{in}_i^! \operatorname{bv}_q a = [w(v) \cap a]_{q-m}|_{F_i}$, where $\operatorname{in}^!$ is the inverse Hopf homomorphism and $[\cdot]_{q-m}$ stands for the $(q - m)$-dimensional component of a nonhomogeneous homology class.*

Proof. The first statement follows from the naturality of the filtration and Lemma A.1.1 applied to $v|_{F_i}$. To prove the second one just note that $\operatorname{in}_i^!$ is the composition of the relativization homomorphism $H_q(Y) \to H_q(T_i, \partial T_i)$ and Thom isomorphism $H_q(T_i, \partial T_i) \to H_{q-m}(F_i)$, and apply Corollary A.2.1. □

Appendix B. 'Generalized Enriques surfaces' with $w_2(E) = 0$

In this section we assume that E satisfies all the axioms of generalized Enriques surfaces (see 1.2) except the requirement $w_2(E) \neq 0$, i.e., E is the orbit space of a generalized $K3$-surface X by a fixed point free holomorphic involution $\tau\colon X \to X$, and $w_2(E) = 0$. These surfaces are closely related to symmetric curves of bi-degree $(4k+2, 4k+2)$ on real quadrics (cf. Introduction). We only state the results, parallel to those of Section 2; proofs are found in [DK2]. (The proofs require some properties of the action of the Steenrod algebra in Kalinin's spectral sequence, which are also studied in [DK2].)

As in the case $w_2 \neq 0$, the components of $E_{\mathbb{R}}$ may be of one of the types S_g, V_g, or T_g (see 1.3). Note that $E_{\mathbb{R}}$ has no components of type V_{2g+1}, as for such a component C_i one would have $[C_i]^2 = 1$. We say that $E_{\mathbb{R}}$ or $E_{\mathbb{R}}^{(i)}$ is of type I if its fundamental class belongs to the image of $\operatorname{Tors}_2 H_2(E; \mathbb{Z})$ in $H_2(E)$.

B.1.1. THEOREM (cf. Theorem 2.1.2). *If $E_{\mathbb{R}}$ is nonorientable, then $E_{\mathbb{R}}$ consists of a single half and the restriction $X_{\mathbb{R}} \to E_{\mathbb{R}}$ of the projection $X \to E$ is the orientation double covering (i.e., there is no components of type T_g). Besides, E is an $(M-d)$-surface, $d \geq 2$, and*

(1) *if $d = 2$, then $\chi(E_{\mathbb{R}}) \equiv \sigma(E)$* (mod 16) *and $E_{\mathbb{R}}$ is of type* I;

(2) *if $d = 3$, then $\chi(E_{\mathbb{R}}) \equiv \sigma(E) \pm 2$* (mod 16);

(3) *if $d = 4$ and $\chi(E_{\mathbb{R}}) \equiv \sigma(E) + 8$* (mod 16), *then $E_{\mathbb{R}}$ is of type I.*

B.1.2. THEOREM (cf. Theorems 2.1.2 and 2.1.3). *If E is an $(M-d)$-surface with orientable real part and either $E_{\mathbb{R}}$ is trivially covered by $X_{\mathbb{R}}$ (i.e., there is no components of type T_g) or $E_{\mathbb{R}}$ consists of a single half, then $d \geq 2$ and*

(1) *if $d = 2$, then $\chi(E_{\mathbb{R}}) \equiv \sigma(E)$* (mod 16) *and $E_{\mathbb{R}}$ is of type* I;

(2) *if $d = 3$, then $\chi(E_{\mathbb{R}}) \equiv \sigma(E) \pm 2$* (mod 16);

(3) *if $d = 4$ and $\chi(E_{\mathbb{R}}) \equiv \sigma(E) + 8$* (mod 16), *then $E_{\mathbb{R}}$ is of type* I.

B.1.3. THEOREM (cf. Theorem 2.1.4). *Let E be an $(M-3)$-surface with $E_{\mathbb{R}} = kS$. Then $E_{\mathbb{R}} = \{4pS\} \sqcup \{(4q+1)S\}$, both the halves being nonempty unless* $k = 1$ (mod 8).

B.1.4. THEOREM (cf. Theorem 2.1.5). *Let $E_{\mathbb{R}} = T_g \sqcup kS$. Suppose that E is an $(M-d)$-surface and $\chi(E_{\mathbb{R}}) \equiv \sigma(E) + 2\delta$* (mod 16). *Then for the values of (d, δ) listed in Table* 1 *in* 2.1 *one has $E_{\mathbb{R}} = \{V_{2g} \sqcup k^{(1)}S\} \sqcup \{k^{(2)}S\}$, where $k^{(2)}$* (mod 4) *takes one of the values given in the table; furthermore, $k^{(2)} \neq 0$ with the possible exception of the case $d = 2$, $\delta = 0$, $E_{\mathbb{R}}$ is of type* I. *Besides, there are the following additional prohibitions:*

(1) *if $d = 0$, then both the halves (as well as $E_{\mathbb{R}}$ itself) are of type* I;

(2) *if $d = 0$, then $k^{(1)} \neq 0$ unless $k \equiv 0$* (mod 8);

(3) *if $d = 1$ and $k^{(1)} = 0$, then either $k \equiv \delta$* (mod 8), *or $k \equiv 0$* (mod 4) *and $E_{\mathbb{R}}^{(2)}$ is of type* I.

Acknowledgements

We would like to thank the *Max-Planck-Institut für Mathematik* and *Universitá di Trento*, where the final parts of the paper were completed. Our special gratitude is to **internet**: without this great innovation the paper would probably never appear.

Added in proof. The proof of Theorem 2.1.4 has a gap: in 7.3.2 one needs to eliminate the case $k = l$ with both $E_{\mathbb{R}}^{(1)}$ and $E_{\mathbb{R}}^{(2)}$ nonempty. It is eliminated by the following lemma: *if $E_{\mathbb{R}}$ is orientable, both the halves are nonempty, and $[E_{\mathbb{R}}^{(1)}] = x + \mathrm{conj}_* x$ with $x \in H_1(E)$, then $x^2 = 0$* mod 2. (This implies that if in 7.3.2 both the halves are nonempty the relation $[E^{(1)}_{\mathbb{R}}] = x + \operatorname{conj}_* x$ holds not only in $H_2(E)$ but also in $H_2(E; \mathbb{Z})$ and, hence, $s_1 + \cdots + s_l$ can be taken to represent one of the halves.) To prove the lemma apply the Pontrjagin square: $P[E^{(1)}_{\mathbb{R}}] = 2P(x) + 2(x \circ \operatorname{conj}_* x)$. Then pick an $s \in H_2(E; \mathbb{Z})$ so that $[E_{\mathbb{R}}] = s + \operatorname{conj}_* s$: such an element exists in $H_2(E)$ as w_2 vanishes in $H^2_2 = H^\infty_2$, and it lifts to $H_2(E; \mathbb{Z})$ since, due to the Arnol'd lemma, $s^2 = s \circ [E_{\mathbb{R}}] - s \circ \operatorname{conj}_* s = 0 \bmod 2$. Due to the Arnol'd lemma again, $x \circ \operatorname{conj}_* x = x \circ (1 + \operatorname{conj}_*)s = (x + \operatorname{conj}_* x) \circ s = s \circ [E^{(1)}_{\mathbb{R}}]$. Thus, $x \circ \operatorname{conj}_* x$ equals $\frac{1}{2}(s + \operatorname{conj}_* s) \circ [E^{(1)}_{\mathbb{R}}] = \frac{1}{2}[E_{\mathbb{R}}] \circ [E^{(1)}_{\mathbb{R}}] = \frac{1}{2}[E^{(1)}_{\mathbb{R}}]^2$ reduced mod 2 and $2x^2$ mod 4 equals $2P(x) = P[E^{(1)}_{\mathbb{R}}] - [E^{(1)}_{\mathbb{R}}]^2 = 0$. □

REFERENCES

[Ar] V. I. Arnol'd, *On the arrangement of ovals of real algebraic plane curves, involutions on 4-dimensional smooth manifolds, and the arithmetic of integral quadratic forms*, Funktsional Anal. i Prilozhen. *5* (1971), no. 3, 1–9; English transl. in Functional Anal. Appl. *5* (1971), no 3, 169–181.

[BPV] W. Barth, C. A. M. Peters, and A. Van de Ven, *Compact Complex Surfaces*, Springer, 1984.

[Bo] A. Borel *et al.*, *Seminar on transformation groups*, Ann. of Math. Studies, vol. 46, Princeton Univ. Press, 1961.

[BH] A. Borel and A. Haefliger, *La classe d'homologie fondamentale d'un espace analitique*, Bull. Soc. Math. France *89* (1961), 461–513.

[B] G. E. Bredon, *Introduction to compact transformation groups*, Academic Press, New York, London, 1972.

[CE] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, 1956.

[Co] A. Comessatti, *Sulla connessione e sui numeri base delle superficie algebriche reali*, Rend. Sem. Mat. Padova *3* (1932), 141–162.

[D] A. Degtyarev, *Stiefel orientations on a real algebraic variety*, Real Algebraic Geometry (Rennes, 1991), Lect. Notes in Math., vol. 1524, Springer-Verlag, 1992, pp. 205–220.

[DK1] A. Degtyarev and V. Kharlamov, *Topological classification of real Enriques surfaces*, Topology *35* (1996), No. 3, 711–729.

[DK2] A. Degtyarev and V. Kharlamov, *Distribution of the components of a real Enriques surface*, Preprint of the Max-Planck Institute, MPI/95-58, 1995 (also available from AMS server as AMSPPS # 199507-14-005).

[DK3] A. Degtyarev and V. Kharlamov, *Around real Enriques surfaces* (to appear in Proc. Segovia RAAG-95; currently available from AMS server as AMSPPS # 199511-14-001).

[E] A. L. Edmonds, *Orientability of fixed point sets*, Proc. AMS *82* (1981), no. 1, 120–124.

[GH] B. H. Gross and J. Harris, *Real algebraic curves*, Ann. Scient. Ec. Norm. Sup. *14* (1982), 157–182.

[HH] F. Hirzebruch and H. Hopf, *Felder von Flächenelementen in 4-dimensionalen Mannigfaltigkeiten*, Math. Ann. *136* (1958), 156–172.

[Ht] N. Hitchin, *Compact four-dimensional Einstein manifolds*, J. Differential Geometry *9* (1974), 435–441.

[Ka] I. Kalinin, *Cohomology characteristics of real algebraic hypersurfaces*, Algebra i Analiz *3* (1991), no. 2, 91–110 (Russian); English transl. in St. Petersburg Math. J. *3* (1992), no. 2, 313–332.

[Kh1] V. M. Kharlamov, *Topological types of nonsingular surfaces of degree 4 in* $\mathbb{R}p^3$, Funktsional Anal. i Prilozhen. *10* (1976), no. 4, 55–68 (Russian); English transl. in Functional Anal. Appl. *10* (1976), no. 4, 295–305.

[Kh2] V. M. KHARLAMOV, *Additional congruences for the Euler characteristic of even-dimensional real algebraic manifolds*, Funktsional Anal. i Prilozhen. *9* (1975), no. 2, 51–60 (Russian); English transl. in Functional Anal. Appl. *9* (1975), no. 2, 134–142.

[Kh3] V. M. KHARLAMOV, *New congruences for the Euler characteristic of real algebraic manifolds*, Funktsional Anal. i Prilozhen. *7* (1973), no. 2, 74–78 (Russian), English transl. in Functional Anal. Appl. *7* (1973), 147–150.

[K] V. A. KRASNOV, *Orientability of real algebraic manifolds*, Konstructivnaia Algebraicheskaia Geometria, Yaroslavl' *194* (1981), 46–57 (Russian).

[M] A. MARIN, *Quelques remarques sur les courbes algébriques planes réelles*, Publ. Math. Univ. Paris VII *9* (1980), 51–86.

[N1] V. V. NIKULIN, *Integer quadratic forms and some of their geometrical applications*, Izv. Akad. Nauk SSSR, Ser. Mat. *43* (1979), 111–117 (Russian); English transl. in Math. USSR–Izv. *43* (1979), 103–167.

[N2] V. V. NIKULIN, *On the topological classification of real Enriques surfaces. I*, Amer. Math. Soc. Transl. (2) *173* (1996), 187–202.

[Sh] E. SHUSTIN, *A new M-curve of order 8*, Matemat. Zametki *43* (1987), no. 2, 180–186 (Russian); English trans. in Math. Notes Acad. Sc. USSR *42* (1987), 606–610.

[S] A. J. SOMMESSE, *Real algebraic spaces*, Annali della Scuola Normale Superiore di Pisa *4* (1977), 599–612.

[VZ] O. VIRO and V. ZVONILOV, *An inequality for the number of nonempty ovals of a curve of odd degree*, Algebra i Analiz *4* (1992) (Russian); English trans. in St. Petersburg Math. J. *4* (1993), 539–548.

[Wi] G. WILSON, *Hilbert's sixteenth problem*, Topology *17* (1978), 53–73.

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Received November 6, 1995