# The rigidity of Clifford torus S1 (...) x Sn-1 (...).

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# The rigidity of Clifford torus $S^1\left(\sqrt{\frac{1}{n}}\right) \times S^{n-1}\left(\sqrt{\frac{n-1}{n}}\right)$

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*Abstract.* In this paper, we prove that if $M$ is an $n$-dimensional closed minimal hypersurface with two distinct principal curvatures of a unit sphere $S^{n+1}(1)$, then $S = n$ and $M$ is a Clifford torus if $n \le S \le n + [2n^2(n+4)/3(n(n+4)+4)]$, where $S$ is the squared norm of the second fundamental form of $M$.

## 1. Introduction

Let $M$ be an $n$-dimensional closed hypersurface in a unit sphere $S^{n+1}(1)$ of dimension $n+1$. Let $S$ denote the squared norm of the second fundamental form of $M$. It is well-known that Chern, do Carmo and Kobayashi [2] and Lawson [3] obtained independently that Clifford tori are the only closed minimal hypersurfaces of the unit sphere with $S = n$. When the scalar curvature of $M$ is constant, there are very nice results on the rigidity of the Clifford torus (see [5] and [6]). On the other hand, Otsuki[4] studied the converse problem for minimal hypersurfaces in $S^{n+1}(1)$. He proved that if $M$ is a closed minimal hypersurface in $S^{n+1}(1)$ with two distinct principal curvatures and the multiplicities of them are at least two, then $M$ is $S^m(\sqrt{m/n}) \times S^{n-m}(\sqrt{(n-m)/n})$ $(1 < m < n-1)$. But for the case in which one of the two principal curvatures is simple, he constructed infinitely many minimal hypersurfaces other than $S^1(\sqrt{1/n}) \times S^{n-1}(\sqrt{(n-1)/n})$ which are not congruent to each other in $S^{n+1}(1)$. When professor K. Shiohama visited China in 1993, he proposed the following interesting problem:

PROBLEM. *Let $M$ be a closed minimal hypersurface in $S^{n+1}(1)$ with two distinct principal curvatures $\lambda_1$ and $\lambda_2$ and one of them be simple (we assume $\lambda_1$). Is there a constant $\epsilon = \epsilon(n)$ such that if $|\lambda_1 - \lambda_{10}| < \epsilon$ and $|\lambda_2 - \lambda_{20}| < \epsilon/(n-1)$ then $M$ is $S^1(\sqrt{1/n}) \times S^{n-1}(\sqrt{(n-1)/n})$, where $\lambda_{i0}$ are the corresponding principal curvatures of $S^1(\sqrt{1/n}) \times S^{n-1}(\sqrt{(n-1)/n})$.*

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This problem is equivalent to whether there is constant $\delta = \delta(n) > 0$ such that if $n - \delta \le S \le n + \delta$, then $M$ is $S^1(\sqrt{1/n}) \times S^{n-1}(\sqrt{(n-1)/n})$.

In this paper, we consider the problem and give a partial answer.

THEOREM. *Let $M$ be an $n$-dimensional closed minimal hypersurface of a unit sphere $S^{n+1}(1)$ with two distinct principal curvatures and one of them be simple. If*

$$n \le S \le n + \frac{2n^2(n+4)}{3[n(n+4)+4]}$$

*then $S = n$ and $M$ is $S^1(\sqrt{1/n}) \times S^{n-1}(\sqrt{(n-1)/n})$.*

COROLLARY. *Let $M$ be an $n$-dimensional closed minimal hypersurface of a unit sphere $S^{n+1}(1)$ with two distinct principal curvatures. If*

$$n \le S \le n + \frac{2n^2(n+4)}{3[n(n+4)+4]}$$

*then $S = n$ and $M$ is a Clifford torus.*

*Proof of Corollary.* This is obvious from the result due to Otsuki and Theorem.

## 2. Local formulae

Let $M$ be an $n$-dimensional closed minimal hypersurface in a unit sphere $S^{n+1}(1)$. We choose a local orthonormal frame field $\{e_1, \dots, e_{n+1}\}$ in $S^{n+1}(1)$, restricted to $M$, so that $e_1, \dots, e_n$ are tangent to $M$. Let $\omega_1, \dots, \omega_n$ denote the dual coframe field on $M$. The connection form $\omega_{ij}$ are characterized by the structure equations

$$d\omega_i + \sum_j \omega_{ij} \wedge \omega_j = 0, \qquad \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, \tag{2.1}$$

$$\Omega_{ij} = \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l,$$

where $\Omega_{ij}$ (resp. $R_{ijkl}$) denotes the curvature form (resp. the components of the curvature tensor) of $M$. The second fundamental form $\alpha$ of $M$ is given by

$$\alpha = \sum_{i,j} h_{ij}\omega_i\omega_j e_{n+1} \quad \text{and} \quad \sum_i h_{ii} = 0. \tag{2.2}$$

Since $\alpha$ is a symmetric tensor, $h_{ij} = h_{ji}$. The Gauss equation, Codazzi equation and Ricci formulas for the second fundamental form and its covariant derivatives are given by

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{2.3}$$

$$h_{ijk} = h_{ikj} = h_{jik}, \tag{2.4}$$

$$h_{ijkl} - h_{ijkl} = \sum_m h_{im}R_{mjkl} + \sum_m h_{mj}R_{mikl}, \tag{2.5}$$

$$h_{ijklm} - h_{ijkml} = \sum_r h_{rjk}R_{rilm} + \sum_r h_{irk}R_{rjlm} + \sum_r h_{ijr}R_{rklm}, \tag{2.6}$$

where $h_{ijk}$, $h_{ijkl}$ and $h_{ijklm}$ are the coefficients of the first, the second and the third covariant derivatives of the second fundamental form of $M$, respectively. The components of the Ricci curvature and the scalar curvature are given by

$$R_{ij} = (n-1)\delta_{ij} - \sum_k h_{ik}h_{jk}, \tag{2.7}$$

$$R = n(n-1) - \sum_{i,j} h_{ij}^2. \tag{2.8}$$

Now we compute some local formulae. For any fixed point $p$ in $M$, we can choose a local frame field $e_1, \ldots, e_n$ such that

$$h_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ \lambda_i & \text{if } i = j. \end{cases} \tag{2.9}$$

The following formulas can be found in [1]. Let

$$S := \sum_{i,j} h_{ij}^2 = \sum_i \lambda_i^2.$$

$$\frac{1}{2}\Delta S = \sum_{i,j,k} h_{ijk}^2 - S(S-n). \tag{2.10}$$

$$\frac{1}{2}\Delta \sum_{i,j,k} h_{ijk}^2 = \sum_{i,j,k,l} h_{ijkl}^2 + (2n+3-S)\sum_{i,j,k} h_{ijk}^2 + 3(2B-A) - \frac{3}{2}|\nabla S|^2, \tag{2.11}$$

where $A = \Sigma_{i,j,k} \lambda_i^2 h_{ijk}^2$ and $B = \Sigma_{i,j,k} \lambda_i \lambda_j h_{ijk}^2$.

$$\frac{1}{3}\Delta f_3 = (n-S)f_3 + 2\sum_{i,j,k} \lambda_i h_{ijk}^2, \tag{2.12}$$

where $f_3 = \Sigma_i \lambda_i^3$.

## 3. Proofs of theorems

At first we give an algebra Lemma which will play a crucial role in the proof of our theorems.

LEMMA. *Let $a_{ij}$ and $b_i$ $(i, j = 1, \ldots, n)$ be real numbers satisfying $\Sigma_i b_i = 0$ and $\Sigma_i b_i^2 = b > 0$, $\Sigma_{i,j} b_i a_{ij} = b(n-b)$ and $\Sigma_{i,j} b_j a_{ij} = 0$. Then*

$$\sum_i a_{ii}^2 + 3\sum_{i\neq j} a_{ij}^2 + 3\sum_{i,j}(b_j^2 b_i - b_i^2 b_j)a_{ij} - 3b(n-b)$$
$$\geq \frac{3b(n-b)^2}{2(n+4)} - \frac{3}{2}\left[b\sum_i b_i^4 - \left(\sum_i b_i^3\right)^2 - 2b^2 + nb\right].$$

*Proof.* We consider $F = \Sigma_i a_{ii}^2 + 3\Sigma_{i\neq j} a_{ij}^2 + 3\Sigma_{i,j}(b_j^2 b_i - b_i^2 b_j)a_{ij} - 3b(n-b)$ as a function of $a_{ij}$. Solve the following problem for the conditional extremum:

$$f = \sum_i a_{ii}^2 + 3\sum_{i\neq j} a_{ij}^2 + 3\sum_{i,j}(b_j^2 b_i - b_i^2 b_j)a_{ij}$$
$$-3b(n-b) + \lambda\left(\sum_{i,j} b_i a_{ij} - b(n-b)\right) + \mu\sum_{i,j} b_j a_{ij}, \tag{3.1}$$

where $\lambda$ and $\mu$ are the Lagrange multipliers. It is obvious that the critical point of $f$ is the minimum point of $f$. Taking derivatives of $f$ with respect to $a_{ij}$, we get

$$f_{a_{ij}} = 6a_{ij} + 3(b_j^2 b_i - b_i^2 b_j) + \lambda b_i + \mu b_j = 0, \qquad \text{for } i \neq j, \tag{3.2}$$

$$f_{a_{ii}} = 2a_{ii} + \lambda b_i + \mu b_i = 0, \qquad \text{for } i = j. \tag{3.3}$$

Hence

$$\sum_i a_{ii} f_{a_{ii}} = 2\sum_i a_{ii}^2 + \lambda \sum_i b_i a_{ii} + \mu \sum_i b_i a_{ii} = 0, \tag{3.4}$$

$$\sum_{i\neq j} a_{ij} f_{a_{ij}} = 6\sum_{i\neq j} a_{ij}^2 + 3\sum_{i,j} (b_j^2 b_i - b_i^2 b_j) a_{ij} + \lambda \sum_{i\neq j} b_i a_{ij} + \mu \sum_{i\neq j} b_j a_{ij} = 0, \tag{3.5}$$

$$\sum_i b_i f_{a_{ii}} = 2\sum_i a_{ii} b_i + \lambda \sum_i b_i^2 + \mu \sum_i b_i^2 = 0, \tag{3.6}$$

$$\sum_{i\neq j} b_i f_{a_{ij}} = 6\sum_{i\neq j} b_i a_{ij} + 3b^2 + \lambda \sum_{i\neq j} b_i^2 + \mu \sum_{i\neq j} b_i b_j = 0, \tag{3.7}$$

$$\sum_{i\neq j} b_j f_{a_{ij}} = 6\sum_{i\neq j} b_j a_{ij} - 3b^2 + \lambda \sum_{i\neq j} b_i b_j + \mu \sum_{i\neq j} b_j^2 = 0. \tag{3.8}$$

(3.4) + (3.5) implies

$$2\left(\sum_i a_{ii}^2 + 3\sum_{i\neq j} a_{ij}^2 + 3\sum_{i,j} (b_j^2 b_i - b_i^2 b_j) a_{ij} - 3b(n-b)\right)$$
$$- 3\sum_{i,j} (b_j^2 b_i - b_i^2 b_j) a_{ij} + \lambda \sum_{i,j} b_i a_{ij} + \mu \sum_{i,j} b_j a_{ij} = -6b(n-b).$$

Thus

$$2f_{\min} = 2\left(\sum_i a_{ii}^2 + 3\sum_{i\neq j} a_{ij}^2 + 3\sum_{i,j} (b_j^2 b_i - b_i^2 b_j) a_{ij} - 3b(n-b)\right)$$
$$= 3\sum_{i,j} (b_j^2 b_i - b_i^2 b_j) a_{ij} - 6b(n-b) - \lambda b(n-b). \tag{3.9}$$

According to (3.6), we get

$$2\sum_i a_{iii}b_i + (\lambda + \mu)b = 0. \tag{3.10}$$

(3.6) + (3.7) + (3.8) yield

$$-4\sum_i a_{iii}b_i + 3b^2 + 6b(n - b) + nb\lambda = 0, \tag{3.11}$$

$$-4\sum_i a_{iii}b_i - 3b^2 + nb\mu = 0. \tag{3.12}$$

Solving the system of the linear equations (3.10), (3.11) and (3.12) with unknown $\lambda$, $\mu$ and $\Sigma_i a_{iii}b_i$, we obtain

$$\lambda - \mu = -6$$

and

$$\lambda + \mu = -\frac{6(n-b)}{(n+4)}.$$

$$-\lambda = 3 + \frac{3(n-b)}{(n+4)}. \tag{3.13}$$

From (3.2) we have

$$-6a_{ij} = 3(b_j^2 b_i - b_i^2 b_j) + \lambda b_i + \mu b_j.$$

Hence

$$\begin{aligned}
&\sum_{i,j} 3(b_j^2 b_i - b_i^2 b_j)a_{ij} \\
&= -\frac{3}{2}\sum_{i,j}(b_j^2 b_i - b_i^2 b_j)^2 - \frac{1}{2}\sum_{i,j}(b_j^2 b_i - b_i^2 b_j)(\lambda b_i + \mu b_j) \\
&= -3\left[b\sum_i b_i^4 - \left(\sum_i b_i^3\right)^2\right] - \frac{1}{2}b^2(\lambda - \mu) \\
&= -3\left[b\sum_i b_i^4 - \left(\sum_i b_i^3\right)^2\right] + 3b^2.
\end{aligned}$$

From (3.9) and (3.13) and the above equality, we conclude

$$f_{\min} = \frac{3b(n-b)^2}{2(n+4)} - \frac{3}{2}\left[b\sum_i b_i^4 - \left(\sum_i b_i^3\right)^2 - 2b^2 + nb\right]. \tag{3.14}$$

Thus we complete the proof of Lemma.

*Proof of theorem.*

$$\begin{aligned}
\frac{1}{3}\sum_{i,j} h_{ij}(f_3)_{ij} &= \frac{1}{3}\sum_i \lambda_i (f_3)_{ii} \\
&= \sum_i \lambda_i\left(\sum_j \lambda_j^2 h_{jjii} + 2\sum_{j,k} \lambda_k h_{jki}^2\right) \\
&= \sum_{i,j} \lambda_i \lambda_j^2 h_{jjii} + 2\sum_{i,j,k} \lambda_i \lambda_j h_{ijk}^2 \\
&= \sum_{i,j} \lambda_i \lambda_j^2 (h_{iijj} + (\lambda_j - \lambda_i)(1 + \lambda_i\lambda_j)) + 2\sum_{i,j,k} \lambda_i\lambda_j h_{ijk}^2 \\
&= \sum_i \frac{\lambda_i^2 S_{ii}}{2} + \sum_{i,j} \lambda_i \lambda_j^2 (\lambda_j - \lambda_i)(1 + \lambda_i\lambda_j) + 2B - A \\
&= \frac{1}{2}\sum_{i,j,k} h_{ik}h_{kj}S_{ij} + \left[S\sum_i \lambda_i^4 - S^2 - \left(\sum_i \lambda_i^3\right)^2\right] + 2B - A.
\end{aligned}$$

Hence

$$\begin{aligned}
0 &= \frac{1}{3}\int_M \sum_{i,j} h_{jji}(f_3)_i \, dM = \frac{1}{3}\int_M \sum_{i,j} h_{ij}(f_3)_{ij} \, dM \\
&= \int_M \left[\frac{1}{3}\sum_{i,j,k} h_{ik}h_{kj}S_{ij} + (2B - A) + Sf_4 - f_3^2 - S^2\right] dM,
\end{aligned}$$

where $f_4$ is defined by $f_4 = \Sigma_i \lambda_i^4$, i.e.,

$$\int_M (A - 2B)\, dM = \int_M \left[Sf_4 - S^2 - f_3^2 - \frac{1}{4}|\nabla S|^2\right] dM. \tag{3.15}$$

$$\begin{aligned}\sum_{i,j,k,l} h_{ijkl}^2 &\geq \sum_i h_{iiii}^2 + 3\sum_{i\neq j} h_{ijij}^2 \\ &= \sum_i h_{iiii}^2 + \frac{3}{4}\sum_{i\neq j}(h_{ijij}+h_{jiji})^2 + \frac{3}{4}\sum_{i\neq j}(h_{ijij}-h_{jiji})^2 \\ &= \sum_i h_{iiii}^2 + 3\sum_{i\neq j} h_{iijj}h_{jjii} + \frac{3}{2}\sum_{i\neq j}(h_{ijij}-h_{jiji})^2 \\ &= \sum_i h_{iiii}^2 + 3\sum_{i\neq j} h_{iijj}h_{jjii} + 3\left[nS - 2S^2 + S\sum_i \lambda_i^4 - \left(\sum_i \lambda_i^3\right)^2\right]. \end{aligned} \tag{3.16}$$

Since

$$\sum_p h_{ijpp} = -(S-n)h_{ij},$$

$\Sigma_i \lambda_i = 0$ and $\Sigma_i \lambda_i^2 = S > 0$, we have

$$\sum_{i,j} h_{iijj}\lambda_i = S(n-S) \quad \text{and} \quad \sum_{i,j} h_{iijj}\lambda_j = 0. \tag{3.17}$$

From Ricci formulas, we have

$$h_{iijj} - h_{jjii} = (\lambda_i - \lambda_j)(1 + \lambda_i\lambda_j).$$

Note that in view of (3.17) $h_{iijj}$ and $\lambda_i$ satisfy the conditions of Lemma. Hence we have

$$\begin{aligned}&\sum_i h_{iiii}^2 + 3\sum_{i\neq j} h_{iijj}h_{jjii} \\ &\quad = \sum_i h_{iiii}^2 + 3\sum_{i\neq j} h_{iijj}^2 + 3\sum_{i,j}(\lambda_j^2\lambda_i - \lambda_i^2\lambda_j)h_{jiij} - 3S(n-S) \\ &\quad \geq \frac{3S(S-n)^2}{2(n+4)} - \frac{3}{2}(Sf_4 - f_3^2 - 2S^2 + nS). \end{aligned} \tag{3.18}$$

Since $M$ has only two distinct principal curvatures $\lambda_1$ and $\lambda_2$ and $\lambda_1$ is simple, we have $\lambda_1 = -(n-1)\lambda_2$ and

$$\frac{3}{2}(Sf_4 - f_3^2 - 2S^2 + nS) = \frac{3}{2n}S(S-n)^2.$$

Thus we get, from (3.16) and the above inequality,

$$\sum_{i,j,k,l} h_{ijkl}^2 \geq 3(Sf_4 - f_3^2 - 2S^2 + nS) - \frac{6S(n-S)^2}{n(n+4)}. \tag{3.19}$$

According to (2.10) and (2.11), we obtain

$$\int_M \sum_{i,j,k} h_{ijk}^2 \, dM = \int_M [S(S-n)] \, dM, \tag{3.20}$$

$$\int_M \sum_{i,j,k,l} h_{ijkl}^2 \, dM = \int_M \left[ -(2n+3-S) \sum_{i,j,k} h_{ijk}^2 - 3(2B - A) + \frac{3}{2} |\nabla S|^2 \right] dM. \tag{3.21}$$

From (3.15), (3.19), (3.20) and (3.21), we infer

$$\int_M \left\{ (2n - S) \sum_{i,j,k} h_{ijk}^2 - \frac{3}{4} |\nabla S|^2 - \frac{6S(n-S)^2}{n(n+4)} \right\} dM \leq 0. \tag{3.22}$$

From (2.10), we get

$$-\int_M \frac{1}{2} |\nabla S|^2 = \int_M \left[ S \sum_{i,j,k} h_{ijk}^2 + (n - S)S^2 \right] dM. \tag{3.23}$$

(3.22) and (3.23) yield

$$\int_M \left\{ \left(2n + \frac{S}{2}\right) \sum_{i,j,k} h_{ijk}^2 - \frac{3}{2} S^2(S-n) - \frac{6}{n(n+4)} S(S-n)^2 \right\} dM \leq 0.$$

Using again (3.20) and the inequality

$$n \leq S \leq n + \frac{2n^2(n+4)}{3[(n(n+4)+4]},$$

we have

$$0 \geq \int_M \left\{\left(2n + \frac{n}{2}\right)\sum_{i,j,k} h_{ijk}^2 + \frac{1}{2}(S-n)\sum_{i,j,k} h_{ijk}^2 - \frac{3}{2}S^2(S-n) - \frac{6}{n(n+4)}S(S-n)^2\right\} dM$$

$$= \int_M \left\{\frac{1}{2}(S-n)\sum_{i,j,k} h_{ijk}^2 + \left(\frac{5}{2}n - \frac{3}{2}S - \frac{6(S-n)}{n(n+4)}\right)S(S-n)\right\} dM \geq 0.$$

Hence

$$\int_M \frac{1}{2}(S-n)\sum_{i,j,k} h_{ijk}^2 \, dM = 0.$$

Since $S$ and $\Sigma_{i,j,k} h_{ijk}^2$ are continuous functions, we have $S = n$. Thus from the assumption of Theorem, $M$ is $S^1(\sqrt{1/n}) \times S^{n-1}(\sqrt{(n-1)/n})$ according to a result due to Chern, do Carmo and Kobayashi [2] or Lawson [3]. We complete the proof of Theorem.


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