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Cohen–Macaulay coordinate rings of blowup schemes

S. Dale Cutkosky and Jürgen Herzog*

Abstract. Suppose that Y is a projective k -scheme with Cohen–Macaulay coordinate ring S . Let $I \subset S$ be a homogeneous ideal of S . I can be blown up to produce a projective k -scheme X which birationally dominates Y . Let I_c be the degree c part of I . Then $k[I_c]$ is a coordinate ring of a projective embedding of X for all c sufficiently large. This paper considers the question of when there exists a constant f such that $k[(I^e)_c]$ is Cohen–Macaulay for $c \geq ef$. A very general result is proved, giving a simple criterion for a linear bound of this type. As a consequence, local complete intersections have this property, as well as many other ideals.

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Introduction

Suppose that Y is a projective k -scheme with Cohen–Macaulay coordinate ring S . Let $I \subset S$ be a homogeneous ideal of S . Then I can be blown up to produce a projective k -scheme X which birationally dominates Y . Let I_c be the degree c part of I . Then $k[I_c]$ is a coordinate ring of a projective embedding of X for all c sufficiently large. In general, $k[I_c]$ is not Cohen–Macaulay even when X is Cohen–Macaulay (a simple example is given in Section 1). Recently, [3], [5], [6], [13] have given criteria for $k[I_c]$ to be Cohen–Macaulay in many important situations.

Powers I^e of I blow up to the same scheme X , and the rings $k[(I^e)_c]$ for $c \gg e > 0$ are coordinate rings of projective embeddings of X .

In Theorem 4.6 [3] an explicit necessary and sufficient linear bound in c and e is given for $k[(I^e)_c]$ to be Cohen–Macaulay, when S is a polynomial ring of dimension n and I is a complete intersection in S . Suppose that the complete intersection ideal I is minimally generated by forms of degree d_1, \dots, d_r . Assume that $c \geq ed + 1$, $d = \max\{d_j | j = 1, \dots, r\}$. Then $k[(I^e)_c]$ is a Cohen–Macaulay ring if and only if $c > \sum_{j=1}^r d_j + (e-1)d - n$.

This leads to the question of when there exists a constant f such that $k[(I^e)_c]$

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is Cohen–Macaulay for $c \geq ef$. In other words, when is there a linear bound on c and e ensuring that $k[(I^e)_c]$ is Cohen–Macaulay?

For instance, it is natural to expect that ideals I that are local complete intersections (that is, IS_p is a complete intersection if p is not the irrelevant ideal of S) will have this property. The Kodaira Vanishing Theorem suggests that there should be a linear bound ensuring that $k[(I^e)_c]$ is Cohen–Macaulay, at least when a regular ideal is blown up in a nonsingular projective variety of characteristic zero.

In this paper, we prove a very general result (Theorem 4.1) giving a simple criterion for a linear bound of this type. As a consequence, we show (Corollary 4.2) that local complete intersections have this property, as well as many other ideals (Corollaries 4.3 and 4.4).

1. Coordinate rings of a blowup

Throughout this paper we will have the following assumptions. Let k be a field, S a noetherian graded k -algebra which is generated in degree 1 with graded maximal ideal M . Then S has a presentation $S = k[x_0, x_1, \dots, x_n]/K$, where each x_i is homogeneous of degree 1. Let $\beta = \dim(S)$, $\bar{n} = \beta - 1$, $I \subset S$ be a homogeneous ideal, and let \tilde{I} be the sheaf associated to I in $Y = \text{Proj}(S)$. Let $X = \text{Proj}(\bigoplus \tilde{I}^n)$ be the blowup of \tilde{I} , with natural map $\pi: X \rightarrow Y$. I_c will denote the c -graded part of I .

Lemma 1.1. *Suppose that I is generated in degree $\leq d$. Then*

- (1) $(I_c) \cdot \mathcal{O}_X = \tilde{I}(c) \cdot \mathcal{O}_X$ for $c \geq d$.
- (2) I_c is a very ample subspace of $\Gamma(X, \tilde{I}(c) \cdot \mathcal{O}_X)$ for $c \geq d + 1$.

Proof. Let I be generated by G_1, \dots, G_m where the G_i are homogeneous of degree $d_i \leq d$. Let

$$R_{ij} = \left(k \left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i} \right] / K_i \right) \left[\frac{G_1 x_i^{d_j - d_1}}{G_j}, \dots, \frac{G_m x_i^{d_j - d_m}}{G_j} \right],$$

$U_{ij} = \text{Spec}(R_{ij})$. $\{U_{ij} : 0 \leq i \leq n, 1 \leq j \leq m\}$ is an affine cover of X . $\Gamma(U_{ij}, \tilde{I}(c) \cdot \mathcal{O}_X) = G_j x_i^{c - d_j} R_{ij}$.

Since $I_c \subset \Gamma(X, \tilde{I}(c) \cdot \mathcal{O}_X)$ and $G_j x_i^{c - d_j} \in I_c$ for $1 \leq j \leq m$ whenever $c \geq d$, we have (1).

To establish (2) we will use the criteria of Proposition II.7.2 [7]. Suppose that $c \geq d + 1$. By (1), I_c gives a morphism of X . I_c is generated over k by $\{G_j x_0^{l_0} x_1^{l_1} \dots x_n^{l_n} : d_j + l_0 + l_1 + \dots + l_n = c\}$. Suppose that $s = G_j x_0^{l_0} x_1^{l_1} \dots x_n^{l_n}$ is one of these generators. Then some $l_i > 0$, and $X_s \subset U_{ij}$.

$$s = G_j x_0^{l_0} x_1^{l_1} \dots x_n^{l_n} = G_j x_i^{c - d_j} \left(\frac{x_0}{x_i} \right)^{l_0} \dots \left(\frac{x_n}{x_i} \right)^{l_n},$$

so that $X_s = \text{Spec}(A)$ where

$$A = R_{ij} \left[\left(\frac{x_i}{x_0} \right)^{l_0} \cdots \left(\frac{x_i}{x_n} \right)^{l_n} \right].$$

We have

$$\begin{aligned} \frac{G_j x_i^{c-d_j}}{G_j x_0^{l_0} \cdots x_n^{l_n}} &= \left(\frac{x_i}{x_0} \right)^{l_0} \cdots \left(\frac{x_i}{x_n} \right)^{l_n} \\ \frac{G_j x_0^{l_0} \cdots x_t^{l_t+1} \cdots x_i^{l_i-1} \cdots x_n^{l_n}}{G_j x_0^{l_0} \cdots x_n^{l_n}} &= \frac{x_t}{x_i} \\ \frac{G_t x_i^{c-d_t}}{G_j x_0^{l_0} \cdots x_n^{l_n}} \cdot \left(\frac{G_j x_0^{l_0+1} \cdots x_i^{l_i-1} \cdots x_n^{l_n}}{G_j x_0^{l_0} \cdots x_n^{l_n}} \right)^{l_0} \cdots \left(\frac{G_j x_0^{l_0} \cdots x_i^{l_i-1} \cdots x_n^{l_n+1}}{G_j x_0^{l_0} \cdots x_n^{l_n}} \right)^{l_n} \\ &= \frac{G_t x_i^{d_j-d_t}}{G_j} \end{aligned}$$

generate A as a k -algebra. □

Let $\mathcal{L} = \tilde{I} \cdot \mathcal{O}_X$, $\mathcal{M} = \pi^* \mathcal{O}_Y(1)$, so that $(I^e)_c \cdot \mathcal{O}_X = \mathcal{L}^e \otimes \mathcal{M}^e$ for $e > 0$ and $c \geq de$, and $X \cong \text{Proj}(k[(I^e)_c])$ for $c \geq de + 1$. In Lemma 1.2 we state the usual exact sequences relating local cohomology and global cohomology (cf. A4.1 [4]).

Lemma 1.2. *Suppose that I is generated in degree $\leq d$, $e > 0$ and $c \geq de + 1$. Let $A = k[(I^e)_c]$, with graded maximal ideal m . There are exact sequences*

$$0 \rightarrow H_m^0(A) \rightarrow A \rightarrow \bigoplus_{s \in \mathbf{Z}} \Gamma(X, \mathcal{L}^{se} \otimes \mathcal{M}^{sc}) \rightarrow H_m^1(A) \rightarrow 0$$

and isomorphisms

$$H_m^{i+1}(A) \cong \bigoplus_{s \in \mathbf{Z}} H^i(X, \mathcal{L}^{se} \otimes \mathcal{M}^{sc})$$

for $i \geq 1$.

Let I^* denote the intersection of the primary components which are not M -primary of an irredundant primary decomposition of I .

Lemma 1.3. *There exists a positive integer f such that $(I^a)_b = (I^a)_b^*$ for all a, b with $b \geq fa$.*

Proof. This is immediate from the Main Theorem of Swanson [14] (cf. also Theorem 1.5 [10]), which states that there exists an integer f such that I^a has an irredundant primary decomposition $I^a = q_1 \cap \cdots \cap q_s$ with $(\sqrt{q_i})^{af} \subset q_i$ for all i . □

Lemma 1.4. *Suppose that no associated prime of S contains I , $\text{depth}_M(S) \geq 2$ and $\pi_*(\tilde{I}^e \cdot \mathcal{O}_X) = \tilde{I}^e$ for all $e \geq 0$. Then there exists a positive integer f such that, with the notation of Lemma 1.2, $H_m^0(A) = 0$ and $H_m^1(A) = 0$ whenever $e > 0$ and $c \geq ef$.*

Proof. Suppose I is generated in degree $\leq d$. By Lemma 1.3 there exists an integer f' such that $(I^t)_s = (I^t)_s^*$ for $s \geq f't$. Set $f = \max(f', d + 1)$.

By consideration of the natural inclusion $A \subset S[It]$, we have $H_M^0(A) = 0$ since $H_I^0(S) = 0$. $H^0(Y, \mathcal{O}_Y(s)) = H_M^1(S)_s = 0$ for all $s < 0$ since $\text{depth}_M(S) \geq 2$. From the inclusions

$$\Gamma(X, \mathcal{L}^{se} \otimes \mathcal{M}^{sc}) \hookrightarrow \Gamma(X, \mathcal{M}^{sc}) = \Gamma(Y, (\pi_* \mathcal{O}_X) \otimes \mathcal{O}_Y(sc)) = \Gamma(Y, \mathcal{O}_Y(sc))$$

where the first equality is by the projection formula (cf. Exercise II.5.1 [7]), we get $\Gamma(X, \mathcal{L}^{se} \otimes \mathcal{M}^{sc}) = 0$ for $s < 0$.

Let Δ be the (c, e) diagonal of \mathbf{Z}^2 ([3]). For $c \geq ed$, we have $k[(I^e)_c] = S[It]_\Delta$, as in Lemma 1.2 of [3].

$$\begin{aligned} \bigoplus_{s \geq 0} H^0(X, \mathcal{L}^{se} \otimes \mathcal{M}^{sc}) &= \bigoplus_{s \geq 0} H^0(Y, \pi_*(\mathcal{L}^{se}) \otimes \mathcal{O}_Y(sc)) \\ &= \bigoplus_{s \in \mathbf{Z}} H^0(Y, \tilde{I}^{se}(sc)) \\ &= (\Gamma(\text{Spec}(S) - M, \tilde{I}))_\Delta \\ &= \left(\bigoplus_{t \geq 0} (I^t)^* \right)_\Delta = S[It]_\Delta. \end{aligned}$$

Now Lemma 1.2 implies $H_m^1(A) = 0$. □

The condition of the existence of a Cohen–Macaulay coordinate ring is somewhat delicate, as shown by the following simple example of a Cohen–Macaulay scheme obtained by blowing up an ideal sheaf on a scheme with a Cohen–Macaulay coordinate ring, which does not have a Cohen–Macaulay coordinate ring. Let T be a nonsingular “irregular” projective surface ($H^1(T, \mathcal{O}_T) \neq 0$). Let $\pi: T \rightarrow U$ be a birational projection onto a hypersurface in \mathbf{P}^3 . π is the blowup of an ideal sheaf on U . The coordinate ring of U is Cohen–Macaulay, and certainly T is Cohen–Macaulay, but no coordinate ring of T can be Cohen–Macaulay since T is irregular.

However, we can give a simple proof of the existence of a linear bound ensuring that $k[(I^e)_c]$ is Cohen–Macaulay when k has characteristic zero, S is Cohen–Macaulay, Y is nonsingular, I is equidimensional and $\text{Proj}(S/I)$ is nonsingular. The proof has three ingredients:

- (1) The Kodaira Vanishing Theorem.

- (2) In this situation (everything nonsingular) $R^i \pi_* \mathcal{O}_X = 0$ for $i > 0$ and $\pi_* \mathcal{O}_X = \mathcal{O}_Y$ (cf. Proposition 10.2 [11]).
- (3) Lemma 1.4.

If \mathcal{N} is an ample invertible sheaf on a smooth projective variety Z of characteristic zero and dimension t , with dualizing sheaf ω_Z , then Kodaira Vanishing states that $H^i(Z, \mathcal{N} \otimes \omega_Z) = 0$ for $i > 0$. The Serre-dual form of Kodaira Vanishing is $H^i(Z, \mathcal{N}^{-1}) = 0$ for $i < t$.

Proposition 1.5. *Suppose that k has characteristic zero, S is Cohen–Macaulay, Y is nonsingular, I is equidimensional and $\text{Proj}(S/I)$ is nonsingular. Then there exists a positive integer f such that*

$$H^i(X, \mathcal{L}^{se} \otimes \mathcal{M}^{sc}) = 0$$

for all $s \in \mathbf{Z}$ if $0 < i < \bar{n}$, $c \geq ef$, $e > 0$.

Proof. Suppose that I has height r , and is generated in degree $< d$. By Lemma 1.1, $\mathcal{L}^a \otimes \mathcal{M}^b$ is very ample if $b > ad$. We immediately get $H^i(X, \mathcal{L}^{se} \otimes \mathcal{M}^{sc}) = 0$ for $c > ed$, $s < 0$ and $i < \bar{n}$, since $\mathcal{L}^e \otimes \mathcal{M}^e$ is then ample.

$H^i(X, \mathcal{O}_X) = 0$ for $0 < i$, by the Leray spectral sequence, since $R^i \pi_* \mathcal{O}_X = 0$ for $i > 0$, $\pi_* \mathcal{O}_X = \mathcal{O}_Y$ and $H^j(Y, \mathcal{O}_Y) = 0$ for $0 < j$ (since S is Cohen–Macaulay).

Let ω_Y be a dualizing sheaf on Y . $\omega_Y^{-1}(g)$ is ample on Y for some $g > 0$. $\omega_X = \mathcal{L}^{1-r} \otimes \omega_Y$ is a dualizing sheaf on X .

$$\mathcal{L}^e \otimes \mathcal{M}^c \cong \mathcal{L}^{e+r-1} \otimes \mathcal{M}^c \otimes \omega_Y^{-1} \otimes \omega_X.$$

$\mathcal{L}^{e+r-1} \otimes \mathcal{M}^c \otimes \omega_Y^{-1}$ is ample if $c > g + (e+r-1)d$, and then $H^i(X, \mathcal{L}^{se} \otimes \mathcal{M}^{sc}) = 0$ for $s > 0$ and $i > 0$. □

Theorem 1.6. *Suppose that k has characteristic zero, S is Cohen–Macaulay, Y is nonsingular, I is equidimensional and $\text{Proj}(S/I)$ is nonsingular. Then there exists a positive integer f such that $k[(I^e)_c]$ is Cohen–Macaulay whenever $e > 0$ and $c \geq ef$.*

Proof. The assumptions of Lemma 1.4 are satisfied by Proposition 10.2 [11] (or Example 2.3) and Proposition III.8.5 [7]. By Lemmas 1.2, 1.4 and Proposition 1.5 there exists a positive integer f such that $k[(I^e)_c]$ has depth $\bar{n} + 1$ at m whenever $e > 0$ and $c \geq ef$. □

Unfortunately, Kodaira Vanishing fails in positive characteristic or if anything is not (almost) nonsingular. However, we obtain a very general result which is sufficient for this global part of the argument in Section 3. The second ingredient is local. We give a very simple argument to generalize this part in Section 2.

2. Local conditions

Lemma 2.1. *Suppose that R is a local ring, essentially of finite type over a field k and $J \subset R$ is an ideal. Let $W = \text{Spec}(R)$, $V = \text{Proj}(\bigoplus_{n \geq 0} J^n)$, $E = \text{Proj}(\bigoplus_{n \geq 0} J^n/J^{n+1})$, $\mathcal{L} = \tilde{J} \cdot \mathcal{O}_V$. Suppose that*

$$\Gamma(E, \mathcal{O}_E(m)) = J^m/J^{m+1} \quad \text{for } m \geq 0$$

and

$$H^i(E, \mathcal{O}_E(m)) = 0 \quad \text{for } i > 0 \text{ and } m \geq 0.$$

Then $\Gamma(V, \mathcal{L}^m) = J^m$ if $m \geq 0$ and $H^q(V, \mathcal{L}^m) = 0$ for $q > 0$ and $m \geq 0$.

Proof. Note that $\mathcal{L} = \mathcal{O}_V(1)$ on V .

We have exact sequences:

$$0 \rightarrow \mathcal{O}_V(m+1) \rightarrow \mathcal{O}_V(m) \rightarrow \mathcal{O}_E(m) \rightarrow 0$$

for all integers m . Thus we have surjections

$$H^i(V, \mathcal{O}_V(m+1)) \rightarrow H^i(V, \mathcal{O}_V(m))$$

for $i > 0$ and $m \geq 0$. Since $\mathcal{O}_V(m)$ is ample, we have $H^i(V, \mathcal{O}_V(m)) = 0$ for all $m \gg 0$, and $i > 0$, so we have all of the desired vanishing. We also have exact sequences:

$$0 \rightarrow \Gamma(V, \mathcal{O}_V(m+1)) \rightarrow \Gamma(V, \mathcal{O}_V(m)) \rightarrow J^m/J^{m+1} \rightarrow 0 \quad (1)$$

for $m \geq 0$. Since R is a localization of a finitely generated k -algebra, $\Gamma(V, \mathcal{O}_V(m)) = J^m$ for $m \gg 0$ (cf. Exercise II.5.9 of [7]). Thus it follows from (1) that $\Gamma(V, \mathcal{O}_V(m)) = J^m$ for all $m \geq 0$. \square

Lemma 2.2. *Let notation be as in Lemma 2.1. Suppose that V is Cohen–Macaulay. Let ω_V be a dualizing sheaf on V and ω_E be a dualizing sheaf on E . Suppose that $H^i(E, \omega_E(m)) = 0$ for $i > 0$ and $m \geq 2$. Then $H^q(V, \omega_V \otimes \mathcal{L}^m) = 0$ for $q > 0$ and $m \geq 1$.*

Proof. The ideal sheaf of E is $I \cdot \mathcal{O}_V \cong \mathcal{O}_V(1)$. By “adjunction” (cf. Proposition 2.4 [1] or Theorem III 7.11 [7]) we have

$$\omega_E \cong \omega_V \otimes \mathcal{O}_E(-1).$$

Since ω_V is Cohen–Macaulay, we deduce from the exact sequence

$$0 \rightarrow \mathcal{O}_V(1) \rightarrow \mathcal{O}_V \rightarrow \mathcal{O}_E \rightarrow 0$$

exact sequences:

$$0 \rightarrow \omega_V(m + 1) \rightarrow \omega_V(m) \rightarrow \omega_E(m + 1) \rightarrow 0$$

for all integers m .

Thus we have surjections

$$H^i(V, \omega_V(m + 1)) \rightarrow H^i(V, \omega_V(m))$$

for $i > 0$ and $m \geq 1$. Since $\mathcal{O}_V(m)$ is ample, we have $H^i(V, \omega_V(m)) = 0$ for all $m \gg 0$ and $i > 0$, so we have all of the desired vanishing. \square

Example 2.3. Suppose that R is a Cohen–Macaulay local ring, essentially of finite type over a field k and $J \subset R$ is an ideal generated by a regular sequence. Then the conclusions of Lemmas 2.1 and 2.2 hold.

Proof. Let f_1, \dots, f_r be a minimal set of generators of I . V is Cohen–Macaulay, $E = \text{Proj}(\bigoplus_{n \geq 0} J^n/J^{n+1}) \cong \mathbf{P}_{R/J}^{r-1}$, $\omega_E \cong (\omega_W/J\omega_W) \otimes_k \mathcal{O}_E(-r)$. Now the assumptions of Lemmas 2.1 and 2.2 follow from the cohomology of projective space and the isomorphisms

$$\begin{aligned} H^i(X, \mathcal{O}_E(m)) &\cong R/J \otimes_k H^i(\mathbf{P}_k^{r-1}, \mathcal{O}(m)), \\ H^i(E, (\omega_W/J\omega_W) \otimes_k \mathcal{O}_E(m)) &\cong (\omega_W/J\omega_W) \otimes_k H^i(\mathbf{P}_k^{r-1}, \mathcal{O}(m)) \end{aligned}$$

by the Künneth formula (cf. p. 77 of [12]). \square

Let

$$T = \bigoplus_{n \geq 0} J^n/J^{n+1}$$

be the associated graded ring of J , and let N be the ideal of positive degree elements of T . Suppose that T is Cohen–Macaulay. Then there is a canonical module W_T of T such that the sheaf associated to W_T is a dualizing sheaf ω_E on E . The vanishing hypotheses of Lemmas 2.1 and 2.2 hold whenever

$$H_N^i(T)_c = 0$$

for $i \geq 0$ and $c \geq 0$ and

$$H_N^i(W_T)_c = 0$$

for $i \geq 2$ and $c \geq 2$.

An ideal J in a ring R is called strongly Cohen–Macaulay if the Koszul homology modules of I with respect to a generating set are Cohen–Macaulay. Let $\mu(J)$ denote the minimal number of generators of an ideal J .

Example 2.4. Suppose that R is a Gorenstein local ring, essentially of finite type over a field k and $J \subset R$ is a strongly Cohen–Macaulay ideal, with $\mu(J_P) \leq$

height(P) for all primes P containing J . Then the conclusions of Lemma 2.1 and 2.2 hold.

Proof. Suppose J is of height g generated by n elements. Let $S = R[X_1, \dots, X_n]$ be a polynomial ring over R . Let $H(J)$ denote the Koszul homology $H(f_1, \dots, f_n, R)$ where f_1, \dots, f_n are generators of J . $W_{R/J} = \text{Ext}^g(R/J, R) \cong H_{n-g}(J)$ is the last non-vanishing $H_i(J)$. The approximation complex \mathcal{M} is

$$0 \rightarrow H_{n-g}(J) \otimes S(-n+g) \rightarrow \cdots \rightarrow H_1(J) \otimes S(-1) \rightarrow H_0(J) \otimes S \rightarrow 0.$$

In [8], it is shown that $H^0(\mathcal{M}) = \oplus J^n/J^{n+1}$. By Theorems 2.5 and 2.6 [8] \mathcal{M} is acyclic and $\oplus J^n/J^{n+1}, \oplus J^n$ are Cohen–Macaulay.

Let $\bar{S} = R/J[X_1, \dots, X_n]$, with canonical module

$$W_{\bar{S}} = W_{R/J} \otimes S(-n) = H_{n-g}(J) \otimes S(-n).$$

$H_N^i(T)$ is dual to $\text{Ext}_{\bar{S}}^{n-i}(T, W_{\bar{S}})$ (cf. Theorem 3.6.19 [2]). We have $\text{Ext}_{\bar{S}}^i(T, W_{\bar{S}}) = 0$ for $i \neq n-g$ since T is a Cohen–Macaulay module of dimension $\dim(R)$, and $\dim(\bar{S}) - \dim(T) = n-g$. By (c) of Theorem 2.6 [8] we can realize W_T as the cokernel of

$$H_1(J) \otimes S(-g-1) \rightarrow H_0(J) \otimes S(-g)$$

so that $W_T \cong T(-g)$.

From this we see that W_T is supported in degree g , so that $H_N^g(T)_j = 0$ for $j > -g$ and $H_N^g(W_T)_j = 0$ for $j > 0$.

By a Theorem of Huneke (Theorem 1.14 [9]), all ideals in the linkage class of a complete intersection in a Gorenstein local ring are strongly Cohen–Macaulay. For instance, codimension 2 perfect and Gorenstein codimension 3 ideals are strongly Cohen–Macaulay.

Remark 2.5. The conclusions of Example 2.4 are true when R is not Gorenstein but only Cohen–Macaulay.

In this case \mathcal{M}^* is acyclic with zeroth homology W_T , and we can use \mathcal{M} and \mathcal{M}^* to compute the desired vanishing.

3. Global conditions

We return to the notation and hypotheses of Section 1.

Proposition 3.1. *Suppose that*

$$R^j \pi_* (\tilde{I}^a \cdot \mathcal{O}_X) = 0 \quad \text{for } 1 \leq a \leq \bar{n} + 1, \quad j > 0.$$

Then there exists a positive integer f such that

$$H^j(X, \mathcal{L}^a \otimes \mathcal{M}^b) = 0 \quad \text{for } j > 0, a > 0 \text{ and } b \geq fa.$$

Proof. After possibly tensoring with an extension field of k , we may suppose that k is an infinite field. Suppose that I is generated in degree $\leq d$. Set $d' = d + 1$. Let $D_1, \dots, D_{\bar{n}}$ be general members of $\Gamma(X, \mathcal{L} \otimes \mathcal{M}^{d'})$. Let

$$L_i = D_1 \cap \dots \cap D_i$$

be the (scheme theoretic) intersection. L_i has dimension $\bar{n} - i$. Set $L_0 = X$.

We have short exact sequences

$$\begin{aligned} 0 \rightarrow (\mathcal{L}^a \otimes \mathcal{M}^b) \otimes \mathcal{O}_{L_i} &\rightarrow (\mathcal{L}^{a+1} \otimes \mathcal{M}^{b+d'}) \otimes \mathcal{O}_{L_i} \\ &\rightarrow (\mathcal{L}^{a+1} \otimes \mathcal{M}^{b+d'}) \otimes \mathcal{O}_{L_{i+1}} \rightarrow 0 \end{aligned} \tag{2}$$

for all integers a and b and $0 \leq i \leq \bar{n} - 1$.

$$R^j \pi_* (\mathcal{L}^a \otimes \mathcal{M}^b) = R^j \pi_* (\tilde{I}^a \cdot \mathcal{O}_X) \otimes \mathcal{O}_Y(b) = 0$$

for $j > 0$ and $1 \leq a \leq \bar{n} + 1$. From the Leray spectral sequence

$$H^i(Y, R^j \pi_* (\mathcal{L}^a \otimes \mathcal{M}^b)) \Rightarrow H^{i+j}(X, \mathcal{L}^a \otimes \mathcal{M}^b)$$

we have

$$H^j(X, \mathcal{L}^a \otimes \mathcal{M}^b) = H^j(Y, \pi_* (\tilde{I}^a \cdot \mathcal{O}_X) \otimes \mathcal{O}_Y(b)).$$

There thus exists an integer $f \geq d'$ such that

$$H^j(X, \mathcal{L}^a \otimes \mathcal{M}^b) = 0 \quad \text{for } j > 0, 1 \leq a \leq \bar{n} + 1 \text{ and } b \geq f \tag{3}$$

since $\pi_* (\tilde{I}^a \cdot \mathcal{O}_X)$ is coherent and $\mathcal{O}_Y(1)$ is ample on Y .

By (3) and induction applied to the long exact cohomology sequences associated to (2) we have

$$H^j(X, \mathcal{L}^a \otimes \mathcal{M}^b \otimes \mathcal{O}_{L_i}) = 0 \text{ for } j > 0, i + 1 \leq a \leq \bar{n} + 1 \text{ and } b \geq f + id'. \tag{4}$$

The following inductive statement (5) \implies (6) can be established by induction using the exact sequences (2), (4) and the equality

$$(a, b) = (a - i - 1)(1, d') + (i + 1, b - (a - i - 1)d').$$

Note that if $a \geq i + 1$ and $b \geq f + (a - 1)d'$ then $b - (a - i - 1)d' \geq f + id'$.

Suppose that

$$H^j(X, \mathcal{L}^a \otimes \mathcal{M}^b \otimes \mathcal{O}_{L_{i+1}}) = 0 \text{ for } j > 0, i + 2 \leq a \text{ and } b \geq f + (a - 1)d'. \quad (5)$$

Then

$$H^j(X, \mathcal{L}^a \otimes \mathcal{M}^b \otimes \mathcal{O}_{L_i}) = 0 \text{ for } j > 0, i + 1 \leq a \text{ and } b \geq f + (a - 1)d'. \quad (6)$$

$L_{\bar{n}}$ has dimension 0, so that (6) is immediate for $i = \bar{n}$. Thus the proposition follows from descending induction on i using the above statement (5) \implies (6). \square

Proposition 3.2. *Suppose that X is a Cohen–Macaulay scheme and*

$$R^j \pi_*(\omega_X \otimes \mathcal{L}^t) = 0 \text{ for } 1 \leq t \leq \bar{n} + 1, j > 0.$$

Then there exists a positive integer f such that

$$H^j(X, \mathcal{L}^{-a} \otimes \mathcal{M}^{-b}) = 0 \text{ for } j < \bar{n}, a > 0 \text{ and } b \geq fa.$$

Proof. After possibly tensoring with an extension field of k , we may suppose that k is an infinite field. Suppose that I is generated in degree $\leq d$. Set $d' = d + 1$. Let $D_1, \dots, D_{\bar{n}}$ be general members of $\Gamma(X, \mathcal{L} \otimes \mathcal{M}^{d'})$. Let

$$L_i = D_1 \cap \dots \cap D_i$$

be the (scheme theoretic) intersection. L_i has dimension $\bar{n} - i$. Set $L_0 = X$.

We have short exact sequences

$$\begin{aligned} 0 \rightarrow (\mathcal{L}^{-a-1} \otimes \mathcal{M}^{-b-d'}) \otimes \mathcal{O}_{L_i} &\rightarrow (\mathcal{L}^{-a} \otimes \mathcal{M}^{-b}) \otimes \mathcal{O}_{L_i} \\ &\rightarrow (\mathcal{L}^{-a} \otimes \mathcal{M}^{-b}) \otimes \mathcal{O}_{L_{i+1}} \rightarrow 0 \end{aligned} \quad (7)$$

for all integers a and b and $0 \leq i \leq \bar{n} - 1$.

By Serre-duality,

$$\begin{aligned} H^j(X, \mathcal{L}^{-a} \otimes \mathcal{M}^{-b}) &= H^{\bar{n}-j}(X, \omega_X \otimes \mathcal{L}^a \otimes \mathcal{M}^b), \\ R^j \pi_*(\omega_X \otimes \mathcal{L}^a \otimes \mathcal{M}^b) &= R^j \pi_*(\omega_X \otimes \mathcal{L}^a) \otimes \mathcal{O}_Y(b) = 0 \end{aligned}$$

for $j > 0$ and $1 \leq a \leq \bar{n} + 1$. From the Leray spectral sequence, we have

$$H^{\bar{n}-j}(X, \omega_X \otimes \mathcal{L}^a \otimes \mathcal{M}^b) = H^{\bar{n}-j}(Y, \pi_*(\omega_X \otimes \mathcal{L}^a) \otimes \mathcal{O}_Y(b)).$$

Hence there exists an integer $f \geq d'$ such that

$$H^j(X, \mathcal{L}^{-a} \otimes \mathcal{M}^{-b}) = 0 \text{ for } j < \bar{n}, 1 \leq a \leq \bar{n} + 1 \text{ and } b \geq f \quad (8)$$

since $\pi_*(\omega_X \otimes \mathcal{L}^a)$ is coherent and $\mathcal{O}_Y(1)$ is ample on Y .

By (8) and induction applied to the long exact cohomology sequences associated to (7) we have

$$H^j(X, \mathcal{L}^{-a} \otimes \mathcal{M}^{-b} \otimes \mathcal{O}_{L_i}) = 0 \text{ for } j < \bar{n} - i, 1 \leq a \leq \bar{n} + 1 - i \text{ and } b \geq f. \quad (9)$$

The following inductive statement (10) \implies (11) can be established by induction using the exact sequences (7), (9) and the equality

$$(a, b) = (a - 1)(1, d') + (1, b - (a - 1)d').$$

Note that if $a \geq 1$ and $b \geq f + (a - 1)d'$ then $b - (a - 1)d' \geq f$.

Suppose that

$$H^j(X, \mathcal{L}^{-a} \otimes \mathcal{M}^{-b} \otimes \mathcal{O}_{L_{i+1}}) = 0 \text{ for } j < \bar{n} - i - 1, 1 \leq a \text{ and } b \geq f + (a - 1)d'. \quad (10)$$

Then

$$H^j(X, \mathcal{L}^{-a} \otimes \mathcal{M}^{-b} \otimes \mathcal{O}_{L_i}) = 0 \text{ for } j < \bar{n} - i, 1 \leq a \text{ and } b \geq f + (a - 1)d'. \quad (11)$$

(11) is immediate for $i = \bar{n}$. Thus the proposition follows from descending induction on i using the above statement (10) \implies (11). \square

4. Linear bounds for Cohen–Macaulay coordinate rings

Let k be a field, S a noetherian graded k -algebra which is generated in degree 1, with graded maximal ideal M . Let $I \subset S$ be a homogeneous ideal, and let \tilde{I} be the sheaf associated to I in $Y = \text{Proj}(S)$. Let $X = \text{Proj}(\bigoplus \tilde{I}^n)$ be the blowup of \tilde{I} , with natural map $\pi: X \rightarrow Y$, and $\mathcal{O}_X(1) = \tilde{I} \cdot \mathcal{O}_X$. Let β be the dimension of S , $\bar{n} = \beta - 1$ be the dimension of Y .

Theorem 4.1. *Suppose that I is an ideal of height > 0 , S is Cohen–Macaulay and X is a Cohen–Macaulay scheme. Let*

$$E = \text{Proj}\left(\bigoplus_{n \geq 0} \tilde{I}^n / \tilde{I}^{n+1}\right)$$

with dualizing sheaf ω_E . Suppose that

$$\begin{aligned} \pi_* \mathcal{O}_E(m) &= \tilde{I}^m / \tilde{I}^{m+1} \text{ for } m \geq 0, \\ R^i \pi_* \mathcal{O}_E(m) &= 0 \text{ for } i > 0 \text{ and } m \geq 0 \text{ and} \\ R^i \pi_* \omega_E(m) &= 0 \text{ for } i > 0 \text{ and } m \geq 2. \end{aligned}$$

Then there exists a positive integer f such that $k[[I^e]_c]$ is Cohen–Macaulay whenever $e > 0$ and $c \geq ef$.

Proof. $R^i\pi_*\mathcal{O}_X = 0$ for $i > 0$ and $\pi_*\mathcal{O}_X = \mathcal{O}_Y$ by Lemma 2.1 (and Proposition III.8.5 [7]). S is Cohen–Macaulay so that $H_M^i(S) = 0$ for $i < \beta$ and $H^i(Y, \mathcal{O}_Y) = 0$ for $0 < i < \bar{n}$. Now by the Leray spectral sequence, $H^i(Y, R^j\pi_*\mathcal{O}_X) \Rightarrow H^{i+j}(X, \mathcal{O}_X)$, we get $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \bar{n}$.

By Lemma 2.1 and Proposition 3.1 we have an f such that $H^j(X, \mathcal{L}^a \otimes \mathcal{M}^b) = 0$ for $j > 0, a > 0, b \geq fa$. By Lemma 2.2 and Proposition 3.2 there exists f such that $H^j(X, \mathcal{L}^{-a} \otimes \mathcal{M}^{-b}) = 0$ for $j < \bar{n}, a > 0, b \geq fa$.

Now the Theorem follows from Lemmas 1.4 and 1.2. □

Corollary 4.2. *Suppose that S is Cohen–Macaulay, I is an ideal of height > 0 and \tilde{I} is locally a complete intersection in $Y = \text{Proj}(S)$. Then there exists a positive integer f such that $k[[I^e]_c]$ is Cohen–Macaulay whenever $e > 0$ and $c \geq ef$.*

Proof. This is immediate from Example 2.3. □

The following Corollary is now immediate from the comments following Example 2.3. By a canonical module W_T we mean a canonical module whose associated sheaf is a dualizing sheaf of $\text{Proj}(T)$. $I_{(P)}^n$ denoted the degree 0 elements of the localization I_P^n .

Corollary 4.3. *Suppose that*

- (1) S is Cohen–Macaulay.
- (2) I is an ideal of height > 0 .
- (3) $\bigoplus_{n \geq 0} I_{(P)}^n$ and $T(P) = \bigoplus_{n \geq 0} (I^n/I^{n+1})_{(P)}$ are Cohen–Macaulay for all $P \in \text{Proj}(S)$.
- (4) $H_{\bar{P}}^i(T(P))_c = 0$ for $i \geq 0$ and $c \geq 0$ and $H_{\bar{P}}^i(W_{T(P)})_c = 0$ for $i \geq 2$ and $c \geq 2$ for all $P \in \text{Proj}(S)$, where $W_{T(P)}$ is the canonical module of $T(P)$, \bar{P} is the maximal ideal of $S_{(P)}$.

Then there exists a positive integer f such that $k[[I^e]_c]$ is Cohen–Macaulay whenever $e > 0$ and $c \geq ef$.

Corollary 4.4. *Suppose that S is Cohen–Macaulay, I is an ideal of height > 0 and $I_{(P)}$ is strongly Cohen–Macaulay with $\mu(I_{(P)}) \leq \text{height}(P)$ for all primes $P \in Y$ containing I . Then there exists a positive integer f such that $k[[I^e]_c]$ is Cohen–Macaulay whenever $e > 0$ and $c \geq ef$.*

Proof. The assumptions of Theorem 4.1 are satisfied by Example 2.4 and Remark 2.5. □

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