Comment. Math. Helv. 73 (1998) 379–399
0010-2571/98/030379-21 \$ 1.50+0.20/0

Commentarii Mathematici Helvetici

Regularity properties of H-graphs

Robert Finn and Jianan Lu

Abstract. It is proved that if $H(u)$ is non-decreasing and if $H(-\infty) \neq H(+\infty)$, then if $u(\mathbf{x})$ describes a graph over a disk $B_R(0)$, with (upward oriented) mean curvature $H(u)$, there is a bound on the gradient $|Du(0)|$ that depends only on R, on $u(0)$, and on the particular function $H(u)$. As a consequence a form of Harnack's inequality is obtained, in which no positivity hypothesis appears. The results are qualitatively best possible, in the senses **a**) that they are false if H is constant, and **b**) the dependences indicated are essential.

The demonstrations are based on an existence theorem for a nonlinear boundary problem with singular data, which is of independent interest.

Mathematics Subject Classification (1991). Primary 53A10, 35J60, 35B45. Secondary 35B50, 35B65, 76B45.

Keywords. Mean curvature, H-graph, gradient estimate, Harnack inequality, moon surface, capillarity.

1. Mise en scène

One of the characteristic properties of solutions of linear elliptic equations is the a priori interior gradient bound: if $u(\mathbf{x})$ is a solution of such an equation in a domain Ω, with $|u(\mathbf{x})| \leq M$ in Ω, if $\mathbf{x}_0 \in \Omega$ has distance $\geq d$ from $\partial\Omega$ and $m = u(\mathbf{x}_0)$, then

$$|Du(\mathbf{x}_0)| \leq \mathcal{F}\left(\frac{m}{M}; \frac{M}{d}\right) \tag{1}$$

where $\mathcal{F}$ depends only on the equation, and not on the particular solution. As an example, we obtain for the Laplace equation $\triangle u = 0$ in a plane domain

$$|Du(\mathbf{x}_0)| \leq \frac{4}{\pi}\frac{M}{d} \cos\frac{\pi}{2}\frac{m}{M}, \tag{2}$$

see, e.g., [F1]. From (2) one derives easily the consequence, that if $u(\mathbf{x}) > 0$ in Ω then

$$|Du(\mathbf{x}_0)| \leq 2\frac{u(\mathbf{x}_0)}{d} \tag{3}$$

and integration of (3) yields a form (inessentially weakened) of Harnack's inequality

$$\left(\frac{d-r}{d}\right)^2 u(\mathbf{x}_0) \le u(\mathbf{x}) \le \left(\frac{d}{d-r}\right)^2 u(\mathbf{x}_0) \qquad (4)$$

for all $\mathbf{x}$ of distance not exceeding $r < d$ from $\mathbf{x}_0$. This result can in turn be extended to an a priori bound above and below over any prescribed compact subdomain of Ω.

If nonlinearities occur in the equation, the solutions can exhibit very different kinds of behavior; however, in [F2,3] it was shown that an estimate of the form (1) and additionally an analogue of (3) hold for the minimal surface equation in the plane

$$\operatorname{div} Tu = 0, \qquad Tu = \frac{1}{W} Du, \qquad W = \sqrt{1 + |Du|^2} \qquad (5)$$

(and more generally for *equations of minimal surface type*); the estimates were later extended in [BG], [K], [L], [S] to equations of prescribed mean curvature

$$\operatorname{div} Tu = 2H(u), \qquad H'(u) \ge 0 \qquad (6)$$

and in [Si] to *equations of mean curvature type.*

The particular nonlinearity in (5), and to a much larger extent the presence of an inhomogeneous term in (6), in fact impose essential changes in the underlying geometry of the solutions, which cannot be evidenced by properties that emulate those of the Laplace equation. Such distinctions are apparent already in the work of Bernstein [B] in 1910, who showed that *there is no surface* $u(\mathbf{x})$ *of mean curvature* $H \ge H_0 > 0$ *that is defined over a disk of radius exceeding* $1/H_0$. (This result was later sharpened by Heinz [H] and by Finn [F4].) Additionally, there is the theorem of Finn [F5] that every isolated singularity of a solution of (6) is removable. Distinctions in behavior with regard to the Harnack inequality were shown by Jenkins and Serrin [JS] for minimal surfaces and later by Serrin [Se] for H graphs with $H \equiv$ constant. More specifically with regard to gradient estimates, it was shown by Finn and Giusti [FG] that *if* $H \equiv const. > 0$*, then there exists* $R_0 = (0.5654062332\ldots)/H$*, and a decreasing function* $\mathcal{G}(RH)$*,* $R_0 H < RH \le 1$*, such that if* $u(\mathbf{x})$ *describes a surface of mean curvature* H *over a disk of radius* $R > R_0$ *centered at the origin, then* $|Du(0)| < \mathcal{G}(RH)$. Thus, in this case the gradient bound depends only on H and on the size of the domain of definition, and in no way on the values achieved by the solution. The result cannot be improved, in the sense that (necessarily) $\mathcal{G}(R_0 H) = \infty$. If $R \ge 1/H$, then necessarily $R = 1/H$ and the surface is a lower hemisphere of radius $1/H$ (see [F4], Theorem 8); thus, $\mathcal{G}(1) = 0$.

If $R \le 1/2H$ the example of a circular cylinder of radius $1/2H$ and with (increasingly) inclined axis, whose lower half covers the disk, shows directly that such

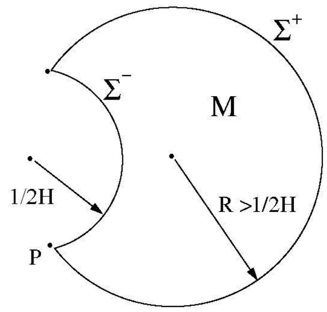

Figure 1.
Extremal domain for H-graph; $H \equiv const. > 0$

an estimate cannot hold for domains with small diameter. In general, extremal configurations over a disk are not achieved, but a least upper bound is provided by a "moonie" (see [F6]), corresponding to (extremal) "capillary" boundary conditions $\nu \cdot Tu = -1$ on Σ^-, $\nu \cdot Tu = +1$ on Σ^+ for the configuration of Figure 1, with $\nu =$ unit exterior normal. The gradient of this solution at the center of Σ^+ (when this center lies in $\mathcal{M}$) can be shown to majorize the gradient at that point of any other solution defined over a disk of radius R with that center. The location of the point P on the two circles is determined by an integrability condition which in turn depends on R, and the condition $R > R_0$ turns out to be precisely the requirement that the center of Σ^+ lies in $\mathcal{M}$.

The present study considers the case in which $H(u)$ is not constant. We expected initially to find a result formally analogous to the one just described; our considerations led us however to a theorem of basically different character, which was for us unexpected and surprising (especially as it is false when $H \equiv$ constant) and for which there seems to be no counterpart in the literature. We intend to prove:

Theorem M. *Suppose* $-\infty \le H^- = \lim_{t\to-\infty} H(t) \le H(u) \le \lim_{t\to+\infty} H(t) = H^+ \le \infty$, $H'(u) \ge 0$, $H^- \ne H^+$. *Then there is a function* $\mathcal{F}(R;\, u_0)$ *such that if* $u(\mathbf{x})$ *satisfies* (6) *in the disk* $B_R(0)$ *with* $u(0) = u_0$, *there holds*

$$|Du(0)| < \mathcal{F}(R;\, u_0). \tag{7}$$

That is, if for any $R > 0$ a solution $u(x)$ is defined in a disk about the origin of radius R, then its gradient at the origin is bounded depending only on R and on the value of u at the single point of evaluation. In this sense the solutions of (6) behave in ways basically different from that of solutions of the Laplace equation $\triangle u = 0$, and even of the superficially more closely related equation $\triangle u = f(u)$. The example of a tilted plane shows that the theorem fails for the minimal surface equation, and the example above of the tilted cylinder shows that it fails more generally whenever $H^- = H^+$. Also, the dependence of $\mathcal{F}$ on u_0 in (7) is essential. We see that from the example in which $H(u) \equiv 1/2$ if $u \geq 0$, $H(u) < 1/2$ if $u < 0$, and $H'(u) \geq 0$. The lower half of a circular cylinder of radius 1, with axis (increasingly) inclined to the plane $z = 0$, lying over the disk $B_1(0)$ and tangent to the boundary of the disk, provides a family of exact solutions defined in $B_1(0)$, for which $u(0)$ and $Du(0)$ become infinite together.

Theorem M leads in turn to a new form of Harnack's inequality, formulated as Corollary M1 in Section 6 below, in which no positivity hypothesis is required.

In the proof of Theorem M, we may assume that $H^+ > 0$. If that is not the case initially, it can be achieved by the transformation $u \to -u$, $H \to -H$. We show now that we may assume additionally that $H^- \neq -\infty$, $H^+ \neq +\infty$.

We note first that if $H^+ = +\infty$ then $M_R = \max\{t : H(t) \leq 1/R\}$ is finite.

Lemma 1.1. *Suppose $H^+ = +\infty$. If $u(\mathbf{x})$ satisfies $\operatorname{div} Tu = 2\,H(u)$ in the disk $B_R(0)$, there follows $u \leq M_R + R$ throughout $B_R(0)$.*

Proof. Choose $R' < R$ and denote by $v(\mathbf{x})$ a lower hemisphere of radius R' over $B_{R'}(0)$. We lift the hemisphere vertically until it lies entirely above the surface $u(\mathbf{x})$, and then lower it until an initial point p of contact occurs. Each such point necessarily lies interior to the hemispherical surface (not on the horizontal equator) as otherwise the hemisphere would penetrate the surface, and since the solution surface $u(\mathbf{x})$ is tangent to the hemisphere at p and lies locally below it, its mean curvature $H(u_p)$ at that point cannot exceed the mean curvature $1/R'$ of the hemisphere. Thus, $u_p \leq M_{R'}$. Since the given surface lies below the hemisphere throughout $B_{R'}(0)$, we find $u(\mathbf{x}) \leq M_{R'} + R'$ in this disk, and the result follows by letting $R' \nearrow R$.

It follows that if $H^+ = +\infty$ and $u(\mathbf{x})$ is a solution in $B_R(0)$ then u is bounded above in that disk; if we now modify $H(u)$ for values of u above that bound, so as to have itself a finite upper bound, we obtain the identical function $u(x)$ as solution of an equation of the same form, but with $H^+ < \infty$. Similarly, we may assume that $H^- > -\infty$. □

Note that the hypothesis $H'(u) \geq 0$ is not needed for Lemma 1.1.

From Lemma 1.1 and the classical estimate in [BG], [K], [L], [S] we obtain

Theorem 1.1. *If $H^- = -\infty$ and $H^+ = +\infty$ then an estimate of the form* (7) *holds, with $\mathcal{F}$ independent of u_0.*

We shall prove Theorem M by a comparison procedure with "moonies" analogous to the one employed in [F6], but with the roles of the two bounding arcs of Figure 1 in a sense interchanged. Notably, we consider a moon domain as in Figure 2, in which the radius $R^+ = 1/H^+$ of the "outer" arc is now *smaller* than the radius $R^- = 1/H^-$ of the "inner" arc. In this figure, a radius is considered

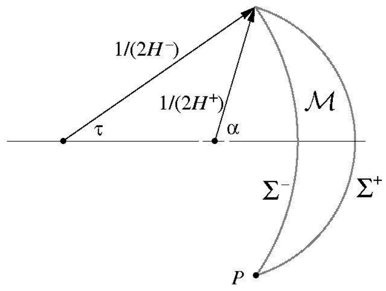

Figure 2.
Moon domain; $H^- \neq H^+$.

positive if the curvature vector is directed to the left as indicated, otherwise it is considered *negative*. The sign of the radius is determined by that of the corresponding H. We note that the relative values of the two radii in Figure 2 are reversed from those of Figure 1, reflecting the fact that different procedures are needed, depending on whether H is constant or not. We will prove that for all $\alpha \leq \pi/2$ there exists a unique solution $v(\mathbf{x})$ of (6) in $\mathcal{M}$, such that

$$\nu \cdot Tv = -1 \quad \text{on } \Sigma^-, \qquad \nu \cdot Tv = +1 \quad \text{on } \Sigma^+. \tag{8}$$

We shall show that this solution increases monotonely on the horizontal line of symmetry, from negative to positive infinity, and has gradient on this line tending uniformly to infinity, as $\alpha \to 0$.

Given any solution $u(\mathbf{x})$ of (6) in $B_R(0)$, we can choose α small enough that the entire configuration of Figure 2 will lie interior to $B_R(0)$ whenever a point on the horizontal symmetry line lies at the origin. We place this line so that the values of u and of v agree at the origin, and then rotate the moon domain about the origin until the two gradient directions coincide. If then $|Du(0)| > |Dv(0)|$, we decrease α (while shifting the point of contact so as to retain equal values at the point) until both gradients are the same at the origin. We will show that any such configuration conflicts with an extended form of the maximum principle, given first in a more limited context in [G1] and in [F6], but equally valid for our present needs. Thus the gradient of u at the origin is bounded by the gradient of any "moonie" v that can be placed interior to $B_R(0)$ as indicated, evaluated at that point on the horizontal symmetry line where $v = u(0) = u_0$. This reasoning shows the existence of a bound of the type asserted; the question of an explicit value for the estimate remains open. We remark however that in the case of constant H described above, an explicit upper bound for the function $\mathcal{G}(RH)$ was obtained by Chua [C].

In view of the above considerations and of the later developments of this paper, we may distinguish, for solutions of equations of the form (6) in a disk $B_R(0)$, four basic kinds of behavior:

I. *Suppose* $H(u) \equiv 0$. *Then there is a gradient bound of the form* (1), *see* [F2]. *However, the Harnack inequality holds only in a restricted sense, see* [JS].

II. *Suppose* $H(u) \equiv const. \neq 0$. *Then if* $R_0 < R < 1/H$ *there holds* $|Du(0)| < \mathcal{G}(RH)$, *with* $\mathcal{G}(RH)$ *decreasing and* $\mathcal{G}(1) = 0$, *see* [FG]; *if* $R \leq R_0$ *there is a bound of the form* (1), *see* [BG], [K], [L], [S]. *The Harnack inequality holds in a restricted sense, see* [Se]; *see also* [F7] *for another proof and interpretation.*

III. *Suppose* $H(u)$ *is not identically constant, but* $|H(u)|$ *is bounded. Then there holds* $|Du(0)| < \mathcal{F}(R;\, u_0)$. *A new kind of Harnack inequality appears, for which no positivity hypothesis is needed, see Section* 6 *below.*

IV. *Suppose* $\lim_{u\to\pm\infty} H(u) = \pm\infty$. *Solutions defined in a disk* $B_R(0)$ *are bounded throughout the disk. There holds* $|Du(0)| < \mathcal{F}(R)$.

With regard to II, Liang [Li] has characterized the largest R^* such that $|Du(\mathbf{x})| < \mathcal{G}^*(RH;\, R^*H)$ whenever $\mathbf{x} \in B_{R^*}(0)$.

2. Existence of comparison surfaces

The motivation for the choice of comparison surfaces connects closely with capillarity theory, in which a solution $v(\mathbf{x})$ of (6) is sought in a domain Ω, such that the surface S defined by v meets the vertical cylinder Z over $\Sigma = \partial\Omega$ in a prescribed angle γ. Some smoothness conditions are needed in order for the boundary condition to be well defined, and for this purpose it suffices to restrict attention to piecewise smooth domains, with no boundary condition imposed at points of discontinuity in the boundary normal. More generally, we consider capillary conditions on a set of smooth relatively open subarcs $\Sigma' \subset \Sigma$, distinguish a set Σ_0 of linear Hausdorff measure zero on which no conditions are prescribed, and define $v(\mathbf{x})$ to be a *variational solution of* (6) *in* Ω *relative to data* γ *on* Σ' *if* $v(\mathbf{x})$ *is locally smooth in* Ω *and if for any* $\eta \in W^{1,1}(\Omega)$ *whose support lies in the complement of* $\Sigma \setminus (\Sigma' \cup \Sigma_0)$ *there holds*

$$2\int_\Omega \eta H(v(\mathbf{x}))\, d\mathbf{x} + \int_\Omega \nabla\eta \cdot Tv\, d\mathbf{x} = \int_{\Sigma'} \eta \cos\gamma\, ds, \tag{9}$$

see [F8] Chapter 7 for further details.

We focus attention on moon domains $\mathcal{M}$ as in Figure 2, in which Σ_0 consists of the two juncture points of the two arcs Σ'. We will allow also configurations in which the sense of curvature of the left hand arc is reversed, as in Figure 3; in such a case we consider H^- to be negative.

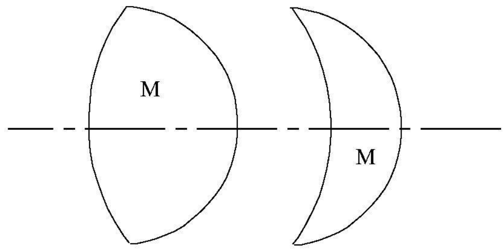

Figure 3.
Admissible moon domains.

Theorem 2.1. *Suppose $H'(u) \geq 0$, $-\infty < H^- = \lim_{t\to-\infty} H(t) \leq H(u) \leq \lim_{t\to+\infty} H(t) = H^+ < \infty$, $H^- \neq H^+$. Then if $\alpha \leq \frac{\pi}{2}$, there exists a variational solution in Ω relative to data $\gamma = 0$ on Σ^+, $\gamma = \pi$ on Σ^-.*

Proof. Our central tool for proving this result is Theorem 7.10 in [F8]. Accordingly, we consider the two functionals on Caccioppoli sets $\mathcal{M}^*$

$$\Phi[\mathcal{M}^*] \equiv \int_{\mathcal{M}} \left| D\chi_{\mathcal{M}^*} \right| - \int_{\Sigma} \beta \chi_{\mathcal{M}^*} + 2H^+|\mathcal{M}^*| \tag{10}$$

$$\Psi[\mathcal{M}^*] \equiv \int_{\mathcal{M}} \left| D\chi_{\mathcal{M}^*} \right| + \int_{\Sigma} \beta \chi_{\mathcal{M}^*} - 2H^-|\mathcal{M}^*| \tag{11}$$

with $\beta = +1$ on Σ^+, $\beta = -1$ on Σ^-, χ the characteristic function of $\mathcal{M}^*$. In the boundary integrals, χ is defined as the trace on Σ, see [G2] Chapter 2 and [EG] Section 5.3. Existence follows if it can be shown that both functionals are positive for any Caccioppoli set $\mathcal{M}^* \subset \mathcal{M}$ with $\mathcal{M}^* \neq \emptyset$.

We begin with the particular case $\mathcal{M}^* = \mathcal{M}$, in which our requirement becomes

$$\Phi[\mathcal{M}] \equiv |\Sigma^-| - |\Sigma^+| + 2H^+|\mathcal{M}| > 0 \tag{12}$$

$$\Psi[\mathcal{M}] \equiv |\Sigma^+| - |\Sigma^-| - 2H^-|\mathcal{M}| > 0. \tag{13}$$

Adopting notation as in Figure 2, we prove initially

Lemma 2.1. *For any moon domain as in Figure 3, if $0 < \alpha \leq \pi/2$, then the inequalities* (12) *and* (13) *are satisfied.*

Proof. We prove only (12) for the case indicated in Figure 2. The remaining cases are analogous. Considering Φ as function of α, we have

$$2H^+\Phi(\alpha) = -\alpha + \tau\frac{H^+}{H^-}\left(2 - \frac{H^+}{H^-}\right) + \frac{H^+}{H^-}\sin\alpha\cos\tau - \sin\alpha\cos\alpha \tag{14}$$

with α and τ related by

$$H^- \sin\alpha = H^+ \sin\tau. \tag{15}$$

We have $\Phi(0) = 0$. A calculation yields

$$H^+\Phi'(\alpha) = \frac{\cos\alpha}{\cos\tau}(1 - \cos(\tau - \alpha)) > 0 \tag{16}$$

if $0 < \alpha < \pi/2$, and the result follows. □

We return to the proof of the theorem. In order to prove (10) and (11) for general Caccioppoli sets $\mathcal{M}^* \subset \mathcal{M}$, we seek to minimize the functionals, and to do that it is helpful to have lower semicontinuity. In this step, a difficulty appears in view of the discontinuity in boundary normal at the juncture points of the two arcs, as the proof of Lemma 6.1 in [F8] does not apply as stated to that configuration with the chosen (extremal) boundary data. We therefore begin by smoothing the boundary with inscribed circular arcs within distance η of the vertices, as indicated in Figure 4, and prescribing data on these arcs that yield a smooth monotonic transition between the two values of β. We thus obtain a sequence of domains $\mathcal{M}^\eta \nearrow \mathcal{M}$, and corresponding boundary data β^η, such that β^η is smooth and locally monotonic, and $\beta^\eta = \beta$ at all common boundary points. In such a configuration the procedure for the proof of Lemma 6.1 in [F8] extends without change, to yield an estimate of the form

$$\left| \int_{\Sigma^\eta} \beta f \, ds \right| \le (1+\epsilon) \int_{\mathcal{A}_\delta^\eta} |Df| + \Upsilon(\mathcal{M}^\eta; \delta; \epsilon) \int_{\mathcal{A}_\delta^\eta} |f| \tag{17}$$

for any $\epsilon > 0$; here $\Sigma^\eta = \partial\mathcal{M}^\eta$ and $\mathcal{A}_\delta^\eta$ is a strip of width δ adjacent to Σ^η in $\mathcal{M}^\eta$.

Using (17) with η fixed, we obtain as in Lemma 6.3 of [F8] the lower semicontinuity of minimizing sequences for the functionals Φ, Ψ and from that, following the discussion in [F8], the existence of minimizing configurations, which could be the entire set $\mathcal{M}^\eta$ or the null set.

Let us consider the functional Φ. If η is sufficiently small, the minimizing set cannot be $\mathcal{M}^\eta$, since by Lemma 2.1 we have $\Phi(\mathcal{M}^\eta) > \frac{1}{2}\Phi(\mathcal{M}) > 0$ for small enough η while $\Phi(\emptyset) = 0$; as in Lemma 6.4 of [F8] it must be either $\emptyset$ or else bounded in $\mathcal{M}^\eta$ by one or more non-intersecting circular arcs Γ^η of radius $1/H^+$. If Γ^η meets Σ^+ it must do so in the angle zero on the side opposite to that into which the curvature vector points, and hence would have to coincide with that arc. Similarly it cannot meet Σ^- in angle π. Thus, any such arc must terminate on one or both of the smoothing arcs near the vertex points, and one sees easily that if the configuration minimizes then for any component of the minimizing set at most one interior boundary arc can extend between the two smoothing arcs. It follows that as $\eta \to 0$ the minimizing sets converge to the null set, with the values Φ^η tending to zero.

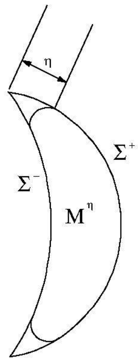

Figure 4.
Construction for Theorem 2.1.

We assert that the null set $\emptyset$ minimizes Φ in $\mathcal{M}$. We have $\Phi(\emptyset) = 0$. Suppose the existence of a Caccioppoli set $E^* \subset \mathcal{M}$ with $\Phi(E^*) = -\omega^2 < 0$. Then for all η small enough, there will hold $\Phi(E^* \cap \mathcal{M}^\eta) < -\frac{1}{2}\omega^2$. But if $\mathcal{M}^*(\eta)$ denotes the minimizing set for $\mathcal{M}^\eta$, there holds $\Phi(\mathcal{M}^*(\eta)) \to 0$ with η, and thus $\Phi(\mathcal{M}^*(\eta)) > -\frac{1}{2}\omega^2$ for small enough η, contradicting the minimizing property of $\mathcal{M}^*(\eta)$.

From this result we conclude that $\emptyset$ is the unique minimizing set for Φ in $\mathcal{M}$. For by Massari's theorem [M] any minimizer is bounded in $\mathcal{M}$ by analytic arcs and by Lemma 6.4 of [F8] these arcs must be circular of radius $1/H^+$, which if they intersect Σ^+ do so in angle zero, and if they intersect Σ^- do so in angle π. That is geometrically not possible for an arc interior to $\mathcal{M}$.

An analogous reasoning establishes the null set as the unique minimizer for Ψ; thus both functionals are positive for any Caccioppoli set $\mathcal{M}^* \subset \mathcal{M}$ with $\mathcal{M}^* \neq \emptyset$, and Theorem 2.1 follows from Theorem 7.10 of [F8]. □

Some comments may be in order on the interpretation of Theorem 7.10 of [F8] in the present context. As presented in [F8], the theorem requires only the positivity of Φ, Ψ for all $\mathcal{M}^* \neq \emptyset$, $\mathcal{M}$. If $H \equiv$ const., then one sees easily that the two functionals necessarily vanish both on $\emptyset$ and on $\mathcal{M}$, and are equivalent with regard to positivity on Caccioppoli subsets $\mathcal{M}^*$; the vanishing on $\mathcal{M}$ was used in [F6] as a basic necessary condition uniquely determining the moon domain, for each given R in the permissible range. If however $H^- \neq H^+$ then the functionals become independent of each other, and for the particular problem considered here both are necessarily positive on $\mathcal{M}$. In our discussion above, we adopted this positivity as a basic necessary condition (Lemma 2.1) in the existence proof for moon surfaces. The parameter R is now replaced by the parameter α, and Lemma 2.1 yields positivity for a range including $(0, \pi/2]$. This permits moon domains of arbitrarily small diameter, which is not possible in the case of constant H.

It should also be remarked that the solutions of Theorem 2.1 can be considered to be living on the "edge of existence" in the sense that if there were any interval of larger curvature on Σ^+ or of lower curvature on Σ^-, then the problem would admit no solution, see Theorem 3 in [CF] or Theorem 6.4 in [F8]. The solutions obtained for the moon domains introduced here can be regarded as the natural analogues, for the given equation, of the infinite vertical cylinders that appear as generalized solutions of the equation (see [Mi 1, 2]) when H is constant.

In Section 4 we shall prove the uniqueness of the solution whose existence is provided by Theorem 2.1

3. A comparison principle

We make essential use in our result of an extended form of the comparison principle, that takes account both of the particular nonlinearity in the equation, and also of the particular singularity in boundary behavior that occurs when $\gamma = 0$ or π, leading to solutions that cannot be expected to lie in $W^{1,1}(\Omega)$. The former consideration is covered by Theorem 5.1 in [F8], according to which arbitrary subsets of Σ of linear Hausdoff measure zero can be neglected when comparing two solutions. Nevertheless, in the proof of that theorem the (truncated) difference of two solutions was chosen as a test function. Since for the cases considered here this difference is not a priori known to be in $W^{1,1}(\Omega)$, we need a more finely tuned version of that theorem. The underlying observation that we use appears first in a particular context in Giusti [G1], and was developed for other configurations in [F6]. In the interest of a unified formulation, we present the result in somewhat more generality than required for our immediate needs. We start with some preliminary observations.

Lemma 3.1. *Let $u(\mathbf{x})$ be a variational solution of (6) in Ω relative to data γ on Σ', and let $\Omega' \subset \Omega$ be bounded by Σ' and by a set of disjoint (in Ω) piecewise smooth arcs $\mathcal{C}' \subset \Omega$, joining the endpoints of the respective arcs of Σ'. Then*

$$2H^-|\Omega'| < \int_{\Sigma'} \cos\gamma \, ds + \int_{\mathcal{C}'} \nu \cdot Tu \, ds < 2H^+|\Omega'|. \tag{18}$$

Proof. For each of the arcs $\mathcal{C}'$ we introduce a strip region $\Omega'_\delta = \{\mathbf{x} \in \Omega' : \ d(\mathbf{x}, \mathcal{C}') < \delta\}$ as in Figure 5. Let $\eta \equiv 1$ in $\Omega' \setminus \Omega'_\delta$, and $\eta \equiv 0$ in $\Omega \setminus \Omega'$, with η tending linearly to zero across the strip. The result follows from (9), taking the limit as $\delta \to 0$. A perhaps necessary non-uniformity in the construction at the endpoints of the arcs causes no difficulty, as $|Tu| < 1$ and H is bounded. □

Corollary 3.1. *Let $u(\mathbf{x})$ be a variational solution of (6) in Ω relative to data γ on Σ', and let $\mathcal{C}'_n$ denote a sequence of simple piecewise smooth curves in Ω joining*

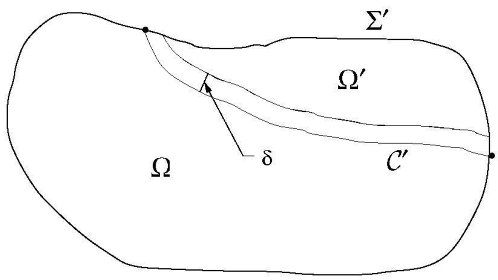

Figure 5.
Construction for Lemma 3.1.

the respective endpoints of Σ'. *Suppose* $\mathcal{C}'_n \to \Sigma'$ *weakly, in the sense that the area* Ω'_n *bounded between* $\mathcal{C}'_n$ *and* Σ' *tends to zero with increasing* n. *Then*

$$\lim_{n\to\infty}\int_{\mathcal{C}'_n} \nu \cdot Tu\, ds = \int_{\Sigma'} \cos\gamma\, ds. \tag{19}$$

Here ν *is directed exterior to* $\Omega\backslash\Omega'_n$.

Lemma 3.2. *Let* $u(\mathbf{x})$ *be a variational solution of* (6) *in* Ω *relative to data* $\gamma = 0$ $(\gamma = \pi)$ *on* Σ'. *Let* $\mathcal{C}'_n$ *denote a sequence of smooth curves in* Ω *joining the respective endpoints of* Σ', *such that* $\mathcal{C}'_n \to \Sigma'$ *pointwise together with normal direction. Then for any* $\epsilon > 0$ *there holds, if* $\gamma = 0$,

$$\lim_{n\to\infty} \mu\{\mathbf{x} \in \mathcal{C}'_n : 1 - \nu \cdot Tu > \epsilon\} = 0.$$

If $\gamma = \pi$, *then*

$$\lim_{n\to\infty} \mu\{\mathbf{x} \in \mathcal{C}'_n : 1 + \nu \cdot Tu > \epsilon\} = 0.$$

Here μ is Lebesgue measure with respect to arc on $\mathcal{C}'_n$.

Proof. By Corollary 3.1 we find, if $\gamma = 0$,

$$\lim_{n\to\infty}\int_{\mathcal{C}'_n}\left[(1 - \nu\cdot Tu) + \left(\frac{ds}{ds_n} - 1\right)\right] ds_n = 0.$$

By hypothesis, $\frac{ds}{ds_n} \to 1$ uniformly. Since $\nu \cdot Tu < 1$, the result follows. The case $\gamma = \pi$ is analogous. □

Lemma 3.3. *Let v, u be variational solutions of* (6) *in Ω relative to data γ_v, γ_u on Σ', with $\gamma_v = 0$ or $\gamma_u = \pi$ on each arc of Σ', and let $\mathcal{C}_n'$ be a sequence of arcs as in Lemma 3.2. Then for any bounded non-negative function η defined in Ω, there holds*

$$\liminf_{n\to\infty} \int_{\mathcal{C}_n'} \eta(\nu \cdot Tv - \nu \cdot Tu)\, ds \geq 0.$$

We note that η is not required to be in the class $W^{1,1}(\Omega)$ for this result. We note also that on each arc of Σ', data are prescribed for only one of the two solutions, with no hypotheses on behavior of the other.

Proof. From Lemma 3.2 we find that for any $\epsilon > 0$

$$\lim_{n\to\infty} \mu\{x \in \mathcal{C}_n' : \nu \cdot Tv - \nu \cdot Tu < -\epsilon\} = 0;$$

the stated assertion follows immediately. □

We may now state:

Theorem 3.1. *Suppose $\Sigma = \partial\Omega$ admits a decomposition*

$$\Sigma = \Sigma_\alpha \cup \Sigma_\beta \cup \Sigma_\beta' \cup \Sigma_0$$

such that Σ_β, Σ_β' consist of smooth arcs, and Σ_0 has linear Hausdorff measure zero. Let v, u be variational solutions of (6) *in Ω relative to data γ_v, γ_u on subarcs whose union is $\Sigma_\beta \cup \Sigma_\beta'$, with $\gamma_v = 0$ or $\gamma_u = \pi$ on each subarc of Σ_β', and suppose that for any non-negative $\eta \in L^\infty(\Omega) \cap W^{1,1}_{loc}(\Omega)$ there is a sequence $\mathcal{C}_n$ tending to Σ_β as in Lemma 3.2, such that* (20) *holds. Suppose further that*

$$\liminf_{x\to\Sigma_\alpha} (v - u) \geq 0. \tag{21}$$

We conclude that either

i) $H(v) \equiv H(u) \equiv const.$, and $u \equiv v + const.$ in Ω, or

ii) $v \geq u$ in Ω, equality holding at any point if and only if $v \equiv u$ in Ω.

We remark that the hypotheses relative to Σ_β are satisfied if γ_v, γ_u are bounded from 0 and π on Σ_β and if u, v have continuous derivatives up to the boundary on these arcs with $\cos\gamma_v = \nu \cdot Tv \geq \nu \cdot Tu = \cos\gamma_u$ on the arcs. In all cases encountered in the present paper, $\Sigma_\beta = \emptyset$.

Proof. The proof follows in outline that of a more limited form of the comparison principle given in [F8], Theorem 5.1. If for some $\mathbf{x} \in \Omega$ there were to hold $u(\mathbf{x}) >$

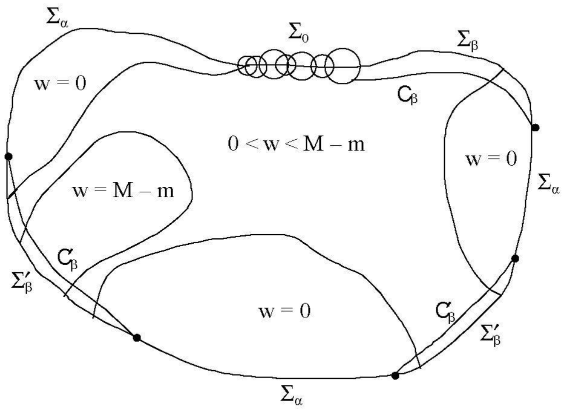

Figure 6.
Construction for comparison principle.

$v(\mathbf{x})$, then for suitable m, M with $0 < m < M$, the set in which $m < u - v < M$ would have positive measure in Ω. The function

$$w(\mathbf{x}) = \begin{cases} M - m, & u - v \geq M \\ u - v - m, & m \leq u - v \leq M \\ 0, & u - v \leq m \end{cases}$$

is then non-negative and bounded, has compact support in the complement of Σ_α, and does not vanish identically. As in the proof of Theorem 5.1 in [F8], we use $w(\mathbf{x})$ as a "test function" in the subdomain of Ω bounded by Σ_α, by the boundary of a union of small disks surrounding Σ_0, by arcs $\mathcal{C}_n$ tending to Σ_β and by arcs $\mathcal{C}'_n$ tending smoothly to Σ'_β (see Figure 6). By hypothesis, the limit contribution from the arcs $\mathcal{C}_n$ will be non-negative; by Lemma 3.3, the same conclusion holds for the arcs $\mathcal{C}'_n$. We may thus follow the proof of Theorem 5.1 in [F8] to conclude that $\nabla w \equiv 0$ throughout Ω, and we find that either Case i) holds or else that $v(x) \geq u(x)$ throughout Ω. Equality at any interior point is excluded by the E. Hopf boundary point lemma. □

4. Properties of moonies

In order to establish the required gradient comparisons, we need information on specific pointwise behavior of the moonies we have constructed. We show first that every moonie is unbounded positive on Σ^+, unbounded negative on Σ^-. We do this by way of the general comparison principle (Theorem 3.1), using a smaller moonie as a barrier.

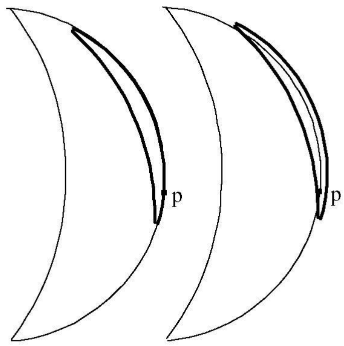

Figure 7.
Auxiliary moon domain: construction (left) and use as barrier (right).

Theorem 4.1. *Let $u(\mathbf{x})$ define a variational solution of* (6) *in a moon domain $\mathcal{M}$ with angle $\alpha \leq \pi/2$, relative to data π and 0 on arcs Σ^- and Σ^+ of curvatures $2H^-$, $2H^+$. Then $u(\mathbf{x}) \to -\infty$ for any interior approach to Σ^-, and $u(\mathbf{x}) \to +\infty$ for any interior approach to Σ^+.*

Proof. Suppose there exists $p \in \Sigma^+$ and a sequence $p_j \to p$ in $\mathcal{M}$, along which $u(p_j) < M < \infty$. Choose $\alpha' < \alpha$ sufficiently small that the corresponding $\mathcal{M}'$ can be situated with either vertex at p and with $\Sigma'^+ \subset \Sigma^+$ (Figure 7). Denote by $v(\mathbf{x})$ the (uniquely defined) moon surface over $\mathcal{M}'$. By a theorem of Finn [F9], there is a dense set of points $q \in \Sigma'^+$ which admit a sequence $q_j \to q$, $q_j \in \mathcal{M}'$, along which $v(q_j) \to \infty$. For each index k, let j_k be the first value of j such that the radial distance from p_{j_k} to Σ^+ is less than that from q_k to Σ'^+. We move $\mathcal{M}'$ rigidly so that q_k coincides with p_{j_k} and the radial directions coincide there; Σ'^+ will then lie exterior to Σ^+ (Figure 7) and Theorem 3.1, applied to the intersection domain, yields that $u(p_{j_k}) > v(q_k)$. Since $v(q_k) \to \infty$, this contradicts the assumed boundedness of the sequence $u(p_j)$. An analogous reasoning shows that $u(\mathbf{x})$ tends to $-\infty$ at interior points of Σ^-. □

Theorem 4.2. *On the symmetry line of any moon domain with $\alpha \leq \pi/2$, the solution $v(\mathbf{x})$ increases monotonely from $-\infty$ to ∞, with gradient becoming uniformly infinite on the entire line as the domain size decreases ($\alpha \to 0$).*

Proof. We may assume that the symmetry line is given by the relations $y = 0$, $a < x < b$. We know from Theorem 4.1 that along this line $v(\mathbf{x})$ tends to $-\infty$ at a and to $+\infty$ at b. If $v(x_1, 0) = v(x_2, 0)$, $a < x_1 < x_2 < b$, we translate $\mathcal{M}$ a distance $h = x_2 - x_1$ to the left, and compare $v(\mathbf{x})$ with the solution $w(\mathbf{x}) = v(\mathbf{x} + h)$. Theorem 3.1 yields $w(\mathbf{x}) > v(\mathbf{x})$, a contradiction. □

Using Theorem 3.1, we see that $v(\mathbf{x})$ is symmetric with respect to the symmetry line; thus, ∇v is directed along this line; by what we have just shown, ∇v is directed from left to right at all points on the line where it doesn't vanish. We compare $v(\mathbf{x})$ with particular solutions $\zeta(x)$ of (6) that are independent of y, and thus satisfy the equation

$$\frac{d}{dx}\left(\frac{\zeta_x}{\sqrt{1+\zeta_x^2}}\right) = 2H(\zeta) \tag{22}$$

in a strip containing $\mathcal{M}$.

Lemma 4.1. *Given x_0, ζ_0, ζ_0', there exists a unique solution $\zeta(x)$ of (22) for which $\zeta(x_0) = \zeta_0$, $\zeta'(x_0) = \zeta_0'$; if $-\infty < H^- \leq H(\zeta) \leq H^+ < \infty$, then this solution can be continued into a strip whose width on either side of x_0 depends only on ζ_0', and on H^-, H^+.*

Proof. The local existence and uniqueness is a standard theorem. Let $H^* = \max\{|H^-|, |H^+|\}$. We integrate (22) from x_0, obtaining

$$\left|\frac{\zeta_x}{\sqrt{1+\zeta_x^2}} - \frac{\zeta_0'}{\sqrt{1+\zeta_0'}}\right| < |x - x_0| H^* \tag{23}$$

from which the result follows. □

The form taken by (23) yields as additional information

Lemma 4.2. *There is a positive lower bound $L(\zeta_0';\ H^*)$ on the width of the strip, on either side of x_0, and $L(\zeta_0';\ H^*)$ decreases in $|\zeta_0'|$.*

We need also

Lemma 4.3. (division lemma) *Let $u(\mathbf{x})$, $v(\mathbf{x})$ be solutions of (6), with $H'(t) \geq 0$ in the interval between u and v, in a domain Ω containing $\mathbf{x}_0$, and such that $u(\mathbf{x}_0) = v(\mathbf{x}_0)$, $\nabla u(\mathbf{x}_0) = \nabla v(\mathbf{x}_0)$. Then there exists a positive integer $k \geq 2$, such that k level curves $w = u - v = 0$ pass through $\mathbf{x}_0$, dividing a neighborhood of $\mathbf{x}_0$ into $2k$ regions in which, alternatingly, $w > 0$ and $w < 0$ (see Figure 8).*

The underlying idea behind this lemma can be traced to Hilbert [Hi]; the lemma has been proved in varying contexts by a number of authors. A proof for the case considered here is given in Hartman [Ha], Section 2.

We return to the proof of Theorem 4.2. Given a positive quantity ζ_0', we choose α small enough that the corresponding moon domain $\mathcal{M}$ lies interior to a strip of total width $2L(\zeta_0';\ H^*)$, when $\mathcal{M}$ is oriented as indicated in Figure 3, and when any point on the symmetry line is placed at $\mathbf{x}_0$. Given a point p interior

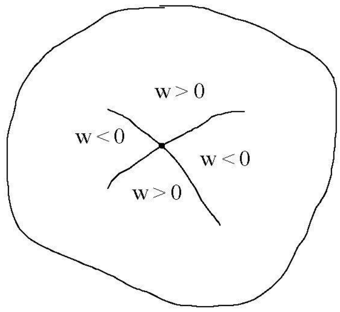

Figure 8.
The division lemma; $k = 2$.

to the symmetry line, we construct the strip solution $\zeta(x)$ with $\mathbf{x}_0$ at p and with $\zeta_0 = v(p)$. We assert that then $|\nabla v(p)| > \zeta_0'$.

If $|\nabla v(p)| \leq \zeta_0'$, we decrease ζ_0' (if necessary) until equality is attained. By Lemma 4.2, $L(\zeta_0'; H^*)$ does not decrease, and thus the moon domain remains interior to the strip. By Lemma 4.3, in the configuration of equality there must be at least two subdomains of $\mathcal{M}$ abutting in x_0, in which $w = v - \zeta$ is positive, and two in which $w < 0$. By Theorem 3.1, each of the former domains must have points of Σ^+ on its boundary, and it follows that there is a (maximal) domain in which $w < 0$, which has no points of Σ^- on its boundary, contradicting Theorem 3.1.

Since the choice of p is arbitrary on the symmetry line, we conclude that $|\nabla v| \to \infty$ uniformly on that line as $\alpha \to 0$, completing the proof of the theorem. □

From Theorems 3.1 and 4.1 we may conclude the uniqueness assertion of the solution constructed in Theorem 2.1. We note first that the angle α completely determines the geometry (see Figure 2). In the construction for Theorem 2.1, we have $\Sigma_\alpha = \Sigma_\beta = \emptyset$, $\Sigma_\beta' = \Sigma^+ \cup \Sigma^-$; Σ_0 consists of the two juncture points of the arcs.

Theorem 4.3. *For given α, at most one moonie can exist.*

Proof. From Theorem 3.1 we conclude that for any two solutions u, v, either $u \equiv v$ or else $H(u) \equiv H(v) \equiv$ *const.* for these particular solutions. In the latter case we note that on any arc joining Σ^- to Σ^+, both u and v vary from $-\infty$ to ∞. That would contradict the hypothesis $H^+ \neq H^-$. □

Theorem 4.4. (continuous dependence) *Let α_j be a decreasing sequence of angles sufficiently small that Theorem 2.1 ensures the existence of moonies v_j, and suppose $\alpha_j \searrow \alpha > 0$, with corresponding base domains $\mathcal{M}_j \to \mathcal{M}$. If $v(\mathbf{x})$ is the moonie over $\mathcal{M}$, then $v_j \to v(\mathbf{x})$, uniformly together with all derivatives, in every compact subdomain of $\mathcal{M}$.*

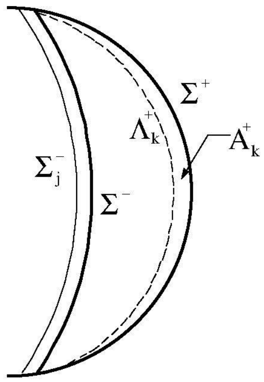

Figure 9.
Proof of continuous dependence.

Proof. We may suppose that Σ^+ remains fixed and that the corresponding sequence Σ_j^- moves to the right, to a limiting position Σ^-, which with Σ^+ bounds $\mathcal{M}$ (Figure 9). The corresponding $\mathcal{M}_j$ form a nested sequence of domains, decreasing to $\mathcal{M}$. Let $\mathbf{x} \in \mathcal{M}$. By the comparison principle Theorem 3.1, $v_j(\mathbf{x})$ forms a decreasing sequence, and $v_j(\mathbf{x}) > v(\mathbf{x}) > -\infty$. In particular, the entire sequence v_j is bounded above and below in every compact subdomain of $\mathcal{M}$. By the result of [BG], [K], [L], [S], $|Dv_j|$ is also bounded in compacta; thus, in every such subdomain the v_j form a sequence of solutions, with bounded gradients, of a uniformly elliptic equation in the plane, and hence a subsequence converging uniformly to a solution can be extracted. By a diagonalization procedure, we may assume that $v_j(\mathbf{x})$ converges throughout $\mathcal{M}$, uniformly in compacta, to a solution $V(p)$ in $\mathcal{M}$. We will show that $V(p) \equiv v(p)$.

Consider a simple arc $\Lambda_k^+ \in \mathcal{M}$ joining the endpoints of Σ^+, approximating Σ^+ pointwise and in direction and thus bounding with Σ^+ an area $\mathcal{A}_k^+$ tending to zero with k. Choosing on Λ_k^+ a unit normal ν directed into $\mathcal{A}_k^+$, we obtain in view of the boundary condition for v_j and Lemma 3.1

$$\int_{\Lambda_k^+} \nu \cdot Tv_j \, ds = 2 \int_{\mathcal{A}_k^+} H(v_j(\mathbf{x}))\, d\mathbf{x} + |\Sigma^+|. \tag{24}$$

In view of the convergence within $\mathcal{M}$ and the inequalities $|\nu \cdot Tv_j| < 1$, $H^- \leq H \leq H^+$, we find

$$\int_{\Lambda_k^+} \nu \cdot TV \, ds = 2 \int_{\mathcal{A}_k^+} H(V(\mathbf{x}))\, d\mathbf{x} + |\Sigma^+|. \tag{25}$$

Since H is bounded in its argument, the integral over $\mathcal{A}_k^+$ vanishes in the limit as $k \to \infty$, and we conclude that $V(p)$ satisfies the same (extremal) boundary condition on Σ^+ as does $v_j(p)$.

We now repeat the entire convergence procedure, this time keeping Σ^- fixed and allowing the arcs Σ_j^+ to move to the left. We obtain (except for rigid translation) the identical sequence of domains, and we choose for $v_j(\mathbf{x})$ the corresponding translates of those already constructed. The analogous reasoning now shows that $V(\mathbf{x})$ satisfies the same (extremal) boundary condition on Σ^- as does $v_j(\mathbf{x})$. The general comparison principle Theorem 3.1 thus yields the identity of the two functions $V(\mathbf{x})$ and $v(\mathbf{x})$, and establishes the asserted continuous dependence. □

5. Proof of Theorem M

Let $u(\mathbf{x})$ be a solution of (6) in $B_R(0)$ with $u(0) = u_0$ and satisfying the hypotheses of Theorem M; as pointed out following the statement of that theorem, we may assume that $H^- \neq -\infty$, $H^+ \neq +\infty$. We choose α small enough that the corresponding moon domain $\mathcal{M}$ will lie interior to $B_R(0)$ whenever a point on the symmetry line is at the origin. Let $v(\mathbf{x})$ denote the moonie over $\mathcal{M}$. By Theorem 4.2, $v(\mathbf{x})$ assumes every real value on the symmetry line exactly once; we put that point p of the line at the origin, at which $v(p) = u_0$, and we then rotate $\mathcal{M}$ about the origin so that the directions of the gradients coincide. The theorem will be proved if we can show that there then holds $|\nabla u(0)| < |\nabla v(0)|$.

Were the converse to hold, we apply Theorem 4.2 to show that by decreasing α, a configuration could be obtained for which $\nabla u(0) = \nabla v(0)$ and $u(0) = v(0)$. We then obtain a contradiction as in the final step of the proof of Theorem 4.2. We are done. □

6. Two corollaries

From the Theorem M, we obtain a kind of Harnack inequality, which takes here in one sense a much stronger form and in another sense a much weaker form, than occurs with harmonic functions.

Corollary M1. *Under the hypotheses of Theorem M, there exists a positive function* $\rho^+(u_0; R) \leq R$ *and a continuous function* $U^+(u_0; R; \rho)$*, with* $U^+(u_0; R; 0) = u_0$*, such that if* $u(\mathbf{x})$ *satisfies* (6) *in* $B_R(0)$ *and* $u(0) = u_0$ *then* $u \leq U^+$ *throughout* $B_\rho(0)$*, for all* $\rho < \rho^+$*. There exists a positive* $\rho^-(u_0; R) \leq R$ *and a continuous* $U^-(u_0; R; \rho)$*, with* $U^-(u_0; R; 0) = u_0$*, such that* $u \geq U^-$ *whenever* $\rho < \rho^-$*.*

Note that the corollary does not require a hypothesis that $u(\mathbf{x})$ have constant sign in $B_R(0)$. In that sense it differs strikingly from the original inequality for harmonic functions, and additionally from a Harnack type inequality obtained by Serrin [Se], see also [F7], for surfaces of constant H; in fact Corollary M1 is false both for harmonic functions and for solutions of (6) when $H \equiv$ *const*. We will give below a heuristic reasoning to show that in general it must be expected, as occurs in Serrin's inequality and in contrast to the behavior of harmonic functions, that there will be upper bounds for the choice of ρ^+ (ρ^-), that go to zero with increasing u_0 (decreasing u_0).

Proof of Corollary M1. In virtue of the continuous dependence property Theorem 4.4, we may assume in (7) that $\mathcal{F}$ is non-increasing in R and nondecreasing in $|u_0|$; thus, $\mathcal{F}$ is bounded in every compact subdomain of the strip $0 < d \leq R$, $-\infty < u < \infty$, and we may replace $\mathcal{F}$ by a locally Lipschitz majorant, which we again denote by $\mathcal{F}$. On a radial segment from the origin we have, by Theorem M,

$$\frac{du}{ds} < \mathcal{F}(R - s; u). \tag{26}$$

There exists locally a solution of (26) with equality sign, such that $u(0) = u_0$. This solution $u = U^+(u_0; R; \rho)$ can be continued throughout an interval $s < \rho^+(u_0; R) \leq R$, thus establishing the stated upper bound. The lower bound follows from the same reasoning, applied to the equation

$$\frac{du}{ds} > -\mathcal{F}(R - s; u). \tag{27}$$

Corollary M1 shows that among all solutions $u(\mathbf{x})$ of (6) in a domain Ω, if $\mathbf{x}_0 \in \Omega$ and $|u(\mathbf{x}_0)| < U_0 < \infty$, there is a disk $B_{\rho_0}(\mathbf{x}_0)$ in which $|u(\mathbf{x})|$ is bounded, depending only on U_0 and on Ω. It then follows from Theorem M that $|Du(\mathbf{x})|$ is also bounded in $B_{\rho_0}(\mathbf{x}_0)$, and thus that (6) is a uniformly elliptic equation in that disk. From the general theory of such equations we are led to

Corollary M2. *Under the hypotheses of Theorem M, let $x_0 \in \Omega$. Corresponding to any family of solutions $u(\mathbf{x})$ of* (6) *in Ω, for which $|u(\mathbf{x}_0| < U_0 < \infty$, there is a function $\rho_0(U_0)$ such that interior to $B_{\rho_0}(\mathbf{x}_0)$ $u(\mathbf{x})$ is bounded in magnitude together with its derivatives of all orders. A subsequence may be chosen which converges uniformly in every compact subset of $B_{\rho_0}(\mathbf{x}_0)$, together with all derivatives, to a solution of* (6) *in $B_{\rho_0}(\mathbf{x}_0)$.*

7. Concluding remark

We show heuristically that the two corollaries cannot be expected to apply to general subdomains of Ω. Our proof of existence of moonies reduced the problem to a general existence theorem, which could be applied (essentially) directly to the prescribed moon domain $\mathcal{M}$. It might in some ways have been preferable to proceed indirectly as was done in [F6], and start with solutions defined in the entire disk obtained by completing Σ^+, with (nearly) extremal data on Σ^+ and appropriately chosen data on the remaining portion of the boundary, and then going to the limit as the data become extremal. We may infer by analogy with [F6] that it would be possible to make a sequence of choices for the remaining data, so that for each choice a solution exists in the entire disk, and that these solutions converge in the disk to a generalized solution in the sense of Miranda [Mi], which is the desired moonie in $\mathcal{M}$ and negative infinity throughout the remaining portion of the disk. In this event, the arc Σ^- would form an entire interior arc of points in the (fixed) disk, adjacent to the domain $\mathcal{M}$ of convergence, and at which the solutions and their gradients converge simultaneously to negative infinity. That behavior could not occur, were it possible to choose either radius ρ^+ or ρ^- in Corollary M1 of a fixed size independent of u_0.

Acknowledgement

This work was supported in part by a grant from the National Aeronautics and Space Administration, and in part by a grant from the National Science Foundation. The former author wishes to express his thanks to Universität Leipzig for its hospitality during completion of the work.

References

[B] S.N. Bernstein, Sur les surfaces définie au moyen de leur courbure moyenne et totale, *Ann. Sci. École Norm. Sup.* **27** (1910), 233–256.
[BG] E. Bombieri and E. Giusti, Local estimates for the gradient of non-parametric surfaces of prescribed mean curvature, *Comm. Pure Applied Math.* **26** (1973), 381–394.
[C] K.-S. Chua, Absolute gradient bound for surfaces of constant mean curvature, *Annali Sc. Norm. Sup.* **21** (1994), 1–9.
[CF] P.Concus and R. Finn, On capillary free surfaces in the absence of gravity, *Acta Math.* **132** (1974), 177–198.
[EG] L.C. Evans and R.F. Gariepy, *Measure Theory and Fine Properties of Functions,* CRC Press, Boca Raton 1992.
[F1] R. Finn, Gradient and Gauss-curvature bounds for H-graphs, pp. 137–150. In: Xin Zhi Liu and David Siegel (Eds.), *Comparison Methods and Stability Theory*, Marcel Dekker 1994.
[F2] R. Finn, On equations of minimal surface type, *Annals of Math.* **60** (1954), 397–416.
[F3] R. Finn, New estimates for equations of minimal surface type, *Arch Rat. Mech. Anal.* **14** (1963), 337–375.
[F4] R. Finn, Remarks relevant to minimal surfaces and to surfaces of prescribed mean curvature, *J. Analyse Math.* **14** (1965), 139–160.
[F5] R. Finn, Partial differential equations whose solutions admit no isolated singularities, *Scripta Math.* **26** (1961), 107–115.
[F6] R. Finn, Moon surfaces, and boundary behaviour of capillary surfaces for perfect wetting and nonwetting, *Proc. London Math. Soc.* **57** (1988), 542–576.
[F7] R. Finn, *The inclination of an H-graph*, Springer-Verlag Lecture Notes **1340** (1988), 40–60.
[F8] R. Finn, *Equilibrium Capillary Surfaces*, Springer-Verlag, New York 1986.
[F9] R. Finn, A note on the capillary problem, *Acta Math.* **132** (1974), 199–206.
[FG] R. Finn and E. Giusti, On nonparametric surfaces of constant mean curvature, *Ann. Scuola Normale Sup. Pisa* **4** (1977), 13–31.

[G1] E. Giusti, On the equation of surfaces of prescribed mean curvature: existence and uniqueness without boundary conditions, *Invent. Math.* **46** (1978), 111–137.
[G2] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation,* Birkhäuser Verlag, Boston 1984.
[H] E. Heinz, Über Flächen mit eindeutiger Projektion auf einer Ebene, deren Krümmungen durch Ungleichungen eingeschränkt sind, *Math. Ann.* **129** (1955), 451–454.
[Ha] P. Hartman, On elliptic partial differential equations and uniqueness theorems for closed surfaces, *J. Math. Mech.* **7** (1958), 377–392.
[Hi] D. Hilbert, Über das Dirichlet'sche Princip, *Jahresber. deutschen Math.-Ver.* **8** (1900), 184–188.
[JS] H. Jenkins and J. Serrin, Variational problems of minimal surface type I, *Arch. Rat. Mech. Anal.* **12** (1963), 185–212.
[K] N.J. Korevaar, An easy proof of the prescribed gradient bound for solutions to the prescribed mean curvature equation, *Proc. Symp. Pure Math.* **45** (1986), Part 2, 81–89.
[L] G.M. Lieberman, Gradient estimates for capillary type problems via the maximum principle, *Comm. in P.D.E.* **13** (1988), 33–59.
[Li] F.-T. Liang, An absolute gradient bound for nonparametric surfaces of prescribed mean curvature, *Ind. Univ. Math. J.* **41** (1992), 569–604.
[M] U. Massari, Esistenza e regolaritá delle ipersuperfici di curvatura media assegnata in $\mathbb{R}^n$, *Arch. Rat. Mech. Anal.* **55** (1974), 257–282.
[Mi1] M. Miranda, Superfici cartesiane generalizzate ed insiemi di perimetro finito sui prodotti cartesiani, *Ann. Scuola Norm. Sup. Pisa* **18** (1964), 515–542.
[Mi2] M. Miranda, Superfici minime illimitate, *Ann. Scuola Norm. Sup. Pisa* **4** (1977), 313–322.
[S] F. Sauvigny, A new proof of the gradient estimate for graphs of prescribed mean curvature, *Manuscr. Math.* **74** (1992), 83–86.
[Se] J.B. Serrin, The Dirichlet problem for surfaces of constant mean curvature, *Proc. Lon. Math. Soc.* **21** (1970), 361–384.
[Si] L. Simon, Equations of mean curvature type in two independent variables, *Pac. J. Math.* **69** (1977), 245–268.

Robert Finn
Mathematics Department
Stanford University
Stanford, CA 94305-2125, USA
e-mail: finn@gauss.stanford.edu

Jianan Lu
IBM Corporation
1055 Joaquin Road
Mountain View, CA 94043, USA
e-mail: jianan@us.ibm.com

(Received: April 22, 1996)