Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 73 (1998)

Artikel: Approximations of stable actions on R-trees
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DOI: https://doi.org/10.5169/seals-55096

Comment. Math. Helv. 73 (1998) 89–121
0010-2571/98/010089-33 $ 1.50+0.20/0

Commentarii Mathematici Helvetici

Approximations of stable actions on $\mathbb{R}$-trees

Vincent Guirardel

Abstract. This article shows how to approximate a *stable* action of a finitely presented group on an $\mathbb{R}$-tree by a *simplicial* one while keeping control over arc stabilizers. For instance, every small action of a hyperbolic group on an $\mathbb{R}$-tree can be approximated by a small action of the same group on a *simplicial* tree. The techniques we use highly rely on Rips's study of stable actions on $\mathbb{R}$-trees and on the dynamical study of exotic components by D. Gaboriau.

Mathematics Subject Classification (1991). 20E08, 20E06, 20F32, 05C05.

Keywords. Stable action, $\mathbb{R}$-trees, Rips theorem, approximation, splitting, Bass-Serre.

M. Bestvina and M. Feighn introduced the notion of *stable* action of a group on an $\mathbb{R}$-tree ([BF2]) by slightly weakening some conditions by E. Rips or by H. Gillet and P. Shalen. Roughly speaking, stable actions are such that the stabilizer of an arc must stabilize when this arc gets smaller and smaller (see section 1 for a formal definition). This condition is true in usual cases: any action with trivial arc stabilizer, the actions coming from iteration of automorphisms of free groups or from degeneracy of hyperbolic structures, and every small action of a hyperbolic group is stable (see section 1).

The main theorem about stable actions is Rips's theorem (see [BF2]): if a finitely presented group Γ has a non trivial stable action on an $\mathbb{R}$-tree, then Γ splits over a group C which is an extension of $\mathbb{Z}^k$ by a subgroup of Γ fixing an arc in T.

On the other hand, M. Cohen and M. Lustig have introduced *very small* actions on $\mathbb{R}$-trees and have shown that the set of free actions of the free group F_n on simplicial $\mathbb{R}$-trees is dense in the space of very small actions of F_n on simplicial $\mathbb{R}$-trees ([CL]). Then, M. Bestvina and M. Feighn showed that every very small action of F_n on an $\mathbb{R}$-tree can be approximated by a very small action on a simplicial $\mathbb{R}$-tree ([BF3]). This showed that the closure of M. Culler and K. Vogtmann's outer space is the projectivised set of very small actions of F_n on $\mathbb{R}$-trees.

Our theorem is both a refinement of E. Rips' splitting theorem, and a generalisation of M. Bestvina and M. Feighn's approximation theorem:

Theorem 1. *Let Γ be a finitely presented group. Every minimal stable action of*

Γ on an $\mathbb{R}$-tree T can be approximated in the equivariant Gromov topology by an action of Γ on a simplicial tree, such that edge stabilizers are an extension of $\mathbb{Z}^k$ by a subgroup of Γ fixing an arc in T.

Moreover, if $\Gamma_1, \ldots, \Gamma_k \subset \Gamma$ are finitely generated subgroups of Γ which fix a point in T, they may be asked to fix a point in the approximation.

Note that edge stabilizers in the approximation can't be shown to be extensions of $\mathbb{Z}$ by a subgroup of Γ fixing an arc in T since there are free actions of $\mathbb{Z}^3$ on $\mathbb{R}$ and that $\mathbb{Z}^3$ doesn't split over $\mathbb{Z}$.

Corollary 1. *Let Γ be a hyperbolic group. Every small action of Γ on an $\mathbb{R}$-tree can be approximated by a small action of Γ on a simplicial tree.*

This corollary is a straightforward consequence of the theorem (see section 1). One may ask if a similar fact is true for very small actions of hyperbolic groups. M. Bestvina and M. Feighn have shown that it is true in the case of the free group in [BF3] essentially using the fact that the group is free. Using techniques similar to [BF3], we could show that very small actions of hyperbolic groups can be approximated by very small simplicial actions if there was a positive answer to the following question in surface theory:

Consider a nonorientable surface. Then, there is a nowhere dense closed subset of the projective set of measured foliations consisting of those foliations containing no 1-sided compact regular leaf (see [DN]). Does the mapping class group of the surface act with dense orbits on this closed invariant subset?

Bounded backtracking property for an action (T, F) of a finitely generated free group ([GJLL]) is a notion that generalizes bounded cancellation (see [Coo]). Namely, (T, F) has bounded backtracking if given $Q \in T$, there exists $C > 0$ such that for any reduced words v, w such that the product vw has no cancellation, then $d(v.Q, [Q, vw.Q]) \leq C$.

Bounded backtracking property is used in [BFH] to link laminations on graphs and actions on $\mathbb{R}$-trees. In [GJLL], it allows to define an injective equivariant map from the boundary at infinity of an action (T, F) to the boundary at infinity of F. In [BFH] is proven the fact that every very small action of F has bounded backtracking (also see [DV,GJLL]). This generalizes to small actions using corollary 1:

Corallary 2. *Every small action of a finitely generated free group F has bounded backtracking.*

Here is a sketch of the proof of Theorem 1, given in sections 2 to 8: starting with a minimal stable action of a finitely presented group Γ on an $\mathbb{R}$-tree T, we approximate it without increasing arc stabilizers by a geometric action built from the "restriction" of the original action on a big finite subtree like in [GLP1] or [BF2]. This leads to a system of isometries, which has a canonical dynamical decomposition into minimal and simplicial components (see [GLP1, BF2]). This provides a graph of actions on $\mathbb{R}$-trees such that its non trivial vertex actions correspond to its minimal components. Using some results of [BF2] coming from the stability hypothesis, an idea of D. Gaboriau shows that there is a quotient of this geometric action with same underlying graph of groups such that the nontrivial vertex actions are given by a morphism from the vertex group Γ_v onto a group Γ'_v acting with trivial arc stabilizers on an $\mathbb{R}$-tree T'_v, the action (T'_v, Γ'_v) being a geometric action corresponding to a minimal system of isometries. So, there remains to approximate the actions (T'_v, Γ'_v).

In the homogeneous case, the tree T'_v is a line (see [BF2, Pau4]) and (T'_v, Γ'_v) can be easily approximated. In the exotic and surface case, a theorem by D. Gaboriau allows to make generators independent. The surface case is then easy to approximate thanks to surface theory. In the exotic case, we perform the pruning process (or process I in [BF2]), and show that we can narrow some band so as to decrease the number of ends of singular leaves and so that the obtained action stays close to the original one. Therefore, after finitely many steps, we'll get a close simplicial action, approximating the original one.

I warmly thank Damien Gaboriau for many improvements and suggestions: the original statement applied only for small actions of hyperbolic groups and he is at the origin of the generalisation of the proof to the stable case. I also thank my advisor Gilbert Levitt for his rigour and his patience in finding out many errors, and Frédéric Paulin for many enriching conversations and helpful comments.

1. Definitions and preliminaries

The actions we consider are all isometric actions of finitely generated groups on $\mathbb{R}$-trees. An action on an $\mathbb{R}$-tree is termed *minimal* if it has no invariant subtree. Note that if an action on an $\mathbb{R}$-tree has no global fixed point, then there is an unique invariant minimal subtree which is the union of all translation axes (see [CuMo]). All the actions we consider are assumed to be minimal. If an action on an $\mathbb{R}$-tree has a dense orbit, then it is termed *transitive*. Since the actions we consider are isometric actions, every orbit of a transitive action is dense. We say that the orbits of an action (T, Γ) are dense in the segments if for all $x \in T$ and for all segment I not reduced to one point, $\Gamma.x \cap I$ is dense in I. Of course, this property implies that (T, Γ) is transitive.

We will identify two actions on $\mathbb{R}$-trees if there exists an equivariant isometry between them. When no confusion is possible, we will simply talk about an *action* or a *tree* to mean a (minimal) action on an $\mathbb{R}$-tree. A simplicial $\mathbb{R}$-tree (or simply a simplicial tree) is a connected and simply connected simplicial 1-complex together with a path metric that makes it an $\mathbb{R}$-tree. We'll also say a *simplicial* action to mean a simplicial action on a simplicial $\mathbb{R}$-tree. Note that the stabilizer of a set should be understood in all this paper as its pointwise stabilizer unless explicitly mentioned.

Now, let's recall the definition of stability ([BF2]). Let (T,Γ) be an action. A subtree K of T is termed *nondegenerate* if it contains more than one point. A nondegenerate subtree K of T is termed *stable* if for any nondegenerate subtree $K' \subset K$, $\operatorname{Stab} K = \operatorname{Stab} K'$.

Definition 1.1. *An action (T,Γ) is termed stable if any nondegenerate segment contains a nondegenerate stable subsegment.*

For instance, any action with trivial arc stabilizers, any simplicial action is stable.

Definition 1.2. *An action of a group Γ on an $\mathbb{R}$-tree T is said to be small if for every nondegenerate arc $I \subset T$, the pointwise stabilizer* $\operatorname{Stab} I$ *of I doesn't contain the nonabelian free group F_2 on 2 generators.*

Note that a small action of a hyperbolic group Γ is stable. As a matter of fact, when a subgroup of a hyperbolic group doesn't contain F_2, it is finite or virtually cyclic. Moreover, there is a bound to the cardinal of finite subgroups of Γ; and a virtually cyclic subgroup of Γ is contained in at most finitely many virtually cyclic subgroups of Γ. Now, given a nondegenerate arc I in an $\mathbb{R}$-tree T endowed with a small action of Γ, we can assume either that the stabilizer of every nondegenerate subinterval of I is finite or that $\operatorname{Stab} I$ is virtually cyclic. In both cases, take a nondegenerate $J \subset I$ such that $\operatorname{Stab} J$ is maximal for inclusion in the finite set of stabilizers of nondegenerate subintervals of I.

The sets of actions of Γ on $\mathbb{R}$-trees modulo equivariant isometry are endowed with the equivariant Gromov topology (see [Paul]). This topology roughly says that two actions are close if they look the same in restriction to a finite set while only considering the action of a finite subset of Γ. More precisely, for two actions (T,Γ) and (T',Γ), given any $\varepsilon > 0$, any finite subset F of Γ, and two finite subsets $\{x_1,\dots,x_p\} \subset T$, $\{x'_1,\dots,x'_p\} \subset T'$, we say that there is an F-equivariant ε-approximation between $\{x_1,\dots,x_p\}$ and $\{x'_1,\dots,x'_p\}$ if

$$\forall g,h \in F\ \forall i,j \in \{1,\dots,p\} \quad |d_T(g.x_i,h.x_j) - d_{T'}(g.x'_i,h.x'_j)| < \varepsilon.$$

Sometimes, we'll term x'_i the approximation point of x_i.

Now, here is a neighbourhood basis for the equivariant Gromov topology: for any $\varepsilon > 0$, any finite subset F of Γ, and any finite subset $\{x_1,\dots,x_p\}$ of T, take the set $V_T(\varepsilon,F,\{x_1,\dots,x_p\})$ consisting of actions (T',Γ) such that there exists $\{x'_1,\dots,x'_p\} \subset T'$ with an F-equivariant ε-approximation between $\{x_1,\dots,x_p\}$ and $\{x'_1,\dots,x'_p\}$.

The usual topology for sets of actions of a given group on $\mathbb{R}$-trees is the *translation length topology.* This topology is based on the *length function* of an action

(T,Γ). It is the function $l_T : \Gamma \to \mathbb{R}_+$ defined by

$$l_T(\gamma) = \inf_{x \in T} d(x, \gamma.x).$$

The translation length topology is the smallest topology that makes continuous the functions $T \mapsto l_T(\gamma)$ for $\gamma \in \Gamma$. For sets of nonabelian (or irreducible) actions of a finitely generated group, (an abelian action is an action whose length function is the absolute value of a morphism $\Gamma \to \mathbb{R}$), this topology is Hausdorff (see [CuMo]), and is equivalent to the equivariant Gromov topology (see [Pau1]). And the equivariant Gromov topology is always finer than the translation length topology. Moreover, the space of small actions of a finitely generated group on $\mathbb{R}$-trees modulo equivariant isometries is closed in the set of all actions for both topologies. Furthermore, the projectivised space of its small actions on $\mathbb{R}$-trees is compact. (see [CuMo]).

In all this paper, we will only deal with approximations for the equivariant Gromov topology unless explicitly mentioned.

Proof of corollary 2. Let (T,Γ) be a small action of a hyperbolic group. As noticed above, (T,Γ) is stable. Hence, we can apply Theorem 1: (T,Γ) can be approximated by a small simplicial action (T',Γ) such that its arc stabilizers are extensions of $\mathbb{Z}^k$ by a small subgroup of Γ. Therefore, (T',Γ) must be small. □

2. Approximation by a geometric action

In this section, we recall some results of [LP]:

Proposition 2.1. [LP] *Let (T,Γ) be a minimal action of a finitely presented group Γ on an $\mathbb{R}$-tree T. We can approximate (T,Γ) by a geometric action $(T_{\bar{\Sigma}_\rho},\Gamma)$ so that arc stabilizers of $T_{\bar{\Sigma}_\rho}$ fix an arc in T.*

Moreover, if $\Gamma_1,\ldots,\Gamma_k \subset \Gamma$ are finitely generated subgroups of Γ which fix a point in T, they may be asked to fix a point in $T_{\bar{\Sigma}_\rho}$.

First, let's fix notations and terminology (see [LP, GLP1, Pau4]). Let $< S || \mathcal{R} >$ be a finite presentation of Γ. Let D be a big finite subtree of T (i. e. the convex hull of a finite number of points). For each generator $g \in S$, consider the partial isometry $\varphi_g = g_{|D\cap g^{-1}(D)}$ which is the maximal restriction of g going from D to D, and denote $X = (D, \{\varphi_g\})$ the system of isometries with domain D and generators $\{\varphi_g\}$. Two points of D are in the same orbit if there exists a word in the generators $\{\varphi_g\}$ and their inverses that takes one to the other. We call $\operatorname{dom}\varphi$ and $\operatorname{Im}\varphi$ the *bases* of a generator φ. We say that a generator is a *singleton* if its bases are reduced to one point. We suppose that D is big enough so that every φ_g

is not a singleton, and each relation in $\mathcal{R}$ "expresses itself" in X, i. e. if r is the relation corresponding to the word $g_1^{\varepsilon_1}\dots g_n^{\varepsilon_n}$, then $\mathrm{dom}(\varphi_1^{\varepsilon_1}\dots\varphi_n^{\varepsilon_n})\neq\emptyset$. If the finitely generated subgroup Γ_i fixes a point x_i in T, we also choose D big enough so that the domains of the words corresponding to a finite generating set of Γ_i contain x_i.

We next build a 2-complex Σ by gluing on D, for each generator φ of X, a band $(\mathrm{dom}\,\varphi)\times[0,1]$ where $(x,0)$ and $(x,1)$ are glued with x and $\varphi(x)$ respectively. Each band is foliated by $\{*\}\times[0,1]$, and we consider the transverse measure which gives to every arc of D a measure equal to its length. Given a point $x\in D$ and a word w in the generators of the system of isometries containing x in its domain, one denotes $[x;w]$ the path in Σ contained in a leaf starting from x, following successively the bands corresponding to the word w and stopping at the point $w(x)$.

Given any base point $*$ in D, the fundamental group of Σ is canonically identified with the free group with free basis S by sending an element φ of S to a path starting at $*$, going inside D to a point $x\in\mathrm{Im}\,\varphi$ then following $[x,\varphi^{-1}]$ and going back to $*$ inside D. So we naturally get a morphism $\rho:\pi_1(\Sigma,*)\to\Gamma$. When no ambiguity is feared, we will denote $\pi_1(\Sigma)$ instead of $\pi_1(\Sigma,*)$. We consider $\bar{\Sigma}_\rho$ the covering space of Σ corresponding to ρ. For every relation $r\in\mathcal{R}$, since it expresses itself in X, we can choose a curve $[x;r]$ contained in a leaf whose free homotopy class represents r in $\pi_1(\Sigma)$. Denote $\mathcal{C}$ the (finite) set of chosen curves. Note that $\ker\rho$ is normally generated by the free homotopy classes of the curves in $\mathcal{C}$. The measured foliation on Σ lifts to $\bar{\Sigma}_\rho$. Now, choose any lift $\bar{D}$ of D. There is a unique Γ-equivariant map $f_{\bar{\Sigma}_\rho}:\bar{\Sigma}_\rho\to T$, which is equal to the covering map p in restriction to $\bar{D}$ and is constant on the leaves of $\bar{\Sigma}_\rho$. Thus, every connected component of $p^{-1}(D)$ isometrically embeds into $T_{\bar{\Sigma}_\rho}$ through the quotient map $\pi:\bar{\Sigma}_\rho\to T_{\bar{\Sigma}_\rho}$.

To sum up, we say we have a resolution of the action (T,Γ):

Definition 2.2. *A resolution of an action (T,Γ) is*

i) a finite metric graph D whose components are 1-connected

ii) a system of isometries X with domain D providing a connected foliated finite 2-complex Σ

iii) a base point $*\in D\subset\Sigma$

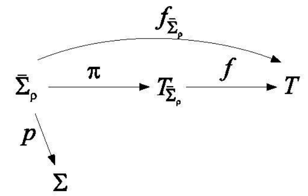
iv) a morphism ρ from $\pi_1(\Sigma,*)$ onto Γ with corresponding covering map $p:\bar{\Sigma}_\rho\to\Sigma$

v) a set $\mathcal{C}$ of curves contained in leaves that normally generate $\ker\rho$ *in $\pi_1(\Sigma)$*

vi) a Γ-equivariant map $f_{\bar{\Sigma}_\rho}:\bar{\Sigma}_\rho\to T$, constant on every leaf, which isometrically embeds any connected component of $p^{-1}(D)\subset\bar{\Sigma}_\rho$ into T.

Note that unlike [BF2, Pau4], the set $\mathcal{C}$ is not required to be finite.

Such a resolution provides an $\mathbb{R}$-tree in this way: there is a pseudo-metric on $\bar{\Sigma}_\rho$ obtained by integration of the transverse measure. According to [Lev3] (also



see [LP]), by making this pseudo-metric Hausdorff, we get an $\mathbb{R}$-tree $T_{\bar{\Sigma}_\rho}$. It is sometimes termed the *leaf space made Hausdorff* of $\bar{\Sigma}_\rho$. It is naturally endowed with an isometric action of Γ. Every connected component of $p^{-1}(D)$ isometrically embeds into $T_{\bar{\Sigma}_\rho}$ through the quotient map $\pi : \bar{\Sigma}_\rho \to T_{\bar{\Sigma}_\rho}$. This allows us to identify the finite tree D with the subtree of $T_{\bar{\Sigma}_\rho}$ given by $\pi(\bar{D})$. Moreover, the map $f_{\bar{\Sigma}_\rho} : \bar{\Sigma}_\rho \to T$ passes to the quotient into $f : T_{\bar{\Sigma}_\rho} \to T$. It is an isometric embedding in restriction to D thus it is a *morphism of* $\mathbb{R}$*-trees*, which means that every segment I of $T_{\bar{\Sigma}_\rho}$ can be subdivided into a finite number of sub-segments which isometrically embed into T through f (see [LP]). Since $\ker \rho$ is generated by free homotopy classes of loops that are contained in leaves, we can apply Lemma 3.4 of [LP] which says that the natural map from the set of leaves of $\bar{\Sigma}_\rho$ to $T_{\bar{\Sigma}_\rho}$ is one to one outside a countable set.

Now, [LP] shows that $T_{\bar{\Sigma}_\rho}$ converges (strongly) to T when D gets bigger. Moreover, since f is a morphism of $\mathbb{R}$-trees if I is an arc in $T_{\bar{\Sigma}_\rho}$, I contains a nondegenerate sub-interval J which isometrically injects into T via f, so

$$\operatorname{Stab} I \subset \operatorname{Stab} J \subset \operatorname{Stab} f(J).$$

Finally, it is clear by construction that the finitely generated subgroups Γ_i fix a point in $T_{\bar{\Sigma}_\rho}$. This shows the proposition.

3. Pure components of Σ

Like in [GLP1, BF2, Pau4], we are first going to perform some *Rips* moves on our resolution. These moves are transformations that preserve the action $(T_{\bar{\Sigma}_\rho}, \Gamma)$ and the map $f : T_{\bar{\Sigma}_\rho} \to T$.

The first Rips move is *the base subdivision.* To carry out this operation, take a generator φ of the system of isometries, and choose a point c in the interior of one of its bases. If necessary, change φ to φ^{-1} so that $c \in \operatorname{dom} \varphi$. Then replace φ by its restrictions $\varphi_1, \dots, \varphi_p$ to the closure of the connected components of $(\operatorname{dom}\varphi) - \{c\}$ and add to $\mathcal{C}$ the $p-1$ curves $[c; \varphi_p^{-1}\varphi_1]$. The base point $*$ remains unchanged, and the new morphism ρ is induced by the collapsing of $[c; \varphi_p^{-1}\varphi_1]$. This gives a new covering space we still denote $\bar{\Sigma}_\rho$. The subgroup $\ker \rho$ is still normally generated by the curves in $\mathcal{C}$, and $T_{\bar{\Sigma}_\rho}$ and f can be seen to be unchanged.

The second Rips move is the *domain cut*: first choose a point c in the interior of the domain D of the system of isometries. Then subdivide each base containing c in its interior (using the Rips move described above). Then cut the domain D at c and denote $(D'_i)_{i=1}^p$ the connected components of the new domain D' containing a copy of c, and denote c_i the copy of c in D'_i. First add, for all $i \neq 1$ a new singleton s_i sending c_1 on c_i so that collapsing the s_i recovers D. Each generator for which none of its bases is reduced to $\{c\}$ has a unique corresponding generator in D'; and given a singleton having a base equal to $\{c\}$, change this base to any of $\{c_i\}$. This provides a new system of isometries on D'. Finally, replace the curves in $\mathcal{C}$ by the natural corresponding curves in the new 2-complex. We don't change the base point when $c \neq *$, and we take $*$ to be any of the c_i otherwise. Collapsing the singletons s_i then gives an identification between the fundamental groups of Σ and Σ' so naturally provides a new morphism $\rho' : \pi_1(\Sigma') \to \Gamma$. This operation can easily be seen to be a Rips move.

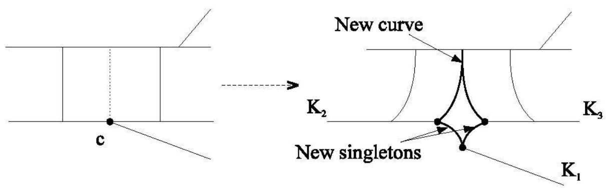


Figure 1.
Domain cut

Recall that a *branch point* of an $\mathbb{R}$-tree T is a point b such that $T - \{b\}$ has at least 3 connected components. For simplicial trees, a branch point is a vertex with valence at least 3. Now, by cutting the domain D of X at every branch point of D, we get a new resolution whose domain is a multi-interval we still denote D. Once we have a system X of isometries on a multi-interval, we can consider the corresponding *open* system of isometries $\mathring{X}$: it has same domain D, and its generators are the restrictions of the generators of X to the interior of their domains. We call an orbit *singular* if it contains a point in the boundary of some base, and we call an $\mathring{X}$-orbit *non orientable* if there is an $\mathring{X}$-word w fixing a point in this orbit and reversing orientation. Let Σ^* denote Σ minus its singletons. Now cut the domain at the points of every finite singular or non orientable $\mathring{X}$-orbit, so that each finite singular $\mathring{X}$-orbit is reduced to one point in ∂D. Thanks to an Imanishi theorem (see [GLP1]), the connected components of Σ^* are either an *orientable family of finite orbits* (each $\mathring{X}$-orbit intersects a component of D in at most one point), or a *minimal component* (each $\mathring{X}$-orbit which intersects a component I of D is dense in I). When a system of isometries satisfies these conditions, we say that it has pure components. See [GLP1] for more details.

Starting with this dynamical decomposition of Σ, we aim to get the following proposition:

Proposition 3.1. *We can construct a finite graph of actions on $\mathbb{R}$-trees $\mathcal{G}$ with fundamental group Γ such that the corresponding action $(T(\mathcal{G}), \Gamma)$ is equivariantly isometric to $(T_{\tilde{\Sigma}_\rho}, \Gamma)$. When a vertex tree T_v is not reduced to one point, the corresponding vertex group Γ_v is the image (by ρ) in Γ of the fundamental group of a minimal component Σ_v; there is a (maybe infinite) set of loops $\mathcal{C}_v$ contained in leaves of Σ_v that normally generate the kernel of the morphism $\rho_v = \rho_{|\pi_1(\Sigma_v)}$; and T_v is the leaf space made Hausdorff of the covering space $\tilde{\Sigma}_v$ of Σ_v corresponding to ρ_v, (T_v, Γ_v) is minimal and the orbits of (T_v, Γ_v) are dense in the segments of T_v.*

Recall (see [Lev4]) that a graph of actions on $\mathbb{R}$-trees consists of

- a metric graph of groups, with vertex groups Γ_v, edge groups Γ_e, and monomorphisms $i_e : \Gamma_e \to \Gamma_{t(e)}$
- an action of each vertex group Γ_v on a (maybe degenerate) $\mathbb{R}$-tree T_v
- a point $p_e \in T_{t(e)}$ fixed by $i_e(\Gamma_e) \subset \Gamma_{t(e)}$ for every oriented edge e.

We use the same notations for graphs as [Ser] section 2.1 p. 22: an edge has terminal vertex $t(e)$, has origin $o(e)$, and $\bar{e}$ is the edge with opposite orientation. Unlike in [Lev4], we allow an edge of the graph of groups to have length 0. Here, the vertex actions we will consider either have dense orbits in the segments (and hence are transitive) or are degenerate (i. e. T_v is a single point).

Proof. Denote by Y the foliated complex obtained from Σ by gluing a disk along the curves of $\mathcal{C}$. These disks should be understood to be contained in a leaf. Recall that Σ^* denotes Σ minus its singletons. Let Y^* be the union of Σ^* and of all the disks D such that $\partial D \subset \Sigma^*$. Now, since Σ^* has pure components, we get a metric graph $\mathcal{G}$ in the following way: take one vertex v for each minimal component Y_v of Y^*, one vertex for each connected component of the closure of $Y - \Sigma^*$, and, for every simplicial component Y_0 of Y^*, add 2 vertices corresponding to the complement in Y_0 of its boundary components. Add an edge between 2 vertices for each component of their intersection, its length being 0 if it is a subset of a leaf and $l > 0$ for an edge corresponding to a simplicial component of width l.

To get the graph of groups $\mathcal{G}$, we want to take as edge and vertex groups the fundamental group of the corresponding subcomplex of Y. So, for each edge or vertex component Y_e, Y_v, choose a base point $*_e \in D \cap Y_e$ and $*_v \in D \cap Y_v$, and take $*_v = *$ (and $*_e = *$) if $* \in Y_v$ (resp. if $* \in Y_e$). Whenever $Y_e \subset Y_v$, choose a path $p_{e\to v}$ in $Y_v \cap \Sigma$ joining $*_e$ to $*_v$ (take the constant path if $*_e = *_v$). Thus, the inclusion gives edge morphisms $i_e : \pi_1(Y_e, *_e) \to \pi_1(Y_v, *_v)$. But we must ensure that the edge morphisms are one-to-one. Thanks to Bass-Serre theory, this will imply that the fundamental group of Y_v injects into $\pi_1(Y)$. That's why we change $\mathcal{C}$ and Y: we add to $\mathcal{C}$ (maybe infinitely many) curves in the edge subcomplexes

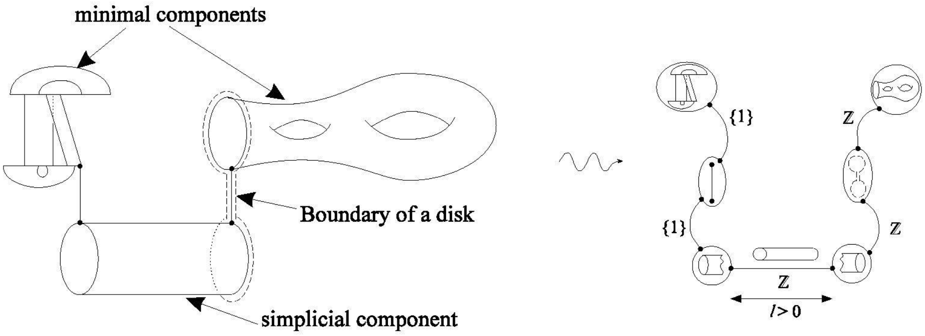


Figure 2.
The graph of groups

Y_e and we glue the corresponding disks to Y so as to kill any loop in Y_e which would be nullhomotopic in Y but not in Y_e. This doesn't change the graph we have constructed. Since an edge subcomplex Y_e is either contained in a leaf or is the interior of a simplicial component $(0, l) \times L$ with product foliation, any set of loops in an edge subcomplex Y_e is homotopic to a set of loops contained in a leaf of Y_e so we can ensure that the loops in $\mathcal{C}$ are contained in a leaf. Note that although there may be infinitely many loops in $\mathcal{C}$, they are contained in a compact graph whose connected components are contained in some leaves of Σ.

Then, choose a maximal subtree τ in $\mathcal{G}$. Using the paths $p_{e \to v}$ for $e \in \tau$, the inclusion provides monomorphisms $\pi_1(Y_v, *_v) \to \pi_1(Y, *)$ and by applying Van Kampen theorem, we get an isomorphism between $\pi_1(\mathcal{G}, \tau)$ and $\pi_1(Y, *) \simeq \Gamma$. Note that the edge groups are finitely generated and that if e is an edge with positive length, Γ_e fixes an arc in T.

For each vertex v and edge e, let Σ_v (resp. Σ_e) be the intersection of Y_v (resp. Y_e) with Σ. Now, the paths $p_{e \to v}$ for $e \in \tau$ also give an identification of $\pi_1(\Sigma_v, *_v)$ with a subgroup of $\pi_1(\Sigma, *)$. Let $\rho_v : \pi_1(\Sigma_v, *_v) \to \Gamma_v$ be the restriction of ρ to $\pi_1(\Sigma_v, *_v)$ and where $\Gamma_v = \rho(\pi_1(\Sigma_v, *_v))$. The set $\mathcal{C}_v$ of loops of $\mathcal{C}$ contained in Σ_v normally generates $\ker \rho_v$.

Let's now explain how to get a graph of actions on $\mathbb{R}$-trees. Each vertex group Γ_v has a natural embedding into $\pi_1(\mathcal{G}, \tau) \simeq \Gamma$. Now choose an orientation A of the set of edges of $\mathcal{G}$. Denote $V(\mathcal{G})$ the set of vertices and $E(\mathcal{G})$ the set of oriented edges of $\mathcal{G}$ whose orientation agree with A. Then each edge group is naturally identified with $i_{\bar{e}}(\Gamma_e) \subset \Gamma_{t(e)}$ for each $e \in E(\mathcal{G})$ oriented according to A. Now consider the unique Γ_v-invariant connected component of $p^{-1}(\Sigma_v \setminus \Sigma_e)$ or of $p^{-1}(\Sigma_v)$ depending whether v is or is not a boundary of a simplicial component e. It is isomorphic to the covering space $\tilde{\Sigma}_v$ of Σ_v associated to the morphism $\rho_v : \pi_1(\Sigma_v) \to \Gamma_v$. Let T_v be its leaf space made Hausdorff. The tree T_v is not a point if and only if v is a minimal component of Σ. Given an oriented edge e of length 0 (resp. of length

$l > 0$) incident on a vertex $v = t(e)$, we consider the unique lift $\bar{\Sigma}_e$ of Σ_e (resp. $\Sigma_v \setminus \Sigma_e$) in $\tilde{\Sigma}_v$ which is Γ_e invariant. Since $\Sigma_e \cap \Sigma_v$ is a subset of a leaf, $\bar{\Sigma}_e$ maps to a point p_e in T_v and it is fixed by $i_e(\Gamma_e)$. So we get a graph of actions on $\mathbb{R}$-trees we also denote $\mathcal{G}$.

Let's show that the natural map $T_v \to T_{\bar{\Sigma}_\rho}$ induced by the inclusion $\tilde{\Sigma}_v \hookrightarrow \bar{\Sigma}_\rho$ is an isometry (this will allow to see T_v as a subtree of $T_{\bar{\Sigma}_\rho}$). Let $x, y \in \tilde{\Sigma}_v$ and let p be a path joining them in $\bar{\Sigma}_\rho$. We aim to find a path in $\tilde{\Sigma}_v$ which is not longer that p relatively to the transverse measure. Consider the Bass-Serre tree for the graph of groups $\mathcal{G}$. The path p provides a loop in this tree starting at the point representing $\tilde{\Sigma}_v$. It can be decomposed as a succession of loops based at the same point and which go forth and back through the same edge. Thus, p is the product of some paths in $\tilde{\Sigma}_v$ and of some path that leave $\tilde{\Sigma}_v$ and come back into $\tilde{\Sigma}_v$ through the same lift $\bar{\Sigma}_e$ of an edge component Σ_e. Since every edge component incident on $\tilde{\Sigma}_v$ is contained in a leaf, we can replace the parts of p outside $\tilde{\Sigma}_v$ by a path contained in a leaf in $\bar{\Sigma}_e \subset \tilde{\Sigma}_v$ and thus get a shorter path which lies in $\tilde{\Sigma}_v$.

Now, to show that (T_v, Γ_v) is minimal and has dense orbits in the segments, we prove the following lemma:

Lemma 3.2. *Let $\bar{\Sigma}_\rho$ be a foliated 2-complex like in the Definition 2.2 of a resolution, coming from a system of isometries with domain D, let $p^{-1}(D)$ be the preimage of D in $\bar{\Sigma}_\rho$, and let $T_{\bar{\Sigma}_\rho}$ be the $\mathbb{R}$-tree obtained by making the leaf space of $\bar{\Sigma}_\rho$ Hausdorff.*

Then, any germ of segment at a point a in $T_{\bar{\Sigma}_\rho}$ may be lifted to $p^{-1}(D)$. More precisely, if $I = [a, b]$ is a nondegenerate interval in $T_{\bar{\Sigma}_\rho}$, there exists an interval $[\bar{a}, \bar{c}]$ in $p^{-1}(D)$ such that the quotient map $\pi : \bar{\Sigma}_\rho \to T_{\bar{\Sigma}_\rho}$ is an isometry from $[\bar{a}, \bar{c}]$ to a nondegenerate subsegment $[a, c]$ of I.

Proof. Let $\bar{a}, \bar{b} \in p^{-1}(D)$ some lifts of a, b. Take α a path in $\bar{\Sigma}_\rho$ joining $\bar{a}$ and $\bar{b}$ such that α is a product of subsegments of $p^{-1}(D)$ and of paths contained in a leaf. Take the last instant t_0 for which $\pi \circ \alpha(t_0) = a$. Because $T_{\bar{\Sigma}_\rho}$ is an $\mathbb{R}$-tree, and since $\pi \circ \alpha$ is a path joining a and b, there exists $\eta > 0$ such that $\pi \circ \alpha([t_0, t_0 + \eta]) \subset I$. There only remains to take η small enough so that $\alpha([t_0, t_0 + \eta]) \subset p^{-1}(D)$ because π isometrically embeds the connected components of $p^{-1}(D)$. □

This lemma implies that the orbits of (T_v, Γ_v) are dense in the segments for any v corresponding to a minimal component of Σ. As a matter of fact, the trace of the Γ_v orbit of a leaf of $\tilde{\Sigma}_v$ on a component of $p^{-1}(D) \cap \tilde{\Sigma}_v$ is dense in this component. This implies that (T_v, Γ_v) has no global fixed point and that any orbit intersects the minimal subtree. Hence (T_v, Γ_v) is minimal.

Now let's explicate the action $(T(\mathcal{G}), \pi_1(\mathcal{G}, \tau))$ corresponding to the graph of actions $\mathcal{G}$ with its maximal subtree τ. We need an orientation A of the set of edges.

Recall that for each oriented edge e, there is a corresponding element $g_e \in \pi_1(\mathcal{G},\tau)$ which is trivial if and only if $e \subset \tau$. Denote $l(e)$ the length of an edge e and let s_v be any section from Γ/Γ_v to Γ such that $s_v(1.\Gamma_v) = 1$. $T(\mathcal{G})$ is by definition the set

$$\left(\coprod_{v\in V(\mathcal{G})} (\Gamma/\Gamma_v)\times T_v\right)\coprod\left(\coprod_{e\in E(\mathcal{G})}(\Gamma/\Gamma_e)\times[0,l(e)]\right)\Big/\sim$$

where $\sim$ is the equivalence relation generated by

$$(g.\Gamma_e, 0) \sim (g.\Gamma_{o(e)}, [s_{o(e)}(g.\Gamma_{o(e)})]^{-1} g.p_{\bar{e}})$$

and

$$(g.\Gamma_e, l(e)) \sim (gg_e.\Gamma_{t(e)}, [s_{t(e)}(gg_e.\Gamma_{t(e)})]^{-1} gg_e.p_e)$$

for all edge e oriented according to A. The action of Γ on $T(\mathcal{G},\tau)$ is given by

$$h.(g.\Gamma_e, x) = (hg.\Gamma_e, x)$$

and

$$h.(g.\Gamma_v, [s_v(g.\Gamma_v)]^{-1} g.x) = (hg.\Gamma_v, [s_v(hg.\Gamma_v)]^{-1} hg.x)$$

which may be put as

$$h.(g.\Gamma_v, x) = (hg.\Gamma_v, [s_v(hg.\Gamma_v)]^{-1} h s_v(g.\Gamma_v).x).$$

This definition is clearly independent of choices modulo equivariant isometry. Since we have a decomposition of $\bar{\Sigma}_\rho$ similar to this decomposition of $T(\mathcal{G})$, we see that $(T(\mathcal{G}),\Gamma)$ is isometric to $(T_{\bar{\Sigma}_\rho},\Gamma)$. □

4. Stability and minimal components

In this section due to D. Gaboriau, we use one of the key facts in the proof of Rips theorem ([BF2, Pau4]) to show the following proposition:

Proposition 4.1. [D. Gaboriau] *There exists a graph of actions $\mathcal{G}'$ with same underlying graph of groups as $\mathcal{G}$ such that*

1. *There exists Γ-equivariant morphisms of $\mathbb{R}$-trees q and f', making the following diagram commutative:*

$$\begin{array}{ccccc} T_{\bar{\Sigma}_\rho} = T(\mathcal{G}) & & \xrightarrow{f} & & T \\ & {\scriptstyle q}\searrow & & \nearrow{\scriptstyle f'} & \\ & & T(\mathcal{G}') & & \end{array}$$

2. *q is induced by Γ_v-equivariant morphisms of $\mathbb{R}$-trees $q_v : T_v \to T'_v$*
3. *The arc stabilizers of (T'_v, Γ_v) lie in the kernel N_v of (T'_v, Γ_v)*
4. *N_v fixes an arc in T*
5. *Consider $\Gamma'_v = \Gamma_v/N_v$ and $\rho'_v : \pi_1(\Sigma_v) \to \Gamma'_v$ the natural morphism. Then the action (T'_v, Γ'_v) is geometric, T'_v is the leaf space made Hausdorff of the covering space $\tilde{\Sigma}'_v$ of Σ_v corresponding to ρ'_v*
6. *(T'_v, Γ'_v) has trivial arc stabilizers and its orbits are dense in the segments of T'_v*
7. *The leaf space of $\tilde{\Sigma}'_v$ is actually Hausdorff.*

Remark 1. If (T, Γ) is a small action of a hyperbolic group, then for every vertex v corresponding to a minimal component, the group N_v must be finite. Indeed, N_v fixes an arc in T, and is normal in Γ_v. So if N_v was virtually cyclic, Γ_v should also be virtually cyclic, and Γ'_v would be finite, which is a contradiction since the orbits of (T'_v, Γ'_v) are dense in the segments.

Remark 2. Since $T(\mathcal{G})$ strongly approximates T, given finite sets $F \subset \Gamma$ and $\{x_i\} \subset T$, we can choose a big enough subtree K of T so that the resulting geometric approximation $T(\mathcal{G})$ is an $\varepsilon = 0$ F-equivariant approximation with points $\{y_i\} \in T(\mathcal{G})$ such that $f(y_i) = x_i$. Therefore, if d_T, $d_{\mathcal{G}}$, and $d_{\mathcal{G}'}$ denote the metrics for the corresponding actions, we get, for $g, h \in F$,

$$\begin{aligned} d_T(g.x_i, h.x_j) &= d_T(f(g.y_i), f(h.y_j)) \\ &\le d_{\mathcal{G}'}(q(g.y_i), q(h.y_j)) \le d_{\mathcal{G}}(g.y_i, h.y_j) = d_T(g.x_i, h.x_j). \end{aligned}$$

The last equality results from the fact that we have a 0-approximation and implies that inequalities are in fact equalities (but we don't need this fact). Therefore, $T_{\mathcal{G}'}$ is closer to T than $T_{\mathcal{G}}$ for the Gromov equivariant topology.

Also note that if a subgroup Γ_i of Γ fixes a point in $T_{\tilde{\Sigma}_\rho} = T(\mathcal{G})$, it will automatically fix a point in $T(\mathcal{G}')$.

Therefore, the proof of the theorem will reduce to the approximation of actions with trivial arc stabilizers coming from pure foliated 2-complexes by simplicial actions with free abelian edge stabilizers thanks to the following reduction lemma.

Reduction lemma. (How to deduce Theorem 1 from Propositions 5.2, 7.2, and 8.1.) *Let $\mathcal{G}'$ be a graph of actions as in prop. 4.1 with vertex actions (T'_v, Γ_v), edge morphisms $i_e : \Gamma_e \to \Gamma_{t(e)}$ and edge points $p'_e \in T'_{t(e)}$. Denote $j_e : \Gamma_e \to \Gamma'_{t(e)}$ the composition of i_e and of the quotient map $\Gamma_{t(e)} \to \Gamma'_{t(e)}$. Assume that we have approximations of (T'_v, Γ'_v) for the Gromov topology by some simplicial action (T''_v, Γ'_v) with free abelian edge stabilizers such that the edge points p'_e can be approximated in T''_v by some points p''_e fixed by the image of the edge groups $j_e(\Gamma_e) \subset \Gamma'_v$.*

Consider the simplicial tree given by the graph of actions $\mathcal{G}''$ with same underlying metric graph of groups as $\mathcal{G}'$, with vertex actions (T''_v, Γ_v) and edge points

$p''_e \in T''_{t(e)}$.

Then $T(\mathcal{G}'')$ is an approximation of $(T(\mathcal{G}'),\Gamma)$ for the Gromov topology, and its arc stabilizers are extensions of free abelian groups by groups fixing an arc in T.

Moreover, assume that any subgroup $\Gamma'_0 \subset \Gamma'_v$ fixing a point in T'_v fixes a point in T''_v, and assume that $\left(g.p'_{e_1} = p'_{e_2}\right) \Rightarrow \left(g.p''_{e_1} = p''_{e_2}\right)$ for $g \in \Gamma'_v$ and edges e_1, e_2 incident on v. Then any subgroup $\Gamma_0 \subset \Gamma$ fixing a point in $T(\mathcal{G}')$ fixes a point in $T(\mathcal{G}')$.

Proof of reduction lemma. Let's first show that $T(\mathcal{G}'')$ is an approximation of $T(\mathcal{G}')$. So, consider a neighbourhood of $T(\mathcal{G}')$ defined by $\varepsilon > 0$, a finite subset F of Γ, and $x'_1, \dots, x'_p \in T(\mathcal{G}')$. We have to approximate the distances between two points $g.x'_i$ and $h.x'_j$ for $g,h \in F$. Using notations of the proof of Proposition 3.1, and assuming for simplicity that x'_i and x'_j lie in the orbit of a vertex tree, denote $g.x'_i = (g_i.\Gamma_{v_i}, y_i)$ and $h.x'_j = (g_j.\Gamma_{v_j}, y_j)$ for some $y_i \in T_{v_i}$, $y_j \in T_{v_j}$, and some $g_i, g_j \in \Gamma$. Now recall that τ is a maximal tree in $\mathcal{G}'$ and write $g = g_j g_i^{-1}$ as a reduced word of $\pi_1(\mathcal{G}',\tau)$: $g = r_0 e_1 r_1 e_2 \dots e_q r_q$ for some path $e_1 \dots e_q$ going from v_i to v_j and some $r_i \in \Gamma_{v_i}$. Then

$$d(g.x'_i, h.x'_j) = d(y_i, r_0.p_{\bar{e}_1}) + l(e_1) + d(p_{e_1}, r_1.p_{\bar{e}_2}) + \cdots + l(e_q) + d(p_{e_q}, r_q.y_j).$$

Note that since $\{x'_i\}$ and F are finite, there are finitely many chosen words that occur. Denote M the maximum of the lengths of the chosen words, and for each vertex v, denote F_v the set of elements $r_p \in \Gamma_v$ that occur in the reduced words, and take as finite set $\{x_{j,v}\}$ of points the set consisting of p_e for edges e incident on v, and of y_i that occur and lie in T_v. Now, for each vertex v, take an F_v-equivariant ε/M-approximation T''_v of T'_v on $\{x_{j,v}\}$ such that $j_e(\Gamma_e)$ fixes p''_e for all edge e incident on v. It is now clear that we get an F-equivariant ε-approximation of $T(\mathcal{G}')$ on $\{x'_i\}$.

Now, let's check the *moreover* part of reduction lemma. Assume that $\Gamma_0 \subset \Gamma$ fixes a point $x \in T(\mathcal{G}')$, say $x = (h.\Gamma_v, y)$ (with notations of the proof of Proposition 3.1 p. 11). Up to conjugation of Γ_0, we may assume that $h = 1$. As above, write any $g \in \Gamma_0$ as a reduced word of the graph of groups $\mathcal{G}'$: $g = r_0 e_1 r_1 e_2 \dots e_q r_q$. Then

$$d(x, gx) = d(y, r_0.p'_{\bar{e}_1}) + l(e_1) + d(p'_{e_1}, r_1.p'_{\bar{e}_2}) + \cdots + l(e_q) + d(p'_{e_q}, r_q.y).$$

Since g fixes x, all those numbers must be 0. If $y \in T'_v$ doesn't lie in the orbit of any p'_e, it implies that g is a one letter word $g = r_0 \in \Gamma_v$, and that r_0 fixes y. Hence, $\Gamma_0 \subset \Gamma_v$ and we are done. Otherwise, up to conjugation, we can assume that $y = p_e$ for some edge e. The lengths of the edges e_i are 0 and $p'_{e_i} = r_i.p'_{\bar{e}_{i+1}}$. Taking $x' \in T''_v$ to be $(1.\Gamma_v, p''_e)$ and using the *moreover* hypothesis, one easily checks that g fixes x'. This shows that Γ_0 fixes a point in $T(\mathcal{G}'')$. □

Proof of Proposition 4.1. We are going to use a somewhat general construction:

Construction lemma 4.2. *Consider a graph of actions $\mathcal{G}$ with corresponding action $(T(\mathcal{G}),\Gamma)$ and for every vertex v, a normal subgroup N_v of Γ_v generated by elements fixing a point in T_v.*

Then there is an action $(T(\mathcal{G}'),\Gamma)$ given by a graph of actions $\mathcal{G}'$ and an onto 1-lipschitz equivariant map $q: T(\mathcal{G}) \to T(\mathcal{G}')$ induced by 1-lipschitz Γ_v-equivariant maps $q_v: T_v \to T'_v$ such that N_v acts trivially in (T'_v,Γ'_v) and which is maximal in the following sense:

Given a 1-lipschitz equivariant map $f: T(\mathcal{G}) \to T$ such that N_v is in the kernel of $(f(T_v),\Gamma_v)$, then there is an equivariant 1-lipschitz map $f': T(\mathcal{G}') \to T$ such that $f = f' \circ q$.

Moreover, if f is a morphism of $\mathbb{R}$-trees, then f' and q are also morphisms of $\mathbb{R}$-trees.

Proof. Fix a vertex v and consider the pseudo-metric on T_v given by

$$\delta_v(N_v.x, N_v.y) = \inf_{g,g' \in N_v} d(g.x, g'.y).$$

Since N_v is generated by elements fixing a point in T_v, th. 1 of [Lev3] tells us that by making this pseudo-metric Hausdorff, we get an $\mathbb{R}$-tree $T'_v = \widehat{T_v/N_v}$. Denote q_v the quotient maps. The graph of actions $\mathcal{G}$ provides some points $p_e \in T_{t(e)}$ fixed by $j_e(\Gamma_e)$ so there is a natural graph of actions $\mathcal{G}'$ with same underlying graph of groups as $\mathcal{G}$ defined by the edge points $p'_e = q_{t(e)}(p_e)$. The quotient maps $q_v: T_v \to T'_v$ naturally induce an equivariant map $q: T(\mathcal{G}) \to T(\mathcal{G}')$. Note that $T(\mathcal{G}')$ may be seen as the metric space obtained by making Hausdorff the following pseudo-metric on $T(\mathcal{G})$:

$$\begin{aligned}\delta(x,x') = \inf\{d(x,x_1) + d(n_1.x_1,x_2) + \ldots + d(n_p.x_p,x')|\\ x_i \in g_i.T_{v_i} \subset T(\mathcal{G}), g_i \in \Gamma, n_i \in g_i.N_{v_i}.g_i^{-1}\}\end{aligned}$$

δ can also be seen to be the maximal equivariant pseudo-metric majorated by d which identifies two points in T_v that are in the same N_v orbit.

Now, given an equivariant map $f: T(\mathcal{G}) \to T$ such that N_v is in the kernel of $(f(T_v),\Gamma_v)$, f must be constant on the 0-balls of (T,δ) so f factors through q into a map f'. This map f' must decrease distances since $d(f(.),f(.))$ is a pseudo-metric which must be majorated by δ thanks to the maximality of δ.

Now assume that $f = f' \circ q$ is a morphism of $\mathbb{R}$-trees. Then any arc in $T(\mathcal{G})$ can be subdivided into sub-arcs that are isometrically embedded through f. Thus, since both f' and q decrease distances, q must isometrically embed those sub-arcs, which means that q is a morphism of $\mathbb{R}$-trees. Now, given an arc $I' = [a',b']$ of $T(\mathcal{G}')$, choose some points a, b in $q^{-1}(a')$, $q^{-1}(b')$ respectively and take $I = [a,b]$. Since $q(I)$ is connected, it must contain I'. Subdividing I into subsegments that are isometrically embedded through f, we see that $q(I)$ is a finite tree containing I'. Therefore, by subdividing $q(I)$ at its branch points and at the images by q of subdividing points of I, the remaining pieces are segments isometrically embedded through f'. □

Let v be a vertex in $\mathcal{G}$ corresponding to a minimal component of Σ^* and let T_{Γ_v} be the minimal subtree of T invariant under the action of Γ_v. We are going to show that the orbits of $(f(T_v), \Gamma_v)$ are dense in the segments. This will imply that Γ_v has no global fixed point in T and that $f(T_v) = T_{\Gamma_v}$ since any orbit must then intersect the minimal subtree. Take a nondegenerate segment J in $f(T_v)$ and let $x \in f(T_v)$. Consider $a, b \in T_v$ some preimages of the endpoints of J and a preimage y of x. Now since f is a morphism of $\mathbb{R}$-trees, $f([a,b])$ is a finite tree, and there is a finite number of nondegenerate subintervals of $[a,b]$ whose union E is such that $f(E) = J$. Now $\Gamma_v.y \cap E$ is dense in E, so $\Gamma_v.x \cap f(E)$ is dense in $f(E) = J$. Therefore, $(f(T_v), \Gamma_v)$ has orbits dense in the segments. Hence it is minimal, and is equal to the minimal subtree T_{Γ_v}.

Now, take N_v the subgroup of Γ_v generated by its elements fixing an arc in T_v. Here is the key fact that requires stability that we use:

Proposition 4.3. ([BF2], cor. 5.9, [Pau4], lem. 2.8) *Under the hypothesis that (T, Γ) is stable, any element of Γ_v fixing (pointwise) an arc in T_{Γ_v} fixes (pointwise) the whole tree T_{Γ_v}.*

This proposition shows that N_v fixes T_{Γ_v} and therefore that N_v fixes an arc in T (which is required in the proposition we are proving). So, consider the graph of actions $\mathcal{G}'$ given by the construction lemma 4.2.

This gives us the points 1, 2, and 4 of the proposition. Now recall that T'_v was defined to be $\widehat{T_v/N_v}$, and that T_v is itself the leaf space made Hausdorff of $\tilde{\Sigma}_v$. Now, to obtain T'_v, starting from $\tilde{\Sigma}_v$, we can kill the action of N_v before killing the leaves, which exactly means that (T'_v, Γ'_v) is the leaf space made Hausdorff of the covering space $\tilde{\Sigma}'_v$ of Σ_v corresponding to $\rho'_v : \pi_1(\Sigma_v) \to \Gamma'_v = \Gamma_v/N_v$. This shows point 5.

Now we want to show that (T'_v, Γ'_v) has trivial arc stabilizers. First recall that the natural map from the space of leaves of $\tilde{\Sigma}_v$ to T_v is one to one outside a countable set (see Lemma 3.4 in [LP]). Therefore, if $g \in \Gamma_v$ fixes an arc in T_v, it fixes a leaf in $\tilde{\Sigma}_v$, so the kernel of the morphism $\pi_1(\Sigma_v) \to \Gamma'_v$ is normally generated by free homotopy classes of loops contained in leaves. Thus, like in the previous argument, we get that an element $g' \in \Gamma'_v$ fixing an arc in T'_v would fix an uncountable number of leaves in the covering of Σ_v corresponding to Γ'_v. So there exists a word in the generators of the system of isometries X_v which is a restriction of the identity on a nondegenerate interval such that the corresponding free homotopy class maps onto the conjugacy class of g' in Γ'_v. Therefore, any element $g \in \Gamma_v$ mapped to g' must fix an arc in T_v so $g \in N_v$ and $g' = 1$. Therefore, the action of Γ'_v on T'_v has trivial arc stabilizers. The last affirmation is a consequence of Theorem 6.3 of [LP] which can be generalized to a group which is not finitely presented using the fact that $\ker\rho$ is generated by free homotopy classes of loops contained in leaves. □

5. Approximation of homogeneous components

Definition 5.1. *A minimal system of isometries with domain D is termed homogeneous if there exists a nondegenerate interval $I \subset D$ and a finitely generated dense subgroup P of* $\mathrm{Isom}\,(\mathbb{R})$ *such that $x, y \in I$ are in the same $\overset{\circ}{X}$-orbit if and only if there exists $\varphi \in P$ mapping x to y.*

Proposition 5.2. *Let X_v be a pure minimal homogeneous system of isometries, Σ_v the corresponding 2-complex, and $\rho'_v : \pi_1(\Sigma_v) \to \Gamma'_v$ be a morphism whose kernel is normally generated by loops contained in leaves of Σ_v. Assume that the corresponding action (T'_v, Γ'_v) has trivial arc stabilizer.*

Then, given a finite set of points $p'_e \in T'_v$ fixed by some subgroups $\Gamma'_e \subset \Gamma'_v$, there is a simplicial approximation (T''_v, Γ'_v) of (T'_v, Γ'_v) such that the approximation point p''_e of p'_e is fixed by Γ'_e in (T''_v, Γ'_v).

Moreover, any subgroup $\Gamma'_0 \subset \Gamma'_v$ fixing a point in T'_v fixes a point in T''_v, and for $g \in \Gamma'_v$ $\left(g.p'_{e_1} = p'_{e_2}\right) \Rightarrow \left(g.p''_{e_1} = p''_{e_2}\right)$.

Proof. The action of Γ'_v on T'_v is minimal and stable since its arc stabilizers are trivial. Therefore, we could apply prop. 2.9 in [BF2] (see also th. 2.9 in [Pau4]) saying that the tree T'_v must be a line. But we give a direct proof in our setting.

The tree T'_v is a line

Denote $\bar{D}_m$ the connected components of $p^{-1}(D) \subset \tilde{\Sigma}'_v$ ordered so that $I_m = \bar{D}_m \cap \mathrm{Sat}\,(\bar{D}_1 \cup \ldots \cup \bar{D}_{m-1})$ contains more than one point, where $\mathrm{Sat}\,(A)$ is the saturate of the set A for the foliation of $\tilde{\Sigma}'_v$, i. e. the union of all leaves intersecting A. Since (T'_v, Γ'_v) has trivial arc stabilizers, lemma 3.5 and 3.4 of [LP] apply, so that I_m are closed intervals, and T'_v is obtained from $\bar{D}_1$ by successively glueing $\bar{D}_m$ isometrically along I_m. Denote K_m the finite trees thus obtained. Our aim is to recursively show that K_m is an interval.

So assume that K_{m-1} is an interval. Let $\bar{\varphi}$ be the glueing isometry between K_{m-1} and $I_m \subset \bar{D}_m$. We want to show that $\bar{\varphi}$ has a maximal domain since it will imply that K_m is a segment. So assume that $\bar{\varphi}$ hasn't a maximal domain and argue towards a contradiction. This means that I_m has an endpoint $\bar{q}$ which neither is an endpoint of $\bar{D}_m$ nor is sent to an endpoint of K_m through $\bar{\varphi}$. Since every I_i is not reduced to a point, $\bar{\varphi}(\bar{q})$ lies in the interior of some $\bar{D}_i$ that we denote by $\bar{D}_0$. Now, $\bar{\varphi}$ gives a partial isometry from a nondegenerate subinterval I'_m of I_m containing $\bar{q}$ to $\bar{D}_0$. Since x and $\bar{\varphi}(x)$ lie in the same leaf, we can choose (maybe not continuously) some path $\bar{p}_x$ from x to $\bar{\varphi}(x)$ contained in a leaf of $\tilde{\Sigma}'_v$.

Downstairs, this situation means that we have a partial isometry φ between two closed intervals I and J of D, together with some paths p_x between $x \in \operatorname{dom}\varphi$ and $\varphi(x)$ contained in a leaf such that the free homotopy class of the loop $[x,x'].p_{x'}.[\varphi(x'),\varphi(x)].p_x^{-1}$ is in the kernel of the morphism ρ'_v which defines the covering. Now apply the following lemma:

Lemma 5.3. *Let X be a pure minimal homogeneous system of isometries on a multi-interval D, and let φ be an isometry between two nondegenerate intervals I and J of $\overset{\circ}{D}$ such that for all $x \in I$, x and $\tau(x)$ are in the same X-orbit. Consider $\Phi : U \to V$ the maximal extension of φ to $\overset{\circ}{D}$.*

Then, for all $x \in U$, x and $\Phi(x)$ are in the same $\overset{\circ}{X}$-orbit, and there exists a finite set of $\overset{\circ}{X}$-words $w_1, \dots, w_p$ which are restrictions of Φ and whose (open) domains cover U.

We keep the proof of this lemma for the end of this section. So consider such words w_i and take w_{i_0} containing q in its (open) domain. Now, take $x \in \operatorname{dom} w_{i_0} \cap I$. Corresponding to the path p_x, there is an unique X-word u_x such that $[x; u_x]$ is homotopic to p_x in the leaf of Σ_v containing x. Since there are only countably many words, there exists two points $x \neq x' \in \operatorname{dom} w_{i_0}$ such that $u_x = u_{x'}$. Therefore, $u_x w_{i_0}^{-1}$ is a restriction of the identity to a nondegenerate interval, hence must fix an arc in T'_v. Since (T'_v, Γ'_v) is assumed to have trivial arc stabilizers, $u_x w_{i_0}^{-1}$ lies in $\ker \rho'_v$. This means that w_{i_0} lifts to a band of leaves joining a neighbourhood of $\bar{q} \in I_m$ to a sub-interval of $\bar{D}_0$. This contradicts the definition of $\bar{q}$. Thus, we have proved that T'_v is a line.

Proof of lemma 5.3. The fact that x and $\Phi(x)$ are in the same $\overset{\circ}{X}$-orbit is a straightforward consequence of the homogeneousness of X. To prove the second statement, we use the so called segment-closed property (which applies even if X isn't homogeneous) :

Segment-closed property. ([GLP1]) *Assume that τ is an isometry between two intervals I and J of D such that for all but countably many $x \in I$, x and $\tau(x)$ are in the same X-orbit. Then there exists a finite number of X-words $w_1, \dots, w_p$ such that:*

- *the domains of the w_i cover I*
- *the w_i and τ coincide on the intersection of their domains.*

Applying segment-closed property to Φ and forgetting the words with degenerate domain provides $\overset{\circ}{X}$-words w_i whose domains cover U but finitely many points. Now take a forgotten point x_0. We consider the uniform restriction $X_{-\varepsilon}$ of X defined by restricting each generator $\varphi : [a,b] \to [c,d]$ of X to $\varphi_{-\varepsilon} : [a+\varepsilon, b-\varepsilon] \to$

$[c+\varepsilon, d-\varepsilon]$. According to [Lev1] or [GLP1] (see rk. p. 418) and thanks to homogeneousness, given any compact set K in $\overset{\circ}{D}$, $X_{-\varepsilon}$ has the same orbits in restriction to K for sufficiently small ε. Take a compact interval A in U containing x_0 in its interior, and let $K = A \cup \Phi(A) \subset \overset{\circ}{D}$. Using segment-closed property for $X_{-\varepsilon}$, we get $X_{-\varepsilon}$-words v_i whose domains cover A and which agree with Φ. We can assume that their domains are nondegenerate. Consider an index i_0 such that the $X_{-\varepsilon}$-word v_{i_0} is defined on x_0. Then the interior of the domain of the X-word v_{i_0} contains x_0. This allows us to fill the holes at forgotten points. □

5.2. Approximation of (T'_v, Γ'_v)

Now, we know that T'_v is a line. Since the arc stabilizers of this action are trivial, the action of Γ'_v is given by a one to one morphism $\Gamma'_v \to \mathrm{Isom}(\mathbb{R})$. Now, identify Γ'_v with its image in $\mathrm{Isom}(\mathbb{R})$. Let $g_1, \ldots, g_n$ be a basis for the orientation preserving part of Γ'_v and s be an orientation reversing element of Γ'_v (if there is some) and take the origin of $\mathbb{R}$ at the centre of s. Now, replacing g_i by close rational translations, we get a morphism $\Gamma'_v \to \mathrm{Isom}(\mathbb{Q})$. Its kernel is a subgroup of $< g_1, \ldots, g_n >$ and therefore is isomorphic to some $\mathbb{Z}^k$. Moreover, since any finitely generated subgroup of $\mathrm{Isom}(\mathbb{Q})$ is discrete, we get a simplicial action of Γ'_v on $\mathbb{R}$.

If $x'_i \in T'_v$ is fixed by a reflection $\sigma \in \mathrm{Isom}(\mathbb{R})$, we approximate it by the centre x''_i of the image of σ in $\mathrm{Isom}(\mathbb{Q})$. Now, among the x'_i with trivial stabilizer, say that two of them are equivalent if they are in the same Γ'_v-orbit. Now take a representant x'_{i_0} of every equivalence class, and take its approximation point to be $x''_{i_0} = x'_{i_0}$. And if $x'_i = g.x'_{i_0}$, take $x''_i = g.x''_{i_0}$. We thus get an approximation of (T'_v, Γ'_v): as a matter of fact, since any $g \in \Gamma'_v \subset \mathrm{Isom}(\mathbb{R})$ can be written as the product of s and of the translations g_i, g stays close to its image in $\mathrm{Isom}(\mathbb{Q})$.

The *moreover* part of the proposition is clear by construction of the x''_i. □

Remark. If (T, Γ) is a small action of a hyperbolic group, this proposition shows there can't be homogeneous components. As a matter of fact, Γ'_v must contain $\mathbb{Z}^2$ since the orbits of Γ'_v are dense in $\mathbb{R}$. Now, we have an exact sequence

$$1 \to N_v \to \Gamma_v \to \Gamma'_v \to 1.$$

So there is a subgroup of Γ_v which is an extension of $\mathbb{Z}^2$ by a finite or virtually cyclic group. Hence, this subgroup can't contain F_2, so must be virtually cyclic or finite since it is a subgroup of the hyperbolic group Γ. This contradicts the fact that this subgroup maps onto $\mathbb{Z}^2$.

6. Getting independent generators in the non homogeneous minimal components

Definition 6.1. *In a system of isometries X, the generators are termed to be independent if no $\overset{\circ}{X}$-word has a fixed point.*

In this section, we start with a vertex v of $\mathcal{G}'$ corresponding to a non homogeneous minimal component Σ_v given by the system of isometries X_v. Using the fact that arc stabilizers are trivial in T'_v, we aim to change the system of isometries by a finite sequence of Rips moves to get a new system of isometries with independent generators (see [Pau4]).

The heart of the argument is the following:

Theorem 6.2. ([Gab2]) *Let X be a system of isometries without homogeneous component on a multi-interval. We can restrict each generator to a (maybe empty) sub-interval of its domain so that the obtained system of isometries X' has the same orbits as X and has independent generators.*

Start with the system of isometries $X = X_v$, whose set of generators is $S = \{\varphi_j\}_{j=1}^n$, and consider thanks to Theorem 6.2 a system X' of isometries with independent generators, with domain D, whose set of generators is $S' = \{\varphi'_j\}_{j=1}^n$ such that X' has the same orbits as X, and φ'_j is a restriction of φ_j to a sub-interval of $\operatorname{dom}\varphi_j$. We next use the Rips base subdivision to cut each generator φ_j at the points of $\partial(\operatorname{dom}\varphi'_j)$ to get a system of isometries X_1 whose set of generators is $S' \cup S_1$. Since X_1 and X' have the same orbits, each generator φ of S_1 is an isometry between two sub-intervals of D such that x and $\varphi(x)$ are in the same X'-orbit for all x in $\operatorname{dom}\varphi$. Now X' has segment closed property (see page 106). Hence, we can cut each $\varphi \in S_1$ into finitely many restrictions φ_k to sub-intervals not reduced to one point such that on each of them, we have a S'-word w'_k which agrees with φ_k on $\operatorname{dom}\varphi_k$.

We now introduce a new Rips move: the base sliding move (see Rips move 4 in [BF2]: the same definition works even if the set of curves of $\mathcal{C}$ is infinite). We are given a generator φ whose range is a subset of the domain of a word w whose letters don't contain φ (or φ^{-1}). We then change φ to $\varphi' = w \circ \varphi$ (this corresponds to sliding the base of φ along w) and we change the set $\mathcal{C}$ of curves naturally.

Here, we slide the φ_k along ${w'_k}^{-1}$ using the sliding Rips move described above. We get a new system of isometries X_2 whose set of generators is $S' \cup S_2$ where the generators in S_2 are restrictions of the identity to non trivial subintervals. Since Γ'_v acts with no arc stabilizer on T'_v, the generators in S_2 must be trivial in Γ'_v, so forgetting them is a Rips move which leads us to a resolution with independent generators.

7. Approximation of exotic components

Definition 7.1. *Let X be a non homogeneous minimal system of isometries, with independent generators. Consider the pruning process described in section 7.1 (also see [Gab1] or process 1 in [BF2]).*

The system of isometries X is called of surface type if the pruning process is finite, and exotic if it is infinite.

Proposition 7.2. *Let X_v be a pure minimal exotic system of isometries with independent generators, Σ_v the corresponding 2-complex, and $\rho'_v : \pi_1(\Sigma_v) \to \Gamma'_v$ be a morphism whose kernel is normally generated by loops contained in leaves of Σ_v. Consider the corresponding action (T'_v, Γ'_v).*

Then, given a finite set of points $p'_e \in T'_v$ fixed by some subgroups $\Gamma'_e \subset \Gamma'_v$, there is a simplicial approximation (T''_v, Γ'_v) of (T'_v, Γ'_v) such that the approximation point p''_e of p'_e is fixed by Γ'_e in (T''_v, Γ'_v).

Moreover, any subgroup $\Gamma'_0 \subset \Gamma'_v$ fixing a point in T'_v fixes a point in T''_v, and for $g \in \Gamma'_v$ $\left(g.p'_{e_1} = p'_{e_2}\right) \Rightarrow \left(g.p''_{e_1} = p''_{e_2}\right)$.

The strategy is the following: because the generators are independent, the curves of $\mathcal{C}_v$ are contained in a finite graph whose components are contained in a leaf. Now, perform the pruning process described for instance in [Gab1] (also see process I of the Rips machine in [BF2]) in order to be able to find a band which doesn't meet the curves in $\mathcal{C}_v$. Then, like in [BF3], we can narrow this band by a small amount and preserve the fact that the curves in $\mathcal{C}_v$ are contained in leaves and normally generate $\ker \rho_v$. This operation turns our exotic component into a finite union of simplicial or exotic components, and the (finite) number of ends of singular leaves has decreased. Since this operation doesn't increase the arc stabilizers, after a finite number of steps, we get a simplicial approximation of T'_v with trivial arc stabilizers.

7.1. The pruning process and the limit lamination

Let $X^0 = X_v$, $\Sigma^0 = \Sigma_v$ and $D^0 = D_v$ denote the (pure) system of isometries, the foliated 2-complex, and the domain of the exotic component corresponding to our vertex v. Recall that the generators are independent in X^0. Also recall that the tree T'_v is the leaf space made Hausdorff of the covering of Σ^0 corresponding to the morphism $\rho'^0 : \pi_1(\Sigma^0) \to \Gamma'_v$ and that its kernel is normally generated by the curves of $\mathcal{C}_v$. Now, $\mathcal{C}_v$ may be infinite but since the generators are independent, the curves of $\mathcal{C}_v$ are contained in the finite graph C, which is the union of all immersed closed curves contained in a leaf.

Denote L^0 the set of points in D^0 that belong to only one base in Σ^0 (by minimality, every point in D^0 belongs to at least one base). L^0 is a finite union of intervals, it is open in D^0, and by purity, the closures of the components of L^0 are disjoint.

Lemma 7.3. ([GLP1]) *If $L^0 = \emptyset$, Σ is a surface component and every point of D^0 but a finite number of them belongs to exactly two bases.*

As a matter of fact since $L^0 = \emptyset$, the pruning process cannot be continued, and Σ is a surface component. Now, since the generators are independent, we have the following theorem:

Theorem 7.4. ([Lev2, GLP1]) *If X is a minimal system of isometries with independent generators, then the total length of its domain D equals the sum of the lengths of the domains of the generators.*

Therefore, if L^0 is empty, it means that every point of D^0 but a finite number of them belongs to exactly two bases. □

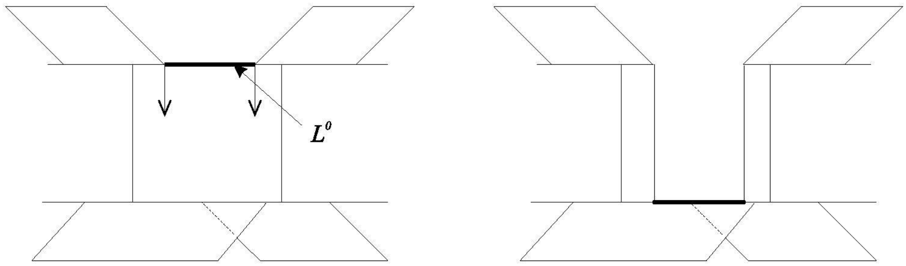


Figure 3.
Interior pruning

Let's now describe the Rips pruning move in Σ: we change D^0 to $D^1 = D^0 - L^0$, the new set of generators consists in the restrictions of the generators to D^1. Denote by X^1 the system of isometries thus obtained. We don't change the base point if $* \notin L^0$, and we slide it along the pruned band otherwise. We thus get an isomorphism between $\pi_1(\Sigma^1)$ and $\pi_1(\Sigma^0)$, so we have a well defined morphism $\rho'^1 : \pi_1(\Sigma^1) \to \Gamma'_v$. Note that the curves of $\mathcal{C}_v$ and C have been preserved since an immersed curve contained in a leaf cannot go through a point belonging to only one base. Therefore, the kernel of $\rho'^1 : \pi_1(\Sigma^1) \to \Gamma'_v$ is normally generated by free homotopy classes of loops contained in leaves. The corresponding action (T^1, γ'_v) is isomorphic to $(T^0, \Gamma'_v) = (T'_v, \Gamma'_v)$.

This pruning operation removes all the points in D^0 which are terminal vertices of their leaf. Also note that the generators remain independent, that X is still pure, and that by Lemma 7.3, L^1 can't be empty (see [Gab1]). So we can iterate this process as long as we like, and we'll denote with an exponent j the sets corresponding to the jth resolution.

For every component I of L^j, we distinguish three kinds of prunings:

1. *total* pruning if $\bar{I}$ is a full component of D^j
2. *boundary* pruning if $\bar{I}$ contains a point in ∂D^j
3. *interior* pruning if $\bar{I} \subset \overset{\circ}{D}{}^j$.

According to [BF2, Gab1], after a finite number of pruning operations on X^0, there will only be interior prunings. By changing X^0 to some X^j, we can assume there are only interior prunings.

Definition 7.5. *$\mathcal{L}^\infty = \bigcap_j \Sigma^j$ is termed the limit lamination and $D^\infty = \bigcap_j D^j$ the limit set.*

From now on, we'll reserve the name *edge* of a band for its edges which are contained in a leaf (recall that its transverse edges are called bases). The fact that there are only interior prunings implies that an edge of a band in Σ^j never can be pruned, and must therefore remain in the limit lamination. There is a natural foliation on $\mathcal{L}^\infty$ given by the trace of the foliation of Σ^j: a leaf of $\mathcal{L}^\infty$ is the intersection of a leaf of Σ^j with $\mathcal{L}^\infty$ and this definition is independent of j. Every $\mathcal{L}^\infty$-leaf is connected and has a natural simplicial structure for which the vertices are the points of intersection of the leaf with D^∞ and the edges are defined in the natural way. A leaf of $\mathcal{L}^\infty$ can be seen as the subset of a Σ^j-leaf corresponding to the union of simplicial immersions of $\mathbb{R}$ in this leaf. We term *trunk* of a leaf L in Σ^j the $\mathcal{L}^\infty$-leaf $L \cap \mathcal{L}^\infty$ (see [Gab1]).

7.2. Narrowing a band

The goal of this section is to find a band which doesn't meet the curves of $\mathcal{C}_v$ and then to narrow it like in [BF3].

First, we want to find a j and a band in Σ^j which doesn't meet the curves in $\mathcal{C}_v$. Recall that these curves are contained in the finite graph C consisting of all immersed closed curves contained in a leaf. Now, the pruning process doesn't change C, but the number of bands in Σ^j keeps growing (since there are only interior prunings). Therefore, after a sufficient number of pruning operations, we can find a band $B = [l, r] \times (0, 1)$ in some Σ^j which doesn't meet C and the curves in $\mathcal{C}_v$.

From now on, we won't do any more pruning so we forget the exponent j in our notations ($D = D^j$, $\Sigma = \Sigma^j$, $\rho : \pi_1(\Sigma) \to \Gamma'_v$ etc.). We consider the new 2-complex Σ_δ depending on a small $\delta > 0$ obtained by narrowing the band B of a width δ on the left (see figure 4: we change $B = [l, r] \times (0, 1)$ to $B_\delta = [l + \delta, r] \times (0, 1)$). The inclusion $\Sigma_\delta \subset \Sigma$ being an homotopy equivalence, ρ provides an onto morphism $\rho_\delta : \pi_1(\Sigma_\delta) \to \Gamma'_v$. Denote $\bar{\Sigma}$ and $\bar{\Sigma}_\delta$ the covering spaces corresponding to ρ and ρ_δ respectively. The kernel of ρ_δ is normally generated by the free homotopy classes of the curves in $\mathcal{C}_v$ which still are contained in leaves. Therefore, we can consider the action (T_δ, Γ'_v) obtained by making Hausdorff the leaf space of $\bar{\Sigma}_\delta$. It has

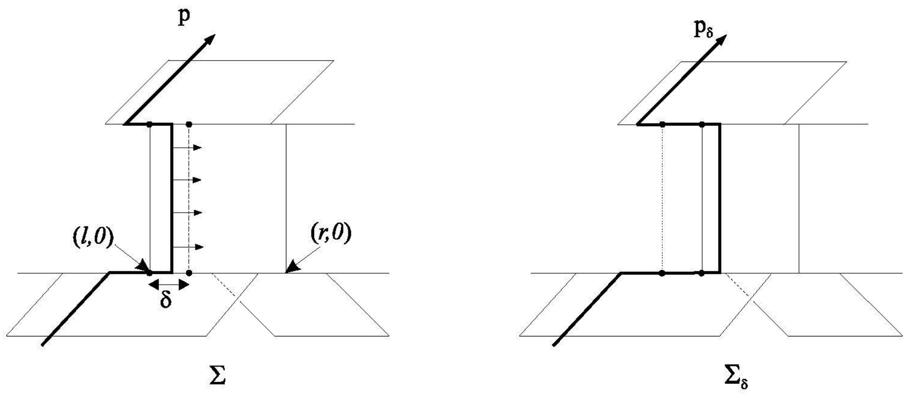


Figure 4.
Narrowing a band

trivial arc stabilizers since there is a natural morphism of $\mathbb{R}$-trees from T_δ to T'_v induced by the inclusion $\bar{\Sigma}_\delta \subset \bar{\Sigma}$.

We now want to prove that, if δ is small enough, T_δ is close to T'_v for the equivariant Gromov topology. So take an $\varepsilon > 0$, a finite set of points $\{x'_i\} \subset T'_v$, and a finite set F of elements in Γ'_v, and choose in $\bar{\Sigma}$ a preimage $\bar{x}_i$ of x'_i which lies in some lift of D. $\bar{x}_i$ can also be seen as a point of $\bar{\Sigma}_\delta$ and if necessary, we change $\bar{x}_i$ to another point in the same $\bar{\Sigma}$-leaf so that the $\bar{\Sigma}_\delta$-leaf containing $\bar{x}_i$ has the same setwise stabilizer as the $\bar{\Sigma}$-leaf containing $\bar{x}_i$ (this is possible because the narrowed band B doesn't cut C). We want to take as approximation of x'_i the image x''_i in T_δ of $\bar{x}_i$. Now, given two points $g.x'_i$ and $h.x'_j$ with $g, h \in F$, we want to estimate the difference between $d_{T'_v}(g.x'_i, h.x'_j)$ and $d_{T_\delta}(g.x''_i, h.x''_j)$. Choose a reference path joining $g.\bar{x}_i$ and $h.\bar{x}_j$ in $\bar{\Sigma}_\delta$ and denote p_0 its projection in Σ_δ or Σ. Given a path p in Σ (resp. Σ_δ), we denote $||\mathrm{p}||_\Sigma$ (resp. $||\mathrm{p}||_{\Sigma_\delta}$) the length of this path relative to the transverse measure of Σ (resp. Σ_δ). Denote $\mathrm{p} \sim \mathrm{p}_0$ when p is homotopic to p_0 (rel. endpoints) modulo $\ker \rho$. One has :

$$d_{T'_v}(g.x'_i, h.x'_j) = \inf_{\substack{\mathrm{p}\sim \mathrm{p}_0 \\ \mathrm{p}\subset\Sigma}} ||\mathrm{p}||_\Sigma$$

and

$$d_{T_\delta}(g.x''_i, h.x''_j) = \inf_{\substack{\mathrm{p}\sim \mathrm{p}_0 \\ \mathrm{p}\subset\Sigma_\delta}} ||\mathrm{p}||_{\Sigma_\delta}$$

Since $\Sigma_\delta \subset \Sigma$, one clearly has

$$d_{T'_v}(g.x'_i, h.x'_j) \leq d_{T_\delta}(g.x''_i, h.x''_j)$$

Now choose a path $p \sim p_0$ in Σ, which is the composition of arcs contained in D and of arcs contained in leaves, and such that :

$$d_{T'_v}(g.x'_i, h.x'_j) \leq ||p||_\Sigma \leq d_{T'_v}(g.x'_i, h.x'_j) + \varepsilon/2.$$

The path p intersects B in a finite number r of vertical segments (contained in leaves). By pushing the segments of $p \cap B$ by at most δ on the right in B, we get a path $p_\delta \subset \Sigma_\delta$ such that :

$$||p_\delta||_{\Sigma_\delta} \leq ||p||_\Sigma + 2r\delta$$

Taking $\delta \leq \varepsilon/4r$, we get

$$d_{T_\delta}(g.x''_i, h.x''_j) \leq d_{T'_v}(g.x'_i, h.x'_j) + \varepsilon.$$

Taking δ sufficiently small so that this estimation holds for all pairs $(g.x'_i, h.x'_j)$, we get that T_δ is close to T'_v.

Now, if $\Gamma'_e \subset \Gamma'_v$ fixes a point in T'_v, it is in the setwise stabilizer of the corresponding leaf in $\bar{\Sigma}$ because its leaf space is Hausdorff (see prop. 4.1 (7)). Hence, the approximation p''_e of a point p'_e fixed by Γ'_e is still fixed by this group because we have chosen $\bar{p}_e$ so that is $\bar{\Sigma}_\delta$-leaf is setwise stabilized by Γ'_e. This also proves that any group $\Gamma'_0 \subset \Gamma'_v$ fixing a point in T'_v must fix a point in T''_v, hence the *moreover* part of the proposition is clear.

Since we want to decrease the number of ends of singular leaves of Σ, we must first ensure not to increase it while narrowing the band. So we choose δ so that the new left edge of the band is in a singular leaf of Σ (this can be done with δ arbitrarily small since every leaf is dense in Σ). Doing so, the number of singular leaves may have increased since some singular leaf could be split into several singular leaves in Σ_δ, but to each end of a singular leaf of Σ_δ injectively corresponds to an end of a singular leaf of Σ. Thus, the number of ends of singular leaves hasn't increased.

7.3. Ends of singular leaves: how to make leaves compact

Let's state the first theorem we'll use in this section:

Theorem 7.6. ([BF2, Gab1]) *Consider an exotic minimal system of isometries on a multi-interval D. Then, every leaf has a finite number of ends and is quasi-isometric to a tree. Moreover, the set of points whose leaf has only one end is a dense G_δ in D, there are uncountably many leaves with 2 ends, and finitely many leaves with 3 ends or more.*

First, after the pruning process and the band narrowing described in sections 7.1 and 7.2, we have created no surface component. As a matter of fact, in a surface component, all but finitely many leaves are lines, and in an exotic component, leaves have generically one end. Since narrowing a band doesn't increase the number of ends of any leave, and since every Rips move we used preserves the number of ends of leaves, we can't have created new surface components. What's more, we have created no homogeneous component since the generators are still independent which is impossible in a homogeneous component (see [Gab2]).

It may happen that Σ_δ hasn't pure components. In this case, we perform finitely many domain cuts so that Σ_δ has pure components. This gives a graph of actions $\mathcal{H}$ as in section 3. The groups corresponding to edges with positive length are trivial and the vertex actions have trivial arc stabilizers since T_δ has trivial arc stabilizers.

If some of the leaves of Σ_δ are non-compact (i. e. if Σ_δ has some exotic components), we can perform again the pruning and narrowing of sections 7.1 and 7.2 on a vertex action. To show that we can get rid of exotic component by a finite number of applications of sections 7.1 and 7.2, we consider the sum $E(\Sigma)$ of the number of ends of all singular leaves in Σ. All the Rips moves we used preserve $E(\Sigma)$ since they don't create new singular leaves (this is clear for pruning and sliding operations, and the base and domain subdivisions we use are done at points whose leaves are already singular). If $V(\mathcal{H})$ denotes the set of vertices of $\mathcal{H}$, this means that $E(\Sigma_\delta) \geq \sum_{w\in V(\mathcal{H})} E(\Sigma_w)$.

So now, there remains to show that $E(\Sigma_\delta) < E(\Sigma)$. Since narrowing a band doesn't increase the number of ends of any leaf, we only have to find a singular leaf whose number of ends has strictly decreased during the narrowing of the band.

Theorem 7.7. ([Gab1]) *D^∞ has no isolated points (it is a Cantor set). Moreover, the union of the trunks of singular leaves of Σ is dense in $\mathcal{L}^\infty$.*

Since the generators are independent, every Σ-leaf is a tree with finitely many extra edges attached on it (i. e. its fundamental group is finitely generated): a loop in a leaf corresponds to a word having a fixed point, so one of the edges of the loop must be singular; and since there are finitely many generators, there are finitely many singular edges. Finally, since every leaf has a finite number of ends, its trunk is the union of a compact graph and of finitely many disjoint semi-lines. Since a pruning operation removes the terminal vertices of a leaf, it is clear that any leaf has the same set of ends as its trunk when its trunk is non-empty. Note that the trunk of a singular leaf must be non-empty if there are no interior prunings to be done.

Now, remember that the left edge of the band B lies in $\mathcal{L}^\infty$ (since in the construction of $\mathcal{L}^\infty$ by the pruning process, there were no more total and boundary prunings). Since D^∞ has no isolated points and empty interior, there will be infinitely many prunings near the left edge of B. So, some edges of bands will accumulate near B. Recall that they lie in the trunk of a singular leaf since only interior prunings occur any more. Since there are finitely many singular leaves and that this number doesn't increase during the pruning process, the trunk $\mathcal{L}$ of a singular leaf must accumulate on the left edge of B. This means that one of the semi-lines in $\mathcal{L}$ accumulates on the left edge of B, and so intersects $[l, l+\delta[\times(0,1)$ an infinite number of times. This implies that the end corresponding to this semi-line doesn't subsist in Σ_δ. So we have shown: $E(\Sigma_\delta) < E(\Sigma)$. $\square$

8. The surface components

Proposition 8.1. *Let X_v be a pure minimal system of isometries of surface type with independent generators, Σ_v the corresponding 2-complex, and $\rho'_v : \pi_1(\Sigma_v) \to \Gamma'_v$ be a morphism whose kernel is normally generated by loops contained in leaves of Σ_v. Consider the corresponding action (T'_v, Γ'_v).*

Then, given a finite set of points $p'_e \in T'_v$ fixed by some subgroups $\Gamma'_e \subset \Gamma'_v$, there is a simplicial approximation (T''_v, Γ'_v) of (T'_v, Γ'_v) such that the approximation point p''_e of p'_e is fixed by Γ'_e in (T''_v, Γ'_v).

Moreover, any subgroup $\Gamma'_0 \subset \Gamma'_v$ fixing a point in T'_v fixes a point in T''_v, and for $g \in \Gamma'_v$ $\left(g.p'_{e_1} = p'_{e_2}\right) \Rightarrow \left(g.p''_{e_1} = p''_{e_2}\right)$.

Proof. Let Σ_v be a such 2-complex. After performing finitely many prunings, all but finitely many points of the domain D lie in two bases (see Prop. 7.3). The purity of Σ_v (every $\mathring{X}$-orbit is dense) implies that Σ_v is a compact surface S with boundary endowed with a minimal measured foliation. Each boundary component is contained in a leaf and purity implies that the only non simply connected leaves are the boundary leaves and that they have cyclic fundamental group. Thus, the curves of $\mathcal{C}_v$ only run in these boundary components.

Now, we approximate the foliation on S by a rational foliation (see [FLP]) with no leaf parallel to the boundary. The foliation thus obtained has compact leaves. We get in this way a simplicial action (T''_v, Γ'_v) close to (T'_v, Γ'_v) for the translation length topology. Its arc stabilizers are the image in Γ of the fundamental group of families of finite orbits. Since S is a surface, its regular leaves are circles, so families of finite orbits have cyclic fundamental group, so the arc stabilizers must be cyclic.

Now, since we have here small actions and since Γ'_v contains F_2, the translation length topology coincide with the equivariant Gromov topology ([Pau1]), so (T''_v, Γ'_v) is an approximation of (T'_v, Γ'_v) in the Gromov topology. So consider an approximation point $p''_e \in T''_v$ of p'_e. If $\mathrm{Stab}_{T'_v}(p'_e) \neq \{1\}$, p'_e is the image in T'_v of a boundary component of $\tilde{\Sigma}'_v$ stabilized by Γ'_e. In place of p''_e, we want to take as approximation point of p'_e the image p'''_e in T''_v of this boundary component. Now, let g is a generator of the (cyclic) global stabilizer in Γ'_v of our boundary component of $\tilde{\Sigma}'_v$. p'''_e is the only fixed point of g because the foliation we consider on S has no leaf parallel to the boundary (and because the leaf space of it covering space is Hausdorff since the leaves of S are compact). Since $g.p'_e = p'_e$, the point

$g.p''_e$ must be close to p''_e. This implies that p''_e must be close to the unique fixed point of g in T''_v which is p'''_e, so we can take p'''_e in place of p''_e as approximation point of p'_e.

The *moreover* part of the proposition is now clear using p'''_e as approximation point of p'_e. □

9. Bounded backtracking

Definition 9.1. ([GJLL]) *An action (T, F) of a finitely generated free group F has bounded backtracking if given $Q \in T$, there exists a constant $C > 0$ such that, if v, w, vw have word length satisfying $|vw| = |v| + |w|$, then $d(v.Q, [Q, vw.Q]) \leq C$.*

In order to prove corollary 2 about bounded backtracking, we first prove the following proposition:

Proposition 9.2. *Let $F =< g_1, \ldots, g_n >$ be a finitely generated free group. Let (T, F) be a small simplicial action of F, and $Q \in T$. Then (T, F) has bounded backtracking with constant $C = \sum_{i=1}^{n} d(Q, g_i.Q)$.*

Proof. Consider T_1 the Cayley graph of F relative to its free basis $\{g_1, \ldots, g_n\}$, and let f_Q be the equivariant map that linearly sends the edge $[w, w.g_i]$ to $[w.Q, wg_i.Q] \subset T$.

Definition 9.3. *A map between $\mathbb{R}$-trees $f : T_1 \to T_2$ has backtracking bounded by C if for any injective path $c : [0, 1] \to T_1$ such that $f \circ c(0) = f \circ c(1)$, the radius*

$$\sup_{t \in [0,1]} d(f \circ c(0), f \circ c(t))$$

of $f \circ c([0, 1])$ is bounded by C.

Lemma 9.4. *If the equivariant map f_Q from the Cayley graph T_1 of F to T has backtracking bounded by C, then (T, F) has backtracking (rel. Q) bounded by C.*

Conversely, if (T, F) has backtracking bounded by C (rel. Q), then f_Q has backtracking bounded by $C + \sup_{i=1}^{n} d(Q, g_i.Q)$.

Proof of Lemma 9.4. If $|vw| = |v| + |w|$, it means that $v \in [1, vw]$ in T_1. Hence there is a subsegment $[A, B] \subset [1, vw]$ containing v such that $f_Q(A) = f_Q(B) \in [Q, vw.Q]$ which implies that $d(v.Q, [Q, vwQ]) \leq C$ so (T, F) has backtracking bounded by C.

Conversely, the fact that (T, F) has backtracking bounded by C rel. Q means that every segment $[A, B]$ between two vertices of T_1 is mapped by f_Q in the C-neighbourhood of $[f_Q(A), f_Q(B)]$. Now, given an injective path c in T_1 such that

$f_Q \circ c(0) = f_Q \circ c(1)$, take the smallest segment $[A, B]$ containing c with endpoints at vertices of T_1. Then, $f_Q(A)$ and $f_Q(B)$ are at distance from $f_Q \circ c(0)$ bounded by $\sup_i d(Q, g_i.Q)$. And $f_Q([A, B])$ is in the C-neighbourhood of $[f_Q(A), f_Q(B)]$. Since $c([0,1]) \subset [A, B]$, this proves the lemma. □

Now, the proof of Proposition 9.2 reduces to show that f_Q has backtracking bounded by $\sum_{i=1}^n d(Q, g_i.Q)$. Note that Definition 9.3 doesn't depend on the metric on T_1. Now, if some $d(Q, g_i.Q)$ are 0, consider T_2 the tree obtained by collapsing the corresponding edges in T_1. The map f_Q factors through the quotient map q into f_1. Since q preserves alignment, f_Q and f_1 have the same backtracking bounds, so the proof reduces to bound the backtracking of f_1. Consider the path length on T_2 which makes f_1 an isometry in restriction to every edge. Now subdivide T so that Q is a vertex of T and subdivide T_2 so that f_1 is simplicial. Note that (T_2, F) has trivial arc stabilizers. Therefore, f_1 satisfy the $(*)$ condition of prop. 9.6. Then, the proof reduces to Proposition 9.6 since $\sum_{i=1}^n d(Q, g_i.Q) = \mathrm{Vol}_f(T_1)$. □

Definition 9.5. *Let (T, F) be a small simplicial action of the finitely generated free group F. The free volume $\mathrm{Vol}_f(T)$ of T is the sum of the lengths of edges of T/F with trivial stabilizer.*

Remark. The quotient graph T/F must be finite because of minimality: the convex hull of Q and $\{g_i.Q\}$ meets every orbit. This means that $\mathrm{Vol}_f(T) \leq \sum_{i=1}^n d(Q, g_i.Q)$.

Proposition 9.6. *Let (T', F) and (T, F) be small minimal simplicial actions and $f : T' \to T$ a simplicial equivariant map which isometrically embeds every edge. Assume that*

$$\begin{aligned}&\textit{For all edges } e \neq e' \textit{ in } T',\\ &\qquad f(e) = f(e') \Rightarrow \mathrm{Stab}_{T'} e = \{1\} \textit{ or } \mathrm{Stab}_{T'} e' = \{1\}\end{aligned} \tag{*}$$

Then f has backtracking bounded by $\mathrm{Vol}_f(T') - \mathrm{Vol}_f(T)$.

Remark. There are silly counterexamples to this proposition if $(*)$ doesn't hold: consider the lifts to the universal cover of the maps between graphs of groups shown on figure 5. They statisfy $\mathrm{Vol}_f(T') = \mathrm{Vol}_f(T)$ and have bounded backtracking but with constant $C \neq 0$.

Proof. First, it is easy to check that if $f = f_1 \circ f_2$ for 1-lipschitz equivariant maps f_i with backtracking bound $BBT(f_i)$, then f has backtracking bound $BBT(f) = BBT(f_1) + BBT(f_2)$.

In light of this remark, we want to decompose f into a product of Stallings' folds satisfying $(*)$. First of all, a Stallings' fold (or fold for short) of a simplicial action is

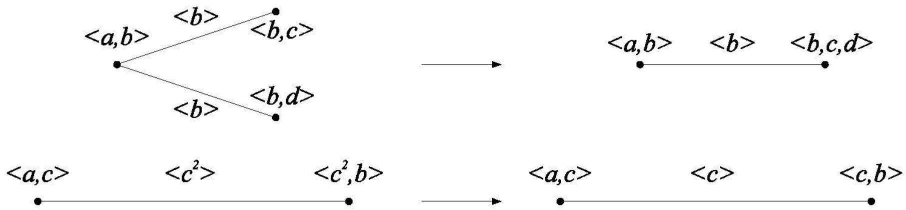


Figure 5.
Silly folds which don't satisfy $(*)$

a map given by equivariantly identifying 2 adjacent edges with same length. [Sta, BF1] show that under general hypotheses, an equivariant simplicial map between two simplicial trees which isometrically embeds edges is a composition of Stallings' folds. Here, we need a little more precise statement: f is a composition of folds $f_i : T_i \to T_{i+1}$ that satisfy $(*)$. Then the proof will reduce to the case of folds.

First step: decomposition into Stallings' folds. By subdividing T (and T'), we may assume that each edge of T embeds into T/F (this prevents inversions and means that the endpoints of an edge can't be in the same orbit). Then since T is minimal, f is onto. If f is one-to-one, there is nothing to prove since it would be isometric. We proceed by induction on the sum of the total number of edges and of the number of edges with trivial stabilizer in T'/F.

Take some vertices $x \neq y$ of T' such that $f(x) = f(y)$. Thus, $f_{|[x,y]}$ cannot be locally injective. This means that there are 2 adjacent edges e_1, e_{-1} in $[x, y]$ with same length that are identified through f. We first assume that e_1 and e_{-1} lie in distinct orbits. Then f factors through the fold $q : T' \to T_1$ which equivariantly identifies e_1 and e_{-1}. Denote $f_1 : T_1 \to T$ the induced map. Note that f_1 is isometric in restriction to edges of T_1. Since $(*)$ holds, so does q. Now take two edges $E \neq E' \in T_1$ such that $f_1(E) = f_1(E')$. Since q is a fold between edges that lie in distinct orbits, E has a preimage in T' with same stabilizer as E and the same is true for E'. Thus, since f satisties $(*)$, so does f_1. The total number of edges in T_1/F is smaller than in T'/F, and the number of edges with trivial stabilizer hasn't increased so the result follows by induction in this case.

Now assume that we cannot perform a fold between edges in distinct orbits. Then any two adjacent edges e_1 and e_{-1} that are identified through f must lie in the same orbit. Since f satisfies $(*)$, we must have $\mathrm{Stab}\, e_1 = \mathrm{Stab}\, e_{-1} = \{1\}$. This means that any segment in T' with nontrivial stabilizer embeds into T. Let g be such that $e_{-1} = g.e_1$ for $g \in F$. First, g fixes the common endpoint M of e_1 and e_{-1} since edges embed into T/F. Now $g \in \mathrm{Stab}_T f(e_1)$, so let h be a generator of $\mathrm{Stab}_T f(e)$. Since g and h commute, g fixes M and $h.M$, so $[M, h.M]$ must embed in T but since $f(M) = f(h.M)$, h must fix M.

Now, consider the fold q that equivariantly identifies e_1 and $h.e_1$. Denote T_1 the quotient tree, and $f_1 : T_1 \to T$ the induced map. The fold q satisfies $(*)$ because f does. Note that $\mathrm{Stab}_{T_1} q(e_1) =< h >$. Assume now that f_1 doesn't satisfy $(*)$ and let $E \neq E'$ be two edges of T_1 with nontrivial stabilizers such that $f_1(E) = f_1(E')$. Since f satisfies $(*)$, we can assume without loss of generality that $E = q(e_1)$ because an edge which is not in the orbit of $q(e_1)$ has a unique preimage in T' and it has same stabilizer. Now, E' has a nontrivial stabilizer, and since $f_1(E') = f(e_1)$, $\mathrm{Stab}_{T_1} E' \subset \mathrm{Stab}_T f(e_1) =< h >$ so some h^k fixes E' $(k \neq 0)$. Then E' can't be in the orbit of $q(e_1)$ since an element of F sending $q(e_1)$ to E' would be in $\mathrm{Stab}_T f(e_1)$ and thus would fix $q(e_1)$. So E' has a unique preimage e' in T', and it is fixed by h^k. Since h^k fixes M, the minimal segment $[M, e']$ in T' containing M and e' is fixed by h^k so must embed in T. Moreover, $[M, e']$ doesn't contain e_1 since e_1 has trivial stabilizer. Now, if $M \notin e'$, we couldn't have $f(e_1) = f(e')$ since f embeds $[M, e']$. This forces e' to be adjacent to e_1. Hence, we have a contradiction because e_1 and e' are two adjacent edges that lie in distinct orbits and which are identified through f, and we could have performed a fold of the first case.

Finally, the number of edges with trivial stabilizer in T_1/F is smaller than in T'/F, and the total number of edges hasn't increased so we proved by induction that f is a product of folds verifying $(*)$.

Second step: Backtracking bound for folds.

Lemma 9.7. *Let $q : T' \to T$ be a fold that equivariantly identifies e_1 and e_{-1}. Denote l the common length of e_1 and e_{-1}. Then that q has backtracking bounded by l.*

Note that this lemma doesn't use the $(*)$ condition. The fact that q satisfies $(*)$ just says that $l = \mathrm{Vol}_f(T') - \mathrm{Vol}_f(T)$ which ends the proof of Proposition 9.6.

We start with the following claim:

Claim. *Let $q : T' \to T$ be the fold identifying e_1 and e_{-1}. For any edge e in T, let $q^{-1}(e)$ be the union of edges e' in T' such that $q(e') = e$. Then $q^{-1}(e)$ is convex.*

Proof of the claim. By definition, q is the quotient map under the smallest equivariant equivalence relation on T which identifies e_1 and e_{-1}. This exactly means that 2 edges E and E' are identified if one has a relation of the type

$$\begin{aligned} w^{-1}.E = e_{\varepsilon_1} \sim e_{-\varepsilon_1} = h_1.e_{\varepsilon_2} \sim h_1.e_{-\varepsilon_2} = h_1 h_2 e_{\varepsilon_3} \sim \cdots \\ \cdots \sim h_1 h_2 \ldots h_{p-1}.e_{-\varepsilon_p} = w^{-1}.E' \end{aligned} \qquad (1)$$

for $h_i \in F$ sending $e_{\varepsilon_{i+1}}$ on $e_{-\varepsilon_i}$. This shows that an edge which is not in the orbit of e_1 or e_{-1} can't be identified with any other one. Moreover, the union of edges identified with e_1 and e_{-1} is connected since two consecutive edges that appear in (1) have a common endpoint. □

Proof of Lemma 9.7. Consider an injective path $c : [0,1] \to T'$ such that $q \circ c(0) =$

$q \circ c(1)$. Denote $P = q \circ c(0) = q \circ c(1)$. We want to prove that $q \circ c([0,1])$ is in the l-neighbourhood of P (recall that l is the length of e_1). Consider $H = (q \circ c)^{-1}(P) \subset [0,1]$, and consider a restriction of c to the closure $[a,b]$ of a component of $[0,1]-H$. We only need to prove that $c_{|[a,b]}$ is in the l-neighbourhood of P. But since $P \notin q \circ c((a,b))$, the initial and terminal germs of $q \circ c_{|[a,b]}$ lie in the same edge e. Thus $c(a), c(b) \in q^{-1}(e)$. The claim says that $q^{-1}(e)$ is convex so $c([a,b]) \subset q^{-1}(e)$. If e is in the orbit of $q(e_1)$, then $q \circ c([a,b]) \subset e$ has diameter at most l. And if it isn't in the orbit of $q(e_1)$, then $q^{-1}(e)$ is a single edge which is embedded by q, thus $q \circ c$ embeds $[a,b]$ which contradicts $q \circ c(a) = q \circ c(b) = P$.□

Corollary 2. *Every small action of a finitely generated free group has bounded backtracking.*

Proof. Let (T,F) be a small action of a finitely generated free group and let $Q \in T$. Theorem 1 implies that (T,F) can be approximated by small simplicial actions for the equivariant Gromov topology. Considering approximations Q' of Q, we see like in [BFH, GJLL] that (T,F) has backtracking bounded by $\sum_{i=1}^{n} d(Q, g_i.Q)$ thanks to Proposition 9.2. □

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(Received: 22 October, 1997)