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On the dilatation of extremal quasiconformal mappings of polygons

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Abstract. A polygon P_N is the unit disk $\mathbb{D}$ with n distinguished boundary points, $4 \le n \le N$. An extremal quasiconformal mapping $f_0 \colon \mathbb{D}_z \to \mathbb{D}_w$ maps each polygon P_N inscribed in $\mathbb{D}_z$ onto a polygon P'_N inscribed in $\mathbb{D}_w$. Let f_N be the extremal quasiconformal mapping of P_N onto P'_N. Let K_N be its dilatation and let K_0 be the maximal dilatation of f_0. Then, evidently $\sup K_N \le K_0$. The problem is, when equality holds. This is completely answered, if f_0 does not have any essential boundary points. For quadrilaterals Q and $Q' = f_0(Q)$ the problem is $\sup(M'/M) = K_0$, with M and M' the moduli of Q and Q' respectively.

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Introduction

1. Let h be a quasisymmetric mapping of the boundary of the unit disk $\mathbb{D}_z$ onto the boundary of $\mathbb{D}_w$ and let f be a quasiconformal extension of h into the disk. It is called extremal and denoted by f_0 if its maximal dilatation K_0 is smallest possible. We always assume $K_0 > 1$. The disk $\mathbb{D}_z$ becomes a quadrilateral Q if we mark four different points z_j, $j = 1, \dots, 4$, in the positive direction on its boundary $\partial\mathbb{D}_z$. The mapping f_0 takes the vertices z_j into points $w_j = f_0(z_j)$ on $\partial\mathbb{D}_w$ and thus the quadrilateral Q into a quadrilateral $Q' = f_0(Q)$ inscribed in $\mathbb{D}_w$. It follows from the definition of quasiconformality that the conformal moduli M and M' of Q and Q' respectively satisfy (for general properties of quasiconformal mappings, see [3])

$$\frac{1}{K_0} M \le M' \le K_0 M. \tag{1}$$

It has been a question for some time, if the bound K_0 is best possible in the inequality (1), in other words, if the maximal dilatation K_0 of the extremal quasiconformal extension f_0 of h can be determined by the ratio of the moduli of inscribed quadrilaterals,

$$\sup \frac{M'}{M} = K_0. \tag{2}$$

The question has recently been answered in the negative by Anderson and Hinkkanen [1] by laborious computations of a counterexample (horizontal stretching of a parallelogram) and by Reich [4] who reduced it to an approximation problem for holomorphic functions. More counterexamples are given in [9].

2. It is easy to find examples where (2) holds; the above solutions consist therefore in the construction of examples where it does not hold. A type of the first kind is a vertical half strip S and its horizontal stretching by K_0. Let $z = x + iy$, $S = \{z;\, 0 < x < a,\ 0 < y\}$, $w = u + iv$, $S' = \{w;\, 0 < u < K_0 a,\ 0 < v\}$. We make S to a quadrilateral by marking the vertices $(0, a, a + ib, ib)$ for arbitrary $b > 0$, and similarly S' by marking the image points $(0, K_0 a, K_0 a + ib, ib)$. Making use of the extremal length definition of the modulus of a quadrilateral ([3], p. 21) as the extremal distance of the vertical sides we easily find the estimates

$$M \leq a/b, \quad M' \geq K_0 a/(b + K_0 a) \tag{3}$$

and thus

$$K_0 \geq \frac{M'}{M} \geq \frac{K_0 a}{b + K_0 a} \cdot \frac{b}{a}, \tag{4}$$

which gives

$$\lim_{b \to \infty} \frac{M'}{M} = K_0. \tag{5}$$

3. The problem with the moduli of quadrilaterals has a different interpretation. We look at the extremal quasiconformal mapping f of Q onto Q'. This is a mapping of $\mathbb{D}_z$ onto $\mathbb{D}_w$ which takes the vertices of Q into those of Q'. Its dilatation is $K = M'/M$, and the question is now what happens with K if we vary the vertices of Q in all possible ways? Of course we always have $K \leq K_0$, but will we have $\sup K = K_0$? In this formulation the problem has a natural generalization to polygons, i.e. disks with an arbitrary finite number $n \geq 4$ of vertices. The basic extremal qc mapping f_0 assigns a polygon P'_n inscribed in $\mathbb{D}_w$ to each polygon P_n inscribed in $\mathbb{D}_z$. The extremal qc mapping f_n of P_n onto P'_n (i.e. of course of $\mathbb{D}_z$ onto $\mathbb{D}_w$, but with the only requirement that the vertices of P_n go into the vertices of P'_n) is a Teichmüller mapping with a complex dilatation $\varkappa_n = k_n(\overline{\varphi_n}/|\varphi_n|)$, $k_n = (K_n - 1)/(K_n + 1)$. The quadratic differential φ_n is rational, with at most first order poles at the vertices of P_n. Moreover, $\varphi_n(z)\, dz^2$ is real along the sides of P_n. Since f_0 also maps the vertices of P_n onto those of P'_n and f_n is extremal with this property, we have $K_n \leq K_0$. The question arises if, by varying the polygon P_n in all possible ways, we have

$$\sup K_n = K_0. \tag{6}$$

4. It follows from general principles of qc mappings (we refer to [3] for the general theory) that this is in fact true if we allow the number n of vertices to become arbitrarily large (for a proof see [5], p. 385, bottom). But how is it, if this number is bounded, $n \leq N$ say? With a certain natural restriction we will characterize the extremal mappings f_0 for which this happens. The proof is an application of the "polygon inequality" ([5], p. 384) and a theorem of R. Fehlmann ([2], p. 567).

The polygon inequality

5. Let f_0 be an extremal qc mapping of $\mathbb{D}_z$ onto $\mathbb{D}_w$ with $f_0 \mid \partial\mathbb{D}_z = h$. Let $\varkappa_0$ with $\|\varkappa_0\|_\infty = k_0$ be its complex dilatation and $K_0 = (1+k_0)/(1-k_0)$ its maximal dilatation. Mark n points z_j, $j = 1, \ldots, n$, on $\partial\mathbb{D}_z$, $4 \leq n \leq N$. The disk $\mathbb{D}_z$ with the marked boundary points z_j is called a polygon P_n. The image of P_n by f_0 is the polygon P'_n, inscribed in $\mathbb{D}_w$, with vertices $w_j = f_0(z_j)$. Let f_n be the extremal qc mapping of P_n onto P'_n, $f_n(z_j) = w_j$, and let φ_n, $\|\varphi_n\| = 1$, denote the associated quadratic differential. The complex dilatation of f_n is $k_n(\overline{\varphi_n}/|\varphi_n|)$. Then, the *Polygon Inequality* holds:

$$\operatorname{Re} \iint_{|z|<1} \frac{\varkappa_0(z)\varphi_n(z)}{1-|\varkappa_0(z)|^2}\, dx\, dy \geq \frac{k_n}{1-k_n} - \iint_{|z|<1} |\varphi_n(z)| \frac{|\varkappa_0(z)|^2}{1-|\varkappa_0(z)|^2}\, dx\, dy. \quad (7)$$

For the proof I refer to ([5], p. 384). In that paper, the inequality was used to prove that the "polygon differentials" φ_n form a Hamilton sequence for $\varkappa_0$ if the number of vertices tends to infinity and the sides of the polygons P_n become arbitrarily short. This led to a proof of the necessity of the Hamilton–Krushkal condition for extremality. Now, on the contrary, we restrict the number of vertices by a fixed number N, and we denote a polygon with $n \leq N$ vertices generically by P_N.

6.

Theorem 1. *Let* $f_0\colon \mathbb{D}_z \to \mathbb{D}_w$ *with complex dilatation* $\varkappa_0$, $\|\varkappa_0\|_\infty = k_0$, *be extremal for its boundary values* h. *Assume that for a fixed number* N *the polygon mappings* $f_N\colon P_N \to P'_N = f_0(P_N)$ *with complex dilatation* $k_N(\overline{\varphi_N}/|\varphi_N|)$ *satisfy*

$$\sup k_N = k_0. \quad (8)$$

(*This is of course equivalent to* $\sup K_N = K_0$.) *Then, there is a sequence of polygon mappings* $f_N^{(i)}$ *the quadratic differentials* $\varphi_N^{(i)}$ *of which,* $\|\varphi_N^{(i)}\| = 1$, *form a Hamilton sequence for* $\varkappa_0$, *i.e.*

$$\operatorname{Re} \iint \varkappa_0(z)\varphi_N^{(i)}(z)\, dx\, dy \to k_0, \quad i \to \infty. \quad (9)$$

Proof. Assume first that f_0 has constant dilatation $|\varkappa_0(z)| = k_0$ a.e. Then, the polygon inequality yields

$$\frac{1}{1-k_0^2} \operatorname{Re} \iint \varkappa_0(z)\varphi_N(z)\, dx\, dy \geq \frac{k_N}{1-k_N} - \frac{k_0^2}{1-k_0^2} \tag{10}$$

for all polygons P_N. Let $P_N^{(i)}$ be a sequence of polygons the extremal mappings $f_N^{(i)}$ of which satisfy $k_N^{(i)} \to k_0$. Then

$$\varliminf_{i\to\infty} \operatorname{Re} \iint \varkappa_0(z)\varphi_N^{(i)}(z)\, dx\, dy \geq \frac{k_0}{1-k_0}(1-k_0^2) - k_0^2 = k_0. \tag{11}$$

On the other hand

$$\operatorname{Re} \iint \varkappa_0(z)\varphi_N^{(i)}(z)\, dx\, dy \leq \left| \iint \varkappa_0(z)\varphi_N^{(i)}(z)\, dx\, dy \right| \leq k_0. \tag{12}$$

This gives the result (9) in the case where $|\varkappa_0(z)| = k_0$ a.e. If $|\varkappa_0(z)|$ is not constant a.e. we proceed as in ([5], p. 386 and p. 382). However, in our present work we only need the case of constant $|\varkappa_0(z)|$. □

Since the number of vertices of the polygons $P_N^{(i)}$ is smaller or equal to N, we can assume, by passing to a further subsequence, that they converge to a finite number $\leq N$ of points on $\partial\mathbb{D}_z$. We write $P_N^{(i)} \to P_N$.

The vertical half strip in the introduction is an example where the given quadrilaterals give rise to a Hamilton sequence for the horizontal stretching (which is uniquely extremal).

Extremal mappings without essential boundary point

7. Let f_0 with complex dilatation $\varkappa_0$, $\|\varkappa_0\|_\infty = k_0$, be extremal for its boundary values h. A boundary point z of $\mathbb{D}_z$ is called essential, if the following is true: For every neighborhood U of z and every qc mapping g of $U \cap \mathbb{D}_z$ which is equal to h on $U \cap \partial\mathbb{D}_z$ the maximal dilatation of g is at least equal to $K_0 = (1+k_0)/(1-k_0)$.

A theorem of R. Fehlmann ([2]), p. 567) says: If the complex dilatation $\varkappa_0$ has a degenerating Hamilton sequence (i.e. which tends to zero locally uniformly in the domain), then f_0 has an essential boundary point.

Combining this result with the considerations in ([7], p. 466) we can say: If f_0 does not have an essential boundary point, then, every Hamilton sequence for $\varkappa_0$ converges in norm to a holomorphic quadratic differential φ_0, $\|\varphi_0\| = 1$, and $\varkappa_0 = k_0(\overline{\varphi_0}/|\varphi_0|)$ is the complex dilatation of f_0.

8. Let us apply this to our case. Every polygon differential $\varphi_N^{(i)}$ can be continued across the boundary $\partial\mathbb{D}_z$ by reflection to a rational differential in the whole plane, of norm two. Therefore the limit φ_0 can be reflected. Since its norm is finite, it has at most first order poles at the $n \leq N$ limits of the vertices of the $P_N^{(i)}$, and $\varphi_0(z)\, dz^2$ is real along the subintervals of $\partial\mathbb{D}_z$ between these limits. Our main result is

Theorem 2. *Let $f_0 : \mathbb{D}_z \to \mathbb{D}_w$ be a qc mapping which is extremal for its boundary values, and assume that it does not have an essential boundary point. For fixed $N \geq 4$ denote the polygons with $4 \leq n \leq N$ vertices inscribed in $\mathbb{D}_z$ generically by P_N. To every P_N the mapping f_0 determines a polygon P'_N inscribed in $\mathbb{D}_w$, simply by mapping the vertices of P_N onto those of P'_N. Assume that the extremal mappings $f_N : P_N \to P'_N$ satisfy $\sup k_N = k_0$. Then, there is a convergent sequence $f_N^{(i)}$ of polygon mappings with $\varphi_N^{(i)} \to \varphi_0$ in norm, where $\varkappa_0 = k_0(\overline{\varphi_0}/|\varphi_0|)$ is the complex dilatation of f_0. f_0 itself is the extremal qc mapping of a polygon with $n \leq N$ vertices, and every maximizing sequence $f_N^{(i)}$, $k_N^{(i)} \to k_0$, tends to f_0 uniformly, $\varphi_N^{(i)} \to \varphi_0$ in norm.*

9. In order to see that the theorem is not empty, let $f : \mathbb{D}_z \to \mathbb{D}_w$ be an extremal polygon mapping and let φ be the associated rational quadratic differential, $\varkappa = k(\overline{\varphi}/|\varphi|)$ the complex dilatation. The vertices z_j are either first order poles or regular points (i.e. $\varphi(z_j) \neq 0$) or zeroes of φ of any order. Along the sides we have $\varphi(z)\, dz^2$ real, and thus the sides are composed of trajectories and orthogonal trajectories.

The first order poles and the zeroes are clearly the only candidates for an essential boundary point of f. In order to find the local maximal dilatation H_z at such a point z we first apply the mapping $\Phi = \int \sqrt{\varphi}$ and then the horizontal stretching by K. The integral Φ maps an interior half neighborhood of z onto an angle with a horizontal and a vertical side. It is a right angle in the case of a first order pole and an angle which is a multiple of $\frac{1}{2}\pi$ in the case of a zero, possibly many sheeted. In the image $\mathbb{D}_w$ we have the same situation, with a quadratic differential ψ and an integral $\Psi = \int \sqrt{\psi}$. The horizontal side of the angle is stretched by K while the vertical side is mapped identically. It is known (and easy to see, using logarithms on both sides, see [6], p. 323) that the local extremal mapping with the given boundary values has dilatation $< K$. Since f itself is extremal with dilatation K, it does not have any essential boundary point, thus satisfying our requirement.

10. Let now $f_0 : P_N \to P'_N$ with complex dilatation $\varkappa_0 = k_0(\overline{\varphi_0}/|\varphi_0|)$ be an extremal polygon mapping. We can clearly take $f_N = f_0$ itself and get $\sup k_N = k_0$. Actually we only need to consider the substantial boundary points of f_0 (= poles of φ_0), since the extremal mapping of the restricted polygon $\widetilde{P}_N$ onto $\widetilde{P}'_N$ is the same as f_0.

Let $\widetilde{N}$ be the number of substantial boundary points of f_0. If, however, we only admit polygons with at most $N' \leq \widetilde{N}-1$ vertices, we find $\sup k_{N'} < k_0$. For, if $\sup k_{N'} = k_0$ we would again arrive, by the same considerations as before, at an extremal polygon mapping $f_{N'}$ with a quadratic differential $\varphi_{N'}$ with at most N' first order poles, whereas φ_0 has $\widetilde{N}$ first order poles. Therefore $\varphi_{N'} \neq \varphi_0$, a contradiction.

11. We started with the following question. Let f_0 with complex dilatation $\varkappa_0$, $\|\varkappa_0\| = k_0$, be a qc mapping of $\mathbb{D}_z$ onto $\mathbb{D}_w$ which is extremal for its boundary values and which does not have an essential boundary point. Inscribe quadrilaterals Q into $\mathbb{D}_z$ and denote their images by f_0 in $\mathbb{D}_w$ by Q'. The image Q' has, as its vertices, the images by f_0 of the vertices of Q. Let M and M' be the moduli of Q and Q' respectively. The question is, if (2) can hold.

Let f with dilatation K be the extremal mapping of Q onto Q'. The equation (2) is equivalent with

$$\sup K = K_0 \tag{13}$$

where the sup is taken over all quadrilaterals Q. This is the special case of (8) for $N = 4$. We find

Theorem 3. *The extremal mapping f_0 satisfies* (13) *for the inscribed quadrilaterals Q if and only if it is the extremal mapping of a quadrilateral itself.*

This means that in all other cases we have inequality in (13). The example of Anderson and Hinkkanen is the horizontal stretching of a parallelogram. This mapping f_0 has no essential boundary point and is, in their situation, not the mapping of quadrilaterals. Therefore $\sup(M'/M) < K_0$.

The example of Reich has analytic boundary values. Therefore we have again $\sup(M'/M) < K_0$.

Clearly, in both examples, we still have inequality in (13) even if we allow any inscribed polygons with an arbitrary fixed bound N for the number of vertices.

Added in Proof. After the completion of this paper I have become aware of two papers with related results: Shanshuang Yang, On dilatations and substantial boundary points of homeomorphisms of Jordan curves, Results Math. **31** (1979), 180–188, and Qi Yi, A problem in extremal quasiconformal extensions, Sci. China Ser. A **41**:11 (1998), 1135–1141.

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