Approximating l2-Betti numbers of an amenable covering by ordinary Betti numbers

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Approximating ℓ_2-Betti numbers of an amenable covering by ordinary Betti numbers

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Abstract. It is shown that the ℓ_2-Betti numbers of an amenable covering of a finite cell-complex can be approximated by ordinary Betti numbers of the finite Følner subcomplexes. This is a new proof, using simple homological arguments, of a recent result of *Dodziuk* and *Mathai*.

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0. Introduction

Let Y be an infinite amenable covering of a finite cell-complex X with covering transformation group G. Then **the ℓ_2-Betti numbers $\overline{\beta_p}(Y)$ can be approximated by the average ordinary Betti numbers of the finite subcomplexes Y_j of a Følner exhaustion of Y.** This has been proved by *Dodziuk* and *Mathai* [D-M]. The purpose of the present paper is to give a simple "homological" proof of that result. It consists in examining the ℓ_2-homology map $H_p(Y_j)\longrightarrow H_p(Y)$ induced by the inclusion $Y_j\longrightarrow Y$.

1. Følner sequence

1.1. We consider a discrete infinite amenable group G and a free cocompact G-space Y. By this we mean a cell complex Y on which G operates freely by permutation of the cells, with finite orbit complex $X = Y/G$. Then Y is a covering of X with covering transformation group G. Since G is a factor group of the fundamental group of X, and X is a finite complex, G is necessarily finitely generated. In short Y is called an infinite amenable covering of X.

1.2. It is known (Cheeger-Gromov [C-G], see also [E] or [D-M]) that in such a situation there exists in Y a Følner sequence (or Følner exhaustion) $Y_j,\ j = 1, 2, 3, \dots$

Here is its description in the form we will need later.

For each closed p-cell σ_p in X we choose an arbitrary lift $\hat{\sigma}_p$ in the corresponding G-orbit. The union of all $\hat{\sigma}_p$, $p \geq 0$, together with its topological closure (i.e. adding if necessary boundary cells of the $\hat{\sigma}_p$) is a closed fundamental domain D for the G-action in Y. The Y_j form an increasing sequence of finite subcomplexes of Y with union Y; each Y_j is a union of N_j distinct translates $x_\nu D$, $\nu = 1, 2, ..., N_j$, $x_\nu \in G$, of D. Let further $\dot{Y}_j$ be the topological boundary of Y_j and $\dot{N}_j$ the number of translates of D which meet $\dot{Y}_j$. From the combinatorial Følner criterion [F] for amenability it follows easily that the sequence Y_j can be chosen such that $\dot{N}_j/N_j \longrightarrow 0$ for $j \longrightarrow \infty$.

2. ℓ_2-chains, restricted trace

2.1. The cellular p-chains of Y with $\mathbb{R}-$coefficients constitute a free $\mathbb{R}G-$module $C_p(Y)$; as basis we can take the lifts (see **1.2**) $\hat{\sigma}_p^i$ of the p-cells σ_p^i of X, $i = 1, 2, ..., \alpha_p$, where α_p is the number of p-cells of X. Each p-cell of Y can be uniquely written as $x\hat{\sigma}_p^i$, $x \in G, i = 1, ..., \alpha_p$, and in each orbit the G-action is by left translation.

2.2. As Y is an infinite complex, one considers besides the ordinary p-chains also ℓ_2-chains, i.e. square-summable real linear combinations of the cells of Y. They constitute a Hilbert space $C_p^{(2)}(Y)$ where all the cells $x\hat{\sigma}_p^i$ as above form an orthonormal basis. We sometimes omit Y and simply write $C_p^{(2)}$. The induced action of G on $C_p^{(2)}$ is isometric.

2.3. For any Hilbert subspace H of $C_p^{(2)}$, not necessarily G-invariant, there is the orthogonal projection

$$\Phi : C_p^{(2)} \longrightarrow C_p^{(2)}$$

with image H. We consider the following "restricted trace" of Φ referring to a finite subcomplex Y_j of Y consisting of N_j translates of the fundamental domain D. Here amenability is not required; it is in **3.4** only that Y_j will refer to a Følner sequence in Y.

Let Π_j be the projection $C_p^{(2)} \longrightarrow C_p^{(2)}$ with image $C_p^{(2)}(Y_j)$. Since Y_j is a finite complex, we have $C_p^{(2)}(Y_j) = C_p(Y_j)$; thus Π_j is projection on a finite dimensional $\mathbb{R}$-subspace of $C_p^{(2)}$ whose basis consists of all cells $x_\nu\hat{\sigma}_p^i$ with $\nu \leq N_j$. One can form the $\mathbb{R}$-trace

$$d_j(H) = trace_{\mathbb{R}} \Pi_j \Phi$$

It will be examined for some special subspaces H. Note that it can be expressed by scalar products in $C_p^{(2)}$ as

$$d_j(H) = \sum_{i=1}^{\alpha_p} \sum_{\nu=1}^{N_j} < \Phi(x_\nu \hat{\sigma}_p^i), x_\nu \hat{\sigma}_p^i > + \sum_{\tau_p} < \Phi(\tau_p), \tau_p > .$$

where the τ_p are cells in $\dot{Y}_j$ not of the form $x_\nu \hat{\sigma}_p^i$.

2.4. Properties of d_j:

1) Since Φ is idempotent and self-adjoint, the scalar products above are equal to $< \Phi(x_\nu \hat{\sigma}_p^i), \Phi(x_\nu \hat{\sigma}_p^i) >$ and $< \Phi(\tau_p), \Phi(\tau_p) >$ respectively and thus ≥ 0: The restricted trace $d_j(H)$ is *non-negative*.

2) Note that one always has

$$d_j(H) \leq \dim_{\mathbb{R}} \Pi_j(H)$$

since

$$\mathrm{tr}_{\mathbb{R}}(\Pi_j \Phi) \leq ||\Pi_j \Phi|| \dim_{\mathbb{R}} \mathrm{im}(\Pi_j \Phi) \leq \dim_{\mathbb{R}} \Pi_j(H).$$

If in particular H is a subspace of $C_p(Y_j)$ then d_j is the same as the trace of the projection of $C_p(Y_j)$ to H. Since these are finite-dimensional vector spaces, the trace is $= \dim_{\mathbb{R}} H$.

3) If H decomposes orthogonally into $H_1 + H_2$ then $d_j(H) = d_j(H_1) + d_j(H_2)$. Just note that then $\Phi = \phi_1 + \phi_2$ where ϕ_i is the projection onto H_i, $i = 1, 2$ and replace Φ in the scalar products above.

4) In case H is G-invariant the projection Φ is G-equivariant and $< \Phi(x_\nu \hat{\sigma}_p^i), x_\nu \hat{\sigma}_p^i >$ is equal to $< \Phi(\hat{\sigma}_p^i), \hat{\sigma}_p^i >$. But $\Sigma_{i=1}^{\alpha_p} < \Phi(\hat{\sigma}_p^i), \hat{\sigma}_p^i >$ is just the *von Neumann dimension* $\dim_G H$ (see e.g. [L] or [E2]). Thus in that case

$$d_j(H) = N_j \dim_G H$$

plus an "error term" T_j coming from the boundary cells τ_p which is $\leq \dim_{\mathbb{R}} C_p(\dot{Y}_j)$.

3. Mapping $H_p(Y_j)$ into $H_p(Y)$

3.1. In the following, homology H_p is to be understood as "reduced" ℓ_2-homology (cycles modulo the closure of boundaries). It can be represented by the orthogonal complement of the space of boundaries in the p-cycle space, i.e. by *harmonic* chains (boundary $\partial = 0$ and coboundary $\delta = 0$). In this sense we will consider $H_p(Y)$ as a Hilbert subspace of $C_p^{(2)}(Y)$ and $H_p(Y_j)$ as a subspace of $C_p(Y_j)$.

3.2. Since the boundary operator ∂ in $C_p^{(2)}$ commutes with the G-action, the homology group $H_p(Y)$ considered as a subspace of $C_p^{(2)}$ is G-invariant. According to **2.4**, 4) we have

$$d_j(H_p(Y)) = N_j \dim_G H_p(Y) + T_j = N_j \, \overline{\beta}_p(Y \mathrm{rel}.G) + T_j,$$

where $\overline{\beta}_p$ denotes the ℓ_2-Betti number and T_j is the error term from **2.4**,4).

As for $H_p(Y_j)$, we have by **2.4**, 2)

$$d_j(H_p(Y_j)) = \dim_{\mathbb{R}} H_p(Y_j) = \beta_p(Y_j),$$

the ordinary p-th Betti number of Y_j.

3.3. The inclusion of Y_j in Y induces a bounded linear map $\phi : H_p(Y_j) \longrightarrow H_p(Y)$. Let K_p be the kernel of ϕ, and K'_p its orthogonal complement in $H_p(Y_j)$; and I_p the image of ϕ, and I'_p its orthogonal complement in $H_p(Y)$.

We will look closer at these harmonic subspaces of $C_p(Y_j)$ and $C_p^{(2)}(Y)$ respectively in order to get estimates for the values of d_j. We recall that ∂ commutes with the inclusion of Y_j in Y but in general not with the the restriction of Y to Y_j, and that for δ things are the other way around. In particular a harmonic chain in Y_j need not be harmonic in Y, but can be made harmonic by adding a well-defined element of the closure of boundaries.

3.4. We decompose the p-chains $c \in C_p^{(2)}$ as $c = \dot{c} + c'$ where all p-cells of $\dot{c}$ intersect the topological boundary $\dot{Y}_j$ and c' does not contain any such cell. This yields an orthogonal decomposition of $C_p^{(2)}$ into $\dot{C}_p$ and C'_p. We now use the amenability of the covering and assume that Y_j is a term of the Følner sequence. Then $\dim_{\mathbb{R}} \dot{C}_p \leq \dot{N}_j \alpha_p$.

1) If $c \in K_p$, with $\partial c = \delta c = 0$ in Y_j, then $c \in \overline{\partial C_{p+1}^{(2)}(Y)}$. If we assume $\dot{c} = 0$, $c = c' \in C'_p$, then δ commutes with the inclusion, i.e. $\delta c = 0$ in Y. But since cocycles are orthogonal to the closure of the space of boundaries, it follows that $c = 0$. Thus $K_p \cap C'_p = 0$, and K_p is isomorphic to a subspace of $\dot{C}_p$. Therefore

$$d_j(K_p) = \dim_{\mathbb{R}} K_p \leq \dim_{\mathbb{R}} \dot{C}_p \leq \dot{N}_j \alpha_p \ .$$

2) As for $d_j(I'_p)$ it does not exceed $\dim_{\mathbb{R}} R_p$ where $R_p = \mathrm{res}_j I'_p$ and res_j is the restriction from Y to Y_j. The chains $c \in I'_p$ fulfill $\partial c = \delta c = 0$. Moreover $< c, z >= 0$ for all p-cycles z in Y_j since $\phi(z) = z + b$, with $b \in \overline{\partial C_{p+1}^{(2)}}$. For $r \in R_p$ the same holds except possibly for $\partial r = 0$. But if $r = \dot{c} + c'$ as above, and if we assume $\dot{c} = 0$ then $\partial r = 0$. From $< r, z >= 0$ for all p-cycles z in Y_j it follows that r is a coboundary in Y_j, $r = \delta s$. Thus $< r, r > = < r, \delta s > = < \partial r, s > = 0$, whence $r = 0$ and $R_p \cap C'_p = 0$. As before this implies $\dim_{\mathbb{R}} R_p \leq \dot{N}_j \alpha_p$ and we get

$$d_j(I'_p) \leq \dim_{\mathbb{R}} R_p \leq \dot{N}_j \alpha_p \ .$$

3.5. K'_p is isomorphic as a (finite-dimensional) vector space to I_p. Their d_j need not be equal, but we show that their difference fulfills an inequality similar to those above. The isomorphism is given by adding to each $c \in K'_p$ a well defined element $b(c) \in \overline{\partial C^{(2)}_{p+1}}(Y)$, in order to get a harmonic chain in Y. If, in particular, $c \in K'_p \cap C'_p$ then $\delta c = 0$ in Y, whence $c \in I_p$. Thus $K'_p \cap C'_p$ is a subspace of I_p which remains unchanged under Π_j. This implies that $d_j(I_p) \geq d_j(K'_p \cap C'_p) = \dim_{\mathbb{R}} K'_p \cap C'_p$ and

$$\dim_{\mathbb{R}} K'_p - d_j(I_p) \leq \dim_{\mathbb{R}} K'_p / K'_p \cap C'_p \ .$$

But $K'_p/K'_p \cap C'_p$ is isomorphic to $(K'_p + C'_p)/C'_p$ which is contained in $C^{(2)}_p/C'_p$ isomorphic to $\dot{C}_p$. Thus its dimension is $\leq \dot{N}_j \alpha_p$ whence

$$d_j(K'_p) - d_j(I_p) \leq \dot{N}_j \alpha_p \ .$$

3.6. Finally we have

$$\beta_p(Y_j) - N_j \overline{\beta_p}(Y \text{rel}.G) = d_j(H_p(Y_j)) - d_j(H_p(Y)) + T_j$$
$$= d_j(K_p) - d_j(I'_p) + (d_j(K'_p) - d_j(I_p)) + T_j$$

where T_j is the error term in **2.4**. By **3.4** and **3.5** and since $T_j \leq \dot{N}_j \alpha_p$ this yields

$$|\frac{1}{N_j}\beta_p(Y_j) - \overline{\beta_p}(Y \text{rel}.G)| \leq 4\alpha_p \frac{\dot{N}_j}{N_j}$$

which goes to 0 with $j \to \infty$. Thus

$$\lim_{j\to\infty} \frac{1}{N_j}\beta_p(Y_j) = \overline{\beta_p}(Y \text{rel}.G).$$

This is the approximation statement mentioned in the introduction.

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