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Covering degrees are determined by graph manifolds involved

Fengchun Yu and Shicheng Wang

Abstract. W.Thurston raised the following question in 1976: Suppose that a compact 3-manifold M is not covered by (surface) $\times S^1$ or a torus bundle over S^1 . If M_1 and M_2 are two homeomorphic finite covering spaces of M , do they have the same covering degree?

For so called geometric 3-manifolds (a famous conjecture is that all compact orientable 3-manifolds are geometric), it is known that the answer is affirmative if M is not a non-trivial graph manifold.

In this paper, we prove that the answer for non-trivial graph manifolds is also affirmative. Hence the answer for the Thurston's question is complete for geometric 3-manifolds. Some properties of 3-manifold groups are also derived.

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Introduction

Definition 1. A 3-manifold M is said to have *Property C* if, whenever M_1, M_2 are homeomorphic finite covering spaces of M , the degrees of the coverings are the same. It has *Property C_r* if the above is true for all regular coverings. (Noted that M has *Property C* if and only if it has *Property C_r* [WW, Lemma 2.1].)

We are mainly interested in the following problem of W.Thurston raised in 1976.

[K, problem 3.16 (a)] Which 3-manifolds have *Property C*? In particular suppose that M is not covered by (surface) $\times S^1$ or a torus bundle over S^1 , does M

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has property C ?

Definition 2. A compact, connected, orientable 3-manifold is said to be *geometric* if it is a Seifert manifold, or a hyperbolic manifold, or a Haken manifold, or a connected sum of such manifolds.

The fundamental conjecture in 3-manifold theory is that all compact, connected, orientable 3-manifolds are geometric. It is known that if M is covered by either $(\text{surface}) \times S^1$ or a torus bundle over S^1 , then it does not have Property C . Actually such manifolds admit non-trivial self-coverings [WW, Theorem 8.6]. Therefore we will concentrate on 3-manifolds in the class \mathcal{G} defined below.

Notation. We use \mathcal{G} to denote the set of all geometric 3-manifolds which are not covered by either $(\text{surface}) \times S^1$ or a torus bundle over S^1 .

Conjecture 1. Every 3-manifold M in \mathcal{G} has Property C .

By using Gromov norm and Kneser-Milnor sphere decomposition theorem, it is proved that Conjecture 1 is true if M is not a non-trivial graph manifold [WW, Theorem 2.5]. So the problem is reduced to the case of non-trivial graph manifolds.

Several partial results on property C of non-trivial graph manifolds had been obtained recently. There are three different covering invariants of graph manifolds introduced by S.C.Wang and Y.Q.Wu [WW], by J.Luecke and Y.Q.Wu [LW] and by W.D. Neumann [N]. So a graph manifold has property C if one of those covering invariants is non-zero. Unfortunately all those three covering invariants are zero for some non-trivial graph manifolds. It is also proved that a non-trivial graph manifold has property C if M has at most three vertex manifolds [WW] or if M is a knot complement in S^3 [LW].

In this paper we prove that non-trivial graph manifolds have property C . Hence a geometric manifold has Property C if and only if it is not covered by either $(\text{surface}) \times S^1$ or a torus bundle over S^1 .

There are four sections after the introduction. In §1, we review the B -matrix $B(M)$ for a non-trivial graph manifold M defined by S.C. Wang and Y.Q.Wu [WW] and its properties, which will be the start point of our further approach. In §2, we introduce a H -matrix $H(M)$ for a non-trivial graph manifold M , and then deduce a matrix equation which expresses the covering degree by B -matrix and H -matrix (Theorem 3). In §3, we prove our main result: non-trivial graph manifolds have Property C (Corollary 5). Some results about maximum eigenvalues of non-negative matrices are used in the proof. In §4, we prove that for a closed geometric 3-manifold M , $\pi_1(M)$ have property I , that is any two isomorphic subgroups of $\pi_1(M)$ have the same index (may be infinite), if and only if M have property C and M is irreducible. This result generalizes the earlier results of F. Gonzales-Acuna and W. Whitten [GW] and of [WW] on the cohopficity of 3-manifold groups.

Remark 1. A non-orientable 3-manifold has Property C if and only if its orientable double cover has property C .

1. Preliminaries

For readers' convenience, we review a matrix invariant defined in [WW] in this section. The notions which are not explained are standard, see [J] and [S].

For a surface S in a 3-manifold M , we use $N(S)$ to denote the regular neighborhood of S and $\eta(S)$ to denote the interior of $N(S)$.

Suppose M is a prime orientable 3-manifold with boundary (possibly empty) a union of tori. Then the torus decomposition of Jaco-Shalen and Johannson cut M open along a minimum collection of embedded incompressible tori \mathcal{T} , the JSJ-surface, into a collection of simple manifolds and Seifert manifolds.

Definition 3. A geometric 3-manifold M is called a graph manifold if the JSJ-surface \mathcal{T} is not empty and \mathcal{T} cuts M into Seifert manifold pieces. A graph manifold is *non-trivial* if it is not covered by a torus bundle over S^1 . (see [WW, §3].) A graph manifold is *non-trivial* if and only if it belongs to \mathcal{G} .

Now consider a non-trivial graph manifold M . In [WW] a new decomposing surface φ of M is defined which consists of those tori in JSJ surface \mathcal{T} which do not bound twisted I -bundles, and those central Klein bottles of twisted I -bundles bounded by tori in \mathcal{T} . If M is a graph manifold in \mathcal{G} , then the decomposing surface exists and is unique up to ambient isotopy [WW, Lemma 3.1].

Comparing with \mathcal{T} , the new decomposition surface φ has two advantages:

Lemma 1. (a) $M - \eta(\varphi)$ consists of Seifert fiber spaces with negative Euler characteristic orbifolds [WW, Lemma 3.2], therefore each Seifert manifold piece of M admits a unique Seifert fibration up to isotopy [S, Theorem 3.9].

(b) Suppose $\phi: \tilde{M} \rightarrow M$ is a finite covering between graph manifolds. If φ is a non-empty decomposing surface of M , then $\tilde{\varphi} = \phi^{-1}(\varphi)$ in \tilde{M} is a decomposing surface for \tilde{M} [WW, Lemma 4.4].

Suppose M is a non-trivial graph manifold and has the decomposing surface φ just defined. We define a graph $\Gamma(M)$ as follows: to each component N_i of $M - \eta(\varphi)$ a vertex v_i is assigned, and to each component S_j of φ an edge e_j is assigned, so that

(1) if S_j is a torus, and $\partial N(S_j)$ has one component in each of N_i and N_k (i may be equal to k), then e_j has endpoints on v_i and v_k ;

(2) if S_j is a Klein bottle and $\partial N(S_j)$ is in N_i , then e_j has both endpoints on v_i .

We define a 'weight' for each vertex or edge of $\Gamma(M)$ as follows. If v_i is a vertex

of $\Gamma(M)$ corresponding to a component N_i of $M - \eta(\varphi)$, let the weight $x_i = x(v_i)$ be the Euler characteristic of the orbifold of N_i . If e is an edge corresponding to a surface S in φ , let the weight $\Delta(e)$ be the fiber intersection number $\Delta(S)$. The weights of vertices are negative rational numbers by Lemma 1 (a), and the weights of edges are all positive integers ([WW, Lemma 4.3]). Since the decomposing surface and the Seifert fibrations on the pieces are unique, this weighted graph is well defined, and is an invariant of M .

For each edge e in the graph $\Gamma(M)$, define

$$\alpha(e) = \frac{1}{\Delta(e)x_ix_j}. \tag{1.1}$$

For all i, j , define

$$a_{ij} = a_{ij}(\Gamma(M)) = \begin{cases} \sum\{\alpha(e)|e \text{ has the endpoints on } v_i \text{ and } v_j\} \\ 0 & \text{if there is no such } e. \end{cases} \tag{1.2}$$

To simplify the formulas, we define

$$b_{ij} = \begin{cases} a_{ij} & \text{if } i \neq j, \\ 2a_{ij} & \text{if } i = j. \end{cases} \tag{1.3}$$

Define \tilde{b}_{pq} in the same way.

Let $B = B(M) = (b_{ij})$. The matrix B depends only on $\Gamma(M)$ and the indexing of its vertices, so up to simultaneous permutations of rows and columns it is well defined. Clearly B is a non-negative symmetric matrix.

Definition 4. $\Gamma = \Gamma(M)$ is called the *weighted graph* of M , and $B = B(M)$ is called the *B-matrix* of M , for a non-trivial graph manifold M .

Consider a regular covering map $\phi: \tilde{M} \rightarrow M$. By Lemma 1 (b), ϕ can induce a map on the graphs $\phi_{\#}: \Gamma(\tilde{M}) \rightarrow \Gamma(M)$ defined in the following way. A vertex \tilde{v}_p is mapped to v_i if the corresponding component \tilde{N}_p in $\tilde{M} - \eta(\tilde{\varphi})$ is mapped to N_i , and an edge \tilde{e}_g is mapped to e_h if the corresponding surface \tilde{S}_g covers S_h .

For each vertex v_i in $\Gamma(M)$, define

$$I_i = \{p|\phi(\tilde{v}_p) = v_i\} \tag{1.4}$$

and use $|I_i|$ to denote the number of elements in I_i .

The result below is the start point for our approach.

Theorem 1. ([WW, Theorem 6.1]) *Suppose $\phi: \tilde{M} \rightarrow M$ is a regular covering over a non-trivial graph manifold. Then the degree d of ϕ satisfies the equation*

$$d \sum_{p \in I_i} \sum_{q \in I_j} \tilde{b}_{pq} = b_{ij}|I_i||I_j|.$$

where (b_{ij}) ((\tilde{b}_{pq})) is the *B-matrix* of M (\tilde{M}).

2. B-matrix, H-matrix and covering degrees

Let M be a non-trivial graph manifold. The symmetry of the B -matrix $B(M)$ will provide some useful information in determining the covering degrees. To clarify the idea, a reduced weighted graph $L(M)$ is defined in [WW, §9] as follows. The graph $L(M)$ has the same vertices as $\Gamma(M)$, and it has one edge e_{ij} for each pair of vertices v_i, v_j (i may be equal to j). Assign b_{ij} as the weight of e_{ij} .

An isometry g of $L(M)$ is an automorphism of the graph which preserves the weights. That is, the weight of e_{ij} equals that of $g(e_{ij})$ for all i, j . Denote the isometry group of $L(M)$ by $G(M)$. It induces an action on the set $V(M)$ of vertices of $L(M)$.

Suppose $\phi: \tilde{M} \rightarrow M$ is a regular covering map. A covering transformation $f: \tilde{M} \rightarrow \tilde{M}$ induces a map $f_{\#}: L(\tilde{M}) \rightarrow L(\tilde{M})$ which is clearly an isometry. Let $F(\tilde{M})$ be the set of isometries induced by covering transformations. Then $F(\tilde{M})$ is a subgroup of $G(\tilde{M})$. Since ϕ is a regular covering, it induces an one-to-one correspondence $V(\tilde{M})/F(\tilde{M}) \rightarrow V(M)$. Since $F(\tilde{M})$ is a subgroup of $G(\tilde{M})$, for each vertex $v \in V(M)$, $\phi^{-1}(v)$ is contained in a G -orbit of $V(\tilde{M})$.

Suppose $V(\tilde{M})$ has m disjoint G -orbits $\tilde{U}_1, \tilde{U}_2, \dots, \tilde{U}_m$, and $V(M)$ has n vertices v_1, v_2, \dots, v_n . For each G -orbit \tilde{U}_α of $V(\tilde{M})$, define its indexing set

$$O_\alpha = \{r | \tilde{v}_r \in \tilde{U}_\alpha\}. \tag{2.0}$$

Each G -orbit of $V(\tilde{M})$ is the union of some $\phi^{-1}(v)$, $v \in V(M)$. Define

$$A_\alpha = \{i | I_i \subset O_\alpha\}.$$

Then we have

$$O_\alpha = \bigcup_{i \in A_\alpha} I_i. \tag{2.1}$$

We use $|O_\alpha|$ to denote the number of elements in O_α , then we have $|O_\alpha| = \sum_{i \in A_\alpha} |I_i|$.
Define

$$\tilde{h}_{\alpha\beta} = \frac{1}{(|O_\alpha||O_\beta|)} \sum_{r \in O_\alpha} \sum_{s \in O_\beta} \tilde{b}_{rs}, \quad \alpha, \beta = 1 \dots m. \tag{2.2}$$

Then $\tilde{H} = H(\tilde{M}) = (\tilde{h}_{\alpha\beta})_{m \times m}$ is a non-negative symmetric matrix determined by $L(\tilde{M})$ the indexing of G -orbit of $\tilde{V}(M)$.

Definition 5. $\tilde{H} = H(\tilde{M})$ is called the H -matrix of \tilde{M} , for a non-trivial graph manifold \tilde{M} .

Lemma 2. *Suppose \tilde{M} is a connected non-trivial graph manifold. Then each column (row) of $H(\tilde{M})$ has at least one non-zero element.*

Proof. Since \tilde{M} is connected, each column (row) of $B(\tilde{M})$ has at least one non-zero element. Then Lemma 2 follows from the definition of $H(\tilde{M})$. \square

Theorem 2. *Suppose $\phi: \tilde{M} \rightarrow M$ is a d -fold regular covering over a non-trivial graph manifold. Then d satisfies the equation*

$$d\tilde{h}_{\alpha\beta}|O_\beta| = \sum_{j \in A_\beta} b_{ij}|I_j| \quad \text{for any } \alpha, \beta = 1 \cdots m, \quad i \in A_\alpha. \quad (2.3)$$

Proof. For any $\tilde{v}_p, \tilde{v}_q \in \tilde{U}_\alpha$, there is an isometry $g \in G(\tilde{M})$ sending \tilde{v}_p to \tilde{v}_q . Suppose g maps \tilde{v}_s to $\tilde{v}_{g(s)}$ for $\tilde{v}_s \in \tilde{U}_\beta$. Since g is an isometry, we have $\tilde{b}_{ps} = \tilde{b}_{qg(s)}$. Since \tilde{U}_β is a G -orbit, when \tilde{v}_s ranges over all vertices of \tilde{U}_β , $\tilde{v}_{g(s)}$ also ranges over all vertices of \tilde{U}_β . Therefore, we have

$$\sum_{s \in O_\beta} \tilde{b}_{ps} = \sum_{s \in O_\beta} \tilde{b}_{qg(s)} = \sum_{s \in O_\beta} \tilde{b}_{qs} = C_{\alpha\beta}. \quad (2.4)$$

$C_{\alpha\beta}$ is a constant determined by α and β , and

$$C_{\alpha\beta} = \frac{1}{|O_\alpha|} \sum_{r \in O_\alpha} \sum_{s \in O_\beta} \tilde{b}_{rs} = \tilde{h}_{\alpha\beta}|O_\beta| \quad (2.5)$$

By Theorem 1, we have

$$b_{ij}|I_i||I_j| = d \sum_{r \in I_i} \sum_{s \in I_j} \tilde{b}_{rs}. \quad (2.6)$$

Assume $i \in A_\alpha$. When j ranges over A_β , (2.6) becomes

$$\begin{aligned} \sum_{j \in A_\beta} b_{ij}|I_i||I_j| &= d \sum_{j \in A_\beta} \sum_{r \in I_i} \sum_{s \in I_j} \tilde{b}_{rs} \\ &= d \sum_{r \in I_i} \sum_{s \in O_\beta} \tilde{b}_{rs} \\ &= d|I_i| \cdot C_{\alpha\beta} \\ &= d|I_i|\tilde{h}_{\alpha\beta}|O_\beta|. \end{aligned} \quad (2.7)$$

The last three equations follow from (2.1) (2.4) and (2.5). Then (2.3) follows. \square

To express Theorem 2 in terms of matrix equation, we have the following theorem.

Theorem 3. *Suppose $\phi: \tilde{M} \rightarrow M$ is a d -fold regular covering over a non-trivial graph manifold. Then d satisfies the equation:*

$$X_{m \times n} B_{n \times n} = d \tilde{H}_{m \times m} W_{m \times n}. \tag{2.8}$$

where B is the B -matrix of M , and \tilde{H} is the H -matrix of \tilde{M} , X and W satisfy:

- (a) both X and W are $m \times n$ non-negative matrices;
- (b) each column of X has only one non-zero element and the row sums of X are constant 1;
- (c) W is obtained from X by substituting the only non-zero element in each column of X for 1. Then the column sums of W are constant 1.

Proof. Define

$$x_{\alpha i} = \begin{cases} \frac{|I_i|}{|O_\alpha|} & \text{if } i \in A_\alpha, \\ 0 & \text{if } i \notin A_\alpha. \end{cases} \tag{2.9}$$

$$w_{\alpha i} = \begin{cases} 1 & \text{if } i \in A_\alpha, \\ 0 & \text{if } i \notin A_\alpha. \end{cases} \tag{2.10}$$

We denote $X = (x_{\alpha i})$, $W = (w_{\alpha i})$. Clearly in the α -th row of X , $x_{\alpha i} \neq 0$ if and only if $i \in A_\alpha$. Since $|O_\alpha| = \sum_{i \in A_\alpha} |I_i|$, we have the row sums of X are constant 1 and each column of X has only one non-zero element. So (b) is true and W is determined by X just like (c).

Assume that $i \in A_\alpha$, by (2.9), (2.3) and the symmetry of B , we have

$$(XB)_{\beta i} = \sum_{j \in A_\beta} x_{\beta j} b_{ji} = \sum_{j \in A_\beta} \frac{|I_j|}{|O_\beta|} b_{ji} = d \tilde{h}_{\alpha\beta}. \tag{2.11}$$

By (2.10) and the symmetry of \tilde{H} we have

$$(d\tilde{H}W)_{\beta i} = d \sum_{k=1}^m \tilde{h}_{\beta k} w_{ki} = d \tilde{h}_{\beta\alpha} = d \tilde{h}_{\alpha\beta} \tag{2.12}$$

So we got the equation (2.8) of Theorem 3. □

3. Nontrivial graph manifolds have Property C

The following Lemma will be used in proving our main result.

Lemma 3. *([HJ, Lemma 8.1.21 and Theorem 8.3.1.]) If $A = (a_{ij})$ is a $m \times m$ non-negative matrix, satisfies the column (row) sums are constant 1, then the*

maximum eigenvalue of A is 1, and there is an non-negative eigenvector $x = (x_1, \dots, x_m)^T$, $x \neq 0$, $x_i \geq 0$, $i = 1 \dots m$ such that $Ax = x$.

Theorem 4. Suppose $\phi, \phi': \tilde{M} \rightarrow M$ are regular coverings over a non-trivial graph manifold with covering degrees d, d' . Then $d = d'$. That is, a non-trivial graph manifold has Property C_r .

Proof. By Theorem 3 we have the following matrix equations.

$$\begin{cases} (X_1)_{m \times n} B_{n \times n} = d \tilde{H}_{m \times m} (W_1)_{m \times n} \\ (X_2)_{m \times n} B_{n \times n} = d' \tilde{H}_{m \times m} (W_2)_{m \times n} \end{cases} \tag{3.0}$$

Then we have

$$\begin{cases} X_1 B X_2^T = d \tilde{H} W_1 X_2^T \\ X_2 B X_1^T = d' \tilde{H} W_2 X_1^T \end{cases} \tag{3.1}$$

Since both B and \tilde{H} are symmetric matrices, we have

$$d \tilde{H} W_1 X_2^T = (d' \tilde{H} W_2 X_1^T)^T = d' X_1 W_2^T \tilde{H}. \tag{3.2}$$

Denote

$$L = X_1 W_2^T, \quad R = W_1 X_2^T, \quad \lambda = \frac{d}{d'}, \tag{3.3}$$

then we have

$$L \tilde{H} = \lambda \tilde{H} R. \tag{3.4}$$

Recall both X_i and W_i are $m \times n$ non-negative matrices, which satisfy the conditions (b) and (c) in Theorem 3 : the row sums of X_i and the column sums of W_i are both constant 1. Then it is easy to see both L and R are non-negative matrices, the row sums of L and column sums of R are both constant 1. (If we denote $J_m = \underbrace{(1 \dots 1)^T}_{m \text{ times}}$, $J_n = \underbrace{(1 \dots 1)^T}_{n \text{ times}}$, then we have $L J_m = X_1 W_2^T J_m = X_1 J_n = J_m$, $R^T J_m = X_2 W_1^T J_m = X_2 J_n = J_m$).

By Lemma 3, the maximum eigenvalues of L and R are both 1, and there is non-negative vector $y = (y_1, \dots, y_m)^T$ satisfies

$$Ry = y, \quad y \neq 0, \quad y_i \geq 0 \quad (i = 1, \dots, m). \tag{3.5}$$

By (3.4) and (3.5), we have

$$L(\tilde{H}y) = \lambda(\tilde{H}y). \tag{3.6}$$

By Lemma 2, each column (row) of \tilde{H} has at least one non-zero element. Since \tilde{H} is a non-negative matrix, y is a non-negative vector and $y \neq 0$, it follows that $\tilde{H}y \neq 0$. So λ is the eigenvalue of L by (3.6). The maximum eigenvalue of L is

1, so we must have $\lambda \leq 1$. That is, $d \leq d'$. By the symmetry of ϕ and ϕ' , we conclude $d' \leq d$ also. Hence $d = d'$ \square

By Theorem 4 and [WW, Lemma 2.1], we have

Corollary 5. *Every non-trivial graph manifold has Property C.*

By Corollary 5, [WW, Theorem 2.5] and [WW, Theorem 8.6], we have

Corollary 6. *A geometric 3-manifold has Property C if and only if it is not covered by either (surface) $\times S^1$ or a torus bundle over S^1 . In particular, Conjecture 1 is true.*

4. Property I of 3-manifold groups

A group G is called cohopfian if each self-monomorphism of G is an isomorphism. Recently when 3-manifolds groups are cohopfian have been studied (see [GW], [WW]).

Definition 6. A group G has *property I* if, whenever G_1 and G_2 are isomorphic subgroups of G , then their embedding indices either are the same or are both infinite.

Remark 2. Both Property C and cohopficity are related indices of embeddings one group to another, the first one restricted on finite index embeddings and the second one restricted on self-embeddings. The notion Property I unifies two notions Property C and cohopficity.

Theorem 7. *Suppose M is a closed geometric 3-manifold. Then the fundamental group $\pi_1(M)$ has Property I if and only if M is irreducible and M is not covered by either (surface) $\times S^1$ or a torus bundle over S^1 .*

Proof. One direction follows directly from [WW, Theorem 8.7] and Remark 2.

Now we prove another direction. Denote $G = \pi_1(M)$, where M is irreducible and it is not covered by either (surface) $\times S^1$ or a torus bundle over S^1 . Then M has property C by Corollary 7. M is aspherical [J], and therefore for any covering \tilde{M} of M , $H_3(\pi_1(\tilde{M})) = H_3(\tilde{M})$.

Let G_1 and G_2 be the isomorphic subgroup of G . Let \tilde{M}_1 and \tilde{M}_2 be the corresponding coverings. Then we have

$$H_3(\tilde{M}_1) = H_3(\pi_1(\tilde{M}_1)) = H_3(G_1) = H_3(G_2) = H_3(\pi_1(\tilde{M}_2)) = H_3(\tilde{M}_2).$$

Now $H_3(\tilde{M}_i) = Z$ if and only if \tilde{M}_i is closed, and this is true if and only if

$[G, G_i]$ is finite. So either both $[G, G_1]$ and $[G, G_2]$ are infinite, or both $[G, G_1]$ and $[G, G_2]$ are finite. In the later case, if G is finite, clearly $[G, G_1] = [G, G_2]$. If G is infinite, then \tilde{M}_i is either a closed hyperbolic manifold, or a closed Haken manifold, or a closed Seifert manifold of infinite π_1 . Since \tilde{M}_1 and \tilde{M}_2 have isomorphic fundamental groups, they are homeomorphic [T]. Since M has property C , the two covering degrees are the same. Hence $[G, G_1] = [G, G_2]$. \square

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