# The decomposition of 3-dimensional Poincaré complexes

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# The decomposition of 3-dimensional Poincaré complexes

John Crisp

**Abstract.** We show that if the fundamental group of an orientable $PD^3$-complex has infinitely many ends then it is either a proper free product or virtually free of finite rank. It follows that every $PD^3$-complex is finitely covered by one which is homotopy equivalent to a connected sum of aspherical $PD^3$-complexes and copies of $S^1 \times S^2$. Furthermore, it is shown that any torsion element of the fundamental group of an orientable $PD^3$-complex has finite centraliser.

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## 1. Introduction

An $n$-dimensional Poincaré complex, or $PD^n$-complex, is a connected finitely dominated CW-complex $P$ with a homomorphism $w : \pi_1(P) \to \{\pm 1\}$ which exhibits the equivariant Poincaré duality of a closed $n$-dimensional manifold with orientation class $w$. (See [15] or [16] for more details). We may regard Poincaré complexes as natural homotopy analogues of closed manifolds. In dimension 3, one has a completely algebraic characterisation of the class of Poincaré complexes due to Turaev [15], and $PD^3$-complexes are distinguished up to homotopy equivalence by their fundamental group, orientation class, and fundamental class [6]. The most interesting and challenging problem in this area is to determine which $PD^3$-complexes are homotopy equivalent to 3-manifolds. With this in mind, we focus in this paper on the connected-sum decomposition of $PD^3$-complexes.

Let $P$ denote an arbitrary 3-dimensional Poincaré complex with fundamental group $\pi = \pi_1(P)$. It is known (see Wall [16]) that if $\pi$ has 0, 1, or 2 ends then $P$ has universal cover $\tilde{P}$ homotopy equivalent to $S^3$, is aspherical ($\tilde{P}$ contractible), or is homotopy equivalent to one of $RP^3 \# RP^3$, $S^1 \times RP^2$, $S^1 \times S^2$ or $S^1 \tilde{\times} S^2$, respectively. Otherwise, $\pi$ has infinitely many ends, and in this case Wall posed the following questions: firstly, whether it follows (for $P$ orientable, i.e: $w$ trivial) that $\pi$ is a proper free product, and secondly, whether such a decomposition of the group $\pi$ would imply a corresponding connected sum decomposition of the complex $P$, whereby one might obtain a decomposition theorem for orientable

$PD^3$-complexes analogous to that for 3-manifolds. Turaev [15] has since answered the second of these questions by showing that if $\pi$ is a proper free product then $P$ is a nontrivial connected sum of $PD^3$-complexes.

In the present paper we approach the first question, and show (Theorem 14) that, for $P$ orientable, if $\pi$ has infinitely many ends then it is either a proper free product or virtually free. Thus the hoped for decomposition into a connected sum of $PD^3$-complexes $P_i$ with $\tilde{P}_i \simeq S^3$, $S^2$ or contractible is at least true of some finite cover of any $PD^3$-complex. In particular, every 3-dimensional Poincaré complex has virtually torsion free fundamental group. Theorem 14 also reduces the question of whether every $PD^3$-complex is virtually homotopy equivalent to a 3-manifold, to the case of aspherical $PD^3$-complexes, namely the problem of realising all $PD^3$-groups as (virtual) 3-manifold groups. Various partial results in this direction were given by Hillman [7], [8], and Thomas [14] in the mid 80's. An analogue of the torus theorem has been given by Kropholler [11], and very recently Bowditch [1] has proved a version of the Seifert Conjecture, namely that a $PD^3$-group which contains an infinite cyclic normal subgroup is the fundamental group of a closed Seifert fibred 3-manifold. However, the problem as stated remains open. We note that there are examples of $PD^3$-complexes which are not homotopy equivalent to manifolds, but these all have finite fundamental group. (Groups with periodic cohomology of period 4 are the fundamental groups of $PD^3$-complexes [16], but Milnor has shown that many of these are not 3-manifold groups, the simplest example being $\mathcal{S}_3$, the symmetric group on three elements).

To completely settle Wall's question one needs to resolve the case that the fundamental group $\pi$ is virtually free. In Theorem 17 we show that if $P$ is orientable then any torsion element of $\pi$ has finite centraliser in $\pi$. Thus, for example, the free product of two finite groups amalgamated over a common normal subgroup which is proper in each group, while being virtually free, cannot be the fundamental group of an orientable $PD^3$-complex. However, this does not resolve every case. For example, the question raised in [9] as to whether $\mathcal{S}_3 *_{C_2} \mathcal{S}_3$ may be the fundamental group of an orientable $PD^3$-complex remains unanswered.

Our approach in this paper is motivated by ideas in Hillman's paper [9]. There the groups $H_*(C, \bar{H}^1(\pi, \mathbb{Z}\pi))$, for $C$ a cyclic subgroup of $\pi$, are known by duality and a spectral sequence argument. Here we show, on the other hand, that these homology groups may be calculated independently of any duality properties. In Section 2 we do this in the general setting of groups acting on trees, where one uses a coefficient module which is "presented" by the tree. In Section 3 we relate this coefficient module to the module $H^1(\pi, \mathbb{Z}\pi)$ via the accessibility of $\pi$, and Chiswell's Mayer-Vietoris sequence for graphs of groups. Comparing the independent calculations leads to the main results in Section 4, where we also recover the main result of [9] as Corollary 18. Finally, in Section 5, we give an extension of our theorems to finite Poincaré pairs.

## 2. Trees with $\infty$-vertices

In this section we introduce the notion of a module $\Pi$ being presented by a tree $X$ with $\infty$-vertices, and proceed to calculate the homology groups $H_*(C,\Pi)$ of a prime order cyclic group $C$ acting by automorphisms on $X$. These turn out to be precisely determined by properties of the subtree of fixed points of $X$ under the action of $C$.

Following [3], we define a *graph* $X$ to be the disjoint union of a pair of sets $EX$ and $VX$, called the edge and vertex sets respectively, together with a pair of functions $o,t : EX \to VX$ which specify for each edge $\epsilon$ an original vertex $o(\epsilon)$, and a terminal vertex $t(\epsilon)$. In practice, however, we shall think of $X$ as an oriented 1-dimensional simplicial complex realised as a topological space. A nonempty graph $X$ is called a *tree* if it is connected and contains no closed loops, that is if it is simply-connected as a topological space. Let $G$ be a group. A tree $X$ together with a left action of $G$ by orientation respecting simplicial automorphisms of $X$ shall be called a *$G$-tree*. Explicitly, each element $g \in G$ acts via a bijection of $X$ such that $g(EX) = EX$, $g(VX) = VX$, and, for $\epsilon \in EX$, $o(g(\epsilon)) = g(o(\epsilon))$ and $t(g(\epsilon)) = g(t(\epsilon))$. Note that any tree shall be considered by default to be a $G$-tree with $G$ the trivial group if not otherwise specified.

**Definition 1.** By a *$G$-tree with $\infty$-vertices* we shall mean a $G$-tree $X$ with a distinguished $G$-invariant subset $V_fX \subset VX$ consisting of vertices with finite valence (i.e: with finitely many adjacent edges). Vertices which do not lie in $V_fX$ are said to be *$\infty$-vertices*. (Note that an $\infty$-vertex need not have infinite valence). Henceforth we shall assume that every $G$-tree $X$ has this extra structure. We shall also assume that the $\infty$-vertex structure of any subtree of $X$ is the one naturally inherited from $X$ by restriction of the set $V_fX$.

To any tree $X$ with $\infty$-vertices we may associate a $\mathbb{Z}$-module $\Pi[X]$, which is said to be *presented* by the tree $X$, as follows. Let $\mathbb{Z}[V_fX]$ and $\mathbb{Z}[EX]$ denote the free $\mathbb{Z}$-modules with bases $V_fX$ and $EX$ respectively. Then $\Pi[X]$ is defined to be the cokernel of the map $\Delta : \mathbb{Z}[V_fX] \to \mathbb{Z}[EX]$ defined for each $\nu \in V_fX$ by the formula

$$\Delta(\nu) = \sum_{\{\epsilon \mid t(\epsilon)=\nu\}} \epsilon \;-\; \sum_{\{\epsilon \mid o(\epsilon)=\nu\}} \epsilon\,.$$

Furthermore, if $X$ is a $G$-tree then $\Pi[X]$ naturally inherits a left $\mathbb{Z}G$-module structure. We write $[\epsilon]_X$ to denote the element of $\Pi[X]$ represented by an edge $\epsilon \in EX$.

**Example 2.** Let $X$ be a $G$-tree, with finite quotient $G\backslash X$, and whose edge stabilizers are finite and vertex stabilizers have at most one end. Take $V_fX$ to be the set of vertices with finite stabilizer under the action of $G$. Then $\Pi[X]$ is isomorphic as a $\mathbb{Z}G$-module to $H^1(G,\mathbb{Z}G)$ (this is shown in Section 3). Such a

$G$-tree exists for any (almost) finitely presented group (see [3], Theorem VI.6.3) and in particular for $G$ the fundamental group of a $PD^3$-complex.

Note that the module $\Pi[X]$ depends (up to isomorphism) only on the unoriented simplicial complex $X$ (together with the $G$-action), the choice of orientation corresponding simply to a choice of canonical generators $\pm[\epsilon]_X$, for each $\epsilon \in X$.

**Definition 3.** Let $X$ be a tree. We define a *geodesic segment*, a *geodesic ray*, and a *geodesic line* in $X$ to be any subcomplex of $X$ homeomorphic, respectively, to a real closed interval $[0, x]$ for $x \geq 0$, a real half-line $[0, \infty)$, and the real line $\mathbb{R}$. These sets correspond to finite, half-infinite and infinite edge paths which are geodesic in the sense of no backtracking. The fact that a tree contains no circuits ensures that the collection of all geodesic segments rays and lines, together with the empty set, is closed under taking finite intersections.

Define the set, $\mathcal{E}X$, of *ends* of $X$ to be the set of equivalence classes of geodesic rays where two rays $\gamma$ and $\gamma'$ are said to be equivalent if $\gamma \cap \gamma'$ is also a geodesic ray.

We make the following observations based on the above definitions and the basic properties of a tree. There is a unique geodesic segment between any pair of vertices $a, b$ in $X$ (that is, having boundary set $\{a, b\}$). There is a unique geodesic ray with given boundary vertex $\nu$, and representing a given end $\varepsilon$, and which we call the geodesic ray from $\nu$ to $\varepsilon$. Finally, between any pair of distinct ends $\varepsilon, \varepsilon' \in \mathcal{E}X$ there is a unique geodesic line which is the union of a (non-unique) pair of rays belonging to $\varepsilon$ and $\varepsilon'$ respectively.

Let $e(X) = |\mathcal{E}X|$ denote the number of ends of $X$, and $\infty(X) = |VX \setminus V_f X|$ the number of $\infty$-vertices, each of which may be an infinite number. Finally write

$$\xi(X) = e(X) + \infty(X) - 1\,.$$

**Theorem 4.** *Let $X$ be a tree with $\infty$-vertices. Then $\Pi[X]$ is free, as a $\mathbb{Z}$-module, with infinite rank whenever $\xi(X)$ is infinite, and finite rank equal to* $\max\{\xi(X), 0\}$ *otherwise.*

*Proof.* Choose an arbitrary vertex $\nu_0$ in $X$. Without loss of generality we may suppose that $X$ is oriented such that, for every edge $\epsilon$, $o(\epsilon)$ lies on the geodesic segment between $\nu_0$ and $t(\epsilon)$. In other words, $t(\epsilon)$ is always further from $\nu_0$ than $o(\epsilon)$. For $\nu \in VX$ write $\gamma_\nu$ for the geodesic segment between $\nu_0$ and $\nu$, and write $X_\nu$ for the subtree of $X$ spanned by the set of vertices $\nu'$ for which $\gamma_{\nu'}$ passes through $\nu$. Finally, write $E^+_\nu$ for the set of edges $\epsilon \in EX$ with $o(\epsilon) = \nu$. That is $E^+_\nu$ contains those edges in $X_\nu$ which are adjacent to $\nu$.

Define $X'$ to be the subgraph of $X$ spanned by $\nu_0$ and those vertices $\nu$ for which $X_\nu$ is either infinite or contains an $\infty$-vertex. If $\nu$ is a vertex of $X'$ other than $\nu_0$ then $\gamma_\nu \subset X'$ since, for each $\nu'$ in $\gamma_\nu$, $X_{\nu'}$ contains $X_\nu$ and so is also either infinite or contains an $\infty$-vertex. Thus $X'$ is connected and hence a subtree of $X$. Note also that the edges of $X'$ are precisely those $\epsilon$ for which $X_{t(\epsilon)}$ is either infinite or has an $\infty$-vertex. Thus, if $\epsilon \in EX \setminus EX'$, then $X_{t(\epsilon)}$ is a finite tree with no $\infty$-vertices and one may easily check that

$$\epsilon = \sum_{\nu \in VX_{t(\epsilon)}} \Delta(\nu) \ , \qquad \text{as an element of } \mathbb{Z}[EX] \, ,$$

and hence $[\epsilon]_X = 0$ in $\Pi[X]$. It follows, easily, that $\Pi[X] \cong \Pi[X']$.

Note that if $X = X_{\nu_0}$ is finite with no $\infty$-vertices then $\xi(X) = -1$, while $X'$ is trivial (consisting only of the vertex $\nu_0$) and so $\Pi[X] \cong 0$. Hence the theorem holds in this case, and we may assume henceforth that $X_{\nu_0}$ is either infinite or contains an $\infty$-vertex, as is already the case for every other vertex in $X'$.

If $\nu \in VX'$ is not an $\infty$-vertex then $E_\nu^+$ is a finite set of edges $\epsilon_1, .., \epsilon_n$ in $EX$, and $EX' \cap E_\nu^+$ must be non-empty, for if each $X_{t(\epsilon_i)}$ were finite with no $\infty$-vertices then the same would be true of $X_\nu$, a contradiction. For each $\nu \in V_f X'$ make an arbitrary choice of edge in $EX' \cap E_\nu^+$ and denote this $\mathrm{succ}(\nu)$. Now define the set $\mathcal{G} = EX' \setminus \{\mathrm{succ}(\nu) \mid \nu \in V_f X'\}$.

We claim that $\Pi[X']$ is freely generated as a $\mathbb{Z}$-module by the subset $\mathcal{G}$ of $EX'$. Consider $\Pi[X']$ as the $\mathbb{Z}$-module presented by the generating set $EX'$ and the relations $\Delta'(\nu) = 0$ for each $\nu \in V_f X'$, where $\Delta'$ is defined as in Definition 1 but with respect to the tree $X'$. The claim follows immediately from the observation that each relation $\Delta'(\nu) = 0$ may be replaced by an equivalent relation which expresses $\mathrm{succ}(\nu)$ as equal to a $\mathbb{Z}$-linear combination of edges in $\mathcal{G}$. This is clearly true if $\nu = \nu_0$. Otherwise $\nu = t(\epsilon)$ for some $\epsilon \in EX'$, and the relation $\Delta'(\nu) = 0$ expresses $\mathrm{succ}(\nu)$ as a $\mathbb{Z}$-linear combination of elements of $\mathcal{G}$ and the edge $\epsilon$ which is either in $\mathcal{G}$ itself, or may be assumed, by induction on the length of $\gamma_{t(\epsilon)}$, to be otherwise expressed as a $\mathbb{Z}$-linear combination of elements of $\mathcal{G}$.

Finally, it suffices to show that $|\mathcal{G}|$ and $\xi(X)$ are either equal (and finite) or both infinite. (We have already dealt with the case where $\xi(X) = -1$). Define $\mathcal{P}$ to be the union of the set of all geodesic rays with boundary vertex $\nu_0$ and the set of all geodesic segments between $\nu_0$ and some $\infty$-vertex. It is clear that $\mathcal{P}$ corresponds bijectively to the set $\mathcal{E}X \cup \{\infty\text{-vertices in } X\}$, so that $|\mathcal{P}| = \xi(X) + 1$. (Note that every segment or ray belonging to $\mathcal{P}$ is contained in $X'$. It will follow from the next step that in fact $X'$ is precisely the union of the elements of $\mathcal{P}$). When $\nu$ is an $\infty$-vertex every edge of $E_\nu^+ \cap EX'$ lies in $\mathcal{G}$, and, when $\nu \in V_f X'$, all but one (namely $\mathrm{succ}(\nu)$). Thus, given any vertex $\nu_1$ of $X'$, there is a unique maximal subcomplex of $X'$, which we call $p_{\nu_1}$, which is a geodesic segment or ray containing the segment $\gamma_{\nu_1}$, with $\nu_0$ as a boundary vertex, but not containing any edges of $\mathcal{G}$ other than those already in $\gamma_{\nu_1}$. In fact, the set $p_{\nu_1}$ is an element of $\mathcal{P}$.

Let $\mathcal{G}^0 = \mathcal{G} \cup \{0\}$ and define the function $\pi : \mathcal{G}^0 \to \mathcal{P}$ such that $\pi(\epsilon) = p_{t(\epsilon)}$ for $\epsilon \in \mathcal{G}$ and $\pi(0) = p_{\nu_0}$. Note that, amongst the edges in $\pi(\epsilon)$ which belong to $\mathcal{G}$, $\epsilon$ is distinguished as the furthest from $\nu_0$, while $\pi(0)$ contains no edge belonging to $\mathcal{G}$.

It follows that $\pi$ is injective. Moreover, when $|\mathcal{G}|$ is finite, $\pi$ is also surjective, for then, given any $\gamma \in \mathcal{P}$, there are only finitely many edges of $\mathcal{G}$ in $\gamma$ and $\gamma = \pi(\epsilon)$ where $\epsilon$ is the furthest of these from $\nu_0$, or $\gamma = \pi(0)$ if there are no such edges. Hence if $|\mathcal{G}|$ is infinite then so is $\xi(X)$, and otherwise $|\mathcal{G}|$ is finite and the bijection gives $|\mathcal{G}| = \xi(X)$. □

Note that when $|\mathcal{G}|$ is infinite it need not have the same cardinality as $\xi(X)$. For example, the infinite tree of valence 3 has countably many edges but uncountably many ends, so that in this case $|\mathcal{G}|$ would be countable but $\xi(X)$ uncountable.

Suppose now that $X$ is a $C$-tree, where $C = \langle g \rangle$ denotes a finite cyclic group of prime order $p$, and write $\Pi = \Pi[X]$ for the $\mathbb{Z}C$-module presented by $X$. Note that the set $X^C$ of fixed points of $X$ under the action of $C$ is a subtree of $X$ (see [13], I.6.1) and so a tree with $\infty$-vertices where we set $V_f X^C = V_f X \cap X^C$. At this point we recall the following standard notation, that, for $M$ a $\mathbb{Z}G$-module, one writes $M^G$ and $M_G$ respectively for the invariant submodule and coinvariant quotient module of $M$. In order to compute the homology $H_q(C, \Pi)$, for $q > 0$, one defines the norm map $\overline{N} : \Pi_C \longrightarrow \Pi^C$ with respect to $C$, which is induced by the map $N : \Pi \longrightarrow \Pi^C$ such that $N(x) = x + g(x) + ... + g^{p-1}(x)$ for $x \in \Pi$. The homology groups $H_q(C, \Pi)$, $q > 0$, are given by the kernel and cokernel of $\overline{N}$ when $q$ is even and odd respectively. These will now be computed purely in terms of the fixed subtree $X^C$.

Let $A$ denote the set of edges of $X$ which are not in $X^C$ but which have a vertex in $X^C$. So $\epsilon \in A$ precisely if one, but not both, of $o(\epsilon)$ or $t(\epsilon)$ lie in $X^C$. Note that each connected component of $X \setminus X^C$ contains the interior of a unique element of $A$. For each $\epsilon \in A$ write $X_\epsilon$ for the tree (with $\infty$-vertices) obtained from the component of $X \setminus X^C$ containing $\mathrm{int}(\epsilon)$ by replacing the missing vertex of $\epsilon$ with an $\infty$-vertex. Define the $\mathbb{Z}C$-module $B = \bigoplus_{\epsilon \in A} \Pi[X_\epsilon]$ with a natural $C$-action induced by the action of $C$ on $X$. Since $g(\Pi[X_\epsilon]) = \Pi[X_{g(\epsilon)}]$ with $g(\epsilon) \neq \epsilon$ for each $\epsilon \in A$, and moreover, by Theorem 4, each $\Pi[X_\epsilon]$ is a free $\mathbb{Z}$-module, it follows that $B$ is a free $\mathbb{Z}C$-module.

We may think of $B$ as the module presented by the edges and vertices of $X$ which lie outside $X^C$. Indeed $\Pi$ is simply the quotient of the $\mathbb{Z}C$-module $B \oplus \mathbb{Z}[EX^C]$ obtained by imposing the remaining relations due to the vertices of $V_f X^C$. Namely, $\Pi \cong \mathrm{coker}(\overline{\Delta} : \mathbb{Z}[V_f X^C] \longrightarrow B \oplus \mathbb{Z}[EX^C])$ where

$$\overline{\Delta}(\nu) = \sum_{\{\epsilon | t(\epsilon) = \nu\}} \overline{\epsilon} \; - \sum_{\{\epsilon | o(\epsilon) = \nu\}} \overline{\epsilon}\,, \qquad \text{and} \qquad \overline{\epsilon} = \begin{cases} \epsilon & \text{if } \epsilon \in EX^C\,, \\ [\epsilon]_{X_\epsilon} & \text{if } \epsilon \in A\,. \end{cases}$$

Let $\phi : B \oplus \mathbb{Z}[EX^C] \longrightarrow \Pi$ denote the corresponding quotient map, and write $D$ for $\overline{\Delta}(\mathbb{Z}[V_f X^C])$ which is the kernel of this map. Significantly, each element of $D$ is fixed by the group $C$, since $g(\overline{\Delta}(\nu)) = \overline{\Delta}(g(\nu)) = \overline{\Delta}(\nu)$ for each $\nu \in V_f X_C$.

Define the map $\mathcal{N}$ on $B \oplus \mathbb{Z}[EX^C]$ by $\mathcal{N}(x) = x + g(x) + .. + g^{p-1}(x)$ for $x \in B \oplus \mathbb{Z}[EX^C]$, and observe that, since $\phi$ is a $\mathbb{Z}C$-homomorphism, $\phi \circ \mathcal{N} = N \circ \phi$. Also since $B$ is a free $\mathbb{Z}C$-module, while $\mathbb{Z}[EX^C]$ is a direct sum of copies of the augmentation module $\mathbb{Z}$, we have that $\ker \mathcal{N} = (g-1)B$ and $\operatorname{im} \mathcal{N} = B^C \oplus p.\mathbb{Z}[EX^C]$.

**Lemma 5.** *Let $K$ denote the submodule of $\Pi$ generated by those edges which do not lie in $X^C$, that is $K = \phi(B)$. Then $K \cap \Pi^C \subset \operatorname{im} N$.*

*Proof.* Suppose that $x \in B$ represents an element $\phi(x)$ of $K \cap \Pi^C$. Then $x - g(x) = \eta$ where $\eta \in D$ and so must be fixed by $g$. Thus

$$p.\eta = \eta + g(\eta) + ... + g^{p-1}(\eta) = \mathcal{N}(x - g(x)) = 0\,,$$

and, since $B$ is free, it follows that $\eta = 0$. Thus $x \in B^C \subset \operatorname{im} \mathcal{N}$, and consequently $\phi(x) \in \operatorname{im} N$. □

**Lemma 6.** *The norm map $\overline{N} : \Pi_C \to \Pi^C$ has cokernel $(\mathbb{Z}/p\mathbb{Z})^R$, where $R = \max\{(\xi(X^C), 0\}$ for $\xi(X^C)$ finite, and $R$ is infinite otherwise.*

*Proof.* Write $\widehat{\Pi}$ for the quotient module $\Pi/K$ and let $\psi : \Pi \to \widehat{\Pi}$ denote the canonical projection. Note that $\widehat{\Pi} \cong \Pi[X^C]$ which, by Theorem 4, is free as a $\mathbb{Z}$-module with rank $R$. Now, one has $\Pi = K + \Pi^C$. So the restriction $\psi^C$ of $\psi$ to $\Pi^C$ is clearly surjective and has kernel $K^C = K \cap \Pi^C$. It now follows that $\operatorname{coker} \overline{N} = \operatorname{coker} N \cong \operatorname{coker}(\psi^C \circ N)$, since, by Lemma 5, one has that $K^C \subset \operatorname{im} N$. Moreover, since $K$ is a $\mathbb{Z}C$-submodule, one has $N(K) \subset K^C$ and hence a well-defined map $\widehat{N} : \widehat{\Pi} \to \widehat{\Pi}$ such that $\widehat{N} \circ \psi = \psi^C \circ N$. Thus $\operatorname{coker}(\psi^C \circ N) = \operatorname{coker} \widehat{N}$ and, since $\operatorname{im} \widehat{N} = p\widehat{\Pi}$, the Lemma is proven. □

**Lemma 7.** *The kernel of the norm map $\overline{N} : \Pi_C \to \Pi^C$ is $\mathbb{Z}/p\mathbb{Z}$ in the case that $\xi(X^C) = -1$, and is trivial otherwise.*

*Proof.* Consider the following commuting square in which $\pi$ denotes the canonical projection of $\Pi$ onto the $C$-coinvariant module.

$$\begin{tikzcd} B \oplus \mathbb{Z}[EX^C] \arrow[r, two heads, "\mathcal{N}"] \arrow[d, two heads, "\pi \circ \phi"'] & \operatorname{im} \mathcal{N} \arrow[d, "\phi|_{\operatorname{im}\mathcal{N}}"] \arrow[dl, dotted, "\phi'"'] \\ \Pi_C \arrow[r, "\overline{N}"] & \Pi^C \end{tikzcd}$$

Note that $D + (g-1)B \subseteq \ker(\pi \circ \phi)$. Conversely, if $\phi(x) \in \ker \pi = (g-1)\Pi$ then $\phi(x) = (g-1)\phi(y) = \phi((g-1)y)$ for some $y \in B \oplus \mathbb{Z}[EX^C]$, and so $x \in D + (g-1)B$.

Hence $\ker(\pi \circ \phi) = D + (g-1)B = D + \ker\mathcal{N}$ and therefore $\pi \circ \phi$ induces a surjective map $\phi' : \mathrm{im}\,\mathcal{N} \to \Pi_C$, as in the diagram, with kernel $\mathcal{N}(D)$ or simply $p.D$ since elements of $D$ are fixed by $g$. It now follows from the diagram that $\ker \overline{N} = \phi'(\ker \phi|_{\mathrm{im}\,\mathcal{N}}) = \phi'(D \cap \mathrm{im}\,\mathcal{N})$ which may be identified with the quotient $(D \cap \mathrm{im}\,\mathcal{N})/p.D$.

Take any element $\overline{a} \in \ker\overline{N}$ with representative $a \in D \cap \mathrm{im}\,\mathcal{N}$ which we may assume to have the form $a = \sum_{\nu \in VX^C} n_\nu \overline{\Delta}(\nu)$ where $0 \le n_\nu < p$ and $n_\nu$ is zero except for finitely many $\nu \in V_f X^C$. Each $\epsilon \in EX^C$ will have coefficient $n_{t(\epsilon)} - n_{o(\epsilon)}$ in this expression, and since $a$ is also an element of $\mathrm{im}\,\mathcal{N} = B^C \oplus p.\mathbb{Z}[EX^C]$ this coefficient must be a multiple of $p$. Since we chose each $n_\nu < p$ it follows that $n_{o(\epsilon)} = n_{t(\epsilon)}$ and, by the connectedness of $X^C$, the coefficients $n_\nu$ take a constant value $n$ over the whole set $VX^C$ of fixed vertices. Now $n$ can be nonzero (that is $\overline{a}$ nontrivial) only if $X^C$ is finite and has no $\infty$-vertices, that is only if $\xi(X^C) = -1$, in which case putting $n = 1$ gives a nontrivial element $\overline{a}$ of order $p$ which clearly generates the whole of $\ker \overline{N}$. □

Given that the homology of $C$ with coefficients in a $ZC$-module may be calculated from the kernel and cokernel of the norm map, the next theorem follows immediately from Lemmas 6 and 7 combined.

**Theorem 8.** *Suppose that the finite cyclic group $C$ of prime order $p$ acts on the tree with $\infty$-vertices $X$, and let $\Pi = \Pi[X]$ be the left $\mathbb{Z}C$-module presented by $X$. Then*

$$H_i(C,\Pi) = \begin{cases} (\mathbb{Z}/p\mathbb{Z})^{R_+} & \text{for } i \text{ odd,} \\ (\mathbb{Z}/p\mathbb{Z})^{R_-} & \text{for } i > 0 \text{ even,} \end{cases}$$

*where $R_+ = \max\{\xi(X^C), 0\}$ if $\xi(X^C)$ is finite, and is infinite otherwise, and $R_- = \max\{-\xi(X^C), 0\}$.* □

## 3. $H^1(G, \mathbb{Z}G)$ for accessible groups

A $G$-tree $X$ is said to be *terminal* (see [3]) if each edge stabilizer is finite and each vertex stabilizer has at most one end. A group $G$ is said to be *accessible* if there exists a terminal $G$-tree. When $G$ is a finitely generated accessible group we may assume, by [3], VI.7.4, that there is a terminal $G$-tree $X$ with quotient graph $G\backslash X$ finite, and in this case we shall adopt the convention of considering $X$ as a tree with $\infty$-vertices by taking $V_f X$ to be precisely the set of vertices with finite stabilizers. For a group $G$ we shall consider the group cohomology $H^1(G, \mathbb{Z}G)$ as a left $\mathbb{Z}G$-module with action defined in terms of the natural right action by $g.x = xg^{-1}$ for $g \in G$ and $x \in H^1(G, \mathbb{Z}G)$.

**Theorem 9.** *Let $G$ be an accessible group, and $X$ a terminal $G$-tree with $G\backslash X$*

*finite. Then the module $\Pi[X]$ presented by $X$ is isomorphic to $H^1(G,\mathbb{Z}G)$ as a left $\mathbb{Z}G$-module.*

*Proof.* Since the statement is evidently true for $G$ finite (in which case $X$ is a finite tree with no $\infty$-vertices), we may assume in what follows that $G$ is an infinite group. Recall that a graph of groups $(\mathcal{G},Y)$ consists of a graph $Y$ together with a collection $\mathcal{G}$ of groups $G_v$, for each $v\in VY$, and subgroups $G_e\subset G_{o(e)}$, for each $e\in EY$, with injective homomorphisms $\phi_e : G_e\to G_{t(e)}$. Fixing a choice of maximal subtree $T$ of $Y$, one defines the fundamental group of $(\mathcal{G},Y)$ to be the group with presentation

$$\langle t_e\ ,\ G_v \mid \mathrm{rel}\ G_v\ ,\ t_e a t_e^{-1}=\phi_e(a) \text{ for } a\in G_e\ ,\ t_e=1 \text{ for } e\in ET\rangle\ ,$$

noting that up to isomorphism this group is independent of the choice of $T$.

Let $X$ be a terminal $G$-tree with finite quotient, as in the statement. By the Bass-Serre structure theorem ([13], I.5.4 Theorem 13) $G$ is the fundamental group of a finite graph of groups $(\mathcal{G},Y)$ where $Y=G\backslash X$ and the edge and vertex groups of $(\mathcal{G},Y)$ are isomorphic to the corresponding edge and vertex stabilizers of $X$ respectively. Furthermore, $X$ is isomorphic to the $G$-tree $\widetilde{X}$ defined with vertex and edge sets

$$V\widetilde{X}=\coprod_{v\in VY} G/G_v\ ,\qquad E\widetilde{X}=\coprod_{e\in EY} G/G_e\ ,$$

such that $o(gG_e)=gG_{o(e)}$ and $t(gG_e)=gt_eG_{t(e)}$ for $g\in G$ and $e\in EY$, and with $G$ acting by left multiplication on cosets. Since $X$ is a terminal $G$-tree the vertex groups of $Y$ have at most one end, so that the cohomology Mayer-Vietoris sequence of Chiswell [2] (see also [13], II.2.8 Proposition 13) gives rise to a short exact sequence of right $\mathbb{Z}G$-modules

$$0\to \bigoplus_{v\in V_fY}\mathbb{Z}[G_v\backslash G]\xrightarrow{\Delta_r}\bigoplus_{e\in EY}\mathbb{Z}[G_e\backslash G]\longrightarrow H^1(G,\mathbb{Z}G)\to 0\,,$$

where $V_fY=\{v\in VY\mid G_v \text{ finite}\}$ or just the set ${}_G\backslash V_f\widetilde{X}$. If we choose to consider this as a sequence of left $\mathbb{Z}G$-modules and left $\mathbb{Z}G$-maps (via the anti-isomorphism $g\mapsto g^{-1}$ of $G$), then the first two modules are naturally isomorphic to $\mathbb{Z}[V_f\widetilde{X}]$ and $\mathbb{Z}[E\widetilde{X}]$ respectively (by taking the coset $G_vg$ to $g^{-1}G_v$ etc..) and one can check that $\Delta_r$ becomes exactly the map $\Delta:\mathbb{Z}[V_f\widetilde{X}]\hookrightarrow\mathbb{Z}[E\widetilde{X}]$ of Definition 1. Thus $\widetilde{X}$, or equivalently $X$, presents $H^1(G,\mathbb{Z}G)$ as a left $\mathbb{Z}G$-module. □

**Corollary 10.** *If $G$ is an infinite group and $X$ a terminal $G$-tree with ${}_G\backslash X$ finite then $e(X)+\infty(X)=1$, $2$ or is infinite according as to whether $G$ has $1$, $2$ or infinitely many ends, respectively.*

*Proof.* This follows immediately from Theorem 9 and Theorem 4, and the fact that $\mathrm{rk}_{\mathbb{Z}}(H^1(G,\mathbb{Z}G))+1$ measures the number of ends of $G$. □

## 4. Results on $PD^3$-complexes

Let $\pi$ be a group equipped with a homomorphism $w : \pi \longrightarrow \{\pm 1\}$. We have in mind, of course, the fundamental group $\pi$ of a $PD^3$-complex with orientation character $w$. If $\Pi$ is a left (resp. right) $\mathbb{Z}\pi$-module, denote by $\overline{\Pi}$ the left $\mathbb{Z}\pi$-module with the same underlying abelian group, and action given by $g.x = w(g)gx$ (resp. $w(g)xg^{-1}$) for all $g \in \pi$, $x \in \Pi$. The only consequence of Poincaré duality which we shall use is the following.

**Lemma 11.** (Hillman, [9]) *Let $P$ be a $PD^3$-complex with infinite fundamental group $\pi$. If $C$ is a finite cyclic subgroup of $\pi$ then there are isomorphisms $H_s(C, \overline{H}^1(\pi, \mathbb{Z}\pi)) \cong H_{s+3}(C, \mathbb{Z})$ for all $s \geq 1$.*

*Proof.* This follows from the spectral sequence for the projection of the universal cover $\widetilde{P}$ onto $\widetilde{P}/C$ given that $H_q(\widetilde{P}; \mathbb{Z}) = \mathbb{Z}, 0, \overline{H}^1(\pi, \mathbb{Z}\pi), 0, \ldots$ which follows from the duality isomorphisms, the fact that $\widetilde{P}$ is simply connected and the assumption that $\pi$ is infinite. (We may assume, without loss, that $P$ is a 3-dimensional CW-complex). □

The thrust of the earlier Sections 2 and 3 was to be able to calculate these homology groups independently of Lemma 11 (in fact without using duality). This is achieved, for prime order cyclic subgroups, by taking Theorem 8 together with the following lemma.

**Lemma 12.** *Given a group $\pi$ and homomorphism $w : \pi \longrightarrow \{\pm 1\}$, let $\Pi$ be a left $\mathbb{Z}\pi$-module and $\overline{\Pi}$ as above. If $C = \langle g \rangle$ is a cyclic subgroup of $\pi$ of prime order $p$ then*

$$H_i(C, \overline{\Pi}) \cong \begin{cases} H_i(C, \Pi) & \text{if } i > 0 \text{ and } w(g) = 1, \\ H_{i+1}(C, \Pi) & \text{if } i > 0 \text{ and } w(g) = -1. \end{cases}$$

*Proof.* The case for $w(g) = 1$ is easy since then $\overline{\Pi} \cong \Pi$ as $\mathbb{Z}C$-modules. In the case $w(g) = -1$ (and $p = 2$ necessarily), $\overline{\Pi}$ and $\Pi$ are distinguished as $\mathbb{Z}C$-modules only by the action of $g$, whereby the homology, $H_*(C, \overline{\Pi})$, of the complex

$$\ldots \longrightarrow \overline{\Pi} \xrightarrow{1-g} \overline{\Pi} \xrightarrow{1+g} \overline{\Pi} \xrightarrow{1-g} \overline{\Pi} \longrightarrow \mathbf{0},$$

is just that of the complex

$$\ldots \longrightarrow \Pi \xrightarrow{1+g} \Pi \xrightarrow{1-g} \Pi \xrightarrow{1+g} \Pi \longrightarrow \mathbf{0},$$

The lemma now follows by comparing this with the complex for $H_*(C, \Pi)$. □

**Remark 13.** Given a $PD^3$-complex $P$ with infinite fundamental group $\pi$, then since $\pi$ is finitely presentable, it is accessible by Theorem VI.6.3 of [3], and there exists a terminal $\pi$-tree $X$, as in Theorem 9, which presents $\Pi \cong H^1(\pi, \mathbb{Z}\pi)$ as a left $\mathbb{Z}\pi$-module. It is now clear, by Lemmas 11, 12 and Theorem 8, that if $g$ is an element of $\pi$ of prime order then exactly one of the following two cases must hold. Either $w(g) = 1$ and $\xi(X^{\langle g\rangle}) = -1$, or $w(g) = -1$ and $\xi(X^{\langle g\rangle}) = 1$. We shall apply this to prove the following two theorems.

**Theorem 14.** *Let $P$ be an orientable $PD^3$-complex with $\pi = \pi_1(P)$. Then $\pi$ either has one end, is a proper free product, or is virtually free of finite rank. That is to say that $P$ is either an aspherical complex, a nontrivial connected sum (by Turaev [15]), or finitely covered by some $\widehat{P} \simeq \#^k(S^1 \times S^2)$, $k \geq 0$.*

*Proof.* Clearly we may assume that $\pi$ is infinite, since a finite group is virtually free of rank 0. Let $X$ be the $\pi$-tree of Remark 13 and $(\mathcal{G}, Y)$ the associated graph of groups, which may be assumed to be finite since $\pi$ is finitely generated ([3], VI.7.4). Assume that $\pi$ is not a proper free product, and hence that the edge groups of $(\mathcal{G}, Y)$ are all nontrivial. It now suffices to show that either the vertex groups of $(\mathcal{G}, Y)$ are all finite, for then $\pi$ must be virtually free of finite rank (see [10], also [13], II.2.6), or $\pi$ has one end.

Suppose that $(\mathcal{G}, Y)$ has an infinite vertex group. Then either there are no edge groups and $\pi$ has one end, or one of $G_{o(e)}$ or $G_{t(e)}$ is infinite for some edge $e \in EY$. But in the latter case we show that both these groups are finite thus reaching a contradiction. Since $G_e$ is nontrivial and finite, we may choose a nontrivial $g \in G_e \subset G_{o(e)}$ of prime order. Now, since $P$ is orientable, $w(g) = 1$ and Remark 13 shows that $\xi(X^{\langle g\rangle}) = -1$ in this case. That is $X^{\langle g\rangle}$ is finite with no $\infty$-vertices, and hence has finite vertex stabilizers. But $G_{o(e)}$ is the stabilizer of some vertex of $X$ which, since $g \in G_{o(e)}$, must lie in $X^{\langle g\rangle}$. Thus $G_{o(e)}$ must be a finite group. Finally take $g \in \phi_e(G_e) \subset G_{t(e)}$ to show similarly that $G_{t(e)}$ is finite. □

**Corollary 15.** *Let $P$ be an orientable $PD^3$-complex. Then $P$ is homotopy equivalent to a connected sum $V\#P_1\#...\#P_m$ where each $P_i$, for $i = 1, .., m$, is an aspherical $PD^3$-complex and $V$ is a $PD^3$-complex with $\pi_1(V)$ virtually free of finite rank.*

*Proof.* This follows from Theorem 14 by Turaev's theorem ([15], Theorem 1) and the fact that $\pi_1(P)$ is finitely presented, and by observing that $A * B$ is a virtually free group of finite rank if both $A$ and $B$ are. This last observation follows from the theorem of [10] which states that a group is virtually free of finite rank if and only if it is the fundamental group of a finite graph of groups with every vertex group finite. □

**Corollary 16.** *Let $P$ be an arbitrary $PD^3$-complex. Then $\pi_1(P)$ is virtually* *torsion free. Indeed, $P$ is finitely covered by an orientable $PD^3$-complex which is homotopy equivalent to a (possibly empty) connected sum of finitely many aspherical $PD^3$-complexes and copies of $S^1 \times S^2$.*

*Proof.* By considering the decomposition of Corollary 15 it is clear that the orientation cover of $P$, and hence $P$ itself, is finitely covered by such a connected sum. Since aspherical $PD^3$-complexes have torsion free fundamental groups, it now follows that $\pi_1(P)$ is virtually torsion free. (It is well known of course that $A * B$ is torsion free if and only if both $A$ and $B$ are. See [12] for instance). □

**Theorem 17.** *Let $P$ be a $PD^3$-complex with $\pi = \pi_1(P)$. If $g \in \pi$ is a nontrivial element of prime order $p$ such that $C_\pi(g)$ is infinite, then $p = 2$, $w(g) = -1$ and $C_\pi(g)$ has 2 ends.*

*Proof.* Again let $X$ be the $\pi$-tree with finite edge stabilizers of Remark 13. Note that $C_\pi(g)$ acts on the subtree $X^{\langle g \rangle}$, also with finite edge stabilizers. (If $x \in C_\pi(g)$, $\epsilon \in X^{\langle g \rangle}$, then $g(x\epsilon) = x(g\epsilon) = x\epsilon$ implies that $x\epsilon \in X^{\langle g \rangle}$ also). If $\xi(X^{\langle g \rangle}) = -1$ then $X^{\langle g \rangle}$ is a finite graph with finite vertex stabilizers in $\pi$ and so in $C_\pi(g)$. This contradicts $C_\pi(g)$ being infinite. Thus $\xi(X^{\langle g \rangle}) \geq 0$. It now follows from the Remark 13 that $w(g) = -1$, $p = 2$ (necessarily) and $\xi(X^{\langle g \rangle}) = 1$. Thus the set $\mathcal{E}X^{\langle g \rangle} \cup \{\infty\text{-vertices in } X^{\langle g \rangle}\}$ contains exactly two elements and there is a unique geodesic segment, ray, or line $\gamma$ joining them in $X^{\langle g \rangle}$. Since $C_\pi(g)$ respects the set of $\infty$-vertices and also acts on $\mathcal{E}X^{\langle g \rangle}$, it must preserve the set $\gamma$. In fact, some subgroup $H$ of index at worst 2 in $C_\pi(g)$ must fix each end or $\infty$-vertex. If there is some $\infty$-vertex involved then the infinite group $H$ stabilizes every edge in $\gamma$ which is a contradiction. Thus $\gamma$ is a geodesic line joining two ends and $H$ acts by translations. For any edge $\epsilon$ in $\gamma$ the quotient $H/\mathrm{stab}_H(\epsilon)$ must be $\mathbb{Z}$. But any surjection $H \to \mathbb{Z}$ splits, and since $\mathrm{stab}_H(\epsilon)$ is finite, $H$ must be virtually $\mathbb{Z}$ and have two ends. Hence $C_\pi(g)$ is also 2-ended as required. □

It follows that if $P$ is an orientable $PD^3$-complex then any torsion element of $\pi_1(P)$ has finite centraliser. Thus, for example $\pi = G_1 *_A G_2$ is not the fundamental group of an orientable $PD^3$-complex if $A$ is finite and a proper subgroup of both $N_{G_1}(A)$ and $N_{G_2}(A)$, since then $N_\pi(A)$ is infinite, and hence so is $C_\pi(a)$ for $a \in A$. As a corollary to this theorem we may also restate the main theorem of [9], which also has application in that paper to 2-knot groups.

**Corollary 18.** (Hillman, [9]) *Let $P$ be a $PD^3$-complex, with infinite fundamental group $\pi$. If $\pi$ has a nontrivial finite normal subgroup $N$, then $P \simeq S^1 \times RP^2$.*

*Proof.* In this case $N$ contains a nontrivial element $g$ of prime order $p$ where $C_\pi(g)$ is a subgroup of finite index in $\pi$. Thus, applying Theorem 17, $P$ is nonorientable, $g$ has order 2, and $\pi$ in fact has two ends. So, by Theorem 4.4 of [16],

$P \simeq S^1 \times RP^2$. □

Ideally we should like to improve Theorem 14 by reducing the case that $\pi$ is virtually free. Beyond applying Theorem 17 as per the examples given, it appears that some other approach is needed to eliminate the cases where $\pi$ is the fundamental group of a graph of groups with finite vertex groups and some nontrivial edge groups. Turaev ([15], Theorem 5) gives an explicit algebraic characterisation of the pair $(\pi_1(X), w(X))$ of a $PD^3$-complex $X$, but it is not clear how to apply this in general, let alone to decide whether the minimal example $\mathcal{S}_3 *_{C_2} \mathcal{S}_3$ proposed in [9] is or is not the fundamental group of an orientable $PD^3$-complex.

## 5. Extension to Poincaré pairs

Let $P$ be a connected finitely dominated CW-complex and $Q$ a subcomplex of $P$ which is a (not necessarily connected) $PD^{n-1}$-complex with orientation class induced (under inclusion) by a homomorphism $w : \pi_1(P) \to \{\pm 1\}$. Then the pair $(P, Q)$ is said to be an $n$-dimensional Poincaré pair if it exhibits the equivariant Lefschetz duality of an $n$-dimensional manifold with boundary (See [16] for details). A "weak loop theorem" due to C.B. Thomas [14] allows us to extend our main results to apply to the fundamental groups of finite Poincaré pairs. However, it is not immediately clear how to extend Turaev's work and deduce topological decompositions in this setting.

**Theorem 19.** *Let $(P, Q)$ be a finite orientable 3-dimensional Poincaré pair with fundamental group $\pi$. Then $\pi$ either has one end, is a proper free product, or is virtually free of finite rank. Furthermore, any torsion element of $\pi$ has finite centraliser in $\pi$.*

*Proof.* Since it is known ([4], [5]) that every $PD^2$-complex is homotopy equivalent to a closed surface we may suppose (by attaching mapping cylinders if necessary) that $Q$ is a disjoint union of closed orientable surfaces $Q_i$ and has a collar neighbourhood $Q \times [0, 1)$ in $P$. By the weak loop theorem of [14], one may find a disjoint collection of simple closed curves in each 'boundary' component $Q_i$ which represent a set of generators whose normal closure is the kernel of the map $\pi_1(Q_i) \to \pi_1(P)$ induced by inclusion. One may then modify $P$ by attaching a copy of $D^2 \times I$ along a neighbourhood of each of these closed curves to obtain a different Poincaré pair with the same fundamental group as $P$. In this way one reduces to the case where each $Q_i$ is $\pi_1$-injective and, by capping off each spherical boundary component with a 3-ball, we may assume moreover that each $Q_i$ is aspherical. In this case the conclusion of Lemma 11 is still valid, and the theorem now follows by precisely the same arguments as used to prove Theorems 14 and 17, since these are otherwise independent of duality properties.

To see that Lemma 11 holds in this context, note that, since each $Q_i$ maps $\pi_1$-injectively into $P$, the universal cover $(\widetilde{P}, \widetilde{Q})$ of the pair $(P, Q)$ has simply connected boundary components. Thus $H_1(\widetilde{Q}) = 0$. Moreover, since each $Q_i$ is aspherical, $H_2(\widetilde{Q}) = 0$ and therefore the relative exact sequence gives an isomorphism $H_2(\widetilde{P}) \cong H_2(\widetilde{P}, \widetilde{Q})$. Now, by Lefschetz duality, $H_2(\widetilde{P}, \widetilde{Q}) \cong \overline{H}^1(\pi, \mathbb{Z}\pi)$ and, since $\widetilde{P}$ is simply-connected and $\pi$ is infinite and we may suppose that $\widetilde{P}$ is a 3-dimensional complex, we have

$$H_q(\widetilde{P}; \mathbb{Z}) = \mathbb{Z}, 0, \overline{H}^1(\pi, \mathbb{Z}\pi), 0, ...$$

as required for the proof of Lemma 11. □

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