Additivity of tunnel number for small knots

⌈Commentarii Mathematici Helvetici

Additivity of tunnel number for small knots

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Abstract. We show that for small knots K_1, K_2, $t(K_1) + t(K_2) - 1 \leq t(K_1 \# K_2) \leq t(K_1) + t(K_2) + 1$, and that for small knots $K_1, \ldots, K_n$, $t(K_1 \# \ldots \# K_n) \geq n$.

Mathematics Subject Classification (2000). 57N10.

Keywords. Small knot, tunnel number, decomposing annulus.

1. Introduction

The tunnel number of a knot has been shown to behave quite erratically under the operation of connected sum. On the one hand, Kobayashi has exhibited knots for which the tunnel number degenerates by an arbitrarily high number under connected sum ([6]) and on the other hand Moriah and Rubinstein and independently Morimoto, Sakuma and Yokota have exhibited knots for which the tunnel number is strictly super-additive under connected sum ([8] and [10]). Restricting attention to connected sums of small knots circumvents some of the possibilities and many of the technical difficulties encountered in the work of Kobayashi concerning torus decompositions of manifolds and of Morimoto concerning the additivity of the tunnel numbers of knots ([7] and [9]). The more general question of how Heegaard genus behaves when two manifolds are glued together along an annulus remains open. This project was inspired by and answers problem 1.70 in [5] for the case of small knots. I wish to thank Andrew Casson and Marty Scharlemann for helpful conversations. I also wish to thank MSRI, where part of this research was carried out.

2. Definitions and general facts

Definition 1. *For any submanifold L of M, $\eta(L)$ denotes an open regular neighborhood of L in M and $N(L)$ denotes a closed regular neighborhood of L in M.*

Definition 2. *Let K be a knot in S^3, then $C(K) = S^3 - \eta(K)$.*

Definition 3. *A knot K is <u>small</u>, if $C(K)$ contains no closed essential surfaces.*

Definition 4. *A <u>tunnel system</u> for a knot K is a collection of disjoint arcs $\mathcal{T} = t_1 \cup \ldots \cup t_n$ embedded in $C(K)$ such that $C(K) - \eta(\mathcal{T})$ is a handlebody. We require one endpoint of t_i to lie on $\partial C(K)$ and the other to either also lie on $\partial C(K)$, or to meet an interior point of t_i. The <u>tunnel number of K</u>, denoted by $t(K)$, is the least number of arcs required in a tunnel system for K.*

Definition 5. *A <u>compression body</u> is a 3-manifold W obtained from a closed orientable surface F by attaching 2-handles to $F \times \{0\} \subset F \times I$ and capping off any resulting 2-sphere boundary components. We denote $F \times \{1\}$ by $\partial_+ W$ and $\partial W - \partial_+ W$ by $\partial_- W$.*

Definition 6. *A <u>spine</u> X of a compression body W is a properly embedded 1-complex in W such that $X \cup \partial_- W$ is connected and such that W collapses to $X \cup \partial_- W$.*

Definition 7. *A <u>Heegaard splitting</u> of a 3-manifold is a pair (W_1, W_2) of compression bodies, such that $W_1 \cap W_2 = \partial_+ W_1 = \partial_+ W_2$ and $M = W_1 \cup W_2$. We call $\partial_+ W_1 = \partial_+ W_2$ the <u>splitting surface</u> or <u>Heegaard surface</u> and denote it by F.*

Definition 8. *A disk D in a compression body W is an essential disk if $(D, \partial D) \subset (W, \partial_+ W)$ and ∂D is an essential curve in $\partial_+ W$. A <u>defining collection of disks for W</u> is a collection of essential disks $\mathcal{D} = D_1 \cup \ldots \cup D_n$ which cut W into $\partial_- W \cup$ 3-balls.*

Definition 9. *A Heegaard splitting is <u>(weakly) reducible</u> if there are essential disks D_1 and D_2, such that $(D_i, \partial D_i) \subset (W_i, \partial_+ W_i)$ and $(\partial D_1 \cap \partial D_2 = \emptyset)$ $\partial D_1 = \partial D_2$. A Heegaard splitting which is not weakly reducible is <u>strongly irreducible</u>.*

Definition 10. *Let $K_1 \# K_2$ be the connected sum of the knots K_1 and K_2. Then there is a sphere S which intersects $K_1 \# K_2$ in exactly two points, the annulus $A = S \cap C(K_1 \# K_2)$ is called a <u>decomposing annulus</u>. Note that cutting $C(K_1 \# K_2)$ along A produces a copy of $C(K_1)$ and of $C(K_2)$.*

The following four lemmata are well known but crucial. We repeat them here for completeness.

Lemma 1. *An incompressible and boundary incompressible surface S with $\partial S \neq \emptyset$ properly embedded in a compression body W is either an essential disk, or an annulus with one boundary component in $\partial_- W$ and the other in $\partial_+ W$.*

Proof. This is [1, Lemma 9]. □

Lemma 2. *An incompressible surface S in a compression body W with $\partial S \subset \partial_+ W$ cuts W into compression bodies.*

Proof. Let $\mathcal{D}$ be a defining collection of disks for W. Isotope S so that $\#|\mathcal{D} \cap S|$ is minimal. Then, in particular, $\mathcal{D} \cap S$ contains no simple closed curves. Cut W along $\mathcal{D}$ to obtain $(\partial_- W \times I) \cup (3-balls)$. Since $\partial S \subset \partial_+ W$, Lemma 1 implies that S is boundary parallel in or disjoint from any given component of $(\partial_- W \times I)$ and hence cuts it into $(closed\ surface) \times I$ and perhaps handlebodies. Furthermore, the incompressibility of S guarantees that S intersects each 3-ball in meridian disks. Therefore the closure of a component of $W - S$ is obtained by attaching $(closed\ surface) \times I$ components, handlebodies and 3-balls along disks. The result is a compression body. □

Lemma 3. *Let M^1 and M^2 be 3-manifolds such that M^i has a Heegaard splitting (V_1^i, V_2^i) of genus g^i and there is an annulus A^i in ∂M^i. Then the manifold M obtained by glueing M^1 and M^2 together along A^1 and A^2 has a Heegaard splitting of genus $g = g^1 + g^2$.*

Proof. Let B^i be the boundary component of M^i containing A^i. We may assume that $B^i \subset V_1^i$. Then by shrinking V_1^i, we may assume that V_1^i consists of a collar of B^i in M^i, denoted by $collar(B^i)$, 1-handles, some of which are attached to $collar(B^i)$ and, perhaps, other collars of boundary components of M^i.

Let α^i be a simple arc in A^i connecting the two components of ∂A^i and let $D^i = A^i - \eta(\alpha^i \cup \partial A^i)$. Let $collar(D^i)$ be the subset of $collar(B^i)$ which is a collar on D^i. We may assume that collar (D^i) does not meet any 1-handles. Glue M^1 to M^2 along A^1 and A^2 so that D^1 and D^2 match up. Define $V_1 = (V_1^1 - collar(D^1)) \cup (V_1^2 - collar(D^2))$ and $V_2 = (V_2^1 \cup collar(D^1)) \cup (V_2^2 \cup collar(D^2))$. Then (V_1, V_2) is a Heegaard splitting of the manifold M obtained by glueing M^1 and M^2 together along A^1 and A^2 and has genus $g = g^1 + g^2$. □

Definition 11. *Suppose an arc α shares an endpoint with an arc β. An arc slide of α over β is the result of replacing α with $\overline{\alpha \cup \beta}$, where $\overline{\alpha \cup \beta}$ is a pushoff of $\alpha \cup \beta$.*

Remark 4. Tunnel systems of a knot correspond to Heegaard splittings of the complement of the knot. Given a tunnel system $\mathcal{T}$ for K, set $V_1 = N(\partial C(K) \cup \mathcal{T})$ and V_2 equal to the closure of $C(K) - V_1$. Conversely, given a Heegaard splitting (V_1, V_2) of $C(K)$, we may assume that $\partial C(K) \subset V_1$. Then, after arc slides, if necessary, a spine X_1 of V_1 is a tunnel system for K. Note that in this correspondence, the number of arcs in $\mathcal{T}$ is exactly one less than the genus of (V_1, V_2).

Corollary 5. $t(K_1 \# K_2) \leq t(K_1) + t(K_2) + 1$.

Proof. This follows from Lemma 3 and Remark 4. $\qquad\square$

Lemma 6. *The splitting surface F of a strongly irreducible Heegaard splitting (V_1, V_2) of M may be isotoped to intersect a properly embedded incompressible surface $S \subset M$ only in essential simple closed curves.*

Proof. Let X_i be a spine of V_i. If X_i is disjoint from F, then, perhaps after isotopy, $F \cap S = \emptyset$. So suppose that $X_i \cap S \neq \emptyset$, for $i = 1, 2$. Let $H : F \times I \to M - \eta(X_1 \sqcup X_2)$ be a foliation, by surfaces isotopic to F, of $M - \eta(X_1 \sqcup X_2)$. We denote $H(F, t)$ by F_t and assume that $F_i = \partial N(X_i)$.

Set $\mathcal{V}_i = \{t \mid F_t \cap S$ contains simple closed curves bounding disks entirely in $V_i \cap S\}$. Then for t close to 1, $t \in \mathcal{V}_1$, whereas for t close to 2, $t \in \mathcal{V}_2$. To better understand $\mathcal{V}_1$ and $\mathcal{V}_2$, consider the singular foliation of $S - \eta(X_1 \sqcup X_2)$ induced by H. If a regular leaf contains a simple closed curve bounding disks entirely in V_i, then so do nearby regular leaves. Furthermore, if the regular leaves limiting on a singular leaf σ from one side all contain simple closed curves bounding disks entirely in V_i, then so does σ. It follows that either $\mathcal{V}_1 \cap \mathcal{V}_2 \neq \emptyset$ or the complement of $\mathcal{V}_1 \cup \mathcal{V}_2$ has nonempty interior. Since (V_1, V_2) is strongly irreducible, the latter must be the case. Choose t in the complement of $\mathcal{V}_1 \cup \mathcal{V}_2$ and so that $F_t \cap S$ is a regular leaf. Then F_t is a copy of F which intersects S only in essential simple closed curves. $\qquad\square$

3. Tunnel systems corresponding to weakly reducible Heegaard splittings

Tunnel systems fall into two types, those corresponding to weakly reducible Heegaard splittings and those corresponding to strongly irreducible Heegaard splittings. We here investigate tunnel systems for $K_1 \# K_2$ which correspond to weakly reducible Heegaard splittings and show how they define tunnel systems for K_1 and K_2. We exploit the ideas in introduced in [2] and extended in [13] and [12], linking weakly reducible Heegaard splittings and incompressible surfaces.

Definition 12. *Let (V_1, V_2) be an irreducible Heegaard splitting of M. We may think of M as being obtained from $\partial_- V_1 \times I$ by attaching all 1-handles in V_1 followed by all 2-handles dual to 1-handles in V_2, followed, perhaps, by 3-handles. An <u>untelescoping</u> of (V_1, V_2) is a rearrangement of the order in which the 1-handles (of V_1) and the 2-handles (dual to the 1-handles of V_2) are attached, so that M is decomposed into submanifolds $M^0, \ldots, M^k$, meeting along surfaces $S^1, \ldots, S^k$, which are incompressible in M, and which inherit, from a subcollection of the original 1-handles and 2-handles, strongly irreducible Heegaard splittings $(V_1^0,$*
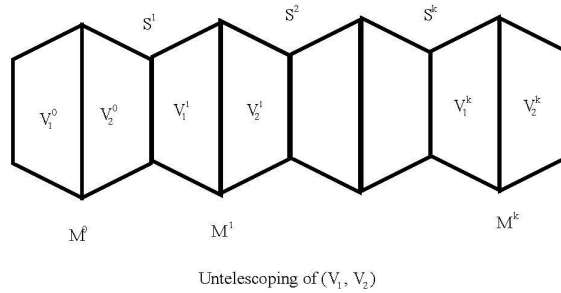
Untelescoping of (V_1, V_2)

Figure 1.

$V_2^0), \ldots, (V_1^k, V_2^k)$. *For details see [13] and [12].* $\square$

Remark 7. The proof of the Main Theorem in [2] shows that if (V_1, V_2) is irreducible but weakly reducible, then $k \geq 1$ in any untelescoping of (V_1, V_2).

Remark 7 shows that we must understand what sort of incompressible surfaces occur in the complement of the connected sum of small knots.
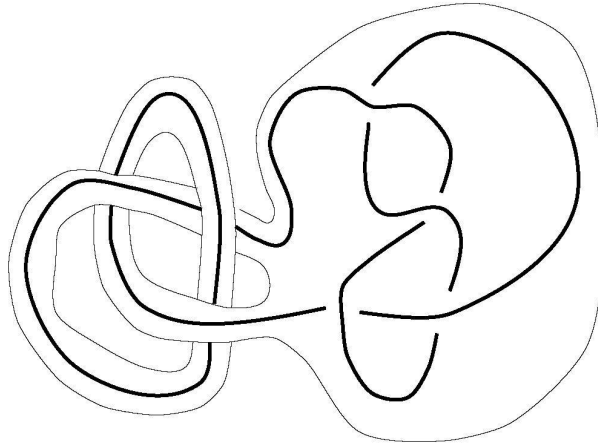
Lemma 8. *Let S be a separating essential surface in $C(K_1 \# K_2)$, where K_i is a small prime knot. Then $S \subset C(K_i)$ for either $i = 1$ or $i = 2$; and S is boundary parallel in $C(K_i)$. (See fig. 7)*

Definition 13. *A surface of the type described is called a <u>swallow follow torus</u>.*

Proof. Let A be the decomposing annulus. If $S \cap A = \emptyset$, then the claim follows. So suppose $S \cap A \neq \emptyset$, and set $S^i = S \cap C(K_i)$. Here $C(K_i)$ is obtained from $C(K_1 \# K_2)$ by cutting along the decomposing annulus. Since ∂S^i consists of meridians, the meridian is a boundary slope. Hence by [3, Theorem 2.0.3], $C(K_i)$ contains a closed essential surface. But this contradicts the fact that $C(K_i)$ is small. $\square$

Theorem 9. *Let K_1, K_2 be small knots, and suppose the tunnel system $\mathcal{T}$ realizing the tunnel number of $K_1 \# K_2$ corresponds to a weakly reducible Heegaard splitting. Then $t(K_1) + t(K_2) \leq t(K_1 \# K_2)$.*

Proof. Let (V_1, V_2) be the Heegaard splitting corresponding to $\mathcal{T}$. Since $\mathcal{T}$ realizes the tunnel number of $K_1 \# K_2$, (V_1, V_2) is irreducible. Consider an untelescoping of (V_1, V_2). By Remark 7, $k \geq 1$. If an S^i were boundary parallel in $C(K_1 \# K_2)$, then the untelescoping of (V_1, V_2) would define a Heegaard splitting of $C(K_1 \# K_2)$ of lower genus, contradicting the choice of $\mathcal{T}$. Hence each S^i must be a swallow follow torus. Since the two distinct isotopy classes of swallow follow tori

A swallow-follow torus

Figure 2.

can't be embedded simultaneously, the S^i must in fact be parallel swallow follow tori.

If we assume, for convenience of notation, that $S^i \subset C(K_1)$ for $i = 1, \ldots, k$, then M_0 is homeomorphic to $C(K_1)$ minus a 2-handle, hence (V_1^0, V_2^0) defines a Heegaard splitting for $C(K_1)$, and M^k is homeomorphic to $C(K_2)$, hence (V_1^k, V_2^k) defines a Heegaard splitting for $C(K_2)$. Counting the 1-handles in V_1^0 and V_1^k, which is the number of arcs in the corresponding tunnel system, shows that $t(K_1) + t(K_2) \leq t(K_1 \# K_2)$. $\square$

Remark 10. By Lemma 3, $k \leq 2$. For more subtle reasons (certain properties of the Heegaard splittings (V_1^i, V_2^i) and of Heegaard splittings of $(surface) \times I$), $k = 1$.

4. Tunnel systems corresponding to strongly irreducible Heegaard splittings

In this section we restrict our attention to tunnel systems of knots which correspond to strongly irreducible Heegaard splittings. Most importantly, we show how to use a tunnel system for a connected sum of knots to construct tunnel systems for its summands when all tunnel systems realizing the tunnel number of the connected sum of knots correspond to strongly irreducible Heegaard splittings. The trick in Definition 14 allows us to choose the original tunnel system so that this process proves bounded degeneration of tunnel number under connected sum.

The results were originally inspired by [9]. However, the arguments changed dramatically due to the use of the notion of strongly irreducible Heegaard splittings and the terminology and techniques available for their study.

Theorem 9 allows us to make certain assumptions. Since these assumptions will be used over and over again, we summarize them here:

Assumption 1: All tunnel systems realizing $t(K_1 \# K_2)$ correspond to strongly irreducible Heegaard splittings.
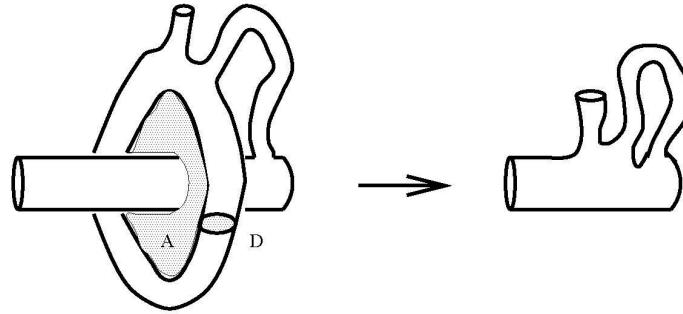
Assumption 2: The tunnel system T realizing $t(K_1 \# K_2)$ is chosen so that the splitting surface of the corresponding Heegaard splitting may be isotoped to intersect the decomposing annulus in the least number of essential curves.

The following definition gives an operation which helps move arcs in a tunnel system away from a decomposing annulus. The trick is due to Marty Scharlemann.

Definition 14. *Let T be a tunnel system for the knot K, let (V_1, V_2) be the corresponding Heegaard splitting of $C(K)$ and suppose there is an annulus $\tilde{A}$ with interior$(\tilde{A}) \subset V_2$, such that one component, $\partial_1 \tilde{A}$, of $\partial \tilde{A}$ lies on ∂V_1 and intersects the boundary of a meridian disk D of V_1 exactly once and such that the other component, $\partial_2 \tilde{A}$, of $\partial \tilde{A}$ lies on the boundary of an arbitrarily small regular neighborhood of $\partial C(K)$. Then setting $\tilde{V}_1 = N(\partial C(K) \cup T \cup \tilde{A})$ and $\tilde{V}_2$ the closure of the complement produces a Heegaard splitting of $C(K)$. To see that $\tilde{V}_1$ is a compression body, consider a collection $\mathcal{D}_1$ of defining disks for V_1 containing D. By [16, Lemma 2.1] $N(D \cup \partial_1 \tilde{A})$ is a solid torus summand of V_1. Let $\tilde{D}$ be an essential disk in V_1 which cuts off this solid torus summand. Then $(\mathcal{D}_1 - D) \cup \tilde{D}$ is a defining collection of disks for $\tilde{V}_1$. $\tilde{V}_2$ is a handlebody by the proof of [1, Lemma 9]). This Heegaard splitting has the same genus as the Heegaard splitting corresponding to T. (For the splitting surface of the former is obtained by cutting the splitting surface of the latter along two essential curves and reconnecting the resulting boundary components along annuli, this operation doesn't change the Euler characteristic of the surface.) The new Heegaard splitting corresponds to a new tunnel system T'. We will say that T' is obtained from T by replacing one of the arcs in T by its <u>dual</u>. (The terminology generalizes that used in the case where the tunnel system contains only one arc.)*

Lemma 11. *Let K_1 and K_2 be small knots. Suppose that $K_1 \# K_2$ satisfies Assumption 1, and the tunnel system T for $K_1 \# K_2$ satisfies Assumption 2. Then, after isotopy, the splitting surface F of the Heegaard splitting (V_1, V_2) corresponding to T intersects A only in essential curves, and $\#| F \cap A| \leq 4$.*

Proof. Suppose that $F \cap A$ consists only of essential curves and that $\#| F \cap A| > 4$. Then one of the components of $V_1 \cap A$ is boundary reducible in V_1. We may assume, by choosing an outermost one, that the boundary reducing disk is disjoint from A. Then after performing the boundary compression along this disk, $V_1 \cap A$

Replacing an arc by its dual

Figure 3.

consists of one disk and at least three annuli. Perform an ambient 2-surgery on F along the disk, and then perform ambient 2-surgery on the resulting surface,along all compressing disks disjoint from A, to obtain a surface F^*, each component of which is separating. We will denote the two sides of F^* by V_1^* and V_2^*. A priori, neither V_1^* nor V_2^* need be connected.

Note that the ambient 2-surgeries may be performed in sequence. Then at each step, the next compressing disk E can be isotoped so that ∂E lies entirely in F. Consider the annulus in F consisting of a bicollar of ∂E. It follows from [14, Lemma 2.6] and the strong irreducibility of (V_1, V_2), that E lies either entirely in V_1 or entirely in V_2. The strong irreducibility further implies, that in this case, E lies entirely in V_1. It follows that $V_1^* \subset V_1$ (whereas $V_2^* = V_2 \cup$ 2-handles). In particular, V_1^* needn't be connected, but V_2^* must be.

Consider a component $\tilde{F}^*$ of $F^* \cap C(K_i)$. If $\tilde{F}^*$ were an essential surface in $C(K_i)$, then the meridian of $C(K_i)$ would be a boundary slope, hence by [3, Theorem 2.0.3], $C(K_i)$ would contain a closed essential surface. Since K_i is small, and since $\tilde{F}^*$ is incompressible in $C(K_i)$, $\tilde{F}^*$ must be boundary parallel in $C(K_i)$. Hence $\tilde{F}^*$ is either parallel into a subannulus of A, or into $\partial C(K_i) \cup$ 2 *subannuli of* A. In the former case, we will call $\tilde{F}^*$ a <u>narrow</u> annulus, and in the latter case a <u>wide</u> annulus. Note that since a component of F^* can be constructed by identifying annuli along their boundary, it must be a torus.

Claim 1: A narrow annulus in F^* cobounds, together with a subannulus of A, a solid torus entirely in V_1^*.

Suppose the solid torus T cobounded by a narrow annulus N_1 in F^* and a subannulus A' of A meets the interior of V_2^*. Then we may assume, by replacing N_1 with a narrow annulus properly embedded in T if necessary, that a collar of N_1 in T lies in V_2^*. The torus T defines an isotopy of N_1 into A'. Since $V_2^* = V_2 \cup$ 2-handles, that is, since V_2^* contains V_2, this isotopy defines an isotopy of F

which reduces $\#|\ F \cap A|$. Since this number was chosen to be minimal, the Claim follows.

Claim 2: There can be at most two nested components of F^*.

This follows from the fact that (the "side") V_2^* (of F^*) is connected.

Claim 3: There is exactly one component of F^* which is parallel into $\partial C(K_1 \# K_2)$.

By Claim 2, there can be no more than two such components. If there were exactly two such components, then, again by Claim 2, $V_1^* \subset V_1$ would contain a copy of $C(K_1 \# K_2)$, but this is impossible. If there were no such component, then the component C of V_1^* containing $\partial C(K_1 \# K_2)$ would be cobounded by tori in F^* none of which was parallel to $\partial C(K_1 \# K_2)$. Since V_1^* is obtained by cutting V_1 along essential disks, a process which produces 3-balls, handlebodies, and compression bodies with only one interior boundary component, this is impossible. This proves Claim 3.

Let $\tilde{T}$ be a component of $V_1^* - C$ (for C as above). Then $\partial \tilde{T}$ is a torus, so, by the same reasoning as above, and Claims 1,2 and 3, $\tilde{T}$ must be a solid torus. It follows that $\partial \tilde{T}$ is comprised of narrow annuli and, by Claims 1, 2, and 3, pairs of adjacent wide annuli.
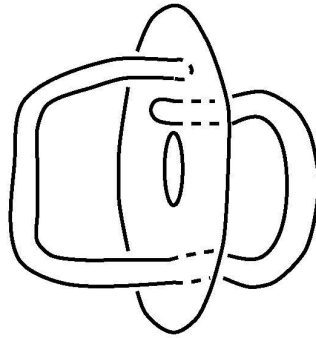
Let $\tilde{A}$ be a subannulus of A connecting $\partial \tilde{T}$ and ∂C. We may assume, by rechoosing $\tilde{T}$ to be outermost if necessary, that $\tilde{A} \subset V_2$. It follows from the preceeding paragraph that $\tilde{T}$ has a meridian disk D consisting of boundary reducing disks in the subtori cut off by narrow annuli and of disks (rectangles) defining a parallelism between the pairs of wide annuli. Now $\#|\partial D \cap \tilde{A}| = 1$. Hence the requirements of Definition 14 are met. So the splitting surface of the Heegaard splitting $(\tilde{V}_1, \tilde{V}_2)$ obtained by replacing the appropriate arc in the tunnel system under consideration into its dual intersects A in two fewer essential curves. It thus corresponds to a tunnel system which violates the minimality assumptions.   □

Theorem 12. *Let K_1 and K_2 be small knots. Suppose $K_1 \# K_2$ satisfies Assumption 1 and the tunnel system $\mathcal{T}$ for $K_1 \# K_2$ satisfies Assumption 2. Then $t(K_1) + t(K_2) - 1 \leq t(K_1 \# K_2)$.*

Proof. Let (V_1, V_2) be the Heegaard splitting corresponding to the tunnel system for $K_1 \# K_2$. Isotope the splitting surface F of (V_1, V_2) so that it intersects the decomposing annulus A for $K_1 \# K_2$ only in essential circles and so that $|F \cap A|$ is minimal.
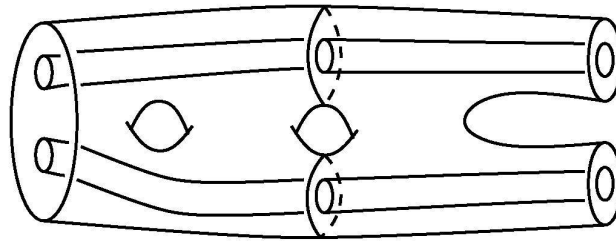
Case 1: $|F \cap A| = 2$.

Cut $C(K_1 \# K_2)$ along A and set $V_1^i = (V_1 \cap C(K_i)) \cup collar(A \cap V_2)$, and let V_2^i be the closure of the complement of V_1^i in $C(K_i)$.

A component of V_1^* intersecting A

Figure 4.
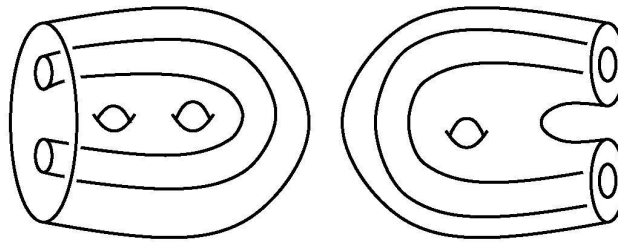

V_i before cutting along A

Figure 5a.

Claim: (V_1^i, V_2^i) is a Heegaard splitting for $C(K_i)$. (See fig. 5a and fig. 5b)

Let $\mathcal{D}_1$ be a defining collection of disks for V_1. After isotopy, we may assume that $\mathcal{D}_1 \cap A = \emptyset$. Then $\mathcal{D}_1^i = \mathcal{D}_1 \cap M^i$ cuts V_1^i into $(closed\ surface) \times I$. (Here $\mathcal{D}$ cuts V_1 into $\partial_- V_1$; then cutting $\partial_- V_1$ along $A' \sqcup A''$ and adding the collar creates $(closed\ surface) \times I$.) Hence V_1^i is a compression body.
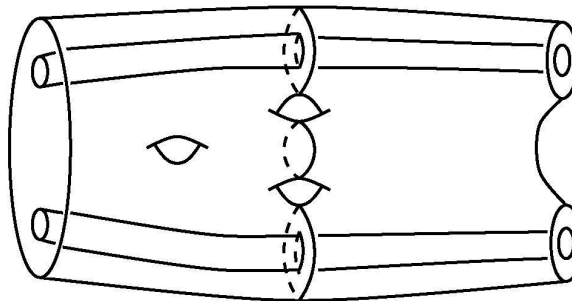
V_2^i is a compression body by Lemma 2.

Let g^i be the genus of (V_1^i, V_2^i) and g the genus of (V_1, V_2). Then, by an Euler characteristic argument, $g^1 + g^2 = g + 1$. Here the spine of V_1^i defines a tunnel system corresponding to the Heegaard splittings (V_1^i, V_2^i), hence $t(K_1) + t(K_2) \leq t(K_1 \# K_2)$.

Case 2: $|F \cap A| = 4$.

V_1^1 and V_1^2

Figure 5b.

V_1 before cutting along A

Figure 6a.

As in the proof of Lemma 11, one of the components of $V_1 \cap A$ is boundary reducible in V_1. We may assume, by choosing an outermost one, that the boundary reducing disk is disjoint from A. Then after performing the boundary compression along this disk, $V_1 \cap A$ consists of two annuli and one disk, and $V_2 \cap A$ is a pair of pants.

As before, cut $C(K_1 \# K_2)$ along A and set $V_1^i = (V_1 \cap C(K_i)) \cup (collar(V_2 \cap A))$ and V_2^i equal to the closure of the complement of V_1^i in $C(K_i)$. (See fig. 6a and fig. 6b.) Then by an argument similar to the one above (add the disk D to $\mathcal{D}_1^i$ as necessary), (V_1^i, V_2^i) is a Heegaard splitting of M^i. By the same reasoning as above, we find that $t(K_1) + t(K_2) - 1 \leq t(K_1 \# K_2)$.

By Lemma 11, this completes the proof.   □

Corollary 13. *Let K_1 and K_2 be small knots, then $t(K_1) + t(K_2) - 1 \leq t(K_1 \# K_2)$.*
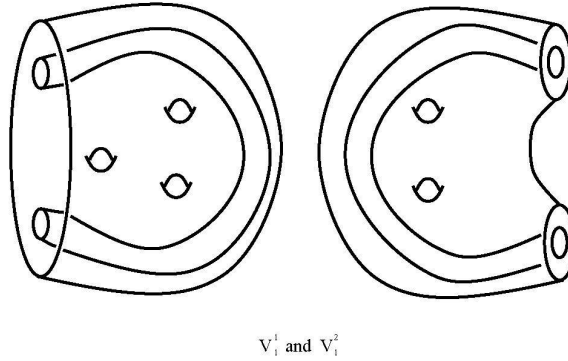
V_i^1 and V_i^2

Figure 6b.

Proof. This follows from Theorem 9 and Theorem 12.   □

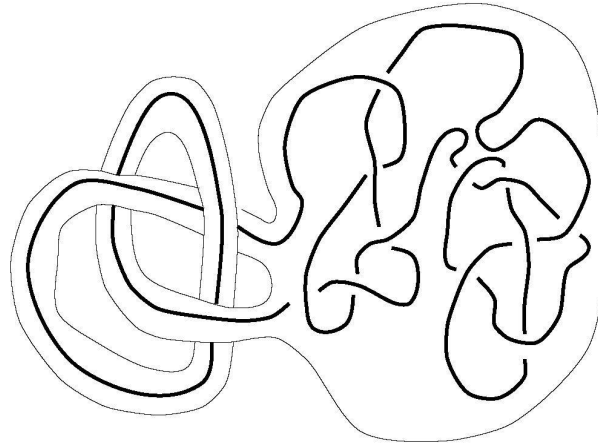## 5. Simultaneous decomposing annuli

The arguments in the two preceeding sections can be tailored to show that for small knots $K_1, \ldots, K_n, t(K_1 \# \ldots \# K_n) \geq n$. In the following we will assume that the decomposing annuli for $C(K_1 \# \ldots \# K_n)$ have been chosen so that each $C(K_i)$ meets at most two decomposing annuli (i.e., we assume that the decomposing spheres are nested in "Matryoshka" fashion).

Lemma 14. *Let S be a separating essential surface in $C(K_1 \# \ldots \# K_n)$, where each K_i is a small knot. Then, after isotopy, for any i, $S \cap C(K_i)$ is either empty, or boundary parallel; i.e., S is a swallow follow torus.*

Proof. Isotope S so that $\#|S \cap \mathcal{A}|$ is minimal. Let $A^1, \ldots, A^{n-1}$ be the decomposing annuli for $C(K_1 \# \ldots \# K_n)$. Consider $C(K_i)$ and suppose it is cut off of $C(K_1 \# \ldots \# K_n)$ by A^i, and perhaps A^{i+1}. If S does not intersect A^i or A^{i+1}, for any l, then S must be disjoint from $C(K_i)$, since K_i is small. If S intersects A^i or A^{i+1}, then S must be boundary parallel in $C(K_i)$ to avoid the contradiction in the proof of Lemma 8.   □

Theorem 15. *Let $K_1, \ldots, K_n$ be small knots, then $t(K_1 \# \ldots \# K_n) \geq n$.*

Proof. Suppose the inequality does not hold, and let $K_1, \ldots, K_n$ be small knots chosen so that n is minimal among such collections of small knots. Let (V_1, V_2) be the Heegaard splitting corresponding to a tunnel system $\mathcal{T}$ which realizes the $t(K_1 \# \ldots \# K_n)$.

A generalized swallow-follow torus

Figure 7.

Claim 1: (V_1, V_2) is strongly irreducible.

Suppose (V_1, V_2) is weakly reducible. Since $\mathcal{T}$ realizes $t(K_1 \# \ldots \# K_n)$, (V_1, V_2) is irreducible. Consider an untelescoping of (V_1, V_2). By Remark 7, $k \geq 1$. Again, if an S^i were boundary parallel in $C(K_1 \# \ldots \# K_n)$, then the untelescoping of (V_1, V_2) would define a Heegaard splitting of $C(K_1 \# \ldots \# K_n)$ of lower genus, contradicting the choice of $\mathcal{T}$. Hence each S^i must be a generalized swallow follow torus. A generalized swallow follow torus separates $C(K_1 \# \ldots \# K_n)$ into $C(K_{i_1} \# \ldots \# K_{i_j})$ minus a 2-handle and $C(K_{i_{j+1}} \# \ldots \# K_{i_n})$ for some partition $\{i_1, \ldots, i_j\} \sqcup \{i_{j+1}, \ldots, i_n\}$ of $\{1, \ldots, n\}$. Thus the untelescoping provides at least one Heegaard splitting corresponding to tunnel systems violating the choice of $K_1, \ldots K_n$.

In the following, we assume that the splitting surface of (V_1, V_2) has been isotoped to intersect the decomposing annuli only in essential curves, and in the least number of such curves.

Claim 2: $\mathcal{T}$ can be chosen so that the splitting surface F of (V_1, V_2) intersects at most one of the decomposing annuli in more than 2 curves and it intersects this decomposing annulus in exactly 4 curves.

The proof of this Claim mimics and extends the proof of Lemma 11. Suppose the Claim is not true. Proceed as in the proof of Lemma 11, performing one boundary compression to produce a disk in the intersection of V_1 with one of the decomposing annuli, then performing an ambient 2-surgery on F along this disk, and then performing ambient 2-surgeries on all compressing disks for the resulting

surface which are disjoint from all the decomposing annuli, to obtain a surface F^*, each component of which is separating. Again, denote the sides of F^* by V_1^* and V_2^*. Again, $V_1^* \subset V_1$ and V_2^* is connected. Again, a component of $F^* \cap C(K_i)$ is boundary parallel. Such a component could either be a narrow annulus as in the proof of Lemma 11, or a wide annulus, but here a wide annulus may either have boundary components on the same decomposing annulus, or on distinct decomposing annuli. Other possibilities would make the meridian a boundary slope and produce the same contradiction as before. Claims 1, 2, and 3 in the proof of Lemma 11 still hold.

Denote the component of V_1^* containing $\partial C(K_1 \# \ldots \# K_n)$ by C, and consider a component T of $V_1^* - C$. As before, T must be a solid torus. A meridian of T can be constructed as before. Again we obtain a new tunnel system $\mathcal{T}'$ for $K_1 \# \ldots \# K_n$, by changing an arc of $\mathcal{T}$ into its dual, which contradicts the minimality assumptions on $\mathcal{T}$. This proves the Claim.

The arguments in Theorem 12 now only give the weaker result: $t(K_1 \# \ldots \# K_n) \geq n - 1$. However, applying the argument in Case 1 of Theorem 12 along all decomposing annuli which intersect F exactly twice, and noting that the remaining composite summand has tunnel number at least 2, since tunnel number 1 knots are prime (see for instance [11]), proves the Theorem. $\square$

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