

Universal octonary diagonal forms over some real quadratic fields

Autor(en): **Kim, Byeong Moon**

Objektyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **75 (2000)**

PDF erstellt am: **05.07.2024**

Persistenter Link: <https://doi.org/10.5169/seals-56626>

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Universal octonary diagonal forms over some real quadratic fields

Byeong Moon Kim

Abstract. In this paper, we will prove there are infinitely many integers n such that $n^2 - 1$ is square-free and $\mathbb{Q}(\sqrt{n^2 - 1})$ admits universal octonary diagonal quadratic forms.

Mathematics Subject Classification (2000). Primary 11E12, Secondary 11E20.

Keywords. Universal quadratic forms, real quadratic fields.

1. Introduction

A universal integral form over totally real number field K is a positive definite quadratic form over the ring of integers of K which represents all the totally positive integers of K . For example, the sum of four squares is universal integral over \mathbb{Q} . In 1917, Ramanujan [8] found there are exactly 54 universal positive diagonal integral quadratic forms over \mathbb{Q} . More concretely, he showed there are 54 diagonal quaternary quadratic forms $f(x, y, z, w) = ax^2 + by^2 + cz^2 + dw^2$ such that $a, b, c, d \in \mathbb{Z}^+$ and the equation $f = n$ is solvable for all $n \in \mathbb{Z}^+$. In 1947, M. Willerding [10] proved there are exactly 178 classic universal integral forms. More concretely, she showed there are 178 quaternary quadratic forms $f(x, y, z, w)$ up to equivalence such that f is positive definite integral quadratic form, the coefficients of cross terms of f are always even and the equation $f = n$ is solvable for all $n \in \mathbb{Z}^+$. On the other hand, the study of positive universal quadratic integral forms over totally real number fields was initiated by F. Götzky [3]. In 1928, he proved that the sum of four squares is universal over $\mathbb{Q}(\sqrt{5})$. H. Maass [6] improved this result. In 1941, he proved the sum of three squares is positive universal over $\mathbb{Q}(\sqrt{5})$. Four years later, Siegel [9] proved $\mathbb{Q}(\sqrt{5})$ is the only totally real number field, other than \mathbb{Q} , over which every (totally) positive integer is a sum of squares. In other words, he showed if a totally real number field K is different from \mathbb{Q} and $\mathbb{Q}(\sqrt{5})$, there is a totally positive algebraic integer α of K which cannot be represented by the sum of any number of squares. For example, if $K = \mathbb{Q}(\sqrt{2})$, $\alpha = 2 + \sqrt{2}$. In 1996, W. K. Chan, M.-H. Kim and S. Raghavan [1]

classified all (totally) positive universal integral ternary lattices over real quadratic fields. Only $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(\sqrt{5})$ admit universal integral ternary lattices and total number of universal integral ternary lattices over real quadratic fields is 11. Recently, the author [5] proved there are only finitely many real quadratic fields which admit universal integral septenary diagonal forms. The content of this paper is to prove if $n^2 - 1$ is square-free, there are universal octonary diagonal forms over $\mathbb{Q}(\sqrt{n^2 - 1})$. So we can prove there are infinitely many real quadratic fields which admit universal integral octonary diagonal forms. Obviously 8 is the minimal rank with this property.

2. Main Theorem

Throughout this chapter, we let $m = n^2 - 1$ be a positive square free integer, $K = \mathbb{Q}(\sqrt{m})$ and \mathcal{O}_K be the ring of algebraic integers of K . Note that $\epsilon = n + \sqrt{m}$ is the fundamental unit of \mathcal{O}_K and is totally positive.

Theorem 1. *The octonary diagonal form $x_1^2 + x_2^2 + x_3^2 + x_4^2 + \epsilon x_5^2 + \epsilon x_6^2 + \epsilon x_7^2 + \epsilon x_8^2$ is universal over \mathcal{O}_K .*

This Theorem is a consequence of following Lemmas.

Lemma 1. *Let $1 \leq b < 2n$. $\alpha = a + b\sqrt{m}$ is totally positive algebraic integer in K if and only if $nb \leq a$.*

Proof. As $nb + b\sqrt{m} = b(n + \sqrt{m})$ is totally positive, the necessity is trivial. For the sufficiency, it suffices to prove $nb - 1 - (b\sqrt{m}) < 0$. This follows from

$$\begin{aligned} (nb - 1)^2 - (b\sqrt{m})^2 &= n^2b^2 - 2nb + 1 - b^2(n^2 - 1) \\ &= (b - n)^2 - n^2 + 1 \leq (n - 1)^2 - n^2 + 1 < 0. \end{aligned}$$

□

Lemma 2. *If $\alpha \in \mathcal{O}_K^+$, α belongs to*

$$S = \{a_0\epsilon^k + a_1\epsilon^{k+1} + \dots + a_l\epsilon^{k+l} \mid k, l \in \mathbb{Z}, a_0, a_1, \dots, a_l \in \mathbb{N}\}.$$

Proof. Suppose $\alpha = a + b\sqrt{m} \notin S$. We may assume that $b > 0$ and $\text{tr}_{K/\mathbb{Q}}(\alpha) \leq \text{tr}_{K/\mathbb{Q}}(\beta)$ for all elements $\beta \in S$. Then, by Lemma 1, we have $b \geq 2n$. Since

$$bn - 1 + b\sqrt{m} = \epsilon^2 + (b - 2n)\epsilon \in S,$$

we also have $a \leq bn - 1$. Then,

$$\alpha\epsilon^{-1} = (a + b\sqrt{m})(n - \sqrt{m}) = an - bm + (bn - a)\sqrt{m}.$$

So

$$\begin{aligned} \operatorname{tr}_{K/\mathbb{Q}}(\alpha\epsilon^{-1}) &= 2(an - bm) \leq 2(n(bn - 1) - b(n^2 - 1)) \\ &= 2(b - n) < 2a = \operatorname{tr}_{K/\mathbb{Q}}(\alpha). \end{aligned}$$

So $\alpha\epsilon^{-1} \in S$. Thus $\alpha \in S$. Contradiction. \square

Lemma 3. For $l \geq 2$, $\epsilon^l = -1 + b_1\epsilon + b_2\epsilon^2 + \dots + b_{l-1}\epsilon^{l-1}$ where $b_1 \geq 2n - 1$ and $b_2, \dots, b_{l-1} \geq 2n - 2$.

Proof. We use induction on l . As $\epsilon^2 = 2n\epsilon - 1$, the assertion holds for $l = 2$. If this Lemma is true for $l = s \geq 2$,

$$\begin{aligned} \epsilon^{s+1} &= \epsilon\epsilon^s = \epsilon(-1 + b_1\epsilon + b_2\epsilon^2 + \dots + b_{s-1}\epsilon^{s-1}) \\ &= -\epsilon + \epsilon^2 + (b_1 - 1)\epsilon^2 + b_2\epsilon^2 + \dots + b_{s-1}\epsilon^s \\ &= -1 + (2n - 1)\epsilon + (b_1 - 1)\epsilon^2 + b_2\epsilon^2 + \dots + b_{s-1}\epsilon^s. \end{aligned}$$

This proves the Lemma. \square

Lemma 4. If $\alpha \in \mathcal{O}_K^+$, $\alpha = p\epsilon^k + q\epsilon^{k+1}$ for some $p, q \in \mathbb{N}$ and $k \in \mathbb{Z}$.

Proof. By Lemma 2, $\alpha = a_k\epsilon^k + \dots + a_{k+l}\epsilon^{k+l}$ with $a_k, \dots, a_{k+l} \geq 0$.

If $l \geq 2$ and $a_{k+l} \leq a_k$,

$$\begin{aligned} \alpha &= a_k\epsilon^k + \dots + a_{k+l-1}\epsilon^{k+l-1} + a_{k+l}\epsilon^k(-1 + b_1\epsilon + \dots + b_{l-1}\epsilon^{l-1}) \\ &= (a_k - a_{k+l})\epsilon^k + (a_{k+1} + a_{k+l}b_1)\epsilon^{k+1} + \dots + (a_{k+l-1} + a_{k+l}b_{l-1})\epsilon^{k+l-1}. \end{aligned}$$

If $l \geq 2$ and $a_{k+l} \geq a_k$,

$$\begin{aligned} \alpha &= a_k\epsilon^k + \dots + a_{k+l-1}\epsilon^{k+l-1} + (a_{k+l} - a_k)\epsilon^{k+l} + a_k\epsilon^k(-1 + b_1\epsilon + \dots + b_{l-1}\epsilon^{l-1}) \\ &= (a_k + a_{k+l}b_1)\epsilon^{k+1} + \dots + (a_k + a_{k+l}b_{l-1})\epsilon^{k+l-1} + (a_{k+l} - a_k)\epsilon^{k+l}. \end{aligned}$$

Repeating the same process, we can obtain the desired expression of α . \square

Proof of Theorem 1. If $\alpha \in \mathcal{O}_K^+$, by Lemma 4, $\alpha = a\epsilon^k + b\epsilon^{k+1}$ for some $a, b \in \mathbb{N}$ and $k \in \mathbb{Z}$. If k is even, by Lagrange's four square theorem, $a\epsilon^k$ is represented by $x_1^2 + x_2^2 + x_3^2 + x_4^2$ and $b\epsilon^{k+1}$ is represented by $\epsilon x_5^2 + \epsilon x_6^2 + \epsilon x_7^2 + \epsilon x_8^2$. So f represents α . Similarly f represents α for odd k . Thus f is universal integral over K . \square

Lemma 5. *There are infinitely many square free integers of the form $n^2 - 1$.*

Proof. If n is even, $n^2 - 1$ is square free if and only if both $n + 1$ and $n - 1$ are square free. It is known that [4] the number of positive square free integers which do not exceed x is $\frac{6x}{\pi^2} + O(\sqrt{x})$. So the number of positive integer n such that $n \leq x$ and both $n + 1$ and $n - 1$ are square free is larger than

$$\left(\frac{6x}{\pi^2} + O(\sqrt{x})\right) + \left(\frac{6x}{\pi^2} + O(\sqrt{x})\right) - x = \frac{12 - \pi^2}{\pi^2}x + O(\sqrt{x}).$$

Since $\frac{12 - \pi^2}{\pi^2} > 0$, there are infinitely many n such that $n \leq x$ and $n^2 - 1$ is square free. \square

Theorem 2. *There are infinitely many real quadratic fields that admit octonary universal forms.*

Proof. This is an immediate consequence of Theorem 1 and Lemma 5. \square

Acknowledgement

The content of this paper is a part of author's thesis. The author wishes to represent his greatest thanks to his advisor Prof. M.-H. Kim of Seoul National University for his kind advice and careful revision of manuscript.

References

- [1] Chan, W. K., Kim, M.-H., Raghavan, S., Ternary Universal Quadratic Forms over Real Quadratic Fields, *Japanese J. Math.* **22** (1996), 263-273.
- [2] Dixon, L. E., Quaternary Quadratic Forms Representing All integers, *Amer. J. Math.* **49** (1927), 39-56.
- [3] Götzky, F., Über eine Zahlentheoretische Anwendung von Modulfunctionen einer Veränderlichen, *Math. Ann.* **100** (1928), 411-437.
- [4] Hardy, G. H., *An introduction to the theory of numbers*, fifth edition, Oxford, 1979.
- [5] Kim, B. M., Finiteness of Real Quadratic Fields which admit a Positive Integral Diagonal Septenary Universal Forms, preprint.
- [6] Maass, H., Über die Darstellung total positiver des Körpers $R(\sqrt{5})$ als Summe von drei Quadraten, *Abh. Math. Sem. Hamburg* **14** (1941), 185-191.
- [7] O'Meara, O. T., *Introduction to quadratic forms*, Springer Verlag, 1973.
- [8] Ramanujan, S., On the Expression of a Number in the Form $ax^2 + by^2 + cz^2 + dw^2$, *Proc. Cambridge Phil. Soc.* **19** (1917), 11-21.
- [9] Siegel, C. L., Sums of m -th Powers of Algebraic Integers, *Ann. Math.* **46** (1945), 313-339.
- [10] Willerding, M. F., Determination of all classes of positive quaternary quadratic forms which represent all (positive) integers, *Bull. Amer. Math. Soc.* **54**, 334-337.

Byeong Moon Kim
Department of Mathematics
College of Natural Science
Kangnung National University
123 Chibyon-Dong Kangnung
Kangwon-do 210-702
Korea
e-mail: kbm@knusun.kangnung.ac.kr

(Received: November 2, 1998)