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Comment. Math. Helv. 76 (2001) 467–500
0010-2571/01/030467-34 $ 1.50+0.20/0

Commentarii Mathematici Helvetici

Modular categories of types B,C and D

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Abstract. We construct four series of modular categories from the two-variable Kauffman polynomial, without use of the representation theory of quantum groups at roots of unity. The specializations of this polynomial corresponding to quantum groups of types B, C and D produce series of pre-modular categories. One of them turns out to be modular and three others satisfy Bruguières' modularization criterion. For these four series we compute the Verlinde formulas, and discuss spin and cohomological refinements.

Mathematics Subject Classification (2000). 57M25, 57R56.

Keywords. Modular category, modular functor, TQFT, 3-manifold, quantum invariants, Verlinde formula

Introduction

Modular categories are tensor categories with additional structure (braiding, twist, duality, a finite set of dominating simple objects satisfying a non-degeneracy axiom). If we remove the last axiom, we get a pre-modular category. A pre-modular category provides invariants of links, tangles, and sometimes of 3-manifolds. Any modular category yields a Topological Quantum Field Theory (TQFT) in dimension three [18].

In this paper we give an elementary construction of modular and pre-modular categories arising from the Kauffman skein relations, without use of the representation theory of quantum groups at roots of unity. Our method is based on the skein-theoretical construction of idempotents in the Birman-Murakami-Wenzl (BMW) algebras given in [2]. This work follows the program of Turaev and Wenzl [19, 20]. We give four specifications of parameters α and s (entering the Kauffman skein relations) which lead to different series of modular categories. In each case, the quantum parameter s is a root of unity and $\pm\alpha$ is a power of s. The order l of s^2 plays a key role in the discussion. When l is odd, then either

$s^l = -1$ or $s^l = 1$. We note that the two cases are quite different: only one of them lead to a modular category, the other one produces a non-modularizable pre-modular category.

It is well-known that the link invariant associated with the fundamental representation of the quantum group of type A_n is a specialization of the Homfly polynomial. Taking the fundamental representations of the quantum groups of types B_n, C_n or D_n one obtains specializations of the Kauffman polynomial [17]. More generally, with each of these quantum groups at a root of unity q a pre-modular category can be associated [9]. The order of q determines the *level* k of the category. It turns out that categories obtained from the quantum groups of types A_n and A_k, where q is $(n+k)$ th root of unity, are isomorphic; here one has to consider either a non standard choice of the framing parameter, or the *projective* subcategory. The isomorphism interchanges the rank n and the level k of the category and it is known as the level-rank duality. This duality has no natural explanation in the context of quantum groups, because the roles of the parameters n and k are completely different there.

In our setting, both parameters n and k serve to restrict the size of the Weyl alcove, and we have natural symmetries interchanging them. Therefore, each of our (pre-)modular categories has its level-rank duality partner. In fact, all our specializations of parameters can be interpreted in two different ways as a quantum group specialization. Accordingly, we denote our categories by pairs of the letters B, C and D (we use just one of them if both coincide). Our main results can be formulated as follows.

- We recover the symplectic (C in our notation) and BC series of modular categories already obtained by Turaev and Wenzl [20]. These series are constructed by killing negligible morphisms in the idempotent completed Kauffman category. In the BC case we further use Bruguières' modularization procedure [7]. This could be avoided here by considering a subcategory (see [20, 9.9]).
- We obtain two new series of modular categories in the orthogonal case: one in the even orthogonal case (D series) and one in the mixed odd-even orthogonal case (BD series). All of them are constructed by using Bruguières' modularization procedure.
- Except for the even orthogonal categories, we describe explicitly the representative sets of simple objects and state the Verlinde formulas, which give the dimensions of the TQFT modules. In the even orthogonal case, the complete description of the set of simple objects depends on a tricky computation which has still to be done.
- We find a correspondence between our categories and categories obtained by the quantum group method. We show that the categories constructed here

give a complete set of 3-manifold invariants that can be obtained from quantum groups of types B, C and D by using non-spin modules.

The paper is organized as follows. In the first section we give the general definitions and theorems concerning pre-modular and modular categories. This includes Bruguières' modularization criterion, and an explicit description of a modularization functor for a modularizable pre-modular category whose *transparent* simple objects are invertible. In the second section we recall the main definitions and properties of the minimal idempotents in the BMW algebras constructed in [2]. In the third section we construct the completed BMW category and use it in order to define series of pre-modular categories. In Section 4, studying transparent objects in these categories, we show that the symplectic category is modular and three other series satisfy Bruguières' modularization criterion. Then for modular categories we describe the representative sets of simple objects, give the Verlinde formulas and discuss spin and cohomological refinements. In the last section we explain how our pre-modular categories can be interpreted in terms of quantum groups.

Conventions. The manifolds throughout this paper are compact, smooth and oriented. By a *link* we mean an isotopy class of an unoriented framed link. Here, a framing is a non-singular normal vector field, up to homotopy. By a *tangle* in a 3-manifold M we mean an isotopy class of a framed tangle relative to the boundary. Here the boundary of the tangle is a finite set of points in ∂M, together with a nonzero vector tangent to ∂M at each point. Note that a framing together with an orientation is equivalent to a trivialization of the normal bundle, up to homotopy. By an *oriented link* we mean an isotopy class of a link together with a trivialization of the normal bundle, up to homotopy. By an *oriented tangle* we mean an isotopy class of a tangle together with a trivialization of the normal bundle, up to homotopy relative to the boundary. Here the boundary of the tangle is a finite set of points in ∂M, together with a trivialization of the tangent space to ∂M at each point. In the figures, a convention using the plane gives the preferred framing (blackboard framing).

1. Pre-modular categories and modularization

1.1. Pre-modular and modular categories

A ribbon category is a category equipped with a tensor product, braiding, twist and duality satisfying compatibility conditions [18]. If we are given a ribbon category A, then we can define an invariant of links whose components are colored by objects of A. This invariant extends to a representation of the A-colored tangle category and more generally to a representation of the category of A-colored ribbon graphs [18, I.2.5]. Using the ribbon structure of A, we get traces of morphisms and dimensions of objects, for which we will use the terminology *quantum trace* and *quantum dimension*. More precisely, for any $X \in \mathrm{O}b(\mathrm{A})$ and $f \in \mathrm{E}nd(X)$ we denote by $\langle f\rangle \in \mathrm{E}nd(\text{trivial object})$ the quantum trace of f and by $\langle X\rangle = \langle \mathbb{1}_X\rangle$ the quantum dimension of X. Throughout this paper $\mathbb{1}_X$ denotes the identity morphism of X.

Let $\mathbf{k}$ be a field. A ribbon category will be said to be $\mathbf{k}$-linear if the Hom sets are $\mathbf{k}$-vector spaces, $\mathrm{E}nd(\text{trivial object}) = \mathbf{k}$, and composition and tensor product are bilinear. We call an object X of A simple if the map $u \mapsto u\,\mathbb{1}_X$ from $\mathbf{k} = \mathrm{E}nd(\text{trivial object})$ to $\mathrm{E}nd(X)$ is an isomorphism.

Definition 1.1. A *modular category* [18], over the field $\mathbf{k}$, is a $\mathbf{k}$-linear ribbon category in which there exists a finite family Γ of simple objects satisfying the four axioms below.

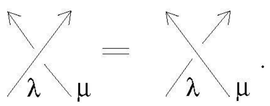
1. (Normalization axiom) The trivial object is in Γ.
2. (Duality axiom) For any object $\lambda \in \Gamma$, its dual λ^* is isomorphic to an object in Γ.
3. (Domination axiom) For any object X of the category there exists a finite decomposition $\mathbb{1}_X = \sum_i f_i\, \mathbb{1}_{\lambda_i} g_i$, with $\lambda_i \in \Gamma$ $f_i \in Hom(X, \lambda_i), g_i \in Hom(\lambda_i, X)$ for every i.
4. (Non-degeneracy axiom) The following matrix is invertible.

$$\mathrm{S} = (\mathrm{S}_{\lambda\mu})_{\lambda,\mu\in\Gamma} \ ,$$

where $\mathrm{S}_{\lambda\mu} \in \mathbf{k}$ is the endomorphism of the trivial object associated with the (λ, μ)-colored, 0-framed Hopf link with linking $+1$.

It follows that Γ is a representative set of isomorphism classes of simple objects. If we remove the last axiom, we get a definition of a *pre-modular* category.

Definition 1.2. An object λ of a pre-modular category A is called *transparent*, if for any object μ in A

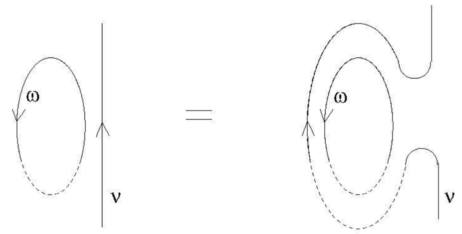


Such an object is also called a central object. It is enough to have the above equality for any μ in a representative set of simple objects. Note that a category containing a nontrivial transparent simple object can not be modular, simply because the row in the S-matrix corresponding to this transparent object is colinear to the row of the trivial one. In the next subsection we show that the absence of nontrivial transparent simple objects implies (under a mild assumption) that the category is modular.

1.2. Properties of pre-modular categories

We will first give some general facts about pre-modular categories. Let A be a pre-modular category and let $\Gamma(\mathrm{A})$ be a representative set of isomorphism classes of its simple objects. We denote by ω the Kirby color, i.e. $\omega = \sum_{\lambda\in\Gamma(\mathrm{A})}\langle\lambda\rangle\lambda$. We use here the same notation as before for traces and dimensions. In addition, we suppose that A has no nontrivial negligible morphisms (we quotient out by negligible morphisms if necessary). Note that a morphism $f \in Hom_{\mathrm{A}}(X,Y)$ is called negligible if for any $g \in Hom_{\mathrm{A}}(Y,X)$ $\langle fg\rangle = 0$.

Proposition 1.1. *(Sliding property) For every $\nu \in \Gamma(\mathrm{A})$, the following holds on* $\mathrm{E}nd_{\mathrm{A}}(\nu)$.

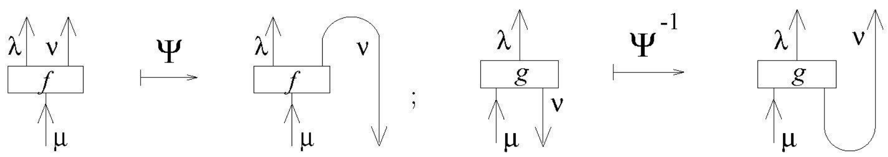


Here the dashed line represents a part of the closed component colored by ω. This part can be knotted or linked with other components of a ribbon graph representing the morphism. Note that the morphism is unchanged if we reverse the orientation of this closed component.

Proof. For $c_i, d_j \in \Gamma(\mathrm{A})$, $i = 1, ..., n$, $j = 1, ..., m$, we put

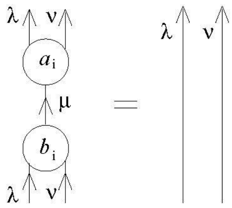
$$Hom_{\mathrm{A}}(c_1 \otimes ... \otimes c_n, d_1 \otimes ... \otimes d_m) := H^{d_1...d_m}_{c_1...c_n}.$$

With this notation the modules $H^{\lambda\nu}_{\mu}$, $H^{\lambda}_{\mu\nu^*}$, $H^{\mu^*\lambda}_{\nu^*}$, $H^{\mu^*}_{\nu^*\lambda^*}$, $H^{\nu\mu^*}_{\lambda^*}$ and $H^{\nu}_{\lambda^*\mu}$ are mutually isomorphic, as well as the modules $H_{\mu\nu^*\lambda^*}$, $H^{\lambda\nu\mu^*}$ and all obtained from them by cyclic permutation of colors. For example, the map $\Psi : H^{\lambda\nu}_{\mu} \to H^{\lambda}_{\mu\nu^*}$ and its inverse are depicted below.

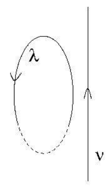
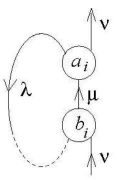


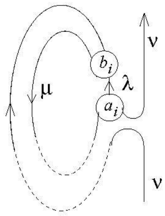
Identifying these modules along the isomorphisms we get a symmetrized multiplicity module $\tilde{H}^{\lambda\nu\mu^*}$; here only the cyclic order of colors is important. We will represent the elements of $\tilde{H}^{\lambda\nu\mu^*}$ by a circle with one incoming line (colored with μ) and two outgoing ones (colored with λ and ν), the cyclic order of lines is

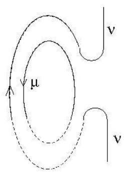
$(\lambda\nu\mu)$. The module $\tilde{H}^{\mu\nu^*\lambda^*}$ is dual to $\tilde{H}^{\lambda\nu\mu^*}$. The natural pairing is non-degenerate, since we have no negligible morphisms. We denote by a_i, $i \in \mathrm{I}^{\lambda\nu\mu^*}$, a basis of $\tilde{H}^{\lambda\nu\mu^*}$, and by b_i the dual basis with respect to this pairing. Applying the domination axiom we get that the natural map $\oplus_\mu \tilde{H}^{\lambda\nu\mu^*} \otimes \tilde{H}^{\mu\nu^*\lambda^*} \to H_{\lambda\nu}^{\lambda\nu}$ is an isomorphism. By writing the identity of $\lambda \otimes \nu$ in the basis corresponding to $(a_i \otimes b_j)$, we get the following decomposition formula (*fusion* formula):

$$\sum_{\mu} \sum_{i \in \mathrm{I}^{\lambda\nu\mu^*}} \langle\mu\rangle$$  . (1)

The calculations below establish the sliding property.

$$\sum_{\lambda \in \Gamma(\mathrm{A})} \langle\lambda\rangle$$  $$= \sum_{\lambda,\mu} \sum_{i \in \mathrm{I}^{\lambda\nu\mu^*}} \langle\lambda\rangle\langle\mu\rangle$$ 

$$= \sum_{\lambda,\mu} \sum_{i \in \mathrm{I}^{\lambda\nu\mu^*}} \langle\lambda\rangle\langle\mu\rangle$$ 

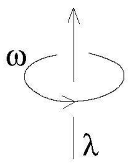
$$= \sum_{\mu \in \Gamma(\mathrm{A})} \langle\mu\rangle$$ 

In the first and third equalities we use the fusion formula, the second equality holds by isotopy.

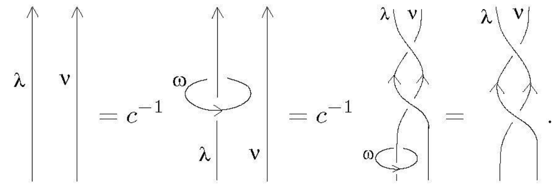
□

A more general statement is shown in [1].

Lemma 1.2. *(Killing property) Suppose that* $\langle\omega\rangle$ *is nonzero. Let* $\lambda \in \Gamma(\mathrm{A})$, *then the following morphism is nonzero in* A *if and only if* λ *is transparent.*



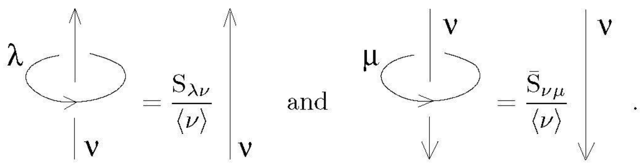
Proof. If λ is transparent, then this morphism is equal to $\langle\omega\rangle\, \mathbb{1}_\lambda$, which is nonzero. Conversely, if this morphism is nonzero, it is equal to $c\,\mathbb{1}_\lambda$ for some $0 \neq c \in \mathbf{k}$. Then, for any $\nu \in \Gamma(\mathrm{A})$, we have



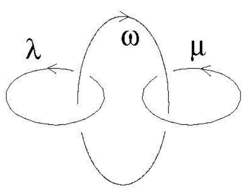
The second equality holds by the sliding lemma. □

Proposition 1.3. *A pre-modular category* A *with* $\langle\omega\rangle \neq 0$ *which has no non-trivial transparent simple object is modular.*

Proof. We have to check the non-degeneracy axiom. Let us denote by $\bar{\mathrm{S}}$ the matrix whose (λ, μ) entry is equal to the value of the 0-framed Hopf link with linking -1 and coloring of the components λ, μ. Then we have that



We deduce that the (λ, μ) entry of the matrix $\mathrm{S}\bar{\mathrm{S}}$ is equal to the invariant of the colored link depicted below.



By using (1) and the killing property we obtain the formula

$$\mathrm{S}\bar{\mathrm{S}} = \langle\omega\rangle \mathrm{I} ,$$

where I is the identity matrix, which proves the invertibility of the S matrix. □

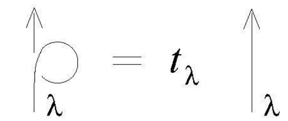
1.3. Bruguières' criterion

A process of constructing modular categories from pre-modular ones is called a modularization. Our reference for such construction is Bruguières' work [7]. See also [13] for an analogous development in the context of $*$-categories. Bruguières considers abelian ribbon linear categories. Direct sums may be defined in a formal way, and a pre-modular category with direct sums is an abelian category. From now on our pre-modular categories are supposed to be equipped with direct sums (we add them if necessary) and hence are abelian.

Definition 1.3. A *modularization* of a pre-modular category A is a modular category $\widetilde{\mathrm{A}}$ together with a ribbon $\mathbf{k}$-linear functor $\mathrm{F}: \mathrm{A} \to \widetilde{\mathrm{A}}$ which is dominant, i.e. any object of $\widetilde{\mathrm{A}}$ is a direct factor of $\mathrm{F}(\lambda)$ for some $\lambda \in \mathrm{O}b(\mathrm{A})$.

Definition 1.4. A simple object λ of a pre-modular category A is *bad* if for any μ in a representative set of simple objects $\Gamma(\mathrm{A})$, one has $\mathrm{S}_{\lambda\mu} = \langle\lambda\rangle\langle\mu\rangle$.

Definition 1.5. For any $\lambda \in \Gamma(\mathrm{A})$, its twist coefficient t_λ is defined by the equality given below.



The following fact was claimed in Corollary 3.5 of [7].

Theorem 1.4 (Bruguières' criterion). *Let* $\mathbf{k}$ *be an algebraically closed field of zero characteristic. Then an abelian pre-modular category* A *over* $\mathbf{k}$ *is modularizable if and only if any bad object* X *is transparent, has twist coefficient* $t_X = 1$ *and quantum dimension* $\langle X\rangle \in \mathbb{N}$.

If A *is modularizable, then its modularization is unique up to equivalence.*

Remark. Clearly, any transparent object is bad. If $\langle\omega\rangle \neq 0$, then any bad object is transparent. This follows from the killing property. Using this fact, Bruguières statement can be slightly simplified [1].

1.4. Modularization functor

We want now to describe the modularization functors explicitly. The main idea consists of adding morphisms to the pre-modular category, that make transparent simple objects isomorphic to the trivial one.

For the remainder of this section we consider a pre-modular category A with $\langle\omega\rangle \neq 0$, whose transparent simple objects have twist coefficient and quantum dimension equal to one. This corresponds to Bruguières' particular case [7, Section 4] and to Müger abelian case [13, Section 5]. The tensor product of two transparent simple objects is then a transparent simple object, and isomorphisms classes of transparent simple objects form a group G under tensor multiplication. We will follow the description of the modularization functor given in the proof of [7, Lemma 4.3]. As before, let $\Gamma(\mathrm{A})$ be the representative set of simple objects of A.

If A is *self-dual* (i.e. any object is isomorphic to its dual), then G is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{|\mathcal{T}|}$, where $\mathcal{T}$ is the set of independent generators of G. This covers all cases considered in the next sections.

In general, G is isomorphic to $\oplus_{i=1}^{p}\mathbb{Z}/k_i\mathbb{Z}$, $k_{i+1}|k_i$, and admits the following presentation by generators and relations: $\mathrm{G} \approx \{t_1,...,t_p; t_i^{k_i} = 1, i = 1,...,p\}$. We fix, for each i, a transparent simple object representing the ith generator of G and denote it by the same letter t_i. Let $\mathcal{T} = \{t_1,...t_p\}$ be the set of generating transparent simple objects. We denote by $\mathrm{G}_{\mathcal{T}}$ the set of representatives of G defined by $\mathcal{T}$, i.e.

$$\mathrm{G}_{\mathcal{T}} = \{\otimes_i t_i^{n_i}; t_i \in \mathcal{T}, 0 \leq n_i < k_i\}.$$

Furthermore, we choose for each i an isomorphism $\Phi_i : t_i^{k_i} \approx \text{trivial object}$.

Let us define a category A' as follows. We set $Ob(\mathrm{A}') = Ob(\mathrm{A})$, we will however use the notation F for the functor from A to A', and

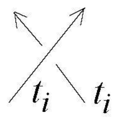
$$Hom_{\mathrm{A}'}(\mathrm{F}(X), \mathrm{F}(Y)) := \oplus_{\mathrm{W}\in \mathrm{G}_{\mathcal{T}}} Hom_{\mathrm{A}}(X, Y \otimes \mathrm{W}) .$$

For composition, we proceed as follows. Let $f \in Hom_{\mathrm{A}}(X, Y \otimes \mathrm{W})$, $g \in Hom_{\mathrm{A}}(Y, \mathrm{Z} \otimes \mathrm{W}')$ with $\mathrm{W}, \mathrm{W}' \in \mathrm{G}_{\mathcal{T}}$. Since the objects of $\mathrm{G}_{\mathcal{T}}$ are transparent, we get a canonical isomorphism $\mathsf{X} : \mathrm{Z} \otimes \mathrm{W}' \otimes \mathrm{W} \to \mathrm{Z} \otimes (\otimes_i t_i^{n_i})$. We define $\mathrm{F}(g)\mathrm{F}(f) := \mathsf{X}(g \otimes \mathbb{1}_{\mathrm{W}})f$, if $n_i < k_i$ for every i; otherwise we compose the right hand side of the previous formula with the isomorphisms $\mathbb{1}_{n_i - k_i} \otimes \Phi_i$ in order to reduce the exponents. Associativity results from the property

$$\Phi_i \otimes \mathbb{1}_{t_i} = \mathbb{1}_{t_i} \otimes \Phi_i \tag{2}$$

which is a consequence of

$$\mathbb{1}_{t_i} \otimes \mathbb{1}_{t_i} = \tag{3}$$



These are properties ($\mathcal{F}$) in [7]; here we use that the transparent simple objects are invertible, so that $t_i \otimes t_i$ is simple, and that their quantum dimensions and twist coefficients are equal to one.

We define the category $\widetilde{\mathrm{A}}$ as the idempotent completion of A'. It results from [7, Section 4] that $\widetilde{\mathrm{A}}$ is a modularization of A.

Remark. The category $\widetilde{\mathrm{A}}$ is called sometimes a modular extension of A by G. Analogously, a modular extension of A by any subgroup G' of G can be constructed. This gives a pre-modular category whose group of transparent objects is G/G'.

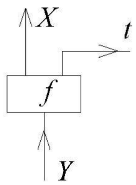
The next problem is to construct a representative set $\Gamma(\widetilde{\mathrm{A}})$ of simple objects of $\widetilde{\mathrm{A}}$. There is an action of the group G on the set $\Gamma(\mathrm{A})$ of simple objects of A by tensor multiplication. For $X \in \Gamma(\mathrm{A})$, the dimension of $\mathrm{E}nd_{\widetilde{\mathrm{A}}}(\mathrm{F}(X))$ is equal to the order d of the stabilizer subgroup $\mathrm{S}tab(X) := \{g \in \mathrm{G}; g \otimes X = X\}$.

If $\mathrm{S}tab(X)$ is cyclic, then the algebra $\mathrm{E}nd_{\widetilde{\mathrm{A}}}(\mathrm{F}(X))$ is abelian; it is isomorphic to the group algebra of $\mathrm{S}tab(X)$, and $\mathrm{F}(X)$ decomposes in the category $\widetilde{\mathrm{A}}$ into d non-isomorphic simple objects.

In the non-cyclic case it can be shown (cf. [13, Section 5]) that $\mathrm{E}nd_{\widetilde{\mathrm{A}}}(\mathrm{F}(X))$ is a twisted group algebra. The computation of the cocycle describing this twisted group algebra has to be done.

1.5. Generalized ribbon graphs

By Turaev's theorem [18, Ch. I, Theorem 2.5] the morphisms of a ribbon category A can be represented by A-colored ribbon graphs with coupons. More precisely, there exists a functor from the category $\mathrm{R}ib_{\mathrm{A}}$ of colored ribbon graphs to the category A which respects the structures. We can extend the category $\mathrm{R}ib_{\mathrm{A}}$ by allowing tangles such that one of the ends of a band colored with an object t of $\mathcal{T}$ is free. This means, it is connected neither to a coupon, nor to the source, nor to the target. An example of such a tangle is depicted below. It is considered as a morphisms from Y to X.



This defines the extended category $\widetilde{\mathrm{R}ib}_{\mathrm{A}}^{\mathcal{T}}$, which is also a ribbon category. We extend the invariant of closed colored graphs, i.e the map $\mathrm{E}nd_{\mathrm{R}ib_{\mathrm{A}}}(\text{trivial}) \to \mathbf{k}$ given by Turaev's functor, in the following way. An extended closed colored graph is sent to zero, if the number of its free ends colored by t_i is not divisible by k_i for some i. Otherwise, it is sent to the invariant of $\mathrm{R}ib_{\mathrm{A}}$ for a graph obtained by closing the free ends with Φ_i.

Using the properties (3), (2), we can show that Turaev's functor extends to a functor from $\widetilde{\mathrm{R}ib}_{\mathrm{A}}^{\mathcal{T}}$ to the modular category $\widetilde{\mathrm{A}}$ which coincides with the invariant described above for closed morphisms.

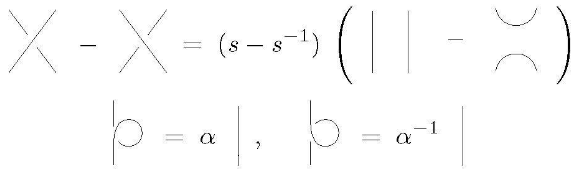
Remark. The modularization can be obtained from the ($\mathbf{k}$ -linear) category $\widetilde{\mathrm{R}ib}_{\mathrm{A}}^{T}$ by first quotienting by negligible morphisms (using the invariant $\mathrm{E}nd_{\mathrm{R}ib_{\mathrm{A}}}(\text{trivial}) \to \mathbf{k}$ described above) and then completing with idempotents. Direct sums are not needed here. This process was sketched in [4].

2. Idempotents of BMW algebras

2.1. Kauffman skein relations

Let M be a 3-manifold (possibly with a given finite set l of points on the boundary, and a nonzero tangent vector at each point). Let $\mathbf{k}$ be a field containing the nonzero elements α and s with $s^2 \neq 1$.

We denote by $\mathcal{S}(\mathrm{M})$ (resp. $\mathcal{S}(\mathrm{M}, l)$) the $\mathbf{k}$-vector space freely generated by links in M (and tangles in M that meet $\partial\mathrm{M}$ in l) modulo the Kauffman skein relations:



$$\mathrm{L} \amalg \bigcirc = \left(\frac{\alpha - \alpha^{-1}}{s - s^{-1}} + 1\right) \mathrm{L}.$$

We call $\mathcal{S}(\mathrm{M})$ the *skein module* of M . For example, $\mathcal{S}(\mathrm{S}^3) \cong \mathbf{k}$.

2.2. Birman-Murakami-Wenzl category

The Birman-Murakami-Wenzl (BMW) category K is defined as follows. An object of K is a standard oriented disc $\mathrm{D}^2 \subset \mathbb{C}$ equipped with a finite set of points and a nonzero tangent vector at each point. Unless otherwise specified, we will use the second vector of the standard basis (the vector $\sqrt{-1}$ in complex notation). If $\beta = (\mathrm{D}^2, l_0)$ and $\gamma = (\mathrm{D}^2, l_1)$ are two such objects, the module $Hom_{\mathrm{K}}(\beta, \gamma)$ is defined as the skein module $\mathcal{S}(\mathrm{D}^2 \times [0,1], l_0 \times 0 \amalg l_1 \times 1)$. Composition is given by stacking of cylinders. We will use the notation $\mathrm{K}(\beta, \gamma)$ for $Hom_{\mathrm{K}}(\beta, \gamma)$ and K_β for $\mathrm{E}nd_{\mathrm{K}}(\beta)$. The tensor product is defined by using $j = j_{-1} \amalg j_1 : \mathrm{D}^2 \amalg \mathrm{D}^2 \hookrightarrow \mathrm{D}^2$, where, for $\epsilon = \pm 1$, $j_\epsilon : \mathrm{D}^2 \hookrightarrow \mathrm{D}^2$ is the embedding which sends z to $\frac{\epsilon}{2} + \frac{1}{4} z$.

The BMW category is a $\mathbf{k}$-linear ribbon category. As before, we denote by $\langle f \rangle \in \mathbf{k}$ the quantum trace of $f \in \mathrm{K}_\beta$. The BMW categories defined using the parameters (α, s) and $(\alpha, -s^{-1})$ are isomorphic.

Let us denote by n the object of K formed with the n points $\{(2j-1)/n-1;\ j=1,...,n\}$ equipped with the standard vector. Composition in the category K provides a $\mathbf{k}$-algebra structure on $\mathrm{K}_n = \mathrm{E}nd_{\mathrm{K}}(n)$, and we get the Birman-Murakami-Wenzl (BMW) algebra.

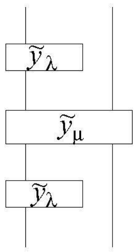
The BMW algebra K_n is a deformation of the Brauer algebra (i.e. the centralizer algebra of the semi-simple Lie algebras of type B,C and D). It is known to be generically semi-simple and its simple components correspond to the partitions $\lambda = (\lambda_1, ..., \lambda_p)$ with $|\lambda| = \sum_i \lambda_i = n-2r$, $r = 0,1,...,[n/2]$.

2.3. Idempotents

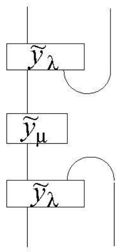
Let λ be a partition with $|\lambda| = n$. We denote by $\square_\lambda$ the object of K formed with one point for each cell of the Young diagram associated with λ. If c has coordinates (i,j) (i-th row, and j-th column), then the corresponding point in D^2 is $\frac{j+i\sqrt{-1}}{n+1}$. In [2] we have constructed minimal idempotents $\tilde{y}_\lambda \in \mathrm{K}_{\square_\lambda}$. Let us recall their main properties in the generic case (i.e. with $\mathbf{k} = \mathbb{Q}(\alpha, s)$).

Branching formula:

$$\tilde{y}_\lambda \otimes \mathbf{1}_1 = \sum_{\substack{\lambda\subset\mu \\ |\mu|=|\lambda|+1}}$$



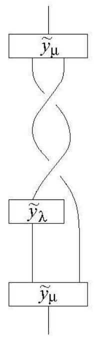
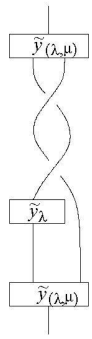
$$+ \sum_{\substack{\mu\subset\lambda \\ |\mu|=|\lambda|-1}} \frac{\langle\mu\rangle}{\langle\lambda\rangle}$$



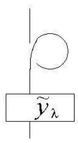
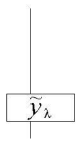
(4)

Here standard isomorphisms are used, in the first tangle between $\square_\lambda \otimes 1$ and $\square_\mu$, in the second tangle between $\square_\mu \otimes 1$ and $\square_\lambda$. The second tangle times $\frac{\langle\mu\rangle}{\langle\lambda\rangle}$ will be further denoted by $\tilde{y}_{(\lambda,\mu)}$. Note that the quantum dimension $\langle\lambda\rangle$ is nonzero in the generic case.

Braiding coefficient: Let *i)* $\mu - \lambda = c$ or *ii)* $\lambda - \mu = c$, where the cell c has coordinates (i,j). Let $cn(c)$ be the content of the cell c: $cn(c) = j - i$. Then

$i)$  $= s^{2cn(c)}\tilde{y}_\mu$; $ii)$  $= \alpha^{-2}s^{-2cn(c)}\tilde{y}_{(\lambda,\mu)}$. (5)

Twist coefficient: A positive 2π-twist of $|\lambda|$ lines with $\tilde{y}_\lambda$ inserted contributes the factor $\alpha^{|\lambda|}s^{2\sum_{c\in\lambda} cn(c)}$.

 $= \alpha^{|\lambda|}s^{2\sum_{c\in\lambda} cn(c)}$  (6)

Quantum dimensions: Let $n \in \mathbb{Z}$, we set

$$[n]_\alpha = \frac{\alpha s^n - \alpha^{-1}s^{-n}}{s - s^{-1}}, \qquad [n] = \frac{s^n - s^{-n}}{s - s^{-1}}.$$

Then the quantum dimension of λ is given by the following formula

$$\langle\lambda\rangle = \langle\lambda\rangle_{\alpha,s} = \prod_{(j,j)\in\lambda} \frac{[\lambda_j - \lambda_j^\vee]_\alpha + [hl(j,j)]}{[hl(j,j)]} \prod_{\substack{(i,j)\in\lambda \\ i\neq j}} \frac{[d_\lambda(i,j)]_\alpha}{[hl(i,j)]}. \tag{7}$$

Here, $hl(i,j)$ denotes the hook-length of the cell (i,j), i.e. $hl(i,j) = \lambda_i + \lambda_j^\vee - i - j + 1$, $\lambda_i^\vee$ is the length of the i-th column of λ and $d_\lambda(i,j)$ is defined by

$$d_\lambda(i,j) = \begin{cases} \lambda_i + \lambda_j - i - j + 1 & \text{if} \quad i \leq j \\ -\lambda_i^\vee - \lambda_j^\vee + i + j - 1 & \text{if} \quad i > j\ . \end{cases}$$

Observe that

$$\langle\lambda\rangle_{\alpha,s} = \langle\lambda\rangle_{-\alpha,-s} = \langle\lambda\rangle_{\alpha^{-1},s^{-1}} = \langle\lambda^\vee\rangle_{\alpha,-s^{-1}}\ . \tag{8}$$

The formula (7) was first proved by Wenzl [21, Theorem 5.5]. If we define $d'_\lambda(i,j)$ by

$$d'_\lambda(i,j) = \begin{cases} \lambda_i + \lambda_j - i - j + 1 & \text{if} \quad i < j \\ -\lambda_i^\vee - \lambda_j^\vee + i + j - 1 & \text{if} \quad i \geq j, \end{cases}$$

then we can write Wenzl's formula as follows.

$$\langle\lambda\rangle = \prod_{(i,j)\in\lambda} \frac{\alpha^{\frac{1}{2}} s^{\frac{1}{2}d_\lambda(i,j)} - \alpha^{-\frac{1}{2}} s^{-\frac{1}{2}d_\lambda(i,j)}}{s^{\frac{1}{2}hl(i,j)} - s^{-\frac{1}{2}hl(i,j)}} \prod_{(i,j)\in\lambda} \frac{\alpha^{\frac{1}{2}} s^{\frac{1}{2}d'_\lambda(i,j)} + \alpha^{-\frac{1}{2}} s^{-\frac{1}{2}d'_\lambda(i,j)}}{s^{\frac{1}{2}hl(i,j)} + s^{-\frac{1}{2}hl(i,j)}} \tag{9}$$

2.4. Idempotents in the non-generic case

By a non-generic case we understand a choice of parameters in the field $\mathbf{k}$ such that s is a root of unity, or $\pm\alpha$ is a power of s. A typical example is given by roots of unity in a cyclotomic field. As in the generic case, the idempotents $\tilde{y}_\lambda$ are obtained recursively by lifting to the BMW category the corresponding idempotent y_λ in the Hecke category. The minimal idempotent y_λ can be defined provided the quantum integers $[m]$ are not zero for $m < \lambda_1 + \lambda_1^\vee$, and $\tilde{y}_\lambda$ can further be obtained provided for *some* $\mu \subset \lambda$, $|\mu| = |\lambda| - 1$, $\tilde{y}_\mu$ is defined and its quantum dimension is not zero. Under the above conditions, Wenzl path idempotent [21] corresponding to a standard tableau t with shapes $\lambda(t) = \lambda$ and $\lambda(t') = \mu$ is defined and could be used here. The minimality property of the idempotent $\tilde{y}_\lambda$ is

$$\tilde{y}_\lambda \mathrm{K}_{\square_\lambda} \tilde{y}_\lambda = \mathbf{k}\tilde{y}_\lambda \ .$$

The generic formulas of the previous subsection hold provided they make sense. In particular the branching formula is valid provided the minimal idempotents exist for all diagrams obtained from λ by adding one cell.

We will consider in the following the case where $\pm\alpha$ is a power of s, and discuss which idempotents are obtained depending if s is a root of unity or not. As explained in [21], in this case if we quotient out the BMW algebra by negligible morphisms (the annihilator of the trace), then we get a semi-simple algebra.

If neither α, nor $-\alpha$ are powers of s but s is a root of unity, then we obtain minimal idempotents corresponding to partitions λ with $\lambda_1 + \lambda_1^\vee < l + 1$, where l is the order of s^2. These diagrams are called l-regular in [21]. If we consider a diagram μ with $\mu_1 + \mu_1^\vee = l + 1$, obtained from an l-regular diagram by adding one cell, then the generic element $\tilde{Y}_\mu = [l]\tilde{y}_\mu$ still can be defined and has nonzero trace. This element satisfies $\tilde{Y}_\mu \mathrm{K}_{\square_\mu} \tilde{Y}_\mu = 0$, since $[l] = 0$ in our specialization.

Lemma 2.1. *The element $\tilde{Y}_\mu$ belongs to the radical of the algebra $\mathrm{K}_{\square_\mu}$ (the intersection of the maximal left ideals).*

Proof. Let J be a maximal left ideal of $\mathrm{K}_{\square_\mu}$. Suppose that J does not contain $\tilde{Y}_\mu$, then, using maximality of J, we get that the left ideal $\mathrm{J} + \mathrm{K}_{\square_\mu}\tilde{Y}_\mu$ is equal to $\mathrm{K}_{\square_\mu}$. We further have that $\mathbb{1}_{\square_\mu} = j + a\tilde{Y}_\mu$, $j \in \mathrm{J}$, $a \in \mathrm{K}_{\square_\mu}$, and so $\tilde{Y}_\mu = \tilde{Y}_\mu j + \tilde{Y}_\mu a \tilde{Y}_\mu = \tilde{Y}_\mu j$ is in the ideal J, which contradicts the hypothesis. $\square$

This shows that the algebra $K_{\square_\mu}$ is not semi-simple in this case and if we quotient out by negligible morphisms we will still have a non semi-simple algebra.

3. The completed BMW categories

In this section we define the completed BMW category and discuss specializations of parameters for which the quotient of the completed BMW category by negligible morphisms is a pre-modular category.

3.1. Completed BMW categories

Let $\mathcal{C}$ be a set of Young diagrams, such that the corresponding minimal idempotents exist. This means that for each element of $\mathcal{C}$ the conditions described in Section 2.4 are satisfied. In each case considered further this set will be the maximal set in which the recursive construction of the idempotents $\tilde{y}_\lambda$ works (this set corresponds to the *affine Weyl alcove* in the quantum group description).

We define the *completed BMW category* $K^{\mathcal{C}}$ as follows. An object of $K^{\mathcal{C}}$ is an oriented disc D^2 equipped with a finite set of points, with a trivialization of the tangent space at each point (usually the standard one), labeled with diagrams from $\mathcal{C}$. Let $\beta = (D^2, l) = (D^2; \lambda^{(1)}, ..., \lambda^{(m)})$ be such an object. Then its expansion $E(\beta) = (D^2, E(l))$ is obtained by embedding the object $\square_{\lambda^{(i)}}$ in a neighborhood of the point labeled by $\lambda^{(i)}$, according to the trivialization. The tensor product $\tilde{y}_{\lambda^{(1)}} \otimes ... \otimes \tilde{y}_{\lambda^{(m)}}$ defines an idempotent $\pi_\beta \in K_\beta$. We define $Hom_{K^{\mathcal{C}}}(\beta, \gamma) := \pi_\beta K(E(\beta), E(\gamma))\pi_\gamma$. We will use the notation $K^{\mathcal{C}}(\beta, \gamma)$ and $K^{\mathcal{C}}_\beta$ similarly as in K.

The duality extends to $K^{\mathcal{C}}$, and we obtain again a $\mathbf{k}$-linear ribbon category. Observe that the dual of an object is isomorphic to itself in a non-canonical way.

The equality of the categories K for the parameters (α, s) and $(\alpha, -s^{-1})$ extends to an isomorphism between the categories $K^{\mathcal{C}}$ and $K^{\mathcal{C}^\vee}$, where $\mathcal{C}^\vee$ is obtained from $\mathcal{C}$ by transposition of diagrams (i.e. exchange of rows and columns). For further discussion of duality, it is useful to note that this change of the parameter s switches a primitive l th root of unity, into a primitive $2l$ th root of unity if l is odd.

We denote by λ the object of $K^{\mathcal{C}}$ formed by a disc with the origin labeled by λ. The minimality property of the idempotent $\tilde{y}_\lambda$ implies that λ is a simple object in $K^{\mathcal{C}}$.

Recall that a morphism $f \in K^{\mathcal{C}}(\alpha, \beta)$ is negligible if for any $g \in K^{\mathcal{C}}(\beta, \alpha)$ one has $\langle fg \rangle = 0$. Negligible morphisms form a tensor ideal in the category, and we obtain a quotient $\mathrm{K}^{\mathcal{C}}/Neg$ which is a $\mathbf{k}$-linear ribbon category. The duality axiom is trivially satisfied here. Our aim is to discuss in which case this quotient category happens to be pre-modular.

We first consider the generic case. Here the set $\mathcal{C}$ contains all Young diagrams. We see from the branching formula that the completed category is semi-simple. Isomorphism classes of simple objects correspond to all Young diagrams, so that the category is not pre-modular. Moreover, from the braiding formula (5) we see that there is no non-trivial transparent simple object, so that we could not get a modularization even if we would consider an extended version of Bruguières' procedure.

We already have considered in Section 2.4 the case where s is a root of unity, but neither α nor $-\alpha$ is a power of s. Here the quotient of the idempotent completed category by negligible morphisms will not be semi-simple, because some endomorphism algebras are not.

We will now consider the specializations where $\pm\alpha$ is a power of s. Recall that $1^{\mathsf{N}+1}$ and $\mathsf{K}+1$ denotes the column and the row Young diagrams with $\mathsf{N}+1$ and $\mathsf{K}+1$ cells, respectively. Let us consider the following system of equations $\langle 1^{\mathsf{N}+1}\rangle = 0$ and $\langle \mathsf{K}+1\rangle = 0$, with N and K minimal. Note that, if $\pm\alpha$ is a power of s, then at least one of these two equations has a solution. The first one is equivalent to $\alpha = -s^{2\mathsf{N}+1}$ or $\alpha = \pm s^{\mathsf{N}-1}$. We have to consider 4 cases.

Case C_n : $\alpha = -s^{2n+1}$ ($\mathsf{N} = n$),
Case B_n : $\alpha = s^{2n}$ ($\mathsf{N} = 2n+1$),
Case B_{-n} : $\alpha = -s^{2n}$ ($\mathsf{N} = 2n+1$),
Case D_n : $\alpha = s^{2n-1}$ ($\mathsf{N} = 2n$),

The interpretation of the notation C_n , B_n , D_n is that the given specialization of the Kauffman polynomial is obtained by using the fundamental representation of the corresponding quantum group. The specializations B_n and B_{-n} are similar, but they are not equivalent; one should think of the fundamental object in the B_{-n} specialization as the deformation of the fundamental representation of $so(2n+1)$, with negative dimension $-(2n+1)$.

The discussion of the equation $\langle \mathsf{K}+1\rangle$ is similar. Note that quantum dimensions are unchanged if we replace s by $-s^{-1}$ and interchange rows with columns. Here are the four cases.

Case C_k : $\alpha = s^{-2k-1}$ ($\mathsf{K} = k$),
Case B_k : $\alpha = s^{-2k}$ ($\mathsf{K} = 2k+1$),
Case B_{-k} : $\alpha = -s^{-2k}$ ($\mathsf{K} = 2k+1$),
Case D_k : $\alpha = -s^{-2k+1}$ ($\mathsf{K} = 2k$),

We observe that, if $\langle 1^{\mathsf{N}+1}\rangle = \langle \mathsf{K}+1\rangle = 0$ for some N , K , then s is a root of unity. We will consider the four cases corresponding to the vanishing of $\langle 1^{\mathsf{N}+1}\rangle$, and then, according to the order of s^2, combine them with the condition corresponding to the lowest K for which $\langle \mathsf{K}+1 \rangle$ vanishes.

The cases $\alpha = \pm 1$, $\alpha = -s$ and $\alpha = s^{-1}$ will be excluded from the general discussion given in the next subsections. If $\alpha = \pm 1$ we get a category with two simple objects: the trivial object and $\lambda = 1$. The second object is transparent and the category is modularizable iff $\alpha = 1$. The corresponding link invariant is trivial. If $\alpha = -s$ or $\alpha = s^{-1}$, then the Kauffman polynomial is zero.

The case $\alpha = s$ (resp. $\alpha = -s^{-1}$) will be included in the general discussion and give the categories $\mathrm{D}^{1,k}$, $\mathrm{DB}^{1,k}$ and $\mathrm{DB}^{1,-k}$ (resp. $\mathrm{D}^{k,1}$, $\mathrm{BD}^{k,1}$ and $\mathrm{BD}^{-k,1}$). Note that the corresponding invariant of a link $\mathrm{L} = (\mathrm{L}_1, \dots, \mathrm{L}_m)$ is equal to $2^{\sharp \mathrm{L}} s^{\sum_i \mathrm{L}_i . \mathrm{L}_i}$. Here $\sharp \mathrm{L} = m$ is the number of components, and $\mathrm{L}_i . \mathrm{L}_i$ is the self linking number (the framing coefficient). The category is modularizable if s is either a primitive root of order $2l$, l even, or a primitive root of odd order l. One can show that the corresponding invariants of 3-manifolds are those known as the $U(1)$ invariants [12].

3.2. The symplectic case

In this subsection let $\alpha = -s^{2n+1}$, $n \geq 1$. (For $n = 1$ the specialized Kauffman polynomial is the Kauffman bracket, and we will recover the TQFT's obtained in [5].)

If s is generic, then we can construct the idempotent $\tilde{y}_\lambda$ for λ in the set

$$\bar{\Gamma}(\mathrm{C}_n) = \{\lambda; \lambda_1^\vee \leq n+1,\ \lambda_2^\vee \leq n\} ,$$

and λ has non-vanishing quantum dimension (see formula (9)) if it belongs to

$$\Gamma(\mathrm{C}_n) = \{\lambda; \lambda_1^\vee \leq n\} .$$

From the branching formula we get that the category $\mathrm{K}^{\bar{\Gamma}(\mathrm{C}_n)}/Neg$ is semi-simple; we will give more details in the proof of Proposition 3.2. A representative set of simple objects is the infinite set $\Gamma(\mathrm{C}_n)$, so that the category is not pre-modular.

The formula for the quantum dimension can be simplified as follows (see [2, Prop. 7.6], compare [8]).

Proposition 3.1. *Let $\alpha = -s^{2n+1}$, with s generic. Then, for a partition $\lambda = (\lambda_1, ..., \lambda_n)$, we have*

$$\langle \lambda \rangle = (-1)^{|\lambda|} \prod_{j=1}^{n} \frac{[2n+2+2\lambda_j - 2j]}{[2n+2-2j]}$$
$$\prod_{1 \leq i < j \leq n} \frac{[2n+2+\lambda_i - i + \lambda_j - j][\lambda_i - i - \lambda_j + j]}{[2n+2-i-j][j-i]} .$$

Let us suppose now that $\alpha = -s^{2n+1}$ with s^2 a primitive l th root of unity and $l \geq 2n+1$. One can check that the above formula for quantum dimensions is still valid provided $l \geq 2n+1$. The condition $l \geq 2n+1$ ensures that 1^{n+1} is the smallest column with vanishing quantum dimension. Note that for $l = 2n+1$ we have $\alpha = \pm 1$, and for $l = 2n+2$, we have $\alpha = -s$. In the following we discuss the equation $\langle \mathsf{K}+1 \rangle = 0$ with K minimal according to $l \geq 2n+3$.

- If $l \geq 2n+4$ is even, then $\mathsf{K} = l/2 - n - 1 = k$, and $\alpha = -s^{2n+1} = s^{-2k-1}$. This will be the C_n - C_k specialization.
- If $l \geq 2n+3$ is odd and $s^l = -1$, then $\mathsf{K} = 2k+1$, $\alpha = -s^{2n+1} = s^{-2k}$. This will be the C_n - B_k specialization.
- If $l \geq 2n+3$ is odd and $s^l = 1$, then $\mathsf{K} = l - 2n = 2k+1$, $\alpha = -s^{2n+1} = -s^{-2k}$. This will be the C_n - B_{-k} specialization.

The specializations C_n - B_k and C_n - B_{-k} are similar because of the symmetry $(\alpha, s) \leftrightarrow (-\alpha, -s)$ for quantum dimensions. Note however that the twist coefficient is not preserved under this symmetry, so that the modularization problems will be distinct. We will show that the C_n - C_k and C_n - B_k specializations lead to modular categories.

$\mathbf{C}^{n,k}$ category. Let us consider the C_n - C_k specialization of parameters with $n, k \geq 1$, i.e. $\alpha = -s^{2n+1} = s^{-2k-1}$ and s is a primitive $2l$ th root of unity with $l = 2n+2k+2$. We will use the following sets of Young diagrams:

$$\bar{\Gamma}(\mathrm{C}^{n,k}) = \{\lambda; \lambda_1 \leq k+1, \lambda_2 \leq k, \lambda_1^\vee \leq n+1, \lambda_2^\vee \leq n\}\ ,$$
$$\Gamma(\mathrm{C}^{n,k}) = \{\lambda; \lambda_1 \leq k, \lambda_1^\vee \leq n\}\ .$$

We can construct the minimal idempotent for each $\lambda \in \Gamma(\mathrm{C}^{n,k})$, since the quantum dimensions of these objects given by Proposition 3.1 do not vanish. Let $\lambda \in \Gamma(\mathrm{C}^{n,k})$. If μ is obtained from λ by adding one cell, then $\tilde{y}_\mu \in \bar{\Gamma}(\mathrm{C}^{n,k})$ can be constructed. Moreover, if μ is not in $\Gamma(\mathrm{C}^{n,k})$, then $\langle \mu \rangle$ vanishes, and so $\tilde{y}_\mu$ is negligible.

The category $\mathrm{C}^{n,k}$ is defined as the quotient of the category $\mathrm{K}^{\bar{\Gamma}(\mathrm{C}^{n,k})}$ by negligible morphisms.

$\mathbf{CB}^{n,k}$ and $\mathbf{CB}^{n,-k}$ categories. In the case of the C_n - B_k (resp. C_n - B_{-k}) specialization with $n, k \geq 1$ we have $\alpha = -s^{2n+1} = s^{-2k}$ and s is a primitive $2l$ th root of unity (resp. $\alpha = -s^{2n+1} = -s^{-2k}$ and s is a primitive l th root of unity), $l = 2n+2k+1$. We proceed as above with

$$\bar{\Gamma}(\mathrm{CB}^{n,k}) = \bar{\Gamma}(\mathrm{CB}^{n,-k}) = \{\lambda; \lambda_1 + \lambda_2 \leq 2k+2, \lambda_1^\vee \leq n+1, \lambda_2^\vee \leq n\},$$
$$\Gamma(\mathrm{CB}^{n,k}) = \Gamma(\mathrm{CB}^{n,-k}) = \{\lambda; \lambda_1 + \lambda_2 \leq 2k+1, \lambda_1^\vee \leq n\},$$
$$\mathrm{CB}^{n,k} = \mathrm{K}^{\Gamma(\mathrm{CB}^{n,k})}/Neg\,, \quad \mathrm{CB}^{n,-k} = \mathrm{K}^{\bar{\Gamma}(\mathrm{CB}^{n,-k})}/Neg\ .$$

Proposition 3.2. *For $n, k \geq 1$, the categories $\mathrm{C}^{n,k}$, $\mathrm{CB}^{n,k}$ and $\mathrm{CB}^{n,-k}$ with representative sets of simple objects $\Gamma(\mathrm{C}^{n,k})$, $\Gamma(\mathrm{CB}^{n,k})$ and $\Gamma(\mathrm{CB}^{n,-k})$, respectively, are pre-modular.*

Proof. We have to prove the dominating property. The proof is the same in all cases, so we will use the notation $\bar{\Gamma}$, Γ for $\bar{\Gamma}(\mathrm{A})$, $\Gamma(\mathrm{A})$ where A is one of the categories mentioned in the claim. It is enough to show that the identity morphism of the object n decomposes using the simple objects in Γ. This is done by induction on n. For the step from n to $n+1$, we have to decompose $1\!\!1_\lambda \otimes 1\!\!1_1$, with $\lambda \in \Gamma$. The key point is that any diagram obtained from λ by adding one cell is in $\bar{\Gamma}$. Hence we have that the branching formula holds and gives the required decomposition, because the idempotents indexed by partitions in $\bar{\Gamma} \setminus \Gamma$ are negligible. □

3.3. The odd orthogonal case

We first consider the B_n specialization $\alpha = s^{2n}$. If s is generic, then we can construct the idempotent $\tilde{y}_\lambda$ for λ in the set

$$\bar{\Gamma}(\mathrm{B}_n) = \{\lambda; \lambda_1^\vee + \lambda_2^\vee \leq 2n+2\} ,$$

and λ has non-vanishing quantum dimension (see formula (9)) if it belongs to

$$\Gamma(\mathrm{B}_n) = \{\lambda; \lambda_1^\vee + \lambda_2^\vee \leq 2n+1\} .$$

As we did before, we get that the category $\mathrm{K}^{\bar{\Gamma}(\mathrm{B}_n)}/Neg$ is semi-simple. A representative set of simple objects is the infinite set $\Gamma(\mathrm{B}_n)$, so that the category is not pre-modular.

We have the following specialized formula for the quantum dimensions (see [2, Prop. 7.6]).

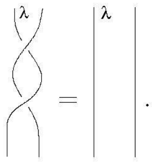
Proposition 3.3. *Let $\alpha = s^{2n}$, with s generic. For a partition $\lambda = (\lambda_1, ..., \lambda_n)$, we have*

$$\langle\lambda\rangle = \prod_{j=1}^{n} \frac{[n+\lambda_j - j + 1/2]}{[n-j+1/2]} \prod_{1\leq i<j\leq n} \frac{[2n+\lambda_i - i + \lambda_j - j + 1][\lambda_i - i - \lambda_j + j]}{[2n-i-j+1][j-i]} .$$

In this case, the object 1^{2n+1} plays a special role.

Lemma 3.4. *Suppose that $\alpha = s^{2n}$, and s is generic. Then the object 1^{2n+1} is transparent and it is the unique nontrivial transparent object in $\Gamma(\mathrm{B}_n)$. Its quantum dimension and twist coefficient are equal to one.*

Proof. An object $\lambda \in \Gamma(\mathrm{B}_n)$ is transparent if and only if for any (non-negligible) μ in the branching formula for λ, the braiding coefficient is equal to one. Indeed, if all braiding coefficients are equal to one, by summing over μ the left hand sides and right hand sides of (5) and applying the branching formula we have



Using this equality repeatedly we conclude that λ is transparent. Conversely, if λ is transparent, its braiding coefficients are trivial.

The object 1^{2n+1} has only one braiding coefficient corresponding to the removal of the last cell, and this coefficient is one. (The two diagrams obtained by adding one cell to 1^{2n+1} are negligible.) It remains to check that any nontrivial $\lambda \in \Gamma(\mathrm{B}_n)$ distinct from 1^{2n+1} has at least one braiding coefficient distinct from 1. If μ is obtained from such λ by adding a cell in the first row, then $\langle\mu\rangle$ is not zero, and the corresponding braiding coefficient in formula (5) is $s^{2\lambda_1} \neq 1$. For a column with j cells, the generic quantum dimension formula reduces to

$$\langle 1^j \rangle = \frac{[0]_\alpha[-1]_\alpha ... [2-j]_\alpha([1-j]_\alpha + [j])}{[j]!}. \tag{10}$$

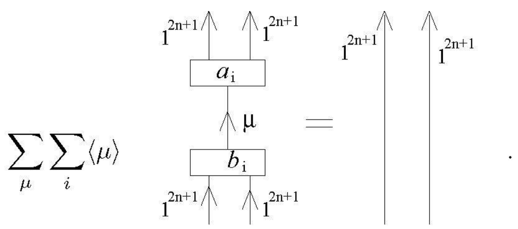
This gives for 1^{2n+1}

$$\langle 1^{2n+1} \rangle = \frac{[2n] \dots [1](0 + [2n+1])}{[2n+1]!} = 1\ .$$

The twist coefficient for 1^{2n+1} is $\alpha^{2n+1}s^{-2n(2n+1)} = 1$. □

Proposition 3.5. *In the category* $\mathrm{K}^{\bar\Gamma(\mathrm{B}_n)}/Neg$ *,*
a) the object $1^{2n+1} \otimes 1^{2n+1}$ *is isomorphic to the trivial object;*
b) the objects $1^{2n+1} \otimes \lambda$ *and* $\tilde\lambda$ *are isomorphic, where* $\lambda \in \Gamma(\mathrm{B}_n)$ *, and* $\tilde\lambda$ *is the Young diagram such that* $\lambda_1^\vee + \tilde\lambda_1^\vee = 2n+1$ *and* $\lambda_j^\vee = \tilde\lambda_j^\vee$ *for* $j > 1$ *,*

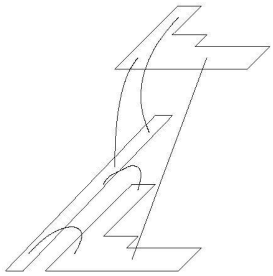
Proof. In the semi-simple category $\mathrm{K}^{\bar\Gamma(\mathrm{B}_n)}/Neg$ we can decompose the identity of the object $1^{2n+1} \otimes 1^{2n+1}$ as we did in formula (1).



Here all simple subobjects μ are transparent and hence have dimension 1. By comparing the dimensions we see that there is only one such μ with multiplicity

1 . It should be trivial, because the duality gives a nonzero morphism from the trivial to $1^{2n+1} \otimes 1^{2n+1}$. We deduce that this duality morphism is an isomorphism, which establishes $a)$.

We consider the morphism from $1^{2n+1} \otimes \lambda$ to $\tilde{\lambda}$ depicted below: the strings corresponding to the points in (the expansion of) 1^{2n+1} are joined to the first columns, the points which are not in the first column of λ and $\tilde{\lambda}$ are joined directly.



One wants to show that this morphism is nonzero. We first consider the case where $\lambda = 1^j$ has only one column. Let $f \in Hom(1^j \otimes 1^{2n+1}, 1^{2n+1-j})$ be the morphism as above and $g \in Hom(1^{2n+1-j}, 1^j \otimes 1^{2n+1})$ be its mirror image with respect to the target plane. Then $\langle gf \rangle = \langle 1^{2n+1} \rangle = 1$. In the general case, if we insert conveniently the isomorphism considered in the particular case between and $\mathbb{1}_{1^{2n+1} \otimes \lambda}$ and $\mathbb{1}_{\tilde{\lambda}}$ we obtain our nontrivial morphism. □

We suppose now that $\alpha = s^{2n}$, with s^2 a primitive l th root of unity, $l \geq 2n+1$. In the following we discuss the equation $\langle \mathsf{K}+1 \rangle = 0$, K minimal. If s has order $2n+1$ and $s^l = 1$, then $\alpha = s^{-1}$ and the Kauffman polynomial is trivial.

- If $l \geq 2n+2$ is even, then $\mathsf{K} = l - 2n + 1 = 2k + 1$, $\alpha = s^{2n} = -s^{-2k}$; this will be the B_n - B_{-k} specialization.
- If $l \geq 2n+1$ is odd and $s^l = -1$, then $\mathsf{K} = l + 1 - 2n = 2k$, $\alpha = s^{2n} = -s^{-2k+1}$; this will be B_n - D_k specialization.
- If $l \geq 2n+3$ is odd and $s^l = 1$, then $\mathsf{K} = \frac{l-1}{2} - n = k$, $\alpha = s^{2n} = s^{-2k-1}$ will be the B_n - C_k specialization.

$\mathbf{B}^{n,-k}$ category. Here we consider the B_n - B_{-k} specialization ($\alpha = s^{2n} = -s^{-2k}$) with $n, k \geq 1$, s is a primitive $2l$ th root of unity, $l = 2n + 2k$. Let

$$\Gamma(\mathrm{B}^{n,-k}) = \{\lambda; \lambda_1 + \lambda_2 \leq 2k+1, \lambda_1^\vee + \lambda_2^\vee \leq 2n+1\} .$$

We can define idempotents for any $\lambda \in \Gamma(\mathrm{B}^{n,-k})$, and they have nonzero quantum dimension. Our general procedure give some more idempotents whose dimension vanishes, namely for each $\lambda \in \bar{\Gamma}(\mathrm{B}^{n,-k}) \setminus \Gamma(\mathrm{B}^{n,-k})$ with

$$\bar{\Gamma}(\mathrm{B}^{n,-k}) = \{\lambda; \lambda_1 + \lambda_2 \leq 2k+2, \lambda_1^\vee + \lambda_2^\vee \leq 2n+2, \lambda_1 + \lambda_1^\vee \leq 2n+2k\}$$

we have $\langle\lambda\rangle = 0$. We define the category $\mathrm{B}^{n,-k}$ as the quotient of the category $\mathrm{K}^{\bar{\Gamma}(\mathrm{B}^{n,-k})}$ by negligible morphisms.

Proposition 3.6. *The category* $\mathrm{B}^{n,-k}$ *is pre-modular.*

Proof. Let $\tilde{\Gamma}(\mathrm{B}^{n,-k}) = \Gamma(\mathrm{B}^{n,-k}) \cup \{1^{2n+1} \otimes 2k+1\}$. We show that $\tilde{\Gamma}(\mathrm{B}^{n,-k})$ is a set of dominating simple objects. As in the proof of Proposition 3.2, we decompose the tensor products $\mathbb{1}_{\mathrm{W}} \otimes \mathbb{1}_1$, for $\mathrm{W} \in \tilde{\Gamma}(\mathrm{B}^{n,-k})$. The sublte point here is that some idempotent in the branching formula for the partition $\mathrm{L} = (2k, 1^{2n-1}) \in \Gamma(\mathrm{B}^{n,-k})$ (i.e. $\mathrm{L}_1 + \mathrm{L}_1^\vee = 2n+2k$) is missing. We will avoid this difficulty by using the isomorphism in Proposition 3.5 which still holds for $\lambda \in \Gamma(\mathrm{B}^{n,-k})$.

More precisely, if $\mathrm{W} = \lambda$ is in $\Gamma(\mathrm{B}^{n,-k}) \setminus \{\mathrm{L}\}$, then the branching formula applies. If $\mathrm{W} = \mathrm{L}$, then we use the isomorphism between L and $1^{2n+1} \otimes 2k$ and we get a decomposition of $\mathrm{L} \otimes 1$ with subobjects $(2k-1, 1^{2n})$, $(2k, 1^{2n-1})$ and $1^{2n+1} \otimes 2k+1$. If $\mathrm{W} = 1^{2n+1} \otimes 2k+1$, then we get an isomorphism between $1^{2n+1} \otimes 2k+1 \otimes 1$ and L. □

$\mathrm{BD}^{n,k}$ category. For the B_n - D_k specialization with $n, k \geq 1$, we put $l = 2n+2k-1$, s is a primitive root of unity of order $2l$, and $\alpha = s^{2n} = -s^{-2k+1}$. Let

$$\Gamma(\mathrm{BD}^{n,k}) = \{\lambda; \lambda_1 + \lambda_2 \leq 2k, \lambda_1^\vee + \lambda_2^\vee \leq 2n+1\}\ .$$

We define the category $\mathrm{BD}^{n,k}$ and prove pre-modularity as we did above.

$\mathrm{BC}^{n,k}$ category. The category $\mathrm{BC}^{n,k}$ for $n, k \geq 1$ with parameters (α, s) is isomorphic to the category $\mathrm{CB}^{k,n}$ with parameters $(\alpha, -s^{-1})$. The isomorphism sends any simple object λ to $\lambda^\vee$. The representative set of simple objects is $\Gamma(\mathrm{BC}^{n,k}) = \{\lambda; \lambda^\vee \in \Gamma(\mathrm{CB}^{k,n})\}$.

The specialization B_{-n}. Let us consider the case $\alpha = -s^{2n}$. If s is generic, we have $\Gamma(\mathrm{B}_n) = \Gamma(\mathrm{B}_{-n})$. The object 1^{2n+1} remains transparent, but its twist coefficient is (-1). Therefore, the categories we get here will be non-modularizable.

Let us suppose that s^2 is a primitive root of unity of order $l \geq 2n+1$. Then we have to consider the following cases.

- If $l \geq 2n+2$ is even, then $\mathsf{K} = l - 2n + 1 = 2k+1$, $\alpha = -s^{2n} = s^{-2k}$; this will be the B_{-n} - B_k specialization.
- If $l \geq 2n+1$ is odd and $s^l = 1$, then $\mathsf{K} = l + 1 - 2n = 2k$, $\alpha = -s^{2n} = -s^{-2k+1}$; this will be B_{-n} - D_k specialization.
- If $l \geq 2n+3$ is odd and $s^l = -1$, then $\mathsf{K} = \frac{l-1}{2} - n = k$, $\alpha = -s^{2n} = s^{-2k-1}$ will be the B_{-n} - C_k specialization.

The categories $\mathrm{B}^{-n,k}$, $\mathrm{BD}^{-n,k}$ and $\mathrm{BC}^{-n,k}$ with $n,k \geq 1$ can be constructed analogously to the previous case. We have $\Gamma(\mathrm{B}^{-n,k}) = \Gamma(\mathrm{B}^{n,-k})$, $\Gamma(\mathrm{BD}^{-n,k}) = \Gamma(\mathrm{BD}^{n,k})$ and $\Gamma(\mathrm{BC}^{-n,k}) = \Gamma(\mathrm{BC}^{n,k})$.

3.4. The even orthogonal case

In this subsection we suppose that $\alpha = s^{2n-1}$, $n \geq 1$. If s is generic, then we can construct the idempotent $\tilde{y}_\lambda$ for λ in the set

$$\bar{\Gamma}(\mathrm{D}_n) = \{\lambda; \lambda_1^\vee + \lambda_2^\vee \leq 2n+1\},$$

and λ has non-vanishing quantum dimension (see formula (9)) if it belongs to

$$\Gamma(\mathrm{B}_n) = \{\lambda; \lambda_1^\vee + \lambda_2^\vee \leq 2n\}.$$

We get that the category $\mathrm{K}^{\bar{\Gamma}(\mathrm{D}_n)}/Neg$ is semi-simple. A representative set of simple objects is the infinite set $\Gamma(\mathrm{D}_n)$.

We have the following specialized formula for the quantum dimension.

Proposition 3.7. *Let $\alpha = s^{2n-1}$, with s generic. For a partition $\lambda = (\lambda_1, ..., \lambda_n)$, we have*

$$\langle\lambda\rangle = \prod_{1\leq i<j\leq n} \frac{[2n+\lambda_i - i + \lambda_j - j][\lambda_i - i - \lambda_j + j]}{[2n-i-j][j-i]} \quad \textit{if } \lambda_n = 0;$$

$$\langle\lambda\rangle = 2\prod_{1\leq i<j\leq n} \frac{[2n+\lambda_i - i + \lambda_j - j][\lambda_i - i - \lambda_j + j]}{[2n-i-j][j-i]} \quad \textit{if } \lambda_n \neq 0.$$

Suppose that s^2 is a primitive lth root of unity with $l \geq 2n$. We discuss the equation $\langle \mathsf{K}+1\rangle = 0$, K minimal.

- If $l \geq 2n$ is even, then $\mathsf{K} = l - 2n + 2 = 2k$, $\alpha = s^{2n-1} = -s^{-2k+1}$; this will be the D_n - D_k specialization.
- If $l \geq 2n+1$ is odd and $s^l = 1$, then $\mathsf{K} = l - 2n + 2 = 2k+1$, $\alpha = s^{2n-1} = s^{-2k}$; this will be D_n - B_k specialization.
- If $l \geq 2n+1$ is odd and $s^l = -1$, then $\mathsf{K} = l - 2n + 2 = 2k+1$, $\alpha = s^{2n-1} = -s^{2k}$ will be the D_n - B_{-k} specialization.

$\mathbf{D}^{n,k}$ category. We consider the D_n - D_k specialization with $n,k \geq 1$. Let

$$\Gamma(\mathrm{D}^{n,k}) = \{\lambda; \lambda_1 + \lambda_2 \leq 2k, \lambda_1^\vee + \lambda_2^\vee \leq 2n\}.$$

We define the category $\mathrm{D}^{n,k}$ and prove pre-modularity as above. The dominating set of simple objects is here $\Gamma(\mathrm{D}^{n,k}) \cup \{1^{2n} \otimes 2k\}$.

3.5. The level-rank duality

As it was already mentioned, the Kauffman polynomial obtained with the parameters (α, s) and $(\alpha, -s^{-1})$ are equal. The corresponding BMW categories are also equal. From this we get an isomorphism between the constructed pre-modular categories. The image of a simple object λ is $\lambda^\vee$. In fact the categories are equal; only the labelling of simple objects has changed. This provides the "level-rank" duality isomorphism

between $\mathrm{C}^{n,k}$ and $\mathrm{C}^{k,n}$, $\mathrm{B}^{n,-k}$ and $\mathrm{B}^{-k,n}$, $\mathrm{D}^{n,k}$ and $\mathrm{D}^{k,n}$;
between $\mathrm{CB}^{n,k}$ and $\mathrm{BC}^{k,n}$, $\mathrm{BD}^{n,k}$ and $\mathrm{DB}^{k,n}$, $\mathrm{CB}^{n,-k}$ and $\mathrm{BC}^{-k,n}$, $\mathrm{BD}^{-n,k}$ and $\mathrm{DB}^{k,-n}$.

Here we use that $\Gamma(\mathrm{DB}^{k,n}) = \{\lambda; \lambda_1+\lambda_2 \le 2n+1; \lambda_1^\vee+\lambda_2^\vee \le 2k\}$. In conclusion, up to the level-rank duality, we have obtained the following seven series of pre-modular categories.

Theorem 3.8. *For $n,k \ge 1$ the categories $\mathrm{C}^{n,k}$, $\mathrm{CB}^{n,k}$, $\mathrm{CB}^{n,-k}$ $\mathrm{B}^{n,-k}$, $\mathrm{BD}^{n,k}$, $\mathrm{BD}^{-n,k}$ and $\mathrm{D}^{n,k}$ are pre-modular.*

4. Modularization of the completed BMW categories

In this section we discuss the modularization question for our series of pre-modular categories.

4.1. Transparent simple objects

Let us first note that $\langle\omega\rangle = \sum_{\mu\in\Gamma(\mathrm{A})} \langle\mu\rangle^2$ is nonzero if A is one of the pre-modular categories constructed in Section 3; the values of $\langle\omega\rangle$ are calculated e.g. in [8]. Therefore, the results of Section 1.2 can be applied.

Lemma 4.1. *i) There is no non-trivial transparent simple object in the category $\mathrm{C}^{n,k}$.*

ii) The non-trivial transparent simple objects are 1^{2n+1}, $2k+1$, $1^{2n+1}\otimes(2k+1)$ in $\mathrm{B}^{n,-k}$ category; $2k$, 1^{2n}, $1^{2n}\otimes 2k$ in $\mathrm{D}^{n,k}$ category; $2k$, 1^{2n+1}, $1^{2n+1}\otimes 2k$ in $\mathrm{BD}^{n,k}$ and $\mathrm{BD}^{-n,k}$ categories; $2k+1$ in $\mathrm{CB}^{n,k}$ and $\mathrm{CB}^{n,-k}$ categories.

The quantum dimensions of these objects are equal to one.

Corollary 4.2. *The category $\mathrm{C}^{n,k}$ with $\Gamma(\mathrm{C}^{n,k})$ as a representative set of simple objects is modular.*

Proof of the Lemma. Recall that a simple object λ is transparent if and only if for any (non-negligible) μ in the branching formula for λ, the braiding coefficient is equal to one. Then $i)$ follows.

For $ii)$ we verify that for each λ mentioned in the lemma all braiding coefficients are equal to one. Let us do it in the $\mathrm{B}^{n,-k}$ category for $\lambda = 2k+1$. Then only $\mu = 2k$ appears in the branching formula for λ. We have $\lambda - \mu = c$, $cn(c) = 2k$ and the braiding coefficient is $\alpha^{-2} s^{-4k} = s^{-4n-4k} = 1$. Other cases can be done analogously. We see that there is no other transparent simple object in these categories.

The quantum dimensions can be calculated directly using (10) and

$$\langle j \rangle = \frac{[0]_\alpha [1]_\alpha ... [j-2]_\alpha ([j-1]_\alpha + [j])}{[j]!} .$$

□

Lemma 4.3. *For pre-modular categories constructed in Section 3 the transparent simple objects form a group under tensor multiplication. This group is isomorphic to* $\mathbb{Z}_2 \times \mathbb{Z}_2$ *for* D, B, *and* BD *series and to* $\mathbb{Z}_2$ *for* CB *series.*

Proof. It is sufficient to show that the transparent simple objects have order 2, i.e. any non-trivial transparent simple object t satisfies the equation: $t \otimes t \approx$ trivial object. Clearly, $t \otimes t$ contains the trivial object and decomposes into a sum of transparent simple ones. Comparing the quantum dimensions on the left and right hand side of this decomposition formula we get the result. □

The twist coefficients of the transparent objects listed in Lemma 4.1 are equal to 1, except for the objects $(2k+1)$ and $1^{2n+1} \otimes (2k+1)$ in the $\mathrm{B}^{n,-k}$ category, 1^{2n+1} and $1^{2n+1} \otimes 2k$ in $\mathrm{BD}^{-n,k}$ category, and $(2k+1)$ in $\mathrm{CB}^{n,-k}$ category, whose twist coefficients are (-1). Applying Bruguières' criterion, we conclude.

Corollary 4.4. *The categories* $\mathrm{D}^{n,k}$, $\mathrm{BD}^{n,k}$, $\mathrm{CB}^{n,k}$ *are modularizable and* $\mathrm{B}^{n,-k}$, $\mathrm{BD}^{-n,k}$, $\mathrm{CB}^{n,-k}$ *are not modularizable.*

Remark. The non-modularizable categories provide invariants of closed framed 3-manifolds (see [15]). Here a framing is a trivialization of the tangent bundle up to isotopy. A choice of a framing is equivalent to the choice of a spin structure and a 2-framing (or p_1-structure) on the 3-manifold.

4.2. Modular categories $\widetilde{\mathbf{CB}}^{n,k}$, $\widetilde{\mathbf{BD}}^{n,k}$ and $\widetilde{\mathbf{D}}^{n,k}$.

Applying the modularization procedure described in Section 1 to the category $\mathrm{CB}^{n,k}$ we get the modular category $\widetilde{\mathrm{CB}}^{n,k}$ with the following representative set of simple objects

$$\Gamma(\widetilde{\mathrm{CB}}^{n,k}) = \{\lambda; \lambda_1 \le k, \lambda_1^\vee \le n\}.$$

The stabilizer subgroup for all elements of $\Gamma(\mathrm{CB}^{n,k})$ is here trivial.

In the $\mathrm{BD}^{n,k}$ case, a simple object λ with $\lambda_1 = k$ has $Stab(\lambda) = \mathbb{Z}_2$. The algebra $\mathrm{E}nd_{\widetilde{\mathrm{BD}}^{n,k}}(\lambda)$ is two-dimensional. It is generated by the tangle a_λ having one free vertex colored by $2k$. We normalize it such that $a_\lambda^2 = 1_\lambda$. The minimal idempotents of $\mathrm{E}nd_{\widetilde{\mathrm{BD}}^{n,k}}(\lambda)$ are $p_\lambda^\pm = 1/2(\mathbb{1}_\lambda \pm a_\lambda)$. We define the simple objects $\lambda_\pm$ by means of idempotents $\tilde{y}_\mu p_\lambda^\pm$. Their quantum dimensions are $\langle\lambda\rangle/2$. As a result,

$$\Gamma(\widetilde{\mathrm{BD}}^{n,k}) = \{\lambda; \lambda_1 < k, \lambda_1^\vee \leq n\} \cup \{\lambda_\pm; \lambda_1 = k, \lambda_1^\vee \leq n\}$$

is the representative set of simple objects for the modular category $\widetilde{\mathrm{BD}}^{n,k}$.

In the $\mathrm{D}^{n,k}$ case, the diagrams belonging to the set $\Gamma_1 = \{\lambda; \lambda_1 < k, \lambda_1^\vee < n\}$ have the trivial stabilizer. An object λ from $\Gamma_2 = \{\lambda; \lambda_1 = k, \lambda_1^\vee < n \ \bigwedge\ \lambda_1 < k, \lambda_1^\vee = n\}$ has the stabilizer equal to $\mathbb{Z}_2$. We decompose it into $\lambda_\pm$ analogously to the previous case. An object from $\Gamma_3 = \{\lambda; \lambda_1 = k, \lambda_1^\vee = n\}$ has the stabilizer of order 4. The algebra $\mathrm{E}nd_{\widetilde{\mathrm{D}}^{n,k}}(\lambda)$, $\lambda \in \Gamma_3$, is either abelian or isomorphic to the algebra of 2×2 matrices.

In the first case, λ will decompose into the direct sum of four non-isomorphic simple objects in the modular category $\widetilde{\mathrm{D}}^{n,k}$. In the second case λ will decompose into two isomorphic simple objects in $\widetilde{\mathrm{D}}^{n,k}$. It is a nontrivial open problem to decide which alternative holds for a given λ. The answer may differ for distinct λ. To any $\lambda \in \Gamma_3$ correspond $m_\lambda \in \{1, 4\}$ simple objects in $\Gamma(\widetilde{\mathrm{D}}^{n,k})$. If $m_\lambda = 1$, we denote the object by $\hat{\lambda}$; if $m_\lambda = 4$, we denote the objects by ${}_\pm\lambda_\pm$. Finally, the representative set of simple objects $\Gamma(\widetilde{\mathrm{D}}^{n,k})$ of the modular category $\widetilde{\mathrm{D}}^{n,k}$ is

$$\mathrm{D}_1 = \Gamma_1 \cup \{\lambda_\pm; \lambda \in \Gamma_2\} \cup \{{}_\pm\lambda_\pm; \lambda \in \Gamma_3,\ m_\lambda = 4\} \cup \{\hat{\lambda}; \lambda \in \Gamma_3,\ m_\lambda = 1\}\ .$$

5. Verlinde formulas

Recall that by Turaev's work any modular category $\widetilde{\mathrm{A}}$ with a set Γ of simple objects gives rise to a TQFT. The dimension of the TQFT module associated with a genus g closed surface is given by the Verlinde formula:

$$d_g(\widetilde{\mathrm{A}}) = \left(\sum_{\lambda\in\Gamma} \langle\lambda\rangle^2\right)^{g-1} \sum_{\lambda\in\Gamma} \langle\lambda\rangle^{2(1-g)}. \tag{11}$$

In this section we calculate the dimensions of TQFT modules arising from the modular categories constructed above.

Let us introduce the notation $[n]_s = s^n - s^{-n}$ for $n \in \mathbb{Z}$.

Theorem 5.1. *i) The genus g Verlinde formulas are*

$$d_g(\mathrm{C}^{n,k}) = (-(2n + 2k + 2))^{n(g-1)} \times$$

$$\times \sum_{n+k\geq l_1>\ldots>l_n>0} \left(\prod_{j=1}^{n} [2l_j]_s \prod_{1\leq i<j\leq n} [l_i+l_j]_s [l_i-l_j]_s \right)^{2(1-g)} ;$$

$$d_g(\widetilde{\mathrm{CB}}^{n,k}) = (-(2n+2k+1))^{n(g-1)} \times$$

$$\times \sum_{n+k\geq l_1>\ldots>l_n>0} \left(\prod_{j=1}^{n} [2l_j]_s \prod_{1\leq i<j\leq n} [l_i+l_j]_s [l_i-l_j]_s \right)^{2(1-g)} ;$$

$$\frac{d_g(\widetilde{\mathrm{BD}}^{n,k})}{(2n+2k-1)^{k(g-1)}} = 2 \sum_{n+k-1\geq \alpha_1>\ldots>\alpha_k>0} \left(\prod_{1\leq i<j\leq k} [\alpha_i+\alpha_j]_s [\alpha_i-\alpha_j]_s \right)^{2(1-g)}$$

$$+ \sum_{n+k-1\geq \alpha_1>\ldots>\alpha_k=0} \left(\prod_{1\leq i<j\leq k} [\alpha_i+\alpha_j]_s [\alpha_i-\alpha_j]_s \right)^{2(1-g)} .$$

$$\frac{d_g(\widetilde{\mathrm{D}}^{n,k})}{(2n+2k-2)^{n(g-1)}} = \sum_{n+k-2\geq l_1>\ldots>l_n=0} \prod_{1\leq i<j\leq n} \left([l_i+l_j]_s [l_i-l_j]_s\right)^{2(1-g)} +$$

$$+ \; 2^{2g-1} \sum_{n+k-1=l_1>\ldots>l_n=0} \prod_{1\leq i<j\leq n} \left([l_i+l_j]_s [l_i-l_j]_s\right)^{2(1-g)} +$$

$$+ \; 2 \sum_{n+k-2\geq l_1>\ldots>l_n>0} \prod_{1\leq i<j\leq n} \left([l_i+l_j]_s [l_i-l_j]_s\right)^{2(1-g)} +$$

$$+ \sum_{n+k-1=l_1>\ldots>l_n>0} (m_{(l-\delta)})^g \prod_{1\leq i<j\leq n} \left([l_i+l_j]_s [l_i-l_j]_s\right)^{2(1-g)} .$$

Here $\delta = (n, n-1, \ldots, 1)$.

ii) We have the following level-rank duality formulas.

$$d_g(\mathrm{C}^{n,k}) = d_g(\mathrm{C}^{k,n}) \qquad d_g(\widetilde{\mathrm{D}}^{n,k}) = d_g(\widetilde{\mathrm{D}}^{k,n})$$

Remark. The Verlinde formula for $\mathrm{C}^{1,2}$ calculates the number of the spin structures with Arf invariant zero on the surface of genus g: $d_g(\mathrm{C}^{1,2}) = 2^{g-1}(1+2^g)$. This fact should be interpreted via the corresponding TQFT, which is the is the one associated with the well known Rochlin invariant of spin 3-manifolds [6].

Proof. i) We substitute Propositions 3.1, 3.3, 3.7 and calculations of Sections 4.4-4.6 in [8] into (11).

Let us consider the $C^{n,k}$ case in details. Here $\alpha = s^{-2k-1}$. By Proposition 3.3 and the calculations of Section 4.5 in [8] we have

$$\sum_{\lambda\in\Gamma(C^{n,k})} \langle\lambda\rangle^2 = \frac{(-(2n+2k+2))^n}{\left(\prod_{j=1}^n [2n+2-2j]_s \prod_{1\le i<j\le n}[2n+2-i-j]_s[j-i]_s\right)^2}.$$

Furthermore,

$$\sum_{\lambda\in\Gamma(C^{n,k})} \langle\lambda\rangle^{2(1-g)}$$

$$= \frac{\sum_{n+k\ge l_1>\ldots>l_n>0} \left(\prod_{j=1}^n [2l_j]_s \prod_{1\le i<j\le n} [l_i+l_j]_s[l_i-l_j]_s\right)^{2(1-g)}}{\left(\prod_{j=1}^n [2n+2-2j]_s \prod_{1\le i<j\le n} [2n+2-i-j]_s[j-i]_s\right)^{2(1-g)}}.$$

Here we used the bijection $\Gamma(C^{n,k}) \to T := \{(l_1,\ldots,l_n), n+k\ge l_1>\ldots>l_n>0\}$ sending λ to $\lambda+(n,n-1,\ldots,1)$. Substituting the last two formulas into (11) we get the result.

For the third formula we use that

$$\sum_{\lambda\in\Gamma(BD^{n,k})} \langle\lambda\rangle^2 = 4 \sum_{\lambda\in\Gamma(\widetilde{BD}^{n,k})} \langle\lambda\rangle^2.$$

This is because the action of the group $\mathbb{Z}_2\times\mathbb{Z}_2$ of the transparent objects on $\{\lambda;\lambda_1<k,\lambda_1^\vee\le n\}$ preserves the quantum dimension and $\langle\lambda_\pm\rangle = 1/2\langle\lambda\rangle$.

ii) By (8) we have for any $p\in\mathbb{Z}$

$$\sum_{\substack{\lambda_1\le k\\ \lambda_1^\vee\le n}} \langle\lambda\rangle^p_{-s^{2n+1},s} = \sum_{\substack{\lambda_1\le k\\ \lambda_1^\vee\le n}} \langle\lambda^\vee\rangle^p_{-s^{2n+1},-s^{-1}} = \sum_{\substack{\lambda_1^\vee\le k\\ \lambda_1\le n}} \langle\lambda\rangle^p_{-s^{2k+1},s}.$$

The second formula can be shown analogously.

□

6. Refinements

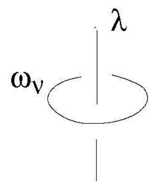
In this section we construct spin and cohomological refinements of the quantum invariants arising from the modular category $C^{n,k}$.

We work here in C^n - C^k specialization, i.e. $\alpha = s^{-2k-1}$, s is a primitive $2l$ th root of unity, $l = 2n+2k+2$. Recall $\Gamma(C^{n,k}) = \{\lambda;\lambda_1\le k,\lambda_1^\vee\le n\}$. Let us introduce a $\mathbb{Z}_2$-grading on the category $\mathrm{C}^{n,k}$ corresponding to the parity of the number of cells in Young diagrams. According to this grading, we decompose the Kirby element: $\omega = \omega_0 + \omega_1$.

Lemma 6.1. *Let $U_\varepsilon(\lambda)$ be the ε-framed unknot colored with λ and $\varepsilon = \pm 1$.*
i) For $kn = 2 \mod 4$, we have $\langle U_\varepsilon(\omega_0)\rangle = 0$.
ii) For $kn = 0 \mod 4$, we have $\langle U_\varepsilon(\omega_1)\rangle = 0$.

Proof. Let us call the *graded sliding property* the equality drawn in Proposition 1.1 by replacing ω on the left-hand side by ω_ν and ω on the right-hand side by $\omega_{\nu+1}$ with $\nu = 0, 1$. The proof of this identity can be adapted from the one of this proposition.

Using twice the graded sliding property, we can see that the morphism drawn below is nonzero only if $\lambda = 0$ or $\lambda = k^n$.



Then

$$\langle U_1(\omega_\nu)\rangle\langle U_{-1}(\omega_\nu)\rangle = \langle H_{1,0}(\omega_0,\omega_\nu)\rangle = (1+\alpha^{kn}s^{nk(k-n)}s^{l\nu})\langle\omega_\nu\rangle \tag{12}$$

where $H_{1,0}(\omega_0,\omega_\nu)$ is the Hopf link whose ω_0-colored component has framing 1 and ω_ν-colored one is 0-framed. The first equality is due to the graded sliding property. In the second one we use the twist and braiding coefficients for $\lambda = 0, k^n$ and the fact that

$$\sum_{c\in k^n} cn(c) = \frac{nk}{2}(k-n) .$$

Substituting the values of α and l into (12) we get the result. □

The following statement is the direct consequence of this lemma and the construction of refined invariants described in [4, Section 4].

Theorem 6.2. *The quantum invariants arising from the modular category $\mathrm{C}^{n,k}$ can be written as sums of refined invariants corresponding to different spin structures if $kn = 2 \mod 4$ and to $\mathbb{Z}_2$-cohomology classes if $kn = 0 \mod 4$.*

One can show by the same method that other categories do not provide refined invariants.

7. Comparison with the quantum group approach

The aim of this section is to find a correspondence between pre-modular categories that have been constructed in Section 3 and those that arise from the quantum group method.

7.1. Modular categories from quantum groups

We keep notation of [10], [11]. Let $(a_{ij})_{1\leq i,j\leq l}$ be the Cartan matrix of a simple complex Lie algebra $\mathfrak{g}$. There are relatively prime integers $d_1,...,d_l$ in $\{1,2,3\}$ such that the matrix $(d_i a_{ij})$ is symmetric. Let $d=max(d_i)$. We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and fundamental roots $\alpha_1,\alpha_2,...,\alpha_l$ in the dual space $\mathfrak{h}^*$. Let $\mathfrak{h}^*_{\mathbb{R}}$ be the $\mathbb{R}$-vector space spanned by the fundamental roots. The root lattice Y is the $\mathbb{Z}$-lattice generated by α_i, $i=1,...,l$. We define an inner product on $\mathfrak{h}^*_{\mathbb{R}}$ by $(\alpha_i|\alpha_j)=d_i a_{ij}$. Then $(\alpha|\alpha)=2$ for every short root α. The inner product normalized such that every long root has length two will be denoted by $(.|.)'$. We have $(.|.)'=(.|.)/d$. Let $\lambda_1,...,\lambda_l$ be the fundamental weights, then $(\lambda_i|\alpha_j)=d_i\delta_{ij}$. The weight lattice X is the $\mathbb{Z}$-lattice generated by $\lambda_1,...,\lambda_l$. Let $\rho=\lambda_1+...+\lambda_l$. The Weyl chamber is defined by $\mathrm{C}=\{x\in\mathfrak{h}^*_{\mathbb{R}};(x|\alpha_i)\geq 0, i=1,...,l\}$. Let us denote by α_0 (resp. β_0) the short (resp. the long) root in the Weyl chamber C.

Let $U_q(\mathfrak{g})$ be the quantum group associated with $\mathfrak{g}$ and q be a primitive root of unity of order r (notation coincides with [11, Section 1]). Let $h^\vee$ be the dual Coxeter number. The case when $r\geq dh^\vee$ is divisible by d was mainly studied in the literature. In that case, simple $U_q(\mathfrak{g})$-modules corresponding to weights in

$$\mathrm{C}_{\mathrm{L}}=\{x\in\mathrm{C};(x|\beta_0)'\leq\mathrm{L}\}$$

form a pre-modular category [9]. Here $\mathrm{L}:=r/d-h^\vee$ is the level of the category. The quantum dimension of $\mu\in X$ is given by

$$\langle\mu\rangle=\prod_{\text{positive roots }\alpha}\frac{v^{(\mu+\rho|\alpha)}-v^{-(\mu+\rho|\alpha)}}{v^{(\rho|\alpha)}-v^{-(\rho|\alpha)}},\tag{13}$$

its twist coefficient is $v^{(\mu+2\rho|\mu)}$, where $v^2=q$. The modularization of these categories was studied in [16].

In the case when $(r,d)=1$ and $r>h$ (h is the Coxeter number), pre-modular categories can also be constructed [10]. The set of simple objects corresponds to weights in

$$\mathrm{C}'_{\mathrm{L}}=\{x\in\mathrm{C};(x|\alpha_0)\leq\mathrm{L}\}$$

with $\mathrm{L} := r - h$. Le showed that if $(r, d\ \det(a_{ij})) = 1$ the set of modules in $\mathrm{C}'_{\mathrm{L}} \cap Y$ generates a modular category.

We say that two pre-modular categories are equivalent if there exists a bijection between their sets of simple objects providing an equality of the corresponding colored link invariants. For modularizable categories this implies that the associated TQFT's are isomorphic (compare [18, III,3.3]).

7.2. Comparison of C cases

Any weight $\mu \in \mathrm{C}$ of C_n is of the form $\mu = \lambda_1 e_1 + ... + \lambda_n e_n$ with $(e_i|e_j) = \delta_{ij}$ and integers $\lambda_1 \geq ... \geq \lambda_n \geq 0$ (compare [14, p.293]). With any μ a Young diagram $\lambda = (\lambda_1, ..., \lambda_n)$ can be associated.

Theorem 7.1. *The pre-modular categories associated with* $U_q(\mathrm{C}_n)$ *and* $U_q(\mathrm{C}_k)$ *at a primitive root of unity* q *of order* $r = 2n + 2k + 2$ *are equivalent to* $\mathrm{C}^{n,k}$.

Proof. A colored m-component link invariant of a pre-modular category A with $\Gamma(\mathrm{A})$ as a representative set of simple objects can be considered as a multilinear function from $\Gamma(\mathrm{A})^m$ to $\mathbf{k}$. Here we supply $\Gamma(\mathrm{A})$ with a ring structure by considering direct sums and tensor products. It is easy to see from the previous discussion, that there exists an isomorphism between such rings in our case. Indeed, for $U_q(\mathrm{C}_n)$ we have $d = 2$, $h^\vee = n+1$, $\mathrm{L} = r/2 - n - 1 = k$, $\beta_0 = 2e_1$, and $\mathrm{C}_{\mathrm{L}} \cap X = \{\mu; \lambda_1 \leq k\}$. After the identification of μ with λ, this coincides with $\Gamma(\mathrm{C}^{n,k})$. The ring structure is preserved under this identification. Furthermore, it is known that these rings are generated by the fundamental module corresponding to $\mu = e_1$ and the object $\lambda = 1$. Therefore, it is sufficient to verify the equality of invariants colored by these two objects. The fact, that the link invariant associated with this fundamental module is a specialization of the Kauffman polynomial was shown in [17]. In order to identify the parameters, compare the quantum dimensions of simple objects given by (13) and Proposition 3.1. We show that $s^2 = q$. The equivalence between $U_q(\mathrm{C}_k)$ and $\mathrm{C}^{k,n}$ can be shown analogously. Then we use the level-rank duality. □

Analogously, the category $\mathrm{CB}^{n,k}$ is equivalent to $U_q(\mathrm{C}_n)$ with $r = l = 2n + 2k + 1$. Indeed, we have $(r, d) = 1$, $h = 2n$, $\mathrm{L} = r - 2n = 2k + 1$, $\alpha_0 = e_1 + e_2$, and $\mathrm{C}'_{\mathrm{L}} \cap X = \{\mu; \lambda_1 + \lambda_2 \leq 2k + 1\}$.

7.3. Comparison of B cases

Any weight $\mu \in \mathrm{C}$ of B_n can be written in the form $\mu = \lambda_1 e_1 + ... + \lambda_n e_n$, where $(e_i|e_j) = 2\delta_{ij}$ and half-integers $\lambda_1 \geq ... \geq \lambda_n \geq 0$. If $\lambda_i \in \mathbb{N}$, $i = 1, ..., n$, the partition $\lambda = (\lambda_1, ..., \lambda_n)$ defines a Young diagram associated with μ . If at least one of λ_i is a half-integer, we call μ a spin module. Our construction of simple objects can be considered as a quantum analog of the Weyl construction and it does not produce spin modules. Let us compare the quantum dimension and/or twist coefficient of a non-spin module μ and the corresponding Young diagram λ given by (13) and Proposition 3.3. They coincide if $v^2 = q = s$.

Let us first consider the case when r is even and $r > 4n$. Here $h^\vee = 2n - 1$. Let $r = 4n + 4k$ with $k \geq 1$. We have $\beta_0 = e_1 + e_2$ and $\mathrm{L} = r/2 - 2n + 1$. Then

$$\mathrm{C_L} \cap X = \{\mu; \lambda_1 + \lambda_2 \leq 2k + 1\}.$$

We conclude that the quotient by spin modules of the pre-modular category for B_n at $(4n + 4k)$ th root of unity is equivalent to the modular extension of $\mathrm{B}^{n,-k}$ by G' generated by 1^{2n+1} . Using the level-rank duality, we get the equivalence of the pre-modular category for B_k at $(4n+4k)$ th root of unity with the modular extension of $\mathrm{B}^{n,-k}$ by G'' generated by $2k + 1$.

Let us put $r = 4n + 4k - 2$ with $k \geq 1$. Then we get analogously that the category $\mathrm{BD}^{n,k}$ is equivalent to the quotient (by spin modules) of the pre-modular category for B_n at $(4n + 4k - 2)$ th root of unity.

For odd $r > h = 2n$ we set $r = 2n + 2k + 1$ with $k \geq 1$. Then $(r, d) = 1$, $\alpha_0 = e_1$ and $\mathrm{L} = r - 2n = 2k + 1$. We have $\mathrm{C}'_\mathrm{L} \cap X = \{\mu; \lambda_1 \leq k + 1/2\}$. We see that the quotient by spin modules of the pre-modular category for B_n on $(2n + 2k + 1)$ th root of unity is equivalent to $\widetilde{\mathrm{BC}}^{n,k}$.

7.4. Comparison of D cases

Any weight of D_n can be written in the form $\mu_\pm = \lambda_1 e_1 + ... + \lambda_{n-1} e_{n-1} \pm \lambda_n e_n$ with $(e_i|e_j) = \delta_{ij}$ and half-integers $\lambda_1 \geq ... \geq \lambda_n \geq 0$. Here we have $v = s$, $d = 1$, $h = 2n - 2$ and $\beta_0 = e_1 + e_2$. Setting $r = 2n + 2k - 2$ with $k \geq 1$ we get

$$\mathrm{C_L} \cap X = \{\mu_\pm; \lambda_1 + \lambda_2 \leq 2k\}$$

For non-spin modules, this coincides with the set of simple objects of the modular extension of $\mathrm{D}^{n,k}$ by G' generated by 1^{2n} . Therefore, the modular categories for D_n and D_k at $(2n + 2k - 2)$ th root of unity are equivalent to $\widetilde{\mathrm{D}}^{n,k}$.

For $r = 2n + 2k - 1$ ($k \geq 1$) we get that the modular extension of $\mathrm{DB}^{n,k}$ (isomorphic to $\mathrm{BD}^{k,n}$) by G' as above is equivalent to the quotient by spin modules of the pre-modular category for D_n .

As a result, any pre-modular category defined in Section 3 is equivalent to a quantum group category. Moreover, our categories produce a complete set of 3-manifold invariants that can be obtained from quantum groups of types B,C and D by using non-spin modules.

Acknowledgements

The first and the second authors thank, respectively, the Laboratoire de Mathématiques de l'Université de Bretagne-Sud in Vannes and the Mathematisches Institut der Universität Basel for their hospitality. The authors also wish to thank Alain Bruguières and Thang Le for explaining algebraic structures relevant to their constructions and for useful remarks on the preliminary version of this paper. This work was supported in part by the Sonderprogramm zur Förderung des akademischen Nachwuchses der Universität Basel.

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(Received: July 1, 2000)