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Complexity of degenerations of modules

R. Aehle, Ch. Riedtmann and G. Zwara

Abstract. A module M over an associative algebra A over an algebraically closed field k is said to degenerate to a module N if N belongs to the closure of the isomorphism class of M in the algebraic variety of d -dimensional A -modules, $d \in \mathbb{N}$. We associate a non-negative integer to a degeneration $M \leq_{\text{deg}} N$, its complexity, and study its properties.

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1. Introduction

Let k be an algebraically closed field, A a finite dimensional associative k -algebra with a unit and $\text{mod } A$ the category of finite dimensional left A -modules. Let $\mathbb{M}_d(k)$ denote the k -algebra of the $d \times d$ -matrices with coefficients in k . We view A as a quotient of a free associative algebra $k\langle X_1, \dots, X_r \rangle$ by a two-sided ideal I . We define the affine variety $\text{mod}_A^d(k)$ as the set of r -tuples (m_1, \dots, m_r) such that $m_i \in \mathbb{M}_d(k)$ and $\rho(m_1, \dots, m_r)$ is the zero matrix for any $\rho \in I$. The general linear group $Gl_d(k)$ acts on $\text{mod}_A^d(k)$ by conjugation.

As an ordinary set, $\text{mod}_A^d(k)$ is just the set $\text{Hom}_{k\text{-alg}}(A, \mathbb{M}_d(k))$ and hence $Gl_d(k)$ -orbits in $\text{mod}_A^d(k)$ correspond bijectively to isomorphism classes of d -dimensional left A -modules.

Let M and N be two d -dimensional A -modules. By definition, M degenerates to N , noted $M \leq_{\text{deg}} N$, if N lies in the closure of the $Gl_d(k)$ -orbit of M in $\text{mod}_A^d(k)$, with respect to the Zariski topology. This defines a partial order on the set of isomorphism classes of d -dimensional A -modules.

Denote by Q the quiver

$$Q = 1 \begin{array}{c} \xrightarrow{a_1} \\ \xleftarrow{b_1} \end{array} 2 \begin{array}{c} \xrightarrow{a_2} \\ \xleftarrow{b_2} \end{array} 3 \dots$$

with vertex set $Q_0 = \mathbb{N} \setminus \{0\}$ and arrows $a_i : i \rightarrow i + 1$, $b_i : i + 1 \rightarrow i$ for every $i \in Q_0$.

We call a representation

$$T = N_1 \begin{matrix} \xrightarrow{\alpha_1} \\ \xleftarrow{\beta_1} \end{matrix} N_2 \cdots N_i \begin{matrix} \xrightarrow{\alpha_i} \\ \xleftarrow{\beta_i} \end{matrix} N_{i+1} \cdots$$

of Q in $\text{mod}A$, the category of finite dimensional A -modules, an *exact tube* if the sequence

$$0 \rightarrow N_i \xrightarrow{\begin{pmatrix} \beta_{i-1} \\ \alpha_i \end{pmatrix}} N_{i-1} \oplus N_{i+1} \xrightarrow{(-\alpha_{i-1}, \beta_i)} N_i \rightarrow 0$$

or equivalently the square

$$\begin{array}{ccc} N_i & \xrightarrow{\alpha_i} & N_{i+1} \\ \downarrow \beta_{i-1} & & \downarrow \beta_i \\ N_{i-1} & \xrightarrow{\alpha_{i-1}} & N_i \end{array}$$

is exact for all $i \geq 1$. Here we set $N_0 = 0$. Note that N_i is an A -module, that α_i, β_i are A -linear and that α_i is injective, β_i is surjective, for all $i \geq 1$. We say that T is an (M, N) -tube if there is a natural number h such that

- (i) $N_1 \xrightarrow[A]{\sim} N$,
- (ii) $N_{h+j+1} \xrightarrow[A]{\sim} N_{h+j} \oplus M$, for all $j \in \mathbb{N}$.

We call the smallest such number h the complexity $\text{cpl}(T)$ of the tube.

Let T be an (M, N) -tube. Note that the sequence

$$0 \rightarrow N_k \xrightarrow{\alpha_k} N_{k+1} \xrightarrow{\beta_1 \cdots \beta_k} N_1 \rightarrow 0$$

is exact for any k . As N_{k+1} is isomorphic to $N_k \oplus M$ for $k \geq \text{cpl}(T)$, there is an exact sequence

$$0 \rightarrow N_k \rightarrow N_k \oplus M \rightarrow N \rightarrow 0,$$

and therefore M degenerates to N [5].

Conversely, whenever M degenerates to N , there exists an (M, N) -tube: Indeed, the third author showed in [7] that there is a short exact sequence

$$0 \rightarrow Z \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} Z \oplus M \rightarrow N \rightarrow 0, \tag{1.1}$$

and in [6] he associated an exact tube $T_{f,g}$ with such a sequence (see also Section 4). In fact, $T_{f,g}$ is the cokernel of the injection $\varphi : X \rightarrow X'$ between the following representations of Q :

$$\begin{array}{ccccccc} X : & & Z & \xleftrightarrow{f} & Z & \xleftrightarrow{f} & \cdots & \xleftrightarrow{f} & Z & \xleftrightarrow{f} & Z & \cdots \\ \varphi \downarrow & & \varphi_1 \downarrow & & \varphi_2 \downarrow & & & & \varphi_i \downarrow & & \varphi_{i+1} \downarrow & \\ X' : & & Z \oplus M & \xleftrightarrow{\begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix}} & Z \oplus M^2 & \xleftrightarrow{\begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix}} & \cdots & \xleftrightarrow{\begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix}} & Z \oplus M^i & \xleftrightarrow{\begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix}} & Z \oplus M^{i+1} \cdots \end{array}$$

with $\varphi_i = (f^i, gf^{i-1}, \dots, g)^t : Z \rightarrow Z \oplus M^i$. Both X and X' are almost exact tubes: they satisfy all requirements except those related to a_1 and b_1 . The only condition left to be checked for $T_{f,g}$ is the exactness of the bottom row in the commutative diagram (Figure 1) with exact columns. This is done by diagram chasing.

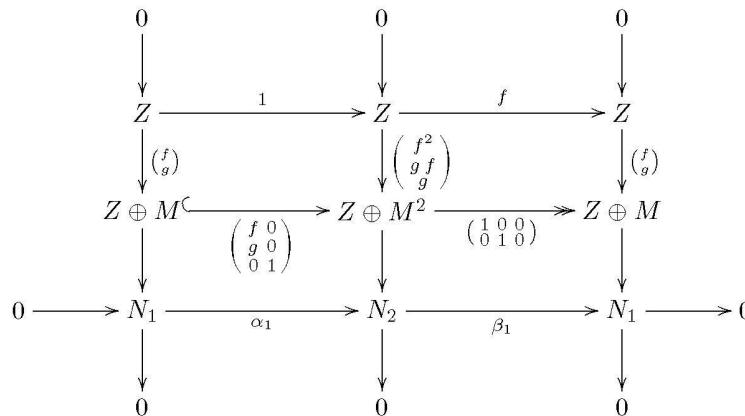


Figure 1

By construction, $N_1 = \text{coker } \varphi_1$ is isomorphic to N . Using Fitting’s lemma in order to replace Z by a direct summand if necessary in the exact sequence (1.1), we may assume that f is nilpotent, say $f^h = 0$. Then φ_{h+j} has the form $\varphi_{h+j} = (0, \dots, 0, gf^{h-1}, \dots, g)^t : Z \rightarrow Z \oplus M^{h+j}$, and its cokernel N_{h+j} is isomorphic to $M^j \oplus N_h$ for $j \geq 0$. We conclude that $T_{f,g}$ is an (M, N) -tube of complexity at most h . In fact, $T_{f,g}$ is an (M, N) -tube even if f is not nilpotent (compare with Proposition 4.2).

We define the *complexity* of a degeneration $M \leq_{\text{deg}} N$ to be

$$\text{cpl}(M, N) = \min \text{cpl}(T),$$

where T ranges over all (M, N) -tubes. This seems to be a good way to measure how “complicated” a degeneration is.

Indeed, we will prove in Sections 3 and 4 that a degeneration $M \leq_{\text{deg}} N$ is of complexity 1 if and only if there exists a non-split exact sequence

$$0 \rightarrow N' \rightarrow M \rightarrow N'' \rightarrow 0$$

with $N \xrightarrow{\sim} N' \oplus N''$. So these are the “simplest” degenerations. In particular, any degeneration to an indecomposable N must have complexity at least 2.

It is quite difficult to compute the complexity of a degeneration. The construc-

tion described before gives an estimate from above: if

$$0 \rightarrow Z \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} Z \oplus M \rightarrow N \rightarrow 0$$

is an exact sequence and $f^h = 0$, then $\text{cpl}(M, N) \leq h$. Conversely, it is easy to show that

$$\text{cpl}(M, N) \geq \frac{\ell\ell(M)}{\ell\ell(N)} - 1,$$

where $\ell\ell(X)$ is the Loewy length of X ; i.e., the smallest number r for which $(\text{rad } A)^r \cdot X = 0$ (see Proposition 3.5). Both bounds are sharp, but in general the complexity differs from both.

The complexity of a degeneration $M \leq_{\text{deg}} N$ obtained from two degenerations $M \leq_{\text{deg}} P \leq_{\text{deg}} N$ seems to be quite unrelated to the sum of the complexities of $M \leq_{\text{deg}} P$ and $P \leq_{\text{deg}} N$. For instance, if we take non-split exact sequences

$$0 \rightarrow A_i \rightarrow B_i \rightarrow C_i \rightarrow 0, \quad i = 1, \dots, r,$$

then there is a sequence of degenerations

$$\begin{aligned} \bigoplus_{i=1}^r B_i &\leq_{\text{deg}} \left(\bigoplus_{i=1}^{r-1} B_i \right) \oplus A_r \oplus C_r \leq_{\text{deg}} \dots \leq_{\text{deg}} \left(\bigoplus_{i=1}^s B_i \right) \oplus \bigoplus_{i=s+1}^r (A_i \oplus C_i) \\ &\leq_{\text{deg}} \dots \leq_{\text{deg}} \bigoplus_{i=1}^r (A_i \oplus C_i), \end{aligned}$$

but the complexity of

$$\bigoplus_{i=1}^r B_i \leq_{\text{deg}} \bigoplus_{i=1}^r (A_i \oplus C_i)$$

is 1. On the other hand, we give an example of a chain of degenerations $M \leq_{\text{deg}} P \leq_{\text{deg}} N$ in Section 5.1 for which $\text{cpl}(M, P) + \text{cpl}(P, N) < \text{cpl}(M, N)$. By Proposition 5.1, a minimal degeneration can have arbitrarily high complexity. A degeneration $M \leq_{\text{deg}} N$ is called minimal if M is not isomorphic to N and moreover $M \leq_{\text{deg}} P \leq_{\text{deg}} N$ implies that P is isomorphic to either M or N .

2. Degenerations, bimodules and exact tubes

The following construction is explained in detail in [7] (compare also [2] and [3], pp. 176–177): If $M \leq_{\text{deg}} N$ is a degeneration, there exists a discrete valuation k -algebra R with maximal ideal \mathfrak{m} and residue class field k and an A - R -bimodule \mathcal{Y} , which is free of rank d over R , such that

- i) $\mathcal{Y}/\mathfrak{m} \cdot \mathcal{Y} \xrightarrow{\sim}_A N$
- ii) \mathcal{Y} contains $R \otimes_k M$ as an A - R -submodule.

These data are related to mapping a curve c to $\text{mod}_A^d(k)$ in such a way that its image lies generically in the orbit of M and intersects the orbit of N . Assuming c to be non-singular and passing to the completion, we may assume that $R = k[[t]]$. The representation $T = (N_i, \alpha_i, \beta_i)$ defined by the setting

$$N_i = \mathcal{Y}/(t^i) \cdot \mathcal{Y}$$

and letting $\alpha_i : N_i \rightarrow N_{i+1}$ and $\beta_i : N_{i+1} \rightarrow N_i$ be induced by multiplication by t and the identity, respectively, is easily seen to be an exact tube, and by [7] it is moreover an (M, N) -tube.

This construction associating an exact tube with a bimodule is an equivalence:

Proposition 2.1. *The category \mathcal{T} of exact tubes is equivalent to the category $\text{mod } A\text{-}k[[t]]$ of $A\text{-}k[[t]]$ -bimodules which are free of finite rank over $k[[t]]$.*

Proof. We just describe a quasi-inverse functor. For an exact tube $T = (N_i, \alpha_i, \beta_i)$ we set

$$\mathcal{Y} = \varprojlim (N_i, \beta_i),$$

and we put

$$t \cdot (n_1, n_2, \dots) = (0, \alpha_1(n_1), \alpha_2(n_2), \dots)$$

for any infinite sequence (n_1, n_2, \dots) with $n_i \in N_i$ and $\beta_i(n_i) = n_{i-1}$ representing an element of \mathcal{Y} . As T is an exact tube, this defines an $A\text{-}k[[t]]$ -bimodule structure on \mathcal{Y} . As t acts without torsion, \mathcal{Y} is free as a $k[[t]]$ -module, and its rank equals $\dim_k N_1$, since clearly $\mathcal{Y}/(t) \cdot \mathcal{Y}$ is isomorphic to N_1 . \square

We give a direct construction of the bimodule corresponding to $T_{f,g}$ for an exact sequence

$$0 \rightarrow Z \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} Z \oplus M \rightarrow N \rightarrow 0 \tag{2.1}$$

with a nilpotent map f . Set

$$\mathcal{Y}_{f,g} = k[[t]] \otimes_k M \oplus Z$$

as an A -module and define the action of t on Z by

$$t \cdot (0, z) = (1 \otimes g(z), f(z)).$$

Clearly, this action of t is torsion free, and $\mathcal{Y}_{f,g}/(t)\mathcal{Y}_{f,g}$ is isomorphic to N , so that $\mathcal{Y}_{f,g}$ actually belongs to $\text{mod } A\text{-}k[[t]]$. It is easy to see that the exact tube associated with $\mathcal{Y}_{f,g}$ is $T_{f,g}$.

We will need the following truncated version of an exact tube:

Definition 2.2. For $m \geq 1$, an exact tube of height m is a representation in $\text{mod } A$

$$N_1 \begin{array}{c} \xrightarrow{\alpha_1} \\ \xleftarrow{\beta_1} \end{array} N_2 \dots N_{m-1} \begin{array}{c} \xrightarrow{\alpha_{m-1}} \\ \xleftarrow{\beta_{m-1}} \end{array} N_m$$

of the full subquiver Q_m of Q whose vertices are $1, 2, \dots, m$, such that the square

$$\begin{array}{ccc} N_i & \xrightarrow{\alpha_i} & N_{i+1} \\ \downarrow \beta_{i-1} & & \downarrow \beta_i \\ N_{i-1} & \xrightarrow{\alpha_{i-1}} & N_i \end{array}$$

is exact for $i = 1, \dots, m - 1$. Again we set $N_0 = 0$.

The category of exact tubes of height m is equivalent to the category of A - $k[t]/(t^m)$ -bimodules which are free of finite rank over $k[t]/(t^m)$.

Obviously, an exact tube T restricts to an exact tube $T_{\leq m}$ of height m for all m . We will see in Section 4 that an M -extendible tube $T = (N_i, \alpha_i, \beta_i)$ of height $h \geq 1$ (see next definition) is always the restriction of an (M, N_1) -tube.

Definition 2.3. A tube $T = (N_i, \alpha_i, \beta_i)$ of height h is called M -extendible if there is a decomposition $N_h = Z \oplus Z'$ and an exact sequence

$$0 \rightarrow Z \xrightarrow{\begin{pmatrix} a \\ b \end{pmatrix}} N_{h-1} \oplus M \xrightarrow{(c \ d)} Z' \rightarrow 0$$

such that $a = \beta_{h-1}|_Z$ and $c = pr_{Z'} \circ \alpha_{h-1}$, where $pr_{Z'} : Z \oplus Z' \rightarrow Z'$ is the natural projection.

We end this section with some questions. We do not know how to describe the full subcategory of $\text{mod } A\text{-}k[[t]]$ corresponding to (M, N) -tubes. Conceivably, its objects are just those bimodules \mathcal{Y} which contain $k[[t]] \otimes_k M$ as a subbimodule. This would follow if we knew that any (M, N) -tube is of the form $\mathcal{Y}_{f,g}$ for some exact sequence (2.1).

3. Complexity

Definition 3.1. We call a map

$$\begin{pmatrix} f \\ g \end{pmatrix} : Z \rightarrow Z \oplus M$$

an (M, N) -monomorphism provided N is isomorphic to $\text{coker} \begin{pmatrix} f \\ g \end{pmatrix}$.

Recall that, for a degeneration $M \leq_{\text{deg}} N$, we defined the complexity as

$$\text{cpl}(M, N) = \min \text{cpl}(T),$$

where T ranges over all (M, N) -tubes. There always are (M, N) -tubes with different complexities. For instance, if $(f, g)^t : Z \rightarrow Z \oplus M$ is an (M, N) -monomorphism and we set

$$f' = \begin{pmatrix} 0 & 1 \\ f & 0 \end{pmatrix} : Z^2 \longrightarrow Z^2, \quad g' = (g \ 0) : Z^2 \longrightarrow M,$$

the map $(f', g')^t$ will be an (M, N) -monomorphism, too, and it is easy to see that

$$\text{cpl}(T_{f',g'}) = 2 \text{cpl}(T_{f,g}).$$

Theorem 3.2. *Let $h \geq 1$ be a natural number and $M \leq_{\text{deg}} N$ a degeneration. The following conditions are equivalent:*

- (i) $\text{cpl}(M, N) \leq h$
- (ii) *There is an exact sequence*

$$0 \longrightarrow Z \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} Z \oplus M \longrightarrow N \longrightarrow 0$$

such that $\text{cpl}(T_{f,g}) \leq h$.

- (iii) *There exists an exact tube $T = (N_i, \alpha_i, \beta_i)$ of height $2h + 1$ with $N \xrightarrow[A]{\sim} N_1$ and such that*

$$N_{h+j+1} \xrightarrow[A]{\sim} N_{h+j} \oplus M$$

for $j = 0, \dots, h$.

- (iv) *There exists an M -extendible exact tube $T = (N_i, \alpha_i, \beta_i)$ of height h with $N \xrightarrow[A]{\sim} N_1$.*

Proof. Most ingredients for the proof will be given in Section 4. Here we indicate how they fit together: The implications (ii) \Rightarrow (i) \Rightarrow (iii) are obvious. The results of Section 4 up to Proposition 4.6 give that (ii) implies (iv), and Proposition 4.8 shows (iv) \Rightarrow (ii). Finally, the implication (iii) \Rightarrow (ii) follows from Proposition 4.9 and the next lemma. □

Lemma 3.3. *Let $T = (N_i, \alpha_i, \beta_i)$ be an (M, N) -tube, and assume that $N_{h+1} \xrightarrow{\sim} N_h \oplus M$ for some $h \geq 1$. Then $\text{cpl}(T) \leq h$.*

Proof. As T is an (M, N) -tube, there exists a natural number $j \geq h$ such that $N_{i+1} \xrightarrow{\sim} N_i \oplus M$ for all $i \geq j$. Take an integer i with $h < i < j$, and consider the

two exact squares

$$\begin{array}{ccccccc}
 N_h \oplus M & \xrightarrow{\sim} & N_{h+1} & \xrightarrow{\alpha_i \dots \alpha_{h+1}} & N_{i+1} & \xrightarrow{\alpha_j \dots \alpha_{i+1}} & N_{j+1} \xrightarrow{\sim} N_j \oplus M \\
 & & \downarrow \beta_h & & \downarrow \beta_i & & \downarrow \beta_j \\
 & & N_h & \xrightarrow{\alpha_{i-1} \dots \alpha_h} & N_i & \xrightarrow{\alpha_{j-1} \dots \alpha_i} & N_j.
 \end{array}$$

The big square splits, and therefore the two small squares split as well. We conclude that N_{i+1} is isomorphic to $N_i \oplus M$. □

As $N_0 = 0$, our theorem takes the following simpler form for $h = 1, 2$:

Corollary 3.4. *Let $M \leq_{\text{deg}} N$ be a degeneration. Then*

i) $\text{cpl}(M, N) \leq 1$ if and only if $N = Z \oplus Z'$ and there exists an exact sequence

$$0 \rightarrow Z \rightarrow M \rightarrow Z' \rightarrow 0.$$

ii) $\text{cpl}(M, N) \leq 2$ if and only if there exist two exact squares

$$\begin{array}{ccc}
 Z & \xrightarrow{a} & N \\
 \downarrow & & \downarrow c \\
 M & \longrightarrow & Z'
 \end{array}
 \qquad
 \begin{array}{ccc}
 N & \longrightarrow & Z \\
 \downarrow c & & \downarrow a \\
 Z' & \longrightarrow & N
 \end{array}$$

Proposition 3.5. *For any degeneration $M \leq_{\text{deg}} N$ we have*

$$\text{cpl}(M, N) \geq \frac{\ell\ell(M)}{\ell\ell(N)} - 1,$$

where $\ell\ell(X)$ denotes the Loewy length of X ; i.e., the smallest integer r such that $(\text{rad } A)^r X = 0$.

Proof. Choose an (M, N) -tube $T = (N_i, \alpha_i, \beta_i)$ of complexity $h = \text{cpl}(M, N)$. Then M is a direct summand of N_{h+1} , and hence $\ell\ell(M) \leq \ell\ell(N_{h+1})$. We claim that, for all $i \geq 1$,

$$\ell\ell(N_i) \leq i \ell\ell(N_1).$$

In fact, for any exact sequence

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

the relation

$$\ell\ell(B) \leq \ell\ell(A) + \ell\ell(C)$$

holds true. Our claim follows by induction, considering the exact sequences

$$0 \rightarrow N_{i-1} \rightarrow N_i \rightarrow N_1 \rightarrow 0. \qquad \square$$

4. Exact tubes from monomorphisms

Throughout this section, $(f, g)^t : Z \rightarrow Z \oplus M$ denotes an (M, N) -monomorphism.

Definition 4.1. We call two exact tubes $T = (N_i, \alpha_i, \beta_i)$ and $T' = (N'_i, \alpha'_i, \beta'_i)$ similar if N_i is isomorphic to N'_i for all $i \geq 1$.

So we do not ask for any compatibility with the maps in the tubes. Note that the property of being an (M, N) -tube is preserved under similarity, and so is complexity.

Proposition 4.2. *There is a direct summand Z' of Z and an exact sequence*

$$0 \rightarrow Z' \xrightarrow{\begin{pmatrix} f|_{Z'} \\ g|_{Z'} \end{pmatrix}} Z' \oplus M \rightarrow N \rightarrow 0$$

such that $f|_{Z'}$ is nilpotent and $T_{f,g}$ is similar to $T_{f|_{Z'}, g|_{Z'}}$. As a consequence, $T_{f,g}$ is an (M, N) -tube.

Proof. By Fitting's lemma, there is a decomposition $Z = Z' \oplus Z''$ of Z as a direct sum which is preserved under f and such that $f' = f|_{Z'}$ is nilpotent and $f'' = f|_{Z''}$ is an automorphism of Z'' . Set $g' = g|_{Z'}$ and $g'' = g|_{Z''}$. Obviously the maps

$$\begin{pmatrix} f'^i & 0 & g' f'^{i-1} & \dots & g' \\ 0 & f''^i & g'' f''^{i-1} & \dots & g'' \end{pmatrix}^t : Z' \oplus Z'' \longrightarrow Z' \oplus Z'' \oplus M^i$$

and

$$(f'^i \ g' f'^{i-1} \ \dots \ g')^t : Z' \longrightarrow Z' \oplus M^i$$

have isomorphic cokernels as $(f'')^i$ is an isomorphism for $i \geq 1$. Since f' is nilpotent, $T_{f',g'}$ is an (M, N) -tube. □

Remark 4.3. Suppose that $f^h = 0$. As

$$\varphi_{h+j} = (0, \dots, 0, g f^{h-1}, \dots, g)^t : Z \longrightarrow Z \oplus M^{h+j},$$

for $j \in \mathbb{N}$, the exact tube $T_{f,g}$ has the following particularly simple form:

$$\begin{aligned} N_{h+j} &= Z \oplus M^j \oplus Z', & N_{h+j+1} &= Z \oplus M^{j+1} \oplus Z', \\ \alpha_{h+j} &= \begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix} : Z \oplus (M^j \oplus Z') \rightarrow Z \oplus M \oplus (M^j \oplus Z'), \\ \beta_{h+j} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & k & l \end{pmatrix} : (Z \oplus M^j) \oplus M \oplus Z' \rightarrow (Z \oplus M^j) \oplus Z', \end{aligned}$$

for $j \in \mathbb{N}$, where Z' is a cokernel of

$$\psi = (g \circ (f^{h-1}, \dots, f, 1))^t : Z \longrightarrow M^h$$

and

$$(k, \ l) : M \oplus Z' \longrightarrow Z'$$

is obtained from the commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & Z & \xrightarrow{\varphi_{h+1}} & Z \oplus M^{h+1} & \longrightarrow & N_{h+1} = Z \oplus M \oplus Z' \longrightarrow 0 \\
 & & \downarrow f & & \downarrow (1 \ 0) & & \downarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & k & l \end{pmatrix} \\
 0 & \longrightarrow & Z & \xrightarrow{\varphi_h} & Z \oplus M^h & \longrightarrow & N_h = Z \oplus Z' \longrightarrow 0
 \end{array}$$

with exact rows.

Our next goal is to show that, up to similarity, we may choose $g \in \text{rad}(Z, M)$. We start with an auxiliary result:

Lemma 4.4. *The tube $T_{f,g}$ is similar to $T_{f',g}$ with $f' = f - hg$, where $h : M \rightarrow Z$ is any homomorphism.*

Proof. It suffices to check the identity $\psi_i \circ \varphi'_i = \varphi_i$, for $i \geq 1$, where

$$\begin{aligned}
 \varphi_i &= (f^i, gf^{i-1}, \dots, g)^t : Z \longrightarrow Z \oplus M^i, \\
 \varphi'_i &= (f'^i, gf'^{i-1}, \dots, g)^t : Z \longrightarrow Z \oplus M^i
 \end{aligned}$$

and

$$\psi_i := \begin{pmatrix} 1 & h & fh & f^2h & \dots & f^{i-1}h \\ 0 & 1 & gh & gfh & \dots & gf^{i-2}h \\ 0 & 0 & 1 & gh & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & gh \\ \vdots & & & & & gh \\ 0 & \dots & \dots & \dots & 0 & 1 \end{pmatrix} : Z \oplus M^i \rightarrow Z \oplus M^i.$$

The key is the equation

$$f^r = f'^r + \sum_{s=0}^{r-1} f^s (hg) f'^{r-1-s}, \quad r \geq 1,$$

which is proved by induction. □

Proposition 4.5. *There exists a direct summand Z' of Z and an exact sequence*

$$0 \rightarrow Z' \xrightarrow{\begin{pmatrix} f' \\ g' \end{pmatrix}} Z' \oplus M \rightarrow N \rightarrow 0 \tag{4.1}$$

with $g' \in \text{rad}(Z', M)$ and such that $T_{f,g}$ is similar to $T_{f',g'}$.

Proof. If $g \in \text{rad}(Z, M)$, there is nothing to be proved. Otherwise, we prove that a sequence (4.1) exists such that $T_{f,g}$ is similar to $T_{f',g'}$ and $\dim Z' < \dim Z$ and then proceed by induction on $\dim Z$. We choose a non-zero direct summand Z_2 of Z for which $g|_{Z_2}$ is a section. Replacing Z by an isomorphic module if

necessary, which leads to an isomorphic tube, we may assume that $Z = Z_1 \oplus Z_2$, $M = M_1 \oplus Z_2$,

$$g = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Applying the preceding lemma for

$$h = \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix},$$

we obtain a monomorphism $\begin{pmatrix} f'' \\ g \end{pmatrix}$ of the form

$$\begin{pmatrix} f'' \\ g \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & 0 \\ q & 0 \\ 0 & 1 \end{pmatrix} : Z_1 \oplus Z_2 \rightarrow Z_1 \oplus Z_2 \oplus M_1 \oplus Z_2.$$

Now we may take $Z' = Z_1$, $M = Z_2 \oplus M_1$, $f' = a$ and $g' = \begin{pmatrix} b \\ q \end{pmatrix}$. □

Proposition 4.6. *Set $h = \text{cpl}(T_{f,g})$, and suppose that $g \in \text{rad}(Z, M)$ and that f is nilpotent. Then $(T_{f,g})_{\leq h}$ is M -extendible.*

Proof. Our assumptions on f and g imply that, for some i , the restriction $\psi|_Z$ of the composition

$$\psi = \begin{pmatrix} \varphi_i & 0 \\ 0 & 1_{M^h} \end{pmatrix} : Z \oplus M^h \rightarrow Z \oplus M^i \oplus M^h$$

of the maps

$$Z \oplus M^h \xrightarrow{\begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix}} Z \oplus M^{1+h} \rightarrow \dots \xrightarrow{\begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix}} Z \oplus M^{i+h}$$

belongs to $\text{rad}(Z, Z \oplus M^{i+h})$. By construction of $T_{f,g}$, the square

$$\begin{array}{ccc} Z \oplus M^h & \xrightarrow{\psi} & Z \oplus M^i \oplus M^h \\ \downarrow \pi_h & & \downarrow \pi_{h+i} \\ N_h & \xrightarrow{\alpha_{i+h-1} \cdots \alpha_h} & N_{i+h} \end{array}$$

is exact, where $\pi_j : Z \oplus M^j \rightarrow N_j$ is the projection to the cokernel of $\varphi_j : Z \rightarrow Z \oplus M^j$, and it splits, since $h = \text{cpl}(T_{f,g})$. Therefore, $\pi_h|_Z$ is a section, and replacing N_h by an isomorphic module, we may assume that

$$N_h = Z \oplus Z', \quad \pi_h = \begin{pmatrix} 1 & * & * \\ 0 & d & * \end{pmatrix} : Z \oplus M \oplus M^{h-1} \longrightarrow Z \oplus Z',$$

where $*$ is an arbitrary map.

Now consider the exact squares

$$\begin{array}{ccc}
 Z \oplus M^{h-1} & \begin{array}{c} \xrightarrow{\begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix}} \\ \xleftarrow{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}} \end{array} & Z \oplus M \oplus M^{h-1} \\
 \downarrow \pi_{h-1} & & \downarrow \pi_h = \begin{pmatrix} 1 & * & * \\ 0 & d & * \end{pmatrix} \\
 N_{h-1} & \begin{array}{c} \xrightarrow{\alpha_{h-1}} \\ \xleftarrow{\beta_{h-1}} \end{array} & Z \oplus Z'
 \end{array}$$

It is easy to see that the square

$$\begin{array}{ccc}
 Z & \xrightarrow{g} & M \\
 \downarrow \pi_{h-1}|_Z & & \downarrow d \\
 N_{h-1} & \xrightarrow{pr_{z'} \circ \alpha_{h-1}} & Z'
 \end{array}$$

is exact as well. Moreover, we have

$$\pi_{h-1}|_Z = \beta_{h-1}|_Z. \quad \square$$

Next we recall a different construction for $T_{f,g}$, which has been presented for the most part in [6]. From $(f, g)^t$ we obtain the commutative diagram (Figure 2) with exact rows and $(k_i, l_i) = \beta_i(k_{i+1}, l_{i+1})$ for $i \leq m - 1$.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & Z & \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} & Z \oplus M & \xrightarrow{(k_1, l_1)} & N_1 = N \longrightarrow 0 \\
 & & \downarrow k_{m-1} & & \downarrow (k_m, l_m) & & \parallel \\
 0 & \longrightarrow & N_{m-1} & \xrightarrow{\alpha_{m-1}} & N_m & \xrightarrow{\beta_1 \dots \beta_{m-1}} & N_1 \longrightarrow 0 \\
 & & \downarrow \beta_{m-2} & & \downarrow \beta_{m-1} & & \parallel \\
 0 & \longrightarrow & N_{m-2} & \xrightarrow{\alpha_{m-2}} & N_{m-1} & \xrightarrow{\beta_1 \dots \beta_{m-2}} & N_1 \longrightarrow 0 \\
 & & \vdots & & \vdots & & \vdots \\
 0 & \longrightarrow & N_2 & \xrightarrow{\alpha_2} & N_3 & \xrightarrow{\beta_1 \beta_2} & N_1 \longrightarrow 0 \\
 & & \downarrow \beta_1 & & \downarrow \beta_2 & & \parallel \\
 0 & \longrightarrow & N_1 & \xrightarrow{\alpha_1} & N_2 & \xrightarrow{\beta_1} & N_1 \longrightarrow 0
 \end{array}$$

Figure 2

The next step is always obtained by squeezing the push-out of the top sequence by k_m between the two top rows.

We claim that the exact tube (N_i, α_i, β_i) of height m thus obtained is isomorphic to the restriction $(T_{f,g})_{\leq m}$ of $T_{f,g}$.

By induction, we obtain the following series of exact squares:

$$\begin{array}{ccccccc}
 Z & \xrightarrow{\begin{pmatrix} f & 0 \\ g & 0 \end{pmatrix}} & Z \oplus M & \xrightarrow{\begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix}} & Z \oplus M^2 & \cdots & Z \oplus M^{m-1} & \xrightarrow{\begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix}} & Z \oplus M^m \\
 \downarrow & & \downarrow \psi_1 & & \downarrow \psi_2 & & \downarrow \psi_{m-1} & & \downarrow \psi_m \\
 0 & \longrightarrow & N_1 & \xrightarrow{\alpha_1} & N_2 & \cdots & N_{m-1} & \xrightarrow{\alpha_{m-1}} & N_m
 \end{array}$$

with $\psi_i = (k_i, l_i, \alpha_{i-1}l_{i-1}, \dots, \alpha_{i-1} \dots \alpha_1 l_1) : Z \oplus M^i \rightarrow N_i$.

Note that the composition of the first i maps of the top row is just $\varphi_i : Z \rightarrow Z \oplus M^i$ and that the sequence

$$0 \rightarrow Z \xrightarrow{\varphi_i} Z \oplus M^i \xrightarrow{\psi_i} N_i \rightarrow 0$$

is exact for $i = 1, \dots, m$. So $N_i \xrightarrow{\sim} \text{coker } \varphi_i$, and the maps α_i are the ones we claim. As for β_i , it suffices to show that

$$\psi_i \circ (1 \ 0) = \beta_i \circ \psi_{i+1}.$$

This follows easily from the explicit formulas for ψ_i, ψ_{i+1} , the equation

$$(k_i, l_i) = \beta_i(k_{i+1}, l_{i+1})$$

and the fact that (N_i, α_i, β_i) is an exact tube of height m . As a consequence we have:

Remark 4.7. Let $(f, g)^t$ be an (M, N) -monomorphism and $T' = (N'_i, \alpha'_i, \beta'_i)$ an exact tube of height m . Then T' is isomorphic to $(T_{f,g})_{\leq m}$ if and only if there exists an exact square

$$\begin{array}{ccc}
 Z & \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} & Z \oplus M \\
 \downarrow \beta'_{m-1} \circ k & & \downarrow (k, l) \\
 N'_{m-1} & \xrightarrow{\alpha'_{m-1}} & N'_m
 \end{array}$$

Proposition 4.8. Any M -extendible exact tube $T = (N_i, \alpha_i, \beta_i)$ of height m with $N_1 \xrightarrow[A]{\sim} N$ is the restriction of the exact tube $T_{f,g}$ to Q_m for some (M, N) -monomorphism $(f, g)^t$.

Proof. Let

$$0 \rightarrow Z \xrightarrow{\begin{pmatrix} a \\ b \end{pmatrix}} N_{m-1} \oplus M \xrightarrow{(cd)} Z' \rightarrow 0$$

be an exact sequence with $N_m = Z \oplus Z'$, $a = \beta_{m-1}|_Z$ and $c = \text{pr}_{Z'} \circ \alpha_{m-1}$. The square

$$\begin{array}{ccc} N_m = Z \oplus Z' & \xrightarrow{\begin{pmatrix} c'a & c'a' \\ b & 0 \\ 0 & 1 \end{pmatrix}} & Z \oplus M \oplus Z' \\ \downarrow (a,a')=\beta_{m-1} & & \downarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & -d & ca' \end{pmatrix} \\ N_{m-1} & \xrightarrow{(c')=\alpha_{m-1}} & N_m = Z \oplus Z' \end{array}$$

is exact. Setting

$$N_{m+1} = Z \oplus M \oplus Z', \quad \alpha_m = \begin{pmatrix} c'a & c'a' \\ b & 0 \\ 0 & 1 \end{pmatrix}, \quad \beta_m = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -d & ca' \end{pmatrix}$$

we may extend T to an exact tube of height $m + 1$. By construction, the map

$$\begin{pmatrix} c'a \\ b \end{pmatrix} : Z \longrightarrow Z \oplus M$$

is an (M, N) -monomorphism, and the square

$$\begin{array}{ccc} Z & \xrightarrow{\begin{pmatrix} c'a \\ b \end{pmatrix}} & Z \oplus M \\ \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} & & \downarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \\ N_m = Z \oplus Z' & \xrightarrow{\alpha_m} & N_{m+1} = Z \oplus M \oplus Z' \end{array}$$

is exact with

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \beta_m \circ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The result now follows from Remark 4.7. □

Proposition 4.9. *Let $T = (N_i, \alpha_i, \beta_i)$ be an exact tube of height $h + m$, for some $h \geq 1$ and $m \geq 1$. Suppose that*

$$N_{h+j+1} \xrightarrow[A]{\sim} N_{h+j} \oplus M$$

for $j \in \{0, \dots, m - 1\}$. Then there is an (M, N) -monomorphism

$$\begin{pmatrix} f \\ g \end{pmatrix} : N_{h+m-1} \longrightarrow N_{h+m-1} \oplus M$$

such that the restrictions $T_{\leq m}$ and $(T_{f,g})_{\leq m}$ are isomorphic.

Proof. We wish to choose

$$\begin{pmatrix} f \\ g \end{pmatrix} = \chi \circ \alpha_{h+m-1}$$

for a suitable isomorphism $\chi : N_{h+m} \rightarrow N_{h+m-1} \oplus M$ and to apply Remark 4.7 to the diagram

$$\begin{array}{ccccc}
 N_{h+m-1} & \xrightarrow{\alpha_{h+m-1}} & N_{h+m} & \xrightarrow{\sim} & N_{h+m-1} \oplus M \\
 \beta = \beta_{m-1} \cdots \beta_{h+m-2} \downarrow & & \beta_m \cdots \beta_{h+m-1} \downarrow & & \swarrow \\
 N_{m-1} & \xrightarrow{\alpha_{m-1}} & N_m & &
 \end{array}$$

In order to do this, we only need to construct a section

$$s : N_{h+m-1} \rightarrow N_{h+m}$$

satisfying

$$\beta_{m-1} \beta_m \cdots \beta_{h+m-1} s = \beta \beta_{h+m-1} s = \beta.$$

By our hypothesis, the square

$$\begin{array}{ccc}
 N_{h+1} & \xrightarrow{\alpha} & N_{h+m} \\
 \beta_h \downarrow & & \beta_{h+m-1} \downarrow \\
 N_h & \xrightarrow{\alpha'} & N_{h+m-1}
 \end{array}$$

splits, where $\alpha = \alpha_{h+m-1} \cdots \alpha_{h+1}$ and $\alpha' = \alpha_{h+m-2} \cdots \alpha_h$. Choose a maximal direct summand A of N_{h+m} for which $\beta_{h+m-1}|_A$ is a section. Replacing T by an isomorphic exact tube, we may assume that we have

$$\beta_{h+m-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma & \delta \end{pmatrix} : A \oplus B \oplus M \rightarrow A \oplus B,$$

$$\alpha' = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix} : C \oplus B \rightarrow A \oplus B$$

for some maps $\gamma, \delta, \varepsilon$. Setting

$$s = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} : A \oplus B \rightarrow A \oplus B \oplus M,$$

we obtain

$$1_{A \oplus B} - \beta_{h+m-1} s = \begin{pmatrix} 0 & 0 \\ 0 & 1 - \gamma \end{pmatrix} : A \oplus B \rightarrow A \oplus B,$$

which factors through α' . But the sequence

$$0 \rightarrow N_h \xrightarrow{\alpha'} N_{h+m-1} \xrightarrow{\beta} N_{m-1} \rightarrow 0$$

is exact, which implies $\beta = \beta \beta_{h+m-1} s$ as required. □

5. Examples

All our examples are representations of quivers with relations. Let Q be a quiver with vertex set $Q_0 = \{1, \dots, n\}$, I an admissible two-sided ideal in the quiver algebra kQ , $\underline{d} = (d_1, \dots, d_n)$ a vector in \mathbb{N}^n , and denote by

$$\text{Rep}(Q, I, \underline{d})$$

the affine algebraic variety of representations X of (Q, I) with $X(i) = k^{d_i}$, $i \in Q_0$. The dimension vector of X in $\text{Rep}(Q, I, \underline{d})$ is \underline{d} . The group $G(\underline{d}) = \prod_{i=1}^n GL(d_i)$ acts on $\text{Rep}(Q, I, \underline{d})$ by

$$(g \cdot X)(\alpha) = g_j \circ X(\alpha) \circ g_i^{-1}$$

for an arrow $\alpha : i \rightarrow j$ and $g = (g_1, \dots, g_n) \in G(\underline{d})$.

If we view M, N in $\text{Rep}(Q, I, \underline{d})$ as modules over kQ/I of dimension $d = \sum_{i=1}^n d_i$, then M degenerates to N if and only if the representation N belongs to the closure of the orbit $G(\underline{d}) \cdot M$ of M in $\text{Rep}(Q, I, \underline{d})$ [1]. This allows us to work with the smaller group $G(\underline{d})$.

5.1. We begin with an example of a degeneration whose complexity is easy to compute: Choose a natural number $n \geq 2$ and let \vec{A}_n be the equioriented quiver with underlying graph A_n :

$$\vec{A}_n = 1 \xleftarrow{\gamma_1} 2 \xleftarrow{\dots} \dots \xleftarrow{\dots} n-1 \xleftarrow{\gamma_{n-1}} n.$$

Denote by X_i the indecomposable representation of \vec{A}_n given by

$$X_i(j) = \begin{cases} k & j \leq i, \\ 0 & j > i, \end{cases}$$

$$X_i(\gamma_j) = \begin{cases} 1 & j < i, \\ 0 & j \geq i. \end{cases}$$

Then $M = X_n$ has a filtration

$$M = X_n \supset X_{n-1} \supset \dots \supset X_2 \supset X_1,$$

and it is well-known that M degenerates to the associated graded module

$$N = \bigoplus_{i=1}^n X_i/X_{i-1},$$

where we set $X_0 = 0$. We wish to compute the complexity $\text{cpl}(M, N)$, thereby

showing again that M actually degenerates to N . Set

$$Z = \bigoplus_{i=1}^{n-1} X_i,$$

$$f = \begin{pmatrix} 0 & & & 0 \\ \iota_2 & \ddots & & \\ & \ddots & \ddots & \\ & & \iota_{n-1} & 0 \end{pmatrix} : Z \longrightarrow Z \quad \text{and}$$

$$g = (0 \cdots 0 \ \iota_n) : Z \longrightarrow M = X_n,$$

where $\iota_i : X_{i-1} \rightarrow X_i$ is the inclusion. It is easy to check that $(f, g)^t$ is an (M, N) -monomorphism. Moreover, $f^{n-1} = 0$, and thus

$$\text{cpl}(M, N) \leq n - 1.$$

On the other hand, the Loewy lengths of M and N are n and 1 , respectively, which implies

$$\text{cpl}(M, N) \geq \frac{\ell\ell(M)}{\ell\ell(N)} - 1 = n - 1$$

by Proposition 3.5. This example shows that there are degenerations of arbitrary complexity.

Note that for $n = 4$ we obtain the following chain of degenerations:

$$M = k \xleftarrow{1} k \xleftarrow{1} k \xleftarrow{1} k \leq_{\text{deg}} P = k \xleftarrow{1} k \xleftarrow{0} k \xleftarrow{1} k$$

$$\leq_{\text{deg}} N = k \xleftarrow{0} k \xleftarrow{0} k \xleftarrow{0} k.$$

The complexities are

$$\text{cpl}(M, P) = 1 = \text{cpl}(P, N) \quad \text{and}$$

$$\text{cpl}(M, N) = 3 > \text{cpl}(M, P) + \text{cpl}(P, N).$$

Comparing with the example given in the introduction, we see that $\text{cpl}(M, P) + \text{cpl}(P, N)$ can be either smaller or greater than $\text{cpl}(M, N)$ for a chain

$$M \leq_{\text{deg}} P \leq_{\text{deg}} N.$$

5.2. Next we give an example of a minimal degeneration of arbitrary complexity: Let Q be the quiver

$$Q = 1 \xrightarrow{\alpha} 2 \begin{matrix} \curvearrowright \\ \curvearrowleft \end{matrix} \beta,$$

choose a natural number $n \geq 2$, and let I be the ideal generated by β^n . Define M and N to be the representations of dimension vector $(1, n)$ given by

$$M(\alpha) = e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad N(\alpha) = e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

and

$$M(\beta) = N(\beta) = J_n, \text{ respectively,}$$

where e_1, \dots, e_n is the standard basis of k^n and J_m is the Jordan block

$$J_m = \begin{pmatrix} 0 & & & 0 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix}$$

in $\mathbb{M}_m(k)$, for $m \in \mathbb{N}$.

Proposition 5.1. *There is a degeneration $M \leq_{\text{deg}} N$, which is minimal, and $\text{cpl}(M, N) = n$.*

Proof. Denote by Z the indecomposable representation with dimension vector $(0, n)$, given by $Z(\beta) = J_n$, and let $(f, g)^t : Z \rightarrow Z \oplus M$ be given by

$$f = (0, J_n), \quad g = (0, 1).$$

It is easy to see that $(f, g)^t$ is an (M, N) -monomorphism, so M degenerates to N . Moreover, we have $f^n = 0$, and therefore $\text{cpl}(M, N) \leq n$. As

$$\dim \text{End } M = 1 \text{ and } \dim \text{End } N = 2,$$

the orbit of N has codimension 1 in the closure of the orbit of M , which implies that the degeneration is minimal.

Suppose $\text{cpl}(M, N) \leq n - 1$, and choose an (M, N) -tube $T = (N_i, \alpha_i, \beta_i)$ with $N_n \xrightarrow{\sim} N_{n-1} \oplus M$. Let $\psi_n : N_n \rightarrow M$ be the surjection obtained from this decomposition.

Claim. *For $i = 1, \dots, n$, there exists a surjection*

$$\psi_i : N_i \rightarrow M^{(i)},$$

where $M^{(i)}$ has dimension vector $(1, i)$ and is given by

$$M^{(i)}(\alpha) = (1, 0, \dots, 0)^t, \quad M^{(i)}(\beta) = J_i.$$

Using the claim for $i = 1$, we obtain a surjection $\psi_1 : N_1 = N \rightarrow M^{(1)}$, which is impossible.

We prove the claim by descending induction on i . Observe that any map from N to $M^{(i)}$ factors through the socle $\text{soc}M^{(i)}$ and that $M^{(i)}/\text{soc}M^{(i)} \xrightarrow{\sim} M^{(i-1)}$. Writing this factorization for $\psi_i \circ \alpha_{i-1} \cdots \alpha_1$, we obtain $\psi_{i-1} : N_{i-1} \rightarrow M^{(i-1)}$ from the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & N & \xrightarrow{\alpha_{i-1} \cdots \alpha_1} & N_i & \xrightarrow{\beta_{i-1}} & N_{i-1} \longrightarrow 0 \\
 & & \downarrow & & \downarrow \psi_i & & \downarrow \psi_{i-1} \\
 0 & \longrightarrow & \text{soc} M^{(i)} & \longrightarrow & M^{(i)} & \longrightarrow & M^{(i-1)} \longrightarrow 0.
 \end{array}$$

As ψ_i is surjective, ψ_{i-1} is as well. □

A version of this argument implies the following result, which we will not use:

$$\text{cpl}(M, M_i) = \left\lceil \frac{n-1}{i-1} \right\rceil + 1, \quad i \geq 2.$$

The representation M_i is given by

$$M_i(\alpha) = e_i, \quad M_i(\beta) = J_n.$$

5.3. We now exhibit a degeneration $M \leq_{\text{deg}} N$ of complexity 2 with the property that $f^2 \neq 0$ for all (M, N) -monomorphisms $(f, g)^t$. Therefore the complexity can be strictly less than the “index of nilpotence of M and N ”; i.e., the number

$$\min\{r : f^r = 0\},$$

where the minimum is taken over all (M, N) -monomorphisms $(f, g)^t$. We stay with the same quiver Q , and we choose I to be generated by β^3 ; i.e., we set $n = 3$ in the preceding example. Note that kQ/I is representation-finite: it admits 29 indecomposables [4].

We let M and N be given by

$$M(\alpha) = e_2, \quad N(\alpha) = e_3, \quad M(\beta) = N(\beta) = J_3,$$

where e_1, e_2, e_3 is the standard basis of k^3 . Choose

$$\begin{array}{l}
 Z' = 0 \longrightarrow k^3 \xrightarrow{J_3} k^3 \\
 f' = (0, J_3) : Z' \longrightarrow Z' \quad \text{and} \\
 g' = (0, 1) : Z' \longrightarrow M.
 \end{array}$$

Then $(f', g')^t$ is an (M, N) -monomorphism. As f'^2 factors through g' , the cokernel N_3 of the map

$$\varphi_3 = (f'^3, g'f'^2, g'f', g')^t : Z' \longrightarrow Z' \oplus M^3$$

used to define the tube $T_{f', g'} = (N_i, \alpha_i, \beta_i)$ is isomorphic to the cokernel of

$$(f'^2, 0, g'f', g')^t : Z' \longrightarrow Z' \oplus M^3$$

and thus isomorphic to $M \oplus N_2$. By Lemma 3.3, we know that

$$\text{cpl}(M, N) \leq 2.$$

On the other hand, as N is indecomposable, the complexity must exceed 1, so

$$\text{cpl}(M, N) = 2.$$

Claim. For any (M, N) -monomorphism

$$(f, g)^t : Z \longrightarrow Z \oplus M,$$

we have $f^2 \neq 0$.

First we show:

Lemma 5.2. For any (M, N) -monomorphism

$$(f, g)^t : Z \longrightarrow Z \oplus M,$$

Z' is a direct summand of Z .

Proof. Consider the exact sequences

$$\Sigma' : 0 \longrightarrow Z' \xrightarrow{\begin{pmatrix} f' \\ g' \end{pmatrix}} Z' \oplus M \xrightarrow{(k', l')} N \longrightarrow 0$$

and

$$\Sigma : 0 \longrightarrow Z \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} Z \oplus M \xrightarrow{(k, l)} N \longrightarrow 0.$$

It is easy to check that

$$\begin{aligned} \dim \text{Hom}(Z', M) &= \dim \text{Hom}(Z', N) = 3, \\ \dim \text{End } M &= \dim \text{Hom}(M, N) = 2. \end{aligned}$$

Therefore the sequence of vector spaces

$$0 \longrightarrow \text{Hom}(Z' \oplus M, Z) \longrightarrow \text{Hom}(Z' \oplus M, Z \oplus M) \longrightarrow \text{Hom}(Z' \oplus M, N) \longrightarrow 0$$

obtained from mapping $Z' \oplus M$ into Σ is exact. In particular, $(k', l') : Z' \oplus M \rightarrow N$ factors through $(k, l) : Z \oplus M \rightarrow N$, and hence we have the following commutative diagram (Figure 3) with exact rows and columns.

$$\begin{array}{ccccccc}
 & & & 0 & & & 0 \\
 & & & \uparrow & & & \uparrow \\
 \Sigma : & 0 & \longrightarrow & Z & \longrightarrow & Z \oplus M & \xrightarrow{(k,l)} & N & \longrightarrow & 0 \\
 & & & \parallel & & & & \uparrow & & \\
 & & & & & & & (k',l') & & \\
 & 0 & \longrightarrow & Z & \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & Z \oplus (Z' \oplus M) & \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} & Z' \oplus M & \longrightarrow & 0 \\
 & & & & & \uparrow & & \uparrow & & \\
 & & & & & \begin{pmatrix} s \\ f' \\ g' \end{pmatrix} & & \begin{pmatrix} f' \\ g' \end{pmatrix} & & \\
 & & & & & Z' & \xlongequal{\quad\quad\quad} & Z' & & \\
 & & & & & \uparrow & & \uparrow & & \\
 & & & & & 0 & & 0 & &
 \end{array}$$

Figure 3

So the middle column splits as well, and since by construction f', g' lie in the radical,

$$s : Z' \longrightarrow Z$$

must be a section. □

Let $(f, g)^t : Z \rightarrow Z \oplus M$ be an (M, N) -monomorphism, suppose $f^2 = 0$, and consider the commutative diagram (Figure 4) with exact rows and columns.

$$\begin{array}{ccccccc}
 & & & 0 & & & 0 \\
 & & & \downarrow & & & \downarrow \\
 0 & \longrightarrow & \ker f & \longrightarrow & M & \longrightarrow & X & \longrightarrow & 0 \\
 & & & \downarrow & & & \downarrow & & \\
 & & & & & & \begin{pmatrix} 0 \\ 1 \end{pmatrix} & & \\
 0 & \longrightarrow & Z & \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} & Z \oplus M & \xrightarrow{(k,l)} & N & \longrightarrow & 0 \\
 & & & \downarrow & & & \downarrow & & \\
 & & & & & & (1,0) & & \\
 0 & \longrightarrow & \text{im } f & \longrightarrow & Z & \longrightarrow & \text{coker } f & \longrightarrow & 0 \\
 & & & \downarrow & & & \downarrow & & \\
 & & & & & & 0 & &
 \end{array}$$

Figure 4

Then X must be a quotient of M and a submodule of N , which is possible in

exactly two ways:

$$(i) \quad X = k \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} k^2 \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} J_2$$

$$(ii) \quad X = 0 \longrightarrow k \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} 0$$

In the first case, we have

$$\ker f = 0 \longrightarrow k \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} 0,$$

and our assumption $f^2 = 0$ implies that $\dim Z(2) \leq 2$. But then Z cannot contain Z' as a direct summand.

In the second case, we see that

$$\ker f = k \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} k^2 \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} J_2.$$

Now $f^2 = 0$ implies that $\dim Z(2) \leq 4$. But then necessarily $Z(\beta\alpha) = 0$, since Z' must be a direct summand of Z , and Z cannot contain $\ker f$ as a submodule.

5.4. As our last example, we find a degeneration $M \leq_{\text{deg}} N$ of complexity 2 for which there exists an exact sequence

$$\Sigma : 0 \longrightarrow N \xrightarrow{\alpha_1 = \begin{pmatrix} f \\ g \end{pmatrix}} N \oplus M \xrightarrow{\beta_1 = (f, -l)} N \longrightarrow 0.$$

So we have an exact tube

$$T = (N_1 = N, N_2 = N \oplus M, \alpha_1, \beta_1)$$

of height 2. If this tube were the restriction of an (infinite) exact tube, the complexity $\text{cpl}(M, N)$ would have to equal 1. So the number $2h + 1$ in condition (iii) of our main theorem cannot be replaced by $2h$.

Choose $A = k[\alpha, \beta]/(\alpha^2, \beta^2)$, let M and N be 4-dimensional with

$$M(\alpha) = N(\alpha) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad M(\beta) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad N(\beta) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

and set

$$f = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad g = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad l = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

It is easy to check that the sequence Σ obtained from these choices is exact. So M degenerates to N . As N is indecomposable and f^2 factors through g , the same argument as in Section 5.3 implies that $\text{cpl}(M, N) = 2$.

This example has another surprising feature: For any degeneration $M \leq_{\text{deg}} N$ we obtain

$$\text{cpl}(M^r, N^r) \leq \text{cpl}(M, N), \quad r \geq 1,$$

by taking for $M^r \leq_{\text{deg}} N^r$ the direct sum of r copies of an (M, N) -tube of minimal complexity. In our example, we have

$$\text{cpl}(M^2, N^2) = 1 < \text{cpl}(M, N) = 2.$$

Indeed, M^2 is a projective cover for N , and the kernel of an epimorphism $M^2 \rightarrow N$ is N again. So there is an exact sequence

$$0 \longrightarrow N \longrightarrow M^2 \longrightarrow N \longrightarrow 0.$$

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