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Commentarii Mathematici Helvetici

Fundamental groups of algebraic fiber spaces

Ichiro Shimada

Abstract. Let $f : E \to B$ be a dominant morphism, where E and B are smooth irreducible complex quasi-projective varieties. Suppose that the general fiber F_b of f is connected. We present an algebro-geometric condition under which the boundary homomorphism $\partial : \pi_2(B) \to \pi_1(F_b)$ is well-defined, and makes the sequence

$$\pi_2(B) \to \pi_1(F_b) \to \pi_1(E) \to \pi_1(B) \to 1$$

exact. As an application, we calculate the fundamental group of the complement to the dual hypersurface of a smooth projective curve.

Mathematics Subject Classification (2000). 14F35, 14D05.

Keywords. Fundamental group, second homotopy group, algebraic fiber space, homotopy exact sequence, dual hypersurface.

1. Introduction

We work over the complex number field $\mathbb{C}$.

Let $\overline{E}$ and B be smooth irreducible quasi-projective varieties, and let

$$\bar{f} : \overline{E} \to B$$

be a projective surjective morphism. Throughout this paper, we assume that the general fiber of $\bar{f}$ is connected. Let Z be a reduced hypersurface of $\overline{E}$. We denote by E the complement $\overline{E} \setminus Z$ to Z in $\overline{E}$, and by f the restriction $\bar{f} \,|\, E$ of $\bar{f}$ to E. Then

$$f : E \to B$$

is a dominant morphism with the general fiber being a smooth irreducible quasi-projective variety. For a point $a \in B$, we denote by $\overline{F}_a$ the fiber $\bar{f}^{-1}(a)$, and by Z_a the *scheme-theoretic* intersection of $\overline{F}_a$ and Z. We put

$$F_a := f^{-1}(a) = \overline{F}_a \setminus Z_a.$$

We choose a general point b of B, and a point $\tilde{b}$ of F_b. Let

$$i : F_b \hookrightarrow E$$

denote the inclusion morphism. We will study the homomorphisms

$$i_* : \pi_1(F_b, \tilde{b}) \to \pi_1(E, \tilde{b}) \quad \text{and} \quad f_* : \pi_1(E, \tilde{b}) \to \pi_1(B, b)$$

on the topological fundamental groups induced by the morphisms i and f. By investigating the homotopy lifting property of the morphism f, we will show the following:

Theorem 1.1. *Suppose that $\bar{f} : \overline{E} \to B$ and Z satisfy the following conditions.*

(C1) *Every irreducible component of the hypersurface Z is mapped surjectively onto B by $\bar{f}$.*

(C2) *The locus* $\operatorname{Sing} \bar{f} \subset \overline{E}$ *of the critical points of $\bar{f}$ is of codimension ≥ 3 in $\overline{E}$.*

(C3) *The locus $\{a \in B \mid Z_a$ is not reduced $\}$ is contained in a Zariski closed subset of codimension ≥ 2 in B.*

Then the boundary homomorphism

$$\partial : \pi_2(B, b) \to \pi_1(F_b, \tilde{b})$$

is well-defined, and the sequence

$$\pi_2(B, b) \xrightarrow{\partial} \pi_1(F_b, \tilde{b}) \xrightarrow{i_*} \pi_1(E, \tilde{b}) \xrightarrow{f_*} \pi_1(B, b) \longrightarrow 1$$

is exact.

In fact, Nori [8, Lemma 1.5 (C)] has proved that, under a condition milder than (C1)–(C3), the sequence

$$\pi_1(F_b, \tilde{b}) \xrightarrow{i_*} \pi_1(E, \tilde{b}) \xrightarrow{f_*} \pi_1(B, b) \longrightarrow 1$$

is exact.

A special case of Theorem 1.1, where B is an affine space $\mathbb{A}^N$, $\overline{E}$ is a product $\overline{F} \times \mathbb{A}^N$ of a smooth irreducible projective variety $\overline{F}$ and $\mathbb{A}^N$, and $\bar{f}$ is the projection, was proved in [10].

As an application, we will calculate the fundamental group of the complement to the dual hypersurface of a smooth projective curve.

Let C be a compact Riemann surface of genus $g > 0$, and $L \to C$ a line bundle of degree $d > 2g + 1$. Then the complete linear system $|L|$ embeds C into the projective space

$$P_L := \mathbb{P}^* H^0(C, L)$$

parameterizing all hyperplanes of the vector space $H^0(C, L)$ of dimension $d - g + 1$. Let C_L denote the image of this embedding. We denote by $C_L^\vee$ the dual hypersurface of C_L in the dual projective space $P_L^\vee$ of P_L. Let $B(C, d)$ be the braid group on d strings on the Riemann surface C; that is, $B(C, d)$ is the fundamental group of the space

$$\operatorname{Sym}^d C \setminus \Delta_C^d,$$

where $\operatorname{Sym}^d C$ is the symmetric product of d copies of C, which parameterizes all effective divisors of degree d on C, and Δ_C^d is the hypersurface of $\operatorname{Sym}^d C$ parameterizing all non-reduced effective divisors of degree d.

Theorem 1.2. *If L corresponds to a general point of the Picard variety* $\mathrm{Pic}^d(C)$ *of isomorphism classes of line bundles of degree d on C, then $\pi_1(P_L^\vee \setminus C_L^\vee)$ is isomorphic to the kernel of the natural homomorphism*

$$B(C,d) \longrightarrow H_1(C,\mathbb{Z}).$$

Theorem 1.2 was stated in [3]. However the proof in [3] seems to be incomplete, because the family of the complements $P_L^\vee \setminus C_L^\vee$ is, in general, not locally trivial over $\mathrm{Pic}^d(C)$. Another proof of Theorem 1.2 was also given in [7], but the proof is sketchy.

A finite presentation of $B(C,d)$ is given in [1], [2] and [9]. A finite presentation of the kernel of the natural homomorphism from $B(C,d)$ to $H_1(C,\mathbb{Z})$ is given in [6] and [7].

This paper is organized as follows. In §2, we collect miscellaneous definitions and lemmas that will be used in this paper. In §3, we state Nori's lemma [8, Lemma 1.5 (C)], and give a proof which is different from Nori's original proof, and uses a similar idea as the proof of Theorem 1.1. In §4, we define the boundary homomorphism ∂ from $\pi_2(B)$ to $\pi_1(F_b)$, and in §5, we prove Theorem 1.1. In §6, we prove Theorem 1.2.

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Notation and terminologies

(1) We consider algebraic varieties with the complex topology unless otherwise stated. An algebraic morphism $\phi : X \to Y$ is said to be locally trivial over Y if it is locally trivial over Y as a continuous map in the complex topology.

(2) The Zariski closure of a subset A of an algebraic variety is denoted by $A^=$.

(3) For an algebraic variety X, we denote by $\mathrm{Sing}\, X$ the singular locus of X. For a morphism $\phi : X \to Y$ with X and Y smooth, we denote by $\mathrm{Sing}\, \phi \subset X$ the locus of critical points of ϕ.

(4) We denote by I the closed interval $[0,1]$ in $\mathbb{R}$. For a subset A of I^k, we denote by A° the interior of A *in* $\mathbb{R}^k$, by $(A)^-$ or $\overline{A}$ the closure of A, and by ∂A the boundary $\overline{A} \setminus A^\circ$ of A.

(5) Let p be a point of a topological space X, and $u : (I^k, \partial I^k) \to (X,p)$ a continuous map. We denote by $[u]$ the element of $\pi_k(X,p)$ represented by u. The constant map from I^k to the point p is denoted by 0_p.

(6) Let $\phi : X \to Y$ and $\psi : Z \to Y$ be continuous maps. We say that ϕ is locally trivial over ψ or over Z if the pull-back

$$\psi^*\phi \,:\, \psi^*X = X \times_Y Z \,\to\, Z$$

of ϕ by ψ is locally trivial over Z.

2. Preliminaries

2.1. Transversality

Let M and N be connected $\mathcal{C}^\infty$ differentiable manifolds, and S a closed submanifold of N, which may have several connected components of various dimensions. Let Y be a smooth irreducible quasi-projective variety, and T a closed, reduced (and possibly reducible) subvariety of Y.

Definition 2.1. (1) We say that a $\mathcal{C}^\infty$-map $\phi : M \to N$ intersects S transversely if, for any $p \in \phi^{-1}(S)$, we have

$$(d\phi)_p(T_pM) + T_{\phi(p)}S = T_{\phi(p)}N.$$

(2) We say that a continuous map $v : I^k \to N$ intersects S transversely if there exists an open subset U of I^k satisfying

$$v^{-1}(S) \subset U \subset \overline{U} \subset (I^k)^\circ$$

such that the restriction

$$v \,|\, U : U \to N$$

of v to U is a $\mathcal{C}^\infty$-map that intersects S transversely in the sense of (1) above.

(3) We say that a $\mathcal{C}^\infty$-map $\varphi : M \to Y$ intersects T transversely if the following hold;

- $\varphi^{-1}(\operatorname{Sing} T) = \emptyset$, so that φ can be regarded as a $\mathcal{C}^\infty$-map to $Y \setminus \operatorname{Sing} T$, which contains $T \setminus \operatorname{Sing} T$ as a closed submanifold, and
- as a $\mathcal{C}^\infty$-map to $Y \setminus \operatorname{Sing} T$, φ intersects the closed submanifold $T \setminus \operatorname{Sing} T$ transversely in the sense of (1) above.

(4) We say that a continuous map $w : I^k \to Y$ intersects T transversely if there exists an open subset U of I^k satisfying

$$w^{-1}(T) \subset U \subset \overline{U} \subset (I^k)^\circ$$

such that the restriction $w \,|\, U$ of w to U is a $\mathcal{C}^\infty$-map that intersects T transversely in the sense of (3) above.

Let A, V and W be open subsets of M such that the closure $\overline{V}$ of V in M is compact and contained in W. Suppose that a distance function

$$d : N \times N \to \mathbb{R}_{\geq 0}$$

on N is given. Using the approximation theorem of continuous maps by $\mathcal{C}^\infty$-maps and the elementary transversality theorem ([5]), we can easily prove the following:

Lemma 2.2. *Let $\phi_0 : M \to N$ be a continuous map. Suppose that the restriction $\phi_0 \,|\, A$ of ϕ_0 to A is a $\mathcal{C}^\infty$-map intersecting S transversely. Then, for any positive real number ε, there exists a continuous map $\phi_1 : M \to N$ with the following properties;*

- *there exists a homotopy from ϕ_0 to ϕ_1 that is stationary on $M \setminus W$,*
- *the restriction $\phi_1 \,|\, (A \cup V)$ of ϕ_1 to $A \cup V$ is a $\mathcal{C}^\infty$-map intersecting S transversely, and*
- *$d(\phi_0(x), \phi_1(x)) \leq \varepsilon$ for any $x \in M$.* □

Starting from $T_0 := T$, we put

$$T_{i+1} := \operatorname{Sing} T_i,$$

where T_i is considered as a subscheme of Y with the reduced structure. Then T_n is empty for n large enough. By definition, $T_i \setminus T_{i+1}$ is a closed submanifold of $Y \setminus T_{i+1}$. Applying Lemma 2.2 to these closed submanifolds repeatedly, we obtain the following:

Corollary 2.3. *Let $\varphi_0 : M \to Y$ be a continuous map such that $\varphi_0 \,|\, A$ is a $\mathcal{C}^\infty$-map intersecting T transversely. If*

$$\dim_{\mathbb{R}} M + 2 \dim \operatorname{Sing} T < 2 \dim Y,$$

then there exists a continuous map $\varphi_1 : M \to Y$ with the following properties;

- *there exists a homotopy from φ_0 to φ_1 that is stationary on $M \setminus W$, and*
- *the restriction $\varphi_1 \,|\, (A \cup V)$ of φ_1 to $A \cup V$ is a $\mathcal{C}^\infty$-map intersecting T transversely.* □

Corollary 2.4. *Let $w_0 : I^k \to Y$ be a continuous map such that $w_0^{-1}(T)$ is contained in $(I^k)^\circ$. If*

$$k + 2 \dim \operatorname{Sing} T < 2 \dim Y,$$

then there is a continuous map $w_1 : I^k \to Y$ intersecting T transversely that is homotopic to w_0 relative to ∂I^k. □

2.2. Local triviality of a $\mathcal{C}^\infty$-map

Let M and N be $\mathcal{C}^\infty$ manifolds, and R a closed submanifold of M, which may have several connected components of various dimensions. Let $\phi : M \to N$ be a $\mathcal{C}^\infty$-map.

Lemma 2.5. *Let K be a non-empty compact subset of M contained in the inverse image $\phi^{-1}(p)$ of a point $p \in N$. Suppose that ϕ is smooth at every point of K, and that the restriction $\phi \,|\, R$ of ϕ to R is smooth at every point of $R \cap K$. Then there exist open neighborhoods V of p in N and U of K in M contained in $\phi^{-1}(V)$ such that the pair of the $\mathcal{C}^\infty$-maps*

$$(\phi \,|\, U, \phi \,|\, U \cap R) : (U, U \cap R) \to V$$

is locally trivial. □

2.3. The topological discriminant locus of an algebraic morphism

Let X and Y be smooth irreducible quasi-projective varieties, and $\phi : X \to Y$ a *dominant* morphism.

Definition 2.6. The topological discriminant locus Σ_ϕ is the minimal Zariski closed subset among the Zariski closed subsets Σ of Y with the following properties;

- Σ contains the locus $\phi(\operatorname{Sing}\phi)$ of the critical values of ϕ, and
- ϕ is locally trivial over $Y \setminus \Sigma$. (See Notation and terminologies (1) and (6).)

There always exists the topological discriminant locus Σ_ϕ (possibly $\Sigma_\phi = \emptyset$) such that $\Sigma_\phi \neq Y$ ([8, Lemma 1.5 (A)]).

2.4. Bertini's Theorem

See [4, Theorem 1.1] for the proof of the following:

Lemma 2.7. *Let $X \subset \mathbb{P}^N$ be an irreducible quasi-projective variety of codimension c in $\mathbb{P}^N$, and let Λ be a general linear subspace of $\mathbb{P}^N$ with dimension $c+1$.*

(1) *The scheme-theoretic intersection $\Lambda \cap X$ is an irreducible curve.*

(2) *Suppose that* $\dim \operatorname{Sing} X \leq \dim X - 2$. *Then $\Lambda \cap X$ is smooth.*

(3) *Suppose that X is smooth. Then the inclusion $\Lambda \cap X \hookrightarrow X$ induces a surjective homomorphism from $\pi_1(\Lambda \cap X)$ to $\pi_1(X)$.* □

3. Nori's lemma

Let $f : E \to B$ be as in §1. We denote by $\Sigma_f \subset B$ the topological discriminant locus of f. We will consider Σ_f as a *reduced* subscheme of B. In particular, we have

$$\dim \operatorname{Sing} \Sigma_f \leq \dim B - 2.$$

Since we have assumed that the general fiber of $\bar{f}$ is connected, F_a is connected for any $a \in B \setminus \Sigma_f$. Since $b \in B$ is general, we can assume $b \notin \Sigma_f$.

Proposition 3.1 ([8], Lemma 1.5 (C)). *Suppose that the Zariski closed subset*

$$\Xi_1 := \{\, a \in B \mid F_a \setminus (F_a \cap \operatorname{Sing} f) = \emptyset \,\}^{-}$$

is of codimension ≥ 2 in B. Then the sequence

$$\pi_1(F_b, \tilde{b}) \xrightarrow{i_*} \pi_1(E, \tilde{b}) \xrightarrow{f_*} \pi_1(B, b) \longrightarrow 1$$

is exact.

We hope that the proof of Proposition 3.1 given below will be helpful in understanding the idea of the proof of Theorem 1.1. First we prove the following lemmas, which will be also used in §4.

Lemma 3.2. *Let $\xi : I \to B \setminus \Sigma_f$ be a path. Suppose that points $p_0 \in F_{\xi(0)}$ and $p_1 \in F_{\xi(1)}$ are given. Then there exists a lift $\widetilde{\xi} : I \to E \setminus f^{-1}(\Sigma_f)$ of ξ that satisfies $\widetilde{\xi}(0) = p_0$ and $\widetilde{\xi}(1) = p_1$.*

Proof. Since f is locally trivial over ξ, the pull-back $\xi^* E \to I$ of f by ξ is trivial. Since the fiber of $\xi^* E \to I$ is connected, the existence of the lift $\widetilde{\xi}$ connecting p_0 and p_1 follows. □

Lemma 3.3. *Let $v : I \times I \to B$ be a continuous map that intersects Σ_f transversely and satisfies $v^{-1}(\Xi_1) = \emptyset$. Suppose that we are given a lift*

$$(v_0)^{\sim} : I \times \{0\} \to E$$

of the restriction $v_0 := v \,|\, I \times \{0\}$ of v to $I \times \{0\}$. Then there exists a lift

$$\widetilde{v} : I \times I \to E$$

of v such that the restriction $\widetilde{v} \,|\, I \times \{0\}$ of $\widetilde{v}$ to $I \times \{0\}$ is equal to $(v_0)^{\sim}$, and that $\widetilde{v}^{-1}(\operatorname{Sing} f)$ is empty.

Proof. Since v intersects Σ_f transversely, $v^{-1}(\Sigma_f)$ is a finite set of points of $(I \times I)^\circ$. In particular, the image of $(v_0)^{\sim}$ is disjoint from $\operatorname{Sing} f$, because $\operatorname{Sing} f \subset f^{-1}(\Sigma_f)$ by the definition of the topological discriminant locus. We put

$$v^{-1}(\Sigma_f) := \{p_1, \ldots, p_N\}.$$

Since $v(p_i) \notin \Xi_1$, there exists at least one point of $F_{v(p_i)}$ at which f is smooth. Hence there is an open neighborhood U_i of $v(p_i)$ on which a holomorphic local section

$$s_i : U_i \to f^{-1}(U_i)$$

of f is defined. Note that $s_i(U_i) \cap \operatorname{Sing} f = \emptyset$. There exists a homeomorphism

$$\psi : I \times I \xrightarrow{\sim} I \times I$$

such that $\psi \,|\, I \times \{0\}$ is the identity map of $I \times \{0\}$, and that

$$\psi(p_i) = (i/(N+1), 1/2) \in (I \times I)^\circ \quad \text{for every} \quad p_i \in v^{-1}(\Sigma_f).$$

We choose a sufficiently small positive real number ρ, and put

$$\Delta'_i := \{ (x, y) \in I \times I \mid (x - i/(N+1))^2 + (y - 1/2)^2 \leq \rho^2 \},$$
$$Z'_i := \{ (i/(N+1), t) \in I \times I \mid t \in [0, 1/2 - \rho] \}.$$

We then put

$$\Delta_i := \psi^{-1}(\Delta'_i), \quad Z_i := \psi^{-1}(Z'_i) \quad \text{and} \quad Q := (I \times \{0\}) \cup \bigcup_{i=1}^{N} (\Delta_i \cup Z_i).$$

Since ρ is small enough, we can assume that $v(\Delta_i) \subset U_i$. Therefore a lift

$$(v \,|\, \Delta_i)^{\sim} \,:\, \Delta_i \to E$$

of $v \,|\, \Delta_i$ can be defined by

$$(v \,|\, \Delta_i)^{\sim} := s_i \circ (v \,|\, \Delta_i).$$

Note that

$$(v \,|\, \Delta_i)^{\sim}(\Delta_i) \cap \operatorname{Sing} f = \emptyset. \tag{3.1}$$

We put

$$(I \times \{0\}) \cap Z_i = \{r_i\}, \quad \Delta_i \cap Z_i = \{q_i\}, \quad \text{and} \quad \tilde{q}_i := (v \,|\, \Delta_i)^{\sim}(q_i) \in E.$$

Since $Z_i \cap v^{-1}(\Sigma_f) = \emptyset$, $v(Z_i)$ is contained in $B \setminus \Sigma_f$. By Lemma 3.2, we have a lift

$$(v \,|\, Z_i)^{\sim} \,:\, Z_i \to E$$

of $v \,|\, Z_i$ such that $(v \,|\, Z_i)^{\sim}(r_i) = (v_0)^{\sim}(r_i)$ and $(v \,|\, Z_i)^{\sim}(q_i) = \tilde{q}_i$. Then a lift

$$(v \,|\, Q)^{\sim} \,:\, Q \to E$$

of $v \,|\, Q$ can be defined by

$$(v \,|\, Q)^{\sim}(p) := \begin{cases} (v_0)^{\sim}(p) & \text{if } p \in I \times \{0\}, \\ (v \,|\, Z_i)^{\sim}(p) & \text{if } p \in Z_i, \\ (v \,|\, \Delta_i)^{\sim}(p) & \text{if } p \in \Delta_i. \end{cases}$$

Note that $v^{-1}(\Sigma_f)$ is contained in the interior of Q. Hence f is locally trivial over the restriction $v \,|\, ((I \times I) \setminus Q^{\circ})$ of v to $(I \times I) \setminus Q^{\circ}$. Since Q is a strong deformation retract of $I \times I$, we can extend $(v \,|\, Q)^{\sim}$ to a lift

$$\widetilde{v} \,:\, I \times I \to E$$

of v. Since $\operatorname{Sing} f$ is contained in $f^{-1}(\Sigma_f)$, we have

$$\widetilde{v}^{-1}(\operatorname{Sing} f) \subset v^{-1}(\Sigma_f) \subset \cup \Delta_i.$$

Since $\widetilde{v} \,|\, \Delta_i$ coincides with $(v \,|\, \Delta_i)^{\sim}$, we have $\widetilde{v}^{-1}(\operatorname{Sing} f) = \emptyset$ by (3.1). By construction, $\widetilde{v} \,|\, I \times \{0\}$ coincides with $(v_0)^{\sim}$. □

Proof of Proposition 3.1. The surjectivity of f_* follows immediately from the connectedness of the general fiber of f. It is also obvious that $\operatorname{Im} i_*$ is contained in $\operatorname{Ker} f_*$. Hence all we have to prove is $\operatorname{Ker} f_* \subseteq \operatorname{Im} i_*$.

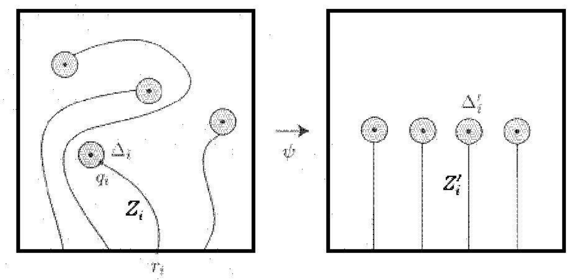

FIG. 3.1. The curve segment Z_i connecting r_i and q_i

Suppose that a loop

$$u : (I, \partial I) \to (E, \tilde{b})$$

represents an element $[u]$ of $\operatorname{Ker} f_*$. By Corollary 2.4, we can assume that $u^{-1}(f^{-1}(\Sigma_f)) = \emptyset$. There is a homotopy

$$w : (I, \partial I) \times I \to (B, b)$$

from the loop

$$f \circ u : (I, \partial I) \to (B, b)$$

to the constant loop 0_b. We have $w^{-1}(\Sigma_f) \subset (I \times I)^\circ$. Note that $\Xi_1 \subset \Sigma_f$. By Corollary 2.4 and the assumption on the codimension of Ξ_1, we can assume that w intersects Σ_f transversely, and that $w^{-1}(\Xi_1)$ is empty. We put

$$K := (\partial I \times I) \cup (I \times \{0\}).$$

We can define a lift

$$(w \,|\, K)^\sim : K \to E$$

of $w \,|\, K$ by

$$(w \,|\, K)^\sim(p) := \begin{cases} \tilde{b} & \text{if } p \in \partial I \times I, \\ u(t) & \text{if } p = (t, 0) \in I \times \{0\}. \end{cases}$$

There is a homeomorphism

$$\zeta : (I \times I, I \times \{0\}) \xrightarrow{\sim} (I \times I, K)$$

that is locally diffeomorphic at each point of $(w \circ \zeta)^{-1}(\Sigma_f)$. Then $w \circ \zeta$ intersects Σ_f transversely. We define a lift of $(w \circ \zeta) \,|\, I \times \{0\}$ to be the composite $(w \,|\, K)^\sim \circ (\zeta \,|\, I \times \{0\})$. This lift can be extended to a lift $(w \circ \zeta)^\sim$ of $w \circ \zeta$ by Lemma 3.3. Hence we can extend $(w \,|\, K)^\sim$ to a lift

$$\widetilde{w} : (I, \partial I) \times I \to (E, \tilde{b})$$

of w by defining $\widetilde{w} := (w \circ \zeta)^\sim \circ \zeta^{-1}$. We define a loop

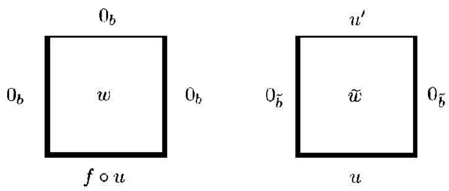

FIG. 3.2. The homotopy w and its lift $\widetilde{w}$

$$u' : (I, \partial I) \to (F_b, \tilde{b})$$

to be the restriction $\widetilde{w} \,|\, I \times \{1\}$ of $\widetilde{w}$ to $I \times \{1\}$. Then $\widetilde{w}$ yields a homotopy from the given loop u in E to the loop u' in F_b. Therefore we have

$$[u] = [u'] \in \operatorname{Im} i_*.$$

Thus $\operatorname{Ker} f_* \subseteq \operatorname{Im} i_*$ is proved. □

4. The boundary homomorphism

We consider the following condition on $f : E \to B$.

Condition (S). There exists a locally closed smooth irreducible subvariety S of E with the following properties.

Let $g : S \to B$ denote the restriction of f to S. For a point $a \in B$, we denote by G_a the fiber $g^{-1}(a)$.

(S1) The morphism g is dominant.

(S2) Let $\Sigma_g \subset B$ be the topological discriminant locus of g. Then the Zariski closed subset

$$\Xi_2 := \Sigma_f \cap \Sigma_g$$

of B is of codimension ≥ 2.

(S3) The Zariski closed subset

$$\Xi_3 := \{\, a \in B \mid G_a \setminus (G_a \cap \operatorname{Sing} f) = \emptyset \,\}^{=}$$

of B is of codimension ≥ 2.

(S4) If $a \in B$ is general, then G_a is connected, and the inclusion of G_a into F_a induces a surjective homomorphism from $\pi_1(G_a)$ to $\pi_1(F_a)$.

Proposition 4.1. *Suppose the following:*

(a) *The Zariski closed subset*

$$\Xi_4 := \{\, a \in B \mid F_a \setminus (F_a \cap \operatorname{Sing} f) \text{ is empty or not connected } \}^{=}$$

of B is of codimension ≥ 2.

(b) *The condition* (S) *is satisfied.*

(c) *The Zariski closed subset* $\operatorname{Sing} f$ *of* E *is of codimension* ≥ 2.

Then there exists a homomorphism ∂ *from* $\pi_2(B,b)$ *to* $\pi_1(F_b,\tilde{b})$ *such that* $\operatorname{Ker} i_*$ *coincides with* $\operatorname{Im}\partial$.

Proof. There exists a proper Zariski closed subset $\Sigma_{(f,g)}$ of B containing the union $\Sigma_f \cup \Sigma_g$ such that the pair

$$(f,g) : (E,S) \to B$$

of the morphisms f and g are locally trivial over $B \setminus \Sigma_{(f,g)}$. Let $\Sigma'_{(f,g)}$ be the union of all irreducible components of $\Sigma_{(f,g)}$ that are not contained in $\Sigma_f \cup \Sigma_g$. We put

$$\Xi_5 := \Sigma'_{(f,g)} \cap (\Sigma_f \cup \Sigma_g),$$

which is a Zariski closed subset of B with codimension ≥ 2. The Zariski closed subset

$$\Xi := \Xi_2 \cup \Xi_3 \cup \Xi_4 \cup \Xi_5$$

of B is of codimension ≥ 2 by the assumptions.

Definition 4.2. We say that a continuous map

$$w : (I^2, \partial I^2) \to (B,b)$$

is *good* if w intersects $\Sigma_{(f,g)}$ transversely, and $w^{-1}(\Xi)$ is empty. A lift

$$\widetilde{w} : I^2 \to E$$

of a good continuous map w is said to be a *good lift* if $\widetilde{w}(K)$ consists of a single point $\tilde{b}$, where $K := (I \times \{0\}) \cup (\partial I \times I)$, and $\widetilde{w}^{-1}(\operatorname{Sing} f)$ is empty.

Let $[w]$ be an element of $\pi_2(B,b)$. We can assume that $[w]$ is represented by a good continuous map w by Corollary 2.4. Because Σ_f is contained in $\Sigma_{(f,g)}$, w intersects Σ_f transversely. Because Ξ_1 in Proposition 3.1 is contained in Ξ_4, we have $w^{-1}(\Xi_1) = \emptyset$. There is a homeomorphism

$$\zeta : (I^2, I \times \{0\}) \xrightarrow{\sim} (I^2, K)$$

such that $w \circ \zeta$ intersects Σ_f transversely. By Lemma 3.3, we can lift $w \circ \zeta$ to

$$(w \circ \zeta)^\sim : (I^2, I \times \{0\}) \to (E, \tilde{b})$$

such that $(w\circ\zeta)^\sim(I^2) \cap \operatorname{Sing} f = \emptyset$. Putting $\widetilde{w} := (w \circ \zeta)^\sim \circ \zeta^{-1}$, we obtain a good lift

$$\widetilde{w} : (I^2, K) \to (E, \tilde{b})$$

of w. We will define $\partial([w])$ to be the element of $\pi_1(F_b, \tilde{b})$ represented by the loop

$$\widetilde{w} \,|\, I \times \{1\} : (I, \partial I) \to (F_b, \tilde{b}).$$

To show that this definition makes sense, we will prove that the homotopy class of the loop $\widetilde{w}\,|\,I \times \{1\}$ in F_b does not depend on the choice of a good continuous map w representing $[w]$, and a good lift $\widetilde{w}$ of w. Let

$$w_0 \,:\, (I^2, \partial I^2) \to (B, b) \quad \text{and} \quad w_1 \,:\, (I^2, \partial I^2) \to (B, b)$$

be good continuous maps representing a same element $[w] \in \pi_2(B, b)$, and let

$$\widetilde{w_0} \,:\, (I^2, K) \to (E, \tilde{b}) \quad \text{and} \quad \widetilde{w_1} \,:\, (I^2, K) \to (E, \tilde{b})$$

be good lifts of w_0 and w_1, respectively. There exists a homotopy

$$h \,:\, (I^2, \partial I^2) \times I \to (B, b)$$

from w_0 to w_1. We choose a sufficiently small positive real number τ, and let

$$\rho \,:\, I^2 \times I \to I^2 \times I$$

be the continuous map defined by

$$\rho(p, t) := \begin{cases} (p, 0) & \text{if } t \in [0, 4\tau], \\ (p, (t - 4\tau)/(1 - 8\tau)) & \text{if } t \in [4\tau, 1 - 4\tau], \\ (p, 1) & \text{if } t \in [1 - 4\tau, 1]. \end{cases}$$

Then the continuous map

$$h \circ \rho \,:\, (I^2, \partial I^2) \times I \to (B, b)$$

is also a homotopy from w_0 to w_1. We replace h by $h \circ \rho$. By the definition of good continuous maps, both of $w_0^{-1}(\Xi)$ and $w_1^{-1}(\Xi)$ are empty. Hence we have

$$h^{-1}(\Xi) \subset I^2 \times (4\tau, 1 - 4\tau).$$

Moreover, there exist open subsets A_0 and A_1 of I^2 satisfying

$$w_\nu^{-1}(\Sigma_{(f,g)}) \subset A_\nu \subset \overline{A_\nu} \subset (I^2)^\circ \qquad (\nu = 0, 1)$$

such that the restrictions

$$w_\nu \,|\, A_\nu \,:\, A_\nu \to B \qquad (\nu = 0, 1)$$

of w_ν to A_ν are $\mathcal{C}^\infty$-maps intersecting $\Sigma_{(f,g)}$ transversely. We put

$$A := (A_0 \times (0, 4\tau)) \cup (A_1 \times (1 - 4\tau, 1)).$$

Then $h\,|\,A$ is a $\mathcal{C}^\infty$-map intersecting $\Sigma_{(f,g)}$ transversely, and $(h\,|\,A)^{-1}(\Xi)$ is empty. There exist open subsets V and W of $I^2 \times I$ satisfying

$$\begin{aligned} h^{-1}(\Sigma_{(f,g)} \cup \Xi) \cap (I^2 \times [3\tau, 1 - 3\tau]) &\subset V \subset \overline{V} \\ &\subset W \subset \overline{W} \subset (I^2)^\circ \times (2\tau, 1 - 2\tau). \end{aligned}$$

By Corollary 2.3, we obtain a continuous map

$$h' \,:\, I^2 \times I \to B$$

with the following properties;

- h and h' are homotopic relative to $(I^2 \times I) \setminus W$, and
- the restriction $h' \,|\, A \cup V$ is a $\mathcal{C}^\infty$-map intersecting $\Sigma_{(f,g)}$ transversely, and its image is disjoint from Ξ.

In particular, h' is again a homotopy from w_0 to w_1 stationary on ∂I^2. We replace h by h'. Then the homotopy h has the following properties;

- $h^{-1}(\Xi) = \emptyset$,
- $h^{-1}(\Sigma_{(f,g)})$ is a one-dimensional manifold with the boundary, and the boundary $\partial h^{-1}(\Sigma_{(f,g)})$ is contained in $(I^2)^\circ \times \partial I$,
- $h(p,t) = h(p,0)$ if $t \in [0, 2\tau]$, while $h(p,t) = h(p,1)$ if $t \in [1-2\tau, 1]$.

Note that $h^{-1}(\Sigma_f)$, $h^{-1}(\Sigma_g)$ and $h^{-1}(\Sigma'_{(f,g)})$ are disjoint, and each of them is a union of connected components of $h^{-1}(\Sigma_{(f,g)})$.

We put

$$L := (K \times I) \cup (I^2 \times \partial I),$$

and define a lift

$$(h \,|\, L)^\sim \; : \; L \to E$$

of $h \,|\, L$ by the following;

$$(h \,|\, L)^\sim(p,t) := \begin{cases} \tilde{b} & \text{if } p \in K, \\ \widetilde{w_0}(p) & \text{if } t = 0, \\ \widetilde{w_1}(p) & \text{if } t = 1. \end{cases}$$

For the well-definedness of ∂, it is enough to show that the lift $(h \,|\, L)^\sim$ extends to a lift

$$\widetilde{h} \; : \; (I^2, K) \times I \to (E, \tilde{b})$$

of h; that is, there exists a lift $\widetilde{h}$ of h such that $\widetilde{h} \,|\, L$ coincides with $(h \,|\, L)^\sim$. Because the restriction of such a lift $\widetilde{h}$ to $I \times \{1\} \times I$ will yield a homotopy from the loop $\widetilde{w_0} \,|\, I \times \{1\}$ to the loop $\widetilde{w_1} \,|\, I \times \{1\}$ in F_b.

We will modify h by a homeomorphism

$$\Psi \; : \; I^2 \times I \xrightarrow{\sim} I^2 \times I.$$

We denote by

$$H \; : \; I^2 \times I \to B$$

the composite $h \circ \Psi$ of h and Ψ. We put

$$\Gamma := H^{-1}(\Sigma_f).$$

For a subset J of I, we put

$$\Gamma_J := \Gamma \cap (I^2 \times J).$$

When J consists of a single point t, we write Γ_t instead of $\Gamma_{\{t\}}$. By choosing an appropriate homeomorphism Ψ, we can assume the following.

(Ψ1) $\Psi^{-1}(L) = I^2 \times \{0\}$.

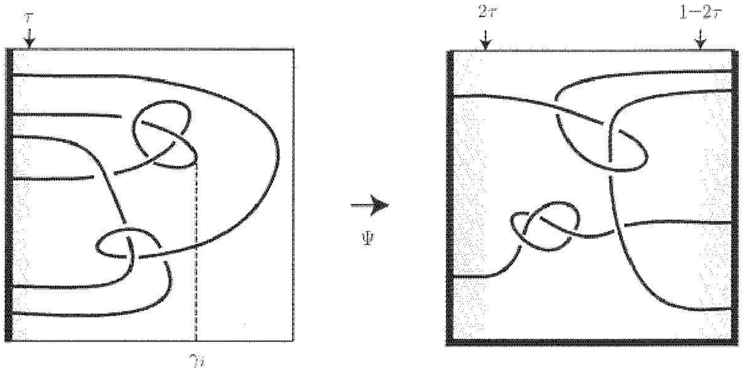

FIG. 4.1. The homeomorphism Ψ

(Ψ2) There exists an open subset A of $I^2 \times \{0\}$ such that
- $\Gamma_0 \subset A \subset \overline{A} \subset (I^2)^\circ \times \{0\}$, and
- $H(p,t) = H(p,0)$ if $(p,0) \in A$ and $t \in [0,\tau]$.

In particular, we have $\Gamma_{[0,\tau]} = \Gamma_0 \times [0,\tau]$.

(Ψ3) The closed subset Γ of $I^2 \times I$ is a closed submanifold of $(I^2)^\circ \times I$ with boundary. From (Ψ1) and (Ψ2) above, the boundary $\partial\Gamma$ is equal to Γ_0, which is a finite subset of points of A. The interior $\Gamma^\circ = \Gamma \setminus \partial\Gamma$ of Γ is a one-dimensional manifold.

(Ψ4) The projection $\varphi : \Gamma^\circ \to I^\circ$ given by $(p,t) \mapsto t$ is a Morse function. If $(q_i, \gamma_i) \in \Gamma^\circ$ and $(q_j, \gamma_j) \in \Gamma^\circ$ are distinct critical points of φ, then the critical values γ_i and γ_j are also distinct.

Let $(q_1, \gamma_1), \dots, (q_N, \gamma_N)$ be the critical points of φ. By renumbering, we have

$$\tau < \gamma_1 < \cdots < \gamma_N < 1.$$

For $t \in I$, we define

$$H_t : I^2 \times [0,t] \to B$$

to be the restriction of H to $I^2 \times [0,t]$. We can define a lift

$$\widetilde{H}_0 : I^2 \times \{0\} \to E$$

of H_0 by

$$\widetilde{H}_0 := (h \,|\, L)^\sim \circ (\Psi \,|\, I^2 \times \{0\}),$$

because Ψ induces a homeomorphism from $I^2 \times \{0\}$ to L by (Ψ1) above. It is enough to show that $\widetilde{H}_0$ extends to a lift

$$\widetilde{H}_1 : I^2 \times I \to E$$

of $H = H_1$, because $\widetilde{H}_1 \circ \Psi^{-1}$ will be the desired lift $\widetilde{h}$ of $h = H \circ \Psi^{-1}$.

Definition 4.3. We say that a lift $\widetilde{H}_t$ of H_t has the *property* (V) if there exists an open subset V_t of $I^2 \times \{t\}$ satisfying

- $\Gamma_t \subset V_t \subset \overline{V}_t \subset (I^2)^\circ \times \{t\}$, and
- $\widetilde{H}_t$ maps V_t to the subvariety S of E.

First we will extend $\widetilde{H}_0$ to a lift

$$\widetilde{H}_\tau \,:\, I^2 \times [0,\tau] \to E$$

of H_τ with the property (V). We put

$$\Gamma_0 = \{(c_1, 0), \dots, (c_m, 0)\}.$$

Then we have

$$H(c_\nu, t) = H(c_\nu, 0) \quad \text{for} \quad t \in [0, \tau].$$

We put

$$a_\nu := H(c_\nu, 0) \in \Sigma_f, \quad \text{and} \quad \alpha_\nu := \widetilde{H}_0(c_\nu, 0) \in F_{a_\nu}.$$

Because $\widetilde{w_0}^{-1}(\operatorname{Sing} f) = \emptyset$ and $\widetilde{w_1}^{-1}(\operatorname{Sing} f) = \emptyset$ by the definition of good lifts, the point $\alpha_\nu = (h\,|\,L)^\sim(\Psi(c_\nu, 0))$ is not contained in $\operatorname{Sing} f$. Because $h^{-1}(\Xi) = \emptyset$, we have $H^{-1}(\Xi) = \emptyset$, and hence $a_\nu \notin \Xi$. This implies the following;

- $F_{a_\nu} \setminus (F_{a_\nu} \cap \operatorname{Sing} f)$ is connected,
- $G_{a_\nu} \not\subset \operatorname{Sing} f$, and
- $a_\nu \notin \Sigma_g$. In particular, g is smooth at every point of G_{a_ν}.

We choose a point $\alpha'_\nu \in G_{a_\nu} \setminus (G_{a_\nu} \cap \operatorname{Sing} f)$. Then we can connect α_ν and α'_ν by a path

$$\xi_\nu \,:\, I \to F_{a_\nu} \setminus (F_{a_\nu} \cap \operatorname{Sing} f).$$

Since $\xi_\nu(I)$ is compact, there exist, by Lemma 2.5, an open neighborhood D_ν of a_ν in B and an open neighborhood M_ν of $\xi_\nu(I)$ in $f^{-1}(D_\nu)$ such that the pair

$$(f\,|\,M_\nu,\, g\,|\,M_\nu \cap S) \,:\, (M_\nu, M_\nu \cap S) \to D_\nu$$

is locally trivial. We choose a small closed disk Δ_ν in I^2 with the center c_ν such that

- $\Delta_\nu \times \{0\} \subset A$, where A is the open subset of $I^2 \times \{0\}$ that has appeared in $(\Psi 2)$, and
- $\widetilde{H}_0(p, 0) \in M_\nu$ for all $p \in \Delta_\nu$.

Then $H(p, 0) \in D_\nu$ for all $p \in \Delta_\nu$, and hence we have

$$H(\Delta_\nu \times [0, \tau]) \subset D_\nu.$$

We can define a lift

$$(H\,|\,(\{c_\nu\} \times [0,\tau]))^\sim \,:\, \{c_\nu\} \times [0,\tau] \to E$$

of $H\,|\,(\{c_\nu\} \times [0,\tau])$ by

$$(H\,|\,(\{c_\nu\} \times [0,\tau]))^\sim(c_\nu, t) := \xi_\nu(t/\tau) \in F_{a_\nu}.$$

This lift satisfies

$$(H\,|\,(\{c_\nu\}\times[0,\tau]))^\sim(c_\nu,0)=\alpha_\nu=\widetilde{H}_0(c_\nu,0),\quad \text{and}$$
$$(H\,|\,(\{c_\nu\}\times[0,\tau]))^\sim(c_\nu,\tau)=\alpha'_\nu\in G_{a_\nu}\cap M_\nu.$$

Since $g\,|\,(M_\nu\cap S)$ is locally trivial over D_ν, and $H(\Delta_\nu\times\{\tau\})$ is contained in D_ν, we have a lift

$$(H\,|\,\Delta_\nu\times\{\tau\})^\sim\;:\;\Delta_\nu\times\{\tau\}\;\to\;M_\nu\cap S$$

of $H\,|\,\Delta_\nu\times\{\tau\}$ with respect to $g\,|\,(M_\nu\cap S)$ such that

$$(H\,|\,\Delta_\nu\times\{\tau\})^\sim(c_\nu,\tau)=\alpha'_\nu.$$

We put

$$Q_\nu:=(\Delta_\nu\times\{0\})\;\cup\;(\{c_\nu\}\times[0,\tau])\;\cup\;(\Delta_\nu\times\{\tau\}).$$

By gluing $\widetilde{H}_0\,|\,\Delta_\nu\times\{0\}$, $(H\,|\,(\{c_\nu\}\times[0,\tau]))^\sim$ and $(H\,|\,\Delta_\nu\times\{\tau\})^\sim$ together, we obtain a lift

$$(H\,|\,Q_\nu)^\sim\;:\;Q_\nu\;\to\;E$$

of $H\,|\,Q_\nu$ such that $(H\,|\,Q_\nu)^\sim(Q_\nu)\subset M_\nu$. Since Q_ν is a strong deformation retract of $\Delta_\nu\times[0,\tau]$, and $\Delta_\nu\times[0,\tau]$ is mapped in D_ν by H, we can use the homotopy lifting property of the locally trivial map $f\,|\,M_\nu:M_\nu\to D_\nu$, and extend $(H\,|\,Q_\nu)^\sim$ to a lift

$$(H\,|\,\Delta_\nu\times[0,\tau])^\sim\;:\;\Delta_\nu\times[0,\tau]\;\to\;E$$

of $H\,|\,\Delta_\nu\times[0,\tau]$. We then put

$$R:=(I^2\times\{0\})\;\cup\;\bigcup_{\nu=1}^{m}(\Delta_\nu\times[0,\tau]).$$

Gluing $\widetilde{H}_0$ and $(H\,|\,\Delta_\nu\times[0,\tau])^\sim$ $(i=1,\dots,m)$ together, we obtain a lift

$$(H\,|\,R)^\sim\;:\;R\;\to\;E$$

of $H\,|\,R$. Since R is a strong deformation retract of $I^2\times[0,\tau]$, and f is locally trivial over

$$H\,|\,((I^2\times[0,\tau])\setminus R^\circ)\;:\;(I^2\times[0,\tau])\setminus R^\circ\;\to\;B,$$

we can extend $(H\,|\,R)^\sim$ to a lift

$$\widetilde{H}_\tau\;:\;I^2\times[0,\tau]\;\to\;E$$

of H_τ. By the construction, $\widetilde{H}_\tau$ is an extension of $\widetilde{H}_0$. Moreover, since $\cup_\nu(\Delta^\circ_\nu\times\{\tau\})$ is an open subset of $I^2\times\{\tau\}$ containing Γ_τ and is mapped into $\cup_\nu(M_\nu\cap S)\subset S$ by $\widetilde{H}_\tau$, this extension $\widetilde{H}_\tau$ has the property (V).

Next we prove the following:

Claim 4.4. *Suppose that we are given a closed interval $[u,v]$ contained in $[\tau,1]$ and a lift*

$$\widetilde{H}_u\;:\;I^2\times[0,u]\;\to\;E$$

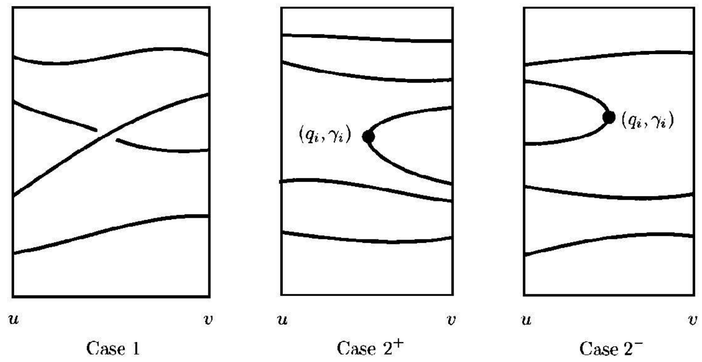

FIG. 4.2. The three cases

of H_u with the property (V). *Then $\widetilde{H}_u$ can be extended to a lift*

$$\widetilde{H}_v \, : \, I^2 \times [0, v] \rightarrow E$$

of H_v with the property (V) *in the following two cases:*

Case 1. The closed interval $[u, v]$ contains no critical values of the Morse function $\varphi : \Gamma^\circ \rightarrow I^\circ$. (See the condition ($\Psi$4).)

Case 2. There exists a critical value γ_i of φ such that $u = \gamma_i - \varepsilon$ and $v = \gamma_i + \varepsilon$, where ε is a sufficiently small positive real number.

Starting from $\widetilde{H}_\tau$ and repeating the extension of $\widetilde{H}_u$ to $\widetilde{H}_v$ in the above two cases, we can extend $\widetilde{H}_\tau$ to a lift $\widetilde{H}_1 : I^2 \times I \rightarrow I$ of $H = H_1$. The well-definedness of the boundary homomorphism $\partial : \pi_2(B) \rightarrow \pi_2(F_b)$ will thus be established.

Proof of Claim 4.4. We divide Case 2 into the following two sub-cases:

Case 2^+. The critical point (q_i, γ_i) of φ is of index $+1$.

Case 2^-. The critical point (q_i, γ_i) of φ is of index -1.

By the property (V) of $\widetilde{H}_u$, we have an open subset $V(\Gamma_u)$ of $I^2 \times \{u\}$ such that

$$\Gamma_u \, \subset \, V(\Gamma_u) \, \subset \, \overline{V(\Gamma_u)} \, \subset \, (I^2)^\circ \times \{u\},$$

and that $\widetilde{H}_u$ maps $V(\Gamma_u)$ into S. We decompose $\Gamma_{[u,v]}$ into the disjoint union of $\Gamma'_{[u,v]}$ and $\Gamma''_{[u,v]}$, where $\Gamma''_{[u,v]}$ is the union of all connected components of $\Gamma_{[u,v]}$ that do *not* contain any critical point of φ, and $\Gamma'_{[u,v]} := \Gamma_{[u,v]} \setminus \Gamma''_{[u,v]}$. In Case 1, $\Gamma'_{[u,v]}$ is empty, while in Case 2, $\Gamma'_{[u,v]}$ consists of a single connected component. In both cases, each connected component of $\Gamma''_{[u,v]}$ is mapped homeomorphically onto $[u, v]$ by the projection φ. We put

$$\Gamma''_u := \Gamma''_{[u,v]} \cap (I^2 \times \{u\}) = \{(c_1, u), \ldots, (c_k, u)\}.$$

There exists a trivialization

$$\begin{array}{ccc} (I^2 \times [u,v], \Gamma''_{[u,v]}) & \xrightarrow[\sigma]{\sim} & (I^2 \times \{u\}, \Gamma''_u) \times [u,v] \\ & \searrow \quad \swarrow & \\ & [u,v] & \end{array}$$

of the pair $(I^2 \times [u,v], \Gamma''_{[u,v]})$ of topological spaces over $[u,v]$ such that the restriction $\sigma \,|\, I^2 \times \{u\}$ of the homeomorphism σ is the identity map of $I^2 \times \{u\}$. For each point (c_ν, u) of Γ''_u, we choose a sufficiently small closed disk Δ_ν in I^2 with the center c_ν such that $\Delta_\nu \times \{u\} \subset V(\Gamma_u)$, and put

$$T_\nu := \sigma^{-1}(\Delta_\nu \times [u,v]), \qquad T := \bigcup_{\nu=1}^{k} T_\nu.$$

Then T is a closed tubular neighborhood of $\Gamma''_{[u,v]}$ in $I^2 \times [u,v]$. Since we have $H^{-1}(\Xi) = \emptyset$, $\Gamma \cap H^{-1}(\Sigma_g)$ is empty. Hence, by taking Δ_ν small enough, we can assume that

$$T \cap H^{-1}(\Sigma_g) = \emptyset.$$

Then g is locally trivial over $H \,|\, T$. Since

$$T_u := T \cap (I^2 \times \{u\}) = (\cup \Delta_\nu) \times \{u\}$$

is contained in $V(\Gamma_u)$, the property (V) of $\widetilde{H}_u$ implies

$$\widetilde{H}_u(T_u) \subset S.$$

Therefore, using the homotopy lifting property of the pull-back of $g : S \to B$ by $H \,|\, T : T \to B$, we can extend $\widetilde{H}_u \,|\, T_u$ to a lift

$$(H \,|\, T)^\sim : T \to S$$

of $H \,|\, T$ with respect to g. Gluing $\widetilde{H}_u$ and $(H \,|\, T)^\sim$ together, we obtain a lift

$$\widetilde{H}_{u,T} : (I^2 \times [0,u]) \cup T \to E$$

of the restriction

$$H \,|\, ((I^2 \times [0,u]) \cup T) : (I^2 \times [0,u]) \cup T \to B$$

of H that satisfies

$$\widetilde{H}_{u,T}(T) \subset S \tag{4.1}$$

Case 1. In this case, we have $\Gamma_{[u,v]} = \Gamma''_{[u,v]}$, and hence

$$(I^2 \times [0,v]) \setminus ((I^2 \times [0,u]) \cup T)^\circ \tag{4.2}$$

is disjoint from Γ. Hence f is locally trivial over the restriction of H to (4.2). Moreover, $(I^2 \times [0,u]) \cup T$ is a strong deformation retract of $I^2 \times [0,v]$. Hence we can extend $\widetilde{H}_{u,T}$ to a lift

$$\widetilde{H}_v : I^2 \times [0,v] \to E$$

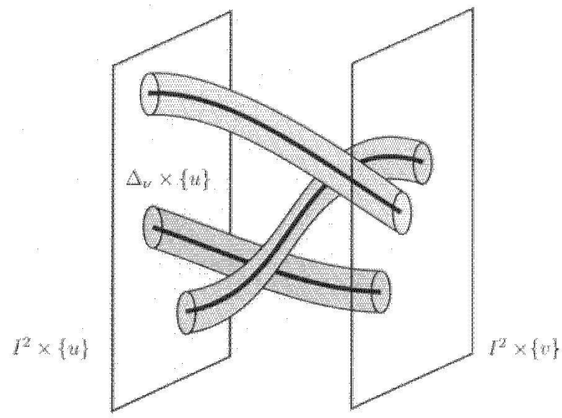

FIG. 4.3. Case 1

of H_v. Since

$$T_v^\circ := T^\circ \cap (I^2 \times \{v\})$$

is an open subset of $I^2 \times \{v\}$ containing Γ_v, and $\widetilde{H}_v(T_v^\circ) = \widetilde{H}_{u,T}(T_v^\circ)$ is contained in S, the lift $\widetilde{H}_v$ thus constructed has the property (V).

Case 2. Note that the critical point (q_i, γ_i) of the Morse function φ in the region $I^2 \times [u, v]$ is not contained in $H^{-1}(\Sigma_g)$, because $H^{-1}(\Xi) = \emptyset$. Note also that, taking the closed discs Δ_ν small enough, we can assume that

$$T \cap \Gamma'_{[u,v]} = \emptyset.$$

Case 2$^+$. Since ε is small enough, there exists a positive real number δ such that the closed subset

$$R := \{\ (p, t) \in I^2 \times [u, v] \ \mid\ |p - q_i| \le \delta,\ \ u + \varepsilon/2 \le t \le v\ \}$$

of $I^2 \times I$ satisfies the following;

$$\Gamma'_{[u,v]} \subset R, \quad T \cap R = \emptyset \quad \text{and} \quad H^{-1}(\Sigma_g) \cap R = \emptyset.$$

We put

$$\Lambda := \{q_i\} \times [u, u + \varepsilon/2],$$

which is a line segment connecting

$$r_0 := (q_i, u) \in I^2 \times \{u\} \quad \text{and} \quad r_1 := (q_i, u + \varepsilon/2) \in R.$$

Since ε is small enough, we can assume that

$$T \cap \Lambda = \emptyset.$$

We put

$$\beta_0 := \widetilde{H}_u(r_0) \in F_{H(r_0)}.$$

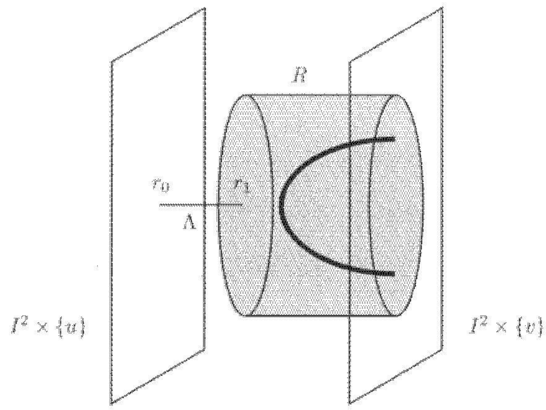

FIG. 4.4. Case 2^+

Since $H(r_1) \notin \Sigma_g$, we have a point

$$\beta_1 \in G_{H(r_1)}.$$

Since $H^{-1}(\Sigma_f) \cap \Lambda = \emptyset$, Lemma 3.2 gives us a lift

$$(H \,|\, \Lambda)^{\sim} \,:\, \Lambda \to E$$

of $H \,|\, \Lambda$ such that

$$(H \,|\, \Lambda)^{\sim}(r_0) = \beta_0 = \widetilde{H}_u(r_0) \quad \text{and} \quad (H \,|\, \Lambda)^{\sim}(r_1) = \beta_1 \in S.$$

Since g is locally trivial over $H \,|\, R$ and $\{r_1\}$ is a strong deformation retract of R, we have a lift

$$(H \,|\, R)^{\sim} \,:\, R \to S$$

of $H \,|\, R$ with respect to g such that

$$(H \,|\, R)^{\sim}(r_1) = \beta_1 = (H \,|\, \Lambda)^{\sim}(r_1).$$

We put

$$Q := (I^2 \times [0, u]) \cup T \cup \Lambda \cup R.$$

Gluing $\widetilde{H}_{u,T}$, $(H \,|\, \Lambda)^{\sim}$ and $(H \,|\, R)^{\sim}$ together, we obtain a lift

$$(H \,|\, Q)^{\sim} \,:\, Q \to E$$

of $H \,|\, Q$. Since the interior of Q contains $\Gamma_{[0,v]}$, f is locally trivial over

$$H \,|\, ((I^2 \times [0, v]) \setminus Q^{\circ}) \,:\, (I^2 \times [0, v]) \setminus Q^{\circ} \to B.$$

Moreover, Q is a strong deformation retract of $I^2 \times [0, v]$. Hence we can extend $(H \,|\, Q)^{\sim}$ to a lift

$$\widetilde{H}_v \,:\, I^2 \times [0, v] \to E$$

of H_v. Note that

$$(R^\circ \cup T^\circ) \cap (I^2 \times \{v\})$$

is an open subset of $I^2 \times \{v\}$ containing Γ_v, and it is mapped to S by $\widetilde{H}_v$ from the construction. Thus the lift $\widetilde{H}_v$ has the property (V).

Case 2⁻. Since $H^{-1}(\Xi) = \emptyset$, we have

$$H(q_i, \gamma_i) \notin \Sigma_g \cup \Sigma'_{(f,g)}.$$

Since ε is small enough, there exist an open ball V in $(I^2 \times I)^\circ$ containing (q_i, γ_i) and coordinates (x, y, z) defined on V such that the following hold:

- $V \supset \Gamma'_{[u,v]}$, $V \cap H^{-1}(\Sigma_g) = \emptyset$, $V \cap H^{-1}(\Sigma'_{(f,g)}) = \emptyset$, $V \cap T = \emptyset$.
- The critical point (q_i, γ_i) is the origin $(0, 0, 0)$ of the coordinates (x, y, z).
- The projection $V \to I$ onto the second factor of $I^2 \times I$ is given by
$$(x, y, z) \mapsto z + \gamma_i.$$
- The real one-dimensional curve $\Gamma'_{[u,v]}$ is given by
$$\{\ (0, s, \gamma_i - s^2)\ \mid\ -\sqrt{\varepsilon} \le s \le \sqrt{\varepsilon}\ \}$$
in terms of (x, y, z).

We introduce the usual Euclidean distance $\|\cdot, \cdot\|$ on V with respect to (x, y, z). Let ρ be a sufficiently small positive real number, and let P be the set of all $(p, t) \in V$ satisfying

- $t \in [u, v]$, and
- there exists a point $(p', t') \in \Gamma'_{[u,v]}$ such that $\|(p, t), (p', t')\| \le \rho$.

Then P is a closed tubular neighborhood of $\Gamma'_{[u,v]}$ in $I^2 \times [u, v]$. Taking ρ small enough, we can assume that

$$P_u := P \cap (I^2 \times \{u\})$$

is contained in the open neighborhood $V(\Gamma_u)$ of Γ_u in $I^2 \times \{u\}$. Let D be the connected component of

$$((V \cap (I^2 \times [u, v])) \setminus P^\circ) \cap \{x = 0\}$$

containing the point $(0, 0, -\varepsilon)$. Then D is homeomorphic to the 2-dimensional closed disk. We put

$$\Lambda_0 := D \cap (I^2 \times \{u\}), \quad \text{and} \quad \Lambda_1 := D \cap P.$$

Let r_0 and r_1 be the end-points of the line segment Λ_0. Then Λ_1 is a curve segment on the boundary of the tube P connecting r_0 and r_1, and the boundary ∂D of the disk D is $\Lambda_0 \cup \Lambda_1$. Since

$$H^{-1}(\Sigma_{(f,g)}) \cap D = (\Gamma \cup H^{-1}(\Sigma_g) \cup H^{-1}(\Sigma'_{(f,g)})) \cap D = \emptyset,$$

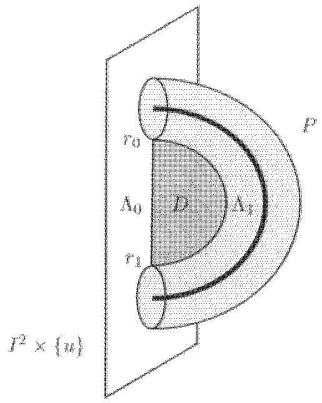

FIG. 4.5. Case 2^-

the pair (f, g) of the morphisms is locally trivial over $H \mid D$. Since D is contractible, we have a trivialization

$$\begin{array}{ccc} ((H \mid D)^* E, (H \mid D)^* S) & \xrightarrow[\psi_D]{\sim} & (F, G) \times D \\ {\scriptstyle (H \mid D)^*(f,g)} \searrow & & \swarrow {\scriptstyle \mathrm{pr}_D} \\ & D & \end{array}$$

of the pull-back of (f, g) by $H \mid D$, where (F, G) is a pair of topological spaces homeomorphic to the general fiber of (f, g). The continuous map

$$\widetilde{H}_u \mid \Lambda_0 \; : \; \Lambda_0 \to E$$

naturally yields a lift

$$s_0 \; : \; \Lambda_0 \to (H \mid D)^* E$$

of $\Lambda_0 \hookrightarrow D$ with respect to $(H \mid D)^* f$. We fix a *homeomorphic* path

$$\lambda \; : \; I \to \Lambda_0$$

from r_0 to r_1, and define a path $s_0' : I \to F$ by

$$s_0'(t) := \mathrm{pr}_F(\psi_D(s_0(\lambda(t)))),$$

where

$$\mathrm{pr}_F \; : \; F \times D \to F$$

is the natural projection. Because r_0 and r_1 are contained in $P_u \subset V(\Gamma_u)$, $\widetilde{H}_u(r_0)$ and $\widetilde{H}_u(r_1)$ are contained in S. Hence we have

$$s_0'(0) \in G, \quad s_0'(1) \in G.$$

Note that both of F and G are path-connected. Moreover, from the item (S4) of the condition (S), the inclusion $G \hookrightarrow F$ induces a surjective homomorphism from $\pi_1(G)$ to $\pi_1(F)$. Since G is path-connected, we have a path

$$s'_2 \,:\, I \,\longrightarrow\, G$$

from $s'_0(1)$ to $s'_0(0)$. Then $s'_0 s'_2$ is a loop in F with the base point $s'_0(0)$. Hence there exists a loop

$$s'_{02} \,:\, (I, \partial I) \,\longrightarrow\, (G, s'_0(0))$$

such that

$$[s'_{02}] = [s'_0 s'_2] \quad \text{in} \quad \pi_1(F, s'_0(0)).$$

We define a path

$$s'_1 \,:\, I \,\longrightarrow\, G$$

from $s'_0(0)$ to $s'_0(1)$ to be the conjunction of the loop s'_{02} and the inverse path $(s'_2)^{-1}$ of s'_2 in G;

$$s'_1 := s'_{02} \cdot (s'_2)^{-1}.$$

Then s'_0 and s'_1 are homotopic in F relative to ∂I. Thus we have a homotopy

$$\eta \,:\, I \times I \,\longrightarrow\, F$$

from s'_0 to s'_1 stationary on ∂I. We have a continuous map

$$\tau \,:\, I \times I \,\longrightarrow\, D$$

such that

- τ maps $I \times \{0\}$ to Λ_0 and $I \times \{1\}$ to Λ_1 homeomorphically,
- τ contracts $\{0\} \times I$ to the point r_0, and $\{1\} \times I$ to the point r_1,
- the homeomorphism $\tau \,|\, I \times \{0\}$ from $I \times \{0\}$ to Λ_0 coincides with the homeomorphic path $\lambda : I \to \Lambda_0$, and
- τ induces a homeomorphism from $I^\circ \times I$ to $D \setminus \{r_0, r_1\}$.

Then there exists a unique continuous map

$$\eta_D \,:\, D \,\longrightarrow\, F$$

such that η factors as $\eta_D \circ \tau$. We define a continuous map

$$\widetilde{\eta}_D \,:\, D \,\longrightarrow\, F \times D$$

by $\widetilde{\eta}_D(p) := (\eta_D(p), p)$. Then we can define a lift

$$(H \,|\, D)^\sim \,:\, D \,\longrightarrow\, E$$

of $H \,|\, D$ by

$$(H \,|\, D)^\sim := \mathrm{pr}_E \circ \psi_D^{-1} \circ \widetilde{\eta}_D,$$

where $\mathrm{pr}_E \,:\, (H \,|\, D)^* E \,\longrightarrow\, E$ is the natural projection, and ψ_D is the homeomorphism from $(H \,|\, D)^* E$ to $F \times D$ that has appeared in the trivialization of $(H \,|\, D)^*(f, g)$ over D. Then we have

$$(H \,|\, D)^\sim \,|\, \Lambda_0 = \widetilde{H}_u \,|\, \Lambda_0,$$

because $(H \,|\, D)^{\sim} \,|\, \Lambda_0$ coincides with

$$\mathrm{pr}_E \circ s_0 \;:\; \Lambda_0 \;\to\; E$$

by the construction. Hence we can glue $\widetilde{H}_{u,T}$ and $(H \,|\, D)^{\sim}$ along Λ_0, and obtain a lift

$$\widetilde{H}_{u,T\cup D} \;:\; (I^2 \times [0,u]) \cup T \cup D \;\to\; E$$

of $H \,|\, ((I^2 \times [0,u]) \cup T \cup D)$. Since s_1' is a path in G, we have

$$\widetilde{H}_{u,T\cup D}(\Lambda_1) \subset S.$$

Since $P_u = P \cap (I^2 \times \{u\})$ is contained in $V(\Gamma_u)$, we have

$$\widetilde{H}_{u,T\cup D}(P_u) \subset S.$$

Since $P_u \cup \Lambda_1$ is a strong deformation retract of P, and g is locally trivial over $H \,|\, P$, we can extend $\widetilde{H}_{u,T\cup D} \,|\, (P_u \cup \Lambda_1)$ to a lift

$$(H \,|\, P)^{\sim} \;:\; P \;\to\; S$$

of $H \,|\, P$ with respect to g. We put

$$R := (I^2 \times [0,u]) \cup T \cup D \cup P.$$

Gluing $\widetilde{H}_{u,T\cup D}$ and $(H \,|\, P)^{\sim}$ together, we obtain a lift

$$(H \,|\, R)^{\sim} \;:\; R \;\to\; E$$

of $H \,|\, R$. Since R is a strong deformation retract of $I^2 \times [0,v]$, and f is locally trivial over $(I^2 \times [0,v]) \setminus R^{\circ}$, we can extend $(H \,|\, R)^{\sim}$ to a lift

$$\widetilde{H}_v \;:\; I^2 \times [0,v] \;\to\; E$$

of H_v. Since $T_v^{\circ} = T^{\circ} \cap (I^2 \times \{v\})$ is an open subset of $I^2 \times \{v\}$ containing Γ_v, and

$$\widetilde{H}_v(T_v^{\circ}) = \widetilde{H}_{u,T}(T_v^{\circ}) \;\subset\; S,$$

the lift $\widetilde{H}_v$ thus constructed have the property (V).

Thus the proof of Claim 4.4 is completed. □

The boundary map $\partial : \pi_2(B,b) \to \pi_1(F_b,\tilde{b})$ is now well-defined. If

$$w \;:\; (I^2, \partial I^2) \;\to\; (B,b) \quad \text{and} \quad w' \;:\; (I^2, \partial I^2) \;\to\; (B,b)$$

are good continuous maps, and

$$\widetilde{w} \;:\; (I^2, K) \;\to\; (E,\tilde{b}) \quad \text{and} \quad \widetilde{w}' \;:\; (I^2, K) \;\to\; (E,\tilde{b})$$

are their good lifts, then the continuous map

$$w + w' \;:\; (I^2, \partial I^2) \;\to\; (B,b)$$

defined by

$$(w+w')(s,t) := \begin{cases} w(2s,t) & \text{if } s \in [0,1/2] \\ w'(2s-1,t) & \text{if } s \in [1/2,1] \end{cases}$$

is also a good continuous map, and the continuous map

$$\widetilde{w} + \widetilde{w}' : (I^2, K) \to (E, \tilde{b})$$

defined in the same way is a good lift of $w + w'$. Hence ∂ is a homomorphism. Now we will prove that $\operatorname{Im}\partial$ coincides with $\operatorname{Ker} i_*$, using the condition (c). It is obvious that $\operatorname{Im}\partial$ is contained in $\operatorname{Ker} i_*$. Suppose that a loop

$$v : (I, \partial I) \to (F_b, \tilde{b})$$

represents an element of $\operatorname{Ker} i_*$. Then there exists a homotopy

$$\widetilde{w} : (I, \partial I) \times I \to (E, \tilde{b})$$

in E from the constant loop $0_{\tilde{b}}$ to v. Note that $\widetilde{w}(K)$ consists of a single point $\tilde{b}$. If $f^{-1}(\Xi)$ had an irreducible component of codimension 1 in E, then the assumption that the codimension of Ξ in B is ≥ 2 implies that this irreducible component of $f^{-1}(\Xi)$ would be contained in $\operatorname{Sing} f$. Hence $f^{-1}(\Xi)$ must be of codimension ≥ 2 in E. By Corollary 2.4, we can assume that

$$\widetilde{w}^{-1}(\operatorname{Sing} f) = \emptyset, \quad \widetilde{w}^{-1}(f^{-1}(\Xi)) = \emptyset,$$

and that $\widetilde{w}$ intersects $f^{-1}(\Sigma_{(f,g)})$ transversely. Then

$$w := f \circ \widetilde{w} : (I^2, \partial I^2) \to (B, b)$$

is a good continuous map, and $\widetilde{w}$ is a good lift of w. Hence we have $[v] = \partial([w])$. Therefore $\operatorname{Ker} i_*$ is contained in $\operatorname{Im}\partial$. □

5. Proof of Theorem 1.1

By Propositions 3.1 and 4.1, it is enough to show that the three conditions (C1)–(C3) in Theorem 1.1 imply the conditions (a)–(c) in Proposition 4.1. (Note that Ξ_1 is contained in Ξ_4.) Since Theorem 1.1 is trivial when $\dim \overline{E} = \dim B$ or $\dim B = 0$, we will assume

$$\dim \overline{E} > \dim B > 0.$$

The condition (c) follows immediately from (C2).

Claim 5.1. *The Zariski closed subset*

$$\Xi_6 := \{\, a \in B \mid \overline{F}_a \text{ is not irreducible } \}^{=}$$

of B is of codimension ≥ 2.

Proof of Claim 5.1. Suppose that Ξ_6 had an irreducible component Ξ_6' of codimension 1 in B. Let ξ be a general point of Ξ_6'. Since $\overline{F}_\xi$ is connected, there exist two irreducible components of $\overline{F}_\xi$ intersecting at a point p. Since ξ is general in Ξ_6', $\bar{f}^{-1}(\Xi_6')$ is not locally irreducible at p. Let A_1 and A_2 be distinct local irreducible components of $\bar{f}^{-1}(\Xi_6')$ at p. Then $A_1 \cap A_2$ is of codimension 2 in $\overline{E}$, because both of A_1 and A_2 are hypersurfaces in the smooth variety $\overline{E}$. Since $A_1 \cap A_2$ is contained in $\operatorname{Sing} f$, we get a contradiction to (C2). □

Claim 5.2. *The Zariski closed subset Ξ_4 of B is of codimension ≥ 2.*

Proof of Claim 5.2. Suppose that Ξ_4 had an irreducible component Ξ_4' of codimension 1 in B. Let ξ be a general point of Ξ_4'. By Claim 5.1, $\overline{F}_\xi$ is irreducible. If F_ξ is empty, then $\overline{F}_\xi$ is contained in Z. Since ξ is general in Ξ_4', the hypersurface $\bar{f}^{-1}(\Xi_4')$ would be contained in Z, and hence we get a contradiction to (C1). If F_ξ is non-empty and contained in $\operatorname{Sing} f$, then $\overline{F}_\xi$ is contained in $\operatorname{Sing} \bar{f}$ and hence we get a contradiction to (C3). Therefore $F_\xi \setminus (F_\xi \cap \operatorname{Sing} f)$ is non-empty. Since $\xi \in \Xi_4$, the Zariski open subset $F_\xi \setminus (F_\xi \cap \operatorname{Sing} f)$ of $\overline{F}_\xi$ must be not connected, which contradicts to the irreducibility of $\overline{F}_\xi$. □

Thus all we have to prove is that the condition (S) is satisfied. Let $\Sigma_f^{(1)}, \dots, \Sigma_f^{(k)}$ be the irreducible components of Σ_f with codimension 1 in B, and let $\zeta(i)$ be a general point of $\Sigma_f^{(i)}$. By Claim 5.1, $\overline{F}_{\zeta(i)}$ is irreducible. By the condition (C2), $\overline{F}_{\zeta(i)} \cap \operatorname{Sing} \bar{f}$ is of codimension ≥ 2 in $\overline{F}_{\zeta(i)}$. By the condition (C3), $Z_{\zeta(i)}$ is reduced.

Since $\overline{E}$ is quasi-projective, we can embed $\overline{E}$ into a projective space $\mathbb{P}^N$. We choose a general linear subspace Λ of $\mathbb{P}^N$ with

$$\dim \Lambda := N - \dim \overline{E} + \dim B + 1.$$

Let a be a general point of B, and let d be the degree of Z_a in $\mathbb{P}^N$. Then $\overline{F}_a \cap \Lambda$ is a smooth connected projective curve, and $Z_a \cap \Lambda$ consists of distinct d points. Moreover, by Lemma 2.7, the inclusion $F_a \cap \Lambda \hookrightarrow F_a$ induces a surjective homomorphism from the fundamental group of the punctured Riemann surface $F_a \cap \Lambda$ to the fundamental group of F_a. Moreover $\overline{F}_{\zeta(i)} \cap \Lambda$ is a smooth connected projective curve disjoint from $\operatorname{Sing} \bar{f}$ with genus equal to the genus of $\overline{F}_a \cap \Lambda$ by Lemma 2.7, and $Z_{\zeta(i)} \cap \Lambda$ consists of distinct d points.

We put

$$S := \Lambda \cap E$$

Then the general fiber G_a of g is a compact Riemann surface minus d distinct points, and its fundamental group is mapped surjectively onto the fundamental group of F_a. Since the genus of the compactification of the fibers of g and the number of the punctured points do not vary locally around $\zeta(i)$, the point $\zeta(i)$ is not contained in Σ_g. Hence $\Xi_2 = \Sigma_f \cap \Sigma_g$ is of codimension ≥ 2 in B.

Let $Z^{(1)}, \dots, Z^{(n)}$ be the irreducible components of Z. Note that

$$\dim Z^{(j)} = \dim \overline{E} - 1 > N - \dim \Lambda,$$

because we have assumed $\dim B > 0$. Since Λ is general, Lemma 2.7 implies that $Z^{(j)} \cap \Lambda$ is irreducible of dimension equal to $\dim B$, and the restriction of $\bar{f}$ to $Z^{(j)} \cap \Lambda$ is generically finite. Therefore the locus

$$\{ \, a \in B \mid \dim(Z_a \cap \Lambda) \geq 1 \, \}$$

is contained in a Zariski closed subset of B with codimension ≥ 2. Hence the locus $\{a \in B \mid G_a = \emptyset\}$ is also contained in a Zariski closed subset of B with codimension ≥ 2. Therefore, if Ξ_3 had an irreducible component of codimension ≤ 1 in B, then it must be contained in $f(\operatorname{Sing} f)$ and hence in Σ_f. Because $G_{\zeta(i)}$ is non-empty and disjoint from $\operatorname{Sing} f$, $\zeta(i)$ is not contained in Ξ_3. Hence Ξ_3 is of codimension ≥ 2 in B. Therefore the condition (S) is satisfied. □

6. Proof of Theorem 1.2

For a line bundle $L \to C$ of degree d on C, we denote by $[L] \in \operatorname{Pic}^d(C)$ the corresponding point of the Picard variety. We consider the natural morphism

$$\bar{\tau} : \operatorname{Sym}^d C \to \operatorname{Pic}^d(C),$$

which is smooth and projective. The fiber $\bar{\tau}^{-1}([L])$ of $\bar{\tau}$ over $[L] \in \operatorname{Pic}^d(C)$ is identified with the projective space

$$P_L^\vee = \mathbb{P}_* H^0(C, L)$$

of one-dimensional linear subspaces of $H^0(C, L)$. The dual hypersurface $C_L^\vee \subset P_L^\vee$ of the curve $C_L \subset P_L$ is the intersection of $\bar{\tau}^{-1}([L])$ with the hypersurface $\Delta_C^d \subset \operatorname{Sym}^d C$. We equip Δ_C^d with the reduced structure. The degree of $C_L^\vee$ does not depend on $[L]$. Hence the scheme-theoretic intersection of $\bar{\tau}^{-1}([L])$ and Δ_C^d is reduced for any $[L] \in \operatorname{Pic}^d(C)$. We denote by τ the restriction of $\bar{\tau}$ to $\operatorname{Sym}^d C \setminus \Delta_C^d$. Then Theorem 1.1 implies that the sequence

$$1 \longrightarrow \pi_1(P_L^\vee \setminus C_L^\vee) \longrightarrow \pi_1(\operatorname{Sym}^d C \setminus \Delta_C^d) \xrightarrow{\tau_*} \pi_1(\operatorname{Pic}^d(C)) \to 1,$$

is exact for a general $[L] \in \operatorname{Pic}^d(C)$, because $\pi_2(\operatorname{Pic}^d(C)) = 0$. Therefore $\pi_1(P_L^\vee \setminus C_L^\vee)$ is isomorphic to the kernel of the natural homomorphism

$$B(C, d) \to H_1(C, \mathbb{Z}),$$

because τ_* is identified with this homomorphism. □

References

[1] J. S. Birman, On braid groups, *Comm. Pure Appl. Math.* **22** (1968), 41–72.

[2] J. S. Birman, Mapping class groups and their relationship to braid groups, *Comm. Pure Appl. Math.* **22** (1969), 213–238.
[3] I. Dolgachev and A. Libgober, On the fundamental group of the complement to a discriminant variety, *Algebraic geometry* (Chicago, Ill., 1980), 1–25, Springer, Berlin, 1981.
[4] W. Fulton and R. Lazarsfeld, Connectivity and its applications in algebraic geometry, *Algebraic geometry* (Chicago, Ill., 1980), 26–92, Springer, Berlin, 1981.
[5] M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Graduate Texts in Mathematics, Vol. 14, Springer-Verlag, New York, 1973.
[6] J. Kaneko, On the fundamental group of the complement to a maximal cuspidal plane curve, *Mem. Fac. Sci. Kyushu Univ. Ser. A* **39** (1985), no. 1, 133–146.
[7] V. S. Kulikov and I. Shimada, *On the fundamental groups of complements to dual hypersurfaces of projective curves*, preprint, MPI 1996 - 32 (1996).
[8] M. V. Nori, Zariski's conjecture and related problems, *Ann. Sci. École Norm. Sup.* (4) **16** (1983), no. 2, 305–344.
[9] G. P. Scott, Braid groups and the group of homeomorphisms of a surface, *Proc. Cambridge Philos. Soc.* **68** (1970), 605–617.
[10] I. Shimada, Fundamental groups of open algebraic varieties, *Topology* **34** (1995), no. 3, 509–531.

Ichiro Shimada
Department of Mathematics
Hokkaido University
Sapporo 060-0810
Japan
e-mail: shimada@math.sci.hokudai.ac.jp

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