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Realizing connected Lie groups as automorphism groups of complex manifolds

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Abstract. We show that every connected real Lie group can be realized as the full automorphism group of a Stein hyperbolic complex manifold.

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1. Introduction

Saerens and Zame, and independently Bedford and Dadok proved that, given a compact real Lie group K there always exists a strictly pseudoconvex bounded domain $D \subset \mathbb{C}^n$ such that $\mathrm{Aut}(D) \simeq K$. By the theorem of Wong–Rosay (which states that every strictly pseudoconvex bounded domain with non-compact automorphism group is isomorphic to the ball) it is clear that an arbitrary non-compact real Lie group can not be realized as the automorphism of a strictly pseudoconvex bounded domain in $\mathbb{C}^n$. However, as we proved in an earlier paper [16], for any connected real Lie group G there does exist a complex manifold X on which G acts effectively. Moreover, X can be chosen in such a way that it enjoys several of the key properties of strictly pseudoconvex bounded domains. Namely, X can be chosen such that it is both Stein and hyperbolic in the sense of Kobayashi.

The purpose of the present note is to prove that it is possible to rule out additional automorphisms, i.e. it is possible to achieve $\mathrm{Aut}(X) \simeq G$.

Theorem 1. *Let G be a connected real Lie group. Then there exists a Stein, complete hyperbolic complex manifold X on which G acts effectively, freely, properly and with totally real orbits such that* $\mathrm{Aut}(X) \simeq G$.

The idea is to follow the strategy of Saerens and Zame: Construct the desired manifold as an open subset of a larger Stein manifold in such a way that the given group acts on this open subset. Ensure that every automorphism of this open subset can be extended to the boundary, then modify the boundary in such a way that this CR-hypersurface simply has no automorphisms other than those from the given group. The latter can be done using the fact that a CR-hypersurface (unlike a complex manifold) does have local invariants. A principal difficulty in this approach is to obtain an extension of automorphisms of the open subset to the boundary. If one is concerned only with compact Lie groups, then one can work with a strictly pseudoconvex bounded domain D. For such a domain it is evident that for every automorphism ϕ of D there exists a sequence $x_n \in D$ such that both x_n and $\phi(x_n)$ converge to a strictly pseudoconvex point in the boundary. This is the starting point for the extension of the automorphism ϕ to the boundary ∂D.

Now, our goal is to obtain a result for arbitrary connected Lie groups, which are not necessarily compact.

This lack of compactness assumption creates some difficulties.

There are two main problems: First, an arbitrary non-compact Lie group is not necessarily linear. For instance, the universal cover of $SL_2(\mathbb{R})$ cannot be embedded into a linear group. Second, as already mentioned, the theorem of Wong–Rosay implies that in general a non-compact Lie group can not be realized as the full automorphism group of a strictly pseudoconvex bounded domain with smooth boundary. Thus we have to work with domains which are not bounded or where the boundary is not everywhere smooth. The trouble is that it is therefore no longer clear that for every automorphism ϕ there exists a sequence x_n in the domain such that both x_n and $\phi(x_n)$ converge to a nice point in the boundary.

In [15] a result similar to ours is claimed for certain Lie groups with a rather sketchy outline of a possible proof.

The first of the aforementioned two problems is dealt with by *assuming* the group G to be linear while the second problem is simply ignored. Since the second problem is in fact a serious obstacle, the proof sketched in [15] can not be regarded as complete.

We proceed in the following way: To deal with the first problem, we note that every Lie algebra is linear by the theorem of Ado. Therefore, in a certain sense, every Lie group is linear up to coverings and the first problem can be attacked by working carefully with coverings.

For the second problem, we use bounded domains whose boundaries are smooth outside an exceptional set E which is small in a certain sense. Exploiting this smallness we prove that for every automorphism ϕ there must exist a sequence x_n such that both x_n and $\phi(x_n)$ converge to a boundary point outside the “bad set” E.

Once this has been verified, we can prove (using arguments similar to those used in[13], [2]) that ϕ extends as holomorphic map near $\lim(x_n)$, and use the theory of Chern–Moser-invariants to deduce that ϕ was in fact given by left multiplication with an element of G.

1.1. Disconnected Lie groups

The result of Bedford and Dadok resp. Saerens and Zame is valid for all compact groups, not only connected ones. However, compactness implies that in this case there are no more than finitely many connected components.

We conjecture that our main theorem is valid for arbitrary real Lie groups, including those with finitely or countably infinitely many connected components.

As a first step regarding disconnected Lie groups, we proved in [17] that the statement of our main theorem does hold for countable discrete groups.

2. Linearization

Given a real Lie group G, we look for a bounded domain on which this group acts. For this purpose we use the theory of hermitian symmetric spaces.

We will need the following:

Proposition 1. *Let $\tilde{G}$ be a simply-connected real Lie group.*

Then there exists a natural number n and a Lie group homomorphism $\xi : \tilde{G} \to Sp(2n, \mathbb{R})$ such that the following conditions are fulfilled:

(1) *ξ has discrete fibers.*

(2) *The image $\xi(\tilde{G})$ is closed in $Sp(2n, \mathbb{R})$.*

Proof. By Ado's theorem there is an injective Lie algebra homomorphism $\mathrm{Lie}(\tilde{G}) \to \mathrm{Lie}\, GL(m, \mathbb{R})$ for some $m \in \mathbb{N}$. Since $\tilde{G}$ is simply-connected, this induces a Lie group homomorphism $\xi_0 : \tilde{G} \to GL(m, \mathbb{R})$ with discrete fibers. Let $V = \mathbb{R}^m$ and $W = V \oplus V^*$ where V^* is the vector space dual of V. Then W carries a natural symplectic structure given by

$$(v, \lambda) \cdot (v', \lambda') = \lambda(v') - \lambda'(v)$$

which is evidently preserved by the natural diagonal action of $GL(V)$ on W. Hence there is an embedding $i : GL(m, \mathbb{R}) \hookrightarrow Sp(2m, \mathbb{R})$.

Let $\xi_1 = i \circ \xi_0 : \tilde{G} \to Sp(2m, \mathbb{R})$, $H = \xi_1(\tilde{G})$ and H' its commutator group. Then H' is already closed in $Sp(2m, \mathbb{R})$. The quotient group H/H' is a connected commutative real Lie group, hence $H/H' \simeq (S^1)^k \times (\mathbb{R})^l$ for some $k, l \in \mathbb{N} \cup \{0\}$. It is easy to see that there is a closed embedding $j : H/H' \hookrightarrow Sp(2m', \mathbb{R})$ for some $m' \in \mathbb{N}$. Furthermore there is an embedding $\zeta : Sp(2m, \mathbb{R}) \times Sp(2m', \mathbb{R}) \hookrightarrow Sp(2n, \mathbb{R})$ with $n = m + m'$. Now let $\tau : H \to H/H'$ denote the natural projection and define $\xi : \tilde{G} \to Sp(2n, \mathbb{R})$ by

$$\xi(g) = \zeta(\xi_1(g), j(\tau(\xi_1(g)))). \qquad \square$$

3. Hermitian symmetric domains

For basic facts on symmetric spaces, see e.g. [9].

Let $S = Sp(2n, \mathbb{R})$ and let K denote a maximal compact subgroup. Then the quotient manifold $D_0 = S/K$ can be endowed with the structure of a hermitian symmetric domain. Furthermore there exist open embeddings ("Cayley transform")

$$D_0 \hookrightarrow \mathbb{C}^N \hookrightarrow Q$$

such that

(1) D_0 is relatively compact in $\mathbb{C}^N$,
(2) Q is a projective manifold (the "compact dual of D") and
(3) the $Sp(2n, \mathbb{R})$-action on D_0 extends to an $Sp(2n, \mathbb{C})$-action on Q.

Lemma 1. *Let Q be a complex manifold on which a complex Lie group $S_{\mathbb{C}}$ acts holomorphically and $D_0 \subset Q$ a non-empty open subset.*

Then there exists a natural number $m \in \mathbb{N}$ and points $p_1, \ldots, p_m \in D_0$ such that

$$\cap_{i=1}^m \{g \in S_{\mathbb{C}} : g(p_i) = p_i\} = \{e\}.$$

Proof. We choose a sequence of points $p_i \in D_0$ recursively. First p_1 is chosen arbitrarily. When $p_1, \ldots, p_k$ are already chosen, we define $I_k = \{g \in S_{\mathbb{C}} : g(p_i) = p_i,\ 1 \le i \le k\}$. Then we proceed as follows: If $\dim(I_k) > 0$, we choose p_{k+1} such that there is an element a_{k+1} in the connected component I_k^0 such that $a_{k+1}(p_{k+1}) \neq p_{k+1}$. This ensures $\dim I_{k+1} < \dim I_k$. If $\dim I_k = 0$, then I_k is countable. Thus

$$\Lambda = \cup_{g \in I_k \setminus \{e\}} \{x \in Q : g(x) = x\}$$

is a countable union of nowhere dense analytic subsets of Q. It follows that Λ is a set of measure zero for any Lebesgue class measure on Q. In particular $\Lambda \cap D_0 \neq D_0$ and we can choose $p_{k+1} \in D_0 \setminus \Lambda$. By the definition of Λ this choice enforces $I_{k+1} = \{e\}$. □

Proposition 2. *Let $\tilde{G}$ be a simply-connected real Lie group. Then there exists a discrete central subgroup Γ such that for $G = \tilde{G}/\Gamma$ the following properties hold:*

There exists a natural number N, a bounded domain $D \subset \mathbb{C}^N$, complex analytic subsets $E \subsetneq \mathbb{C}^N$, $Z \subset D$ and a G-action on Z such that

(1) *There is a G-invariant non-empty open subset Ω of Z such that G acts freely, properly, and with totally real orbits on Ω.*
(2) *The topological closure $\bar{\Omega}$ of Ω in $\mathbb{C}^N$ is contained inside $Z \cup E$.*
(3) $\Omega \cap E = \{\}$.
(4) $\dim_{\mathbb{C}}(\Omega) \ge 3$.

Proof. By prop. 1, there is a discrete central subgroup Γ of $\tilde{G}$ such that $G = \tilde{G}/\Gamma$ can be embedded into some $Sp(2n, \mathbb{R})$ as closed Lie subgroup. Let $D_0 =$

$Sp(2n,\mathbb{R})/K$ be the associated hermitian symmetric space and Q and $S_{\mathbb{C}} = Sp(2n,\mathbb{C})$ as described in the beginning of this section.

By lemma 1 there is a natural number m and a point $p \in D = D_0^m$ such that the diagonal $S_{\mathbb{C}}$-orbit in Q^m through p is free. Now let $\bar{E} = \{x \in Q^m : \dim S_{\mathbb{C}}(x) < \dim S_{\mathbb{C}}\}$, $Z' = S_{\mathbb{C}}(p)$ and $Z = Z' \cap D$. Because the $S_{\mathbb{C}}$-action on Q^m is algebraic, the $S_{\mathbb{C}}$-action on Q^m is algebraic as well. In particular every $S_{\mathbb{C}}$-orbit in Q^m is Zariski open in its closure. This implies in particular that $\overline{Z'} \subset Z' \cup E$.

Now G is closed in $Sp(2n,\mathbb{R})$ and $Sp(2n,\mathbb{R})$ is closed in $Sp(2n,\mathbb{C}) = S_{\mathbb{C}}$. We obtain a fiber bundle $\tau : S_{\mathbb{C}} \to G\backslash S_{\mathbb{C}}$, where $G\backslash S_{\mathbb{C}}$ denotes the quotient of $S_{\mathbb{C}}$ by the left action of G. Let $U \subset G\backslash S_{\mathbb{C}}$ be a relatively compact open contractible subset and $\Omega = \{g \cdot p : g \in \tau^{-1}(U)\}$.

Then Ω has the desired properties. (Concerning property (4), observe that $\dim_{\mathbb{C}}(\Omega) = \dim_{\mathbb{C}}(S_{\mathbb{C}}) \geq 3$ by our construction.) □

4. Chern–Moser-invariants

4.1. Chern–Moser-invariants

For every real-analytic strictly pseudoconvex CR-hypersurface M in a complex manifold X and every point $p \in M$ there is a system of local coordinates

$$(w; z) = (w; z_1, \dots, z_n)$$

($w, z_i \in \mathbb{C}$, $n+1 = \dim_{\mathbb{C}}(X)$) such that M can be written as $M = \{\rho < 0\}$ where ρ is a real-analytic function whose power series development is given as

$$\rho(w; z) = \Im(w) + ||z||^2 + \sum_{k,l \geq 2} F_{k,l,r}(\Re(w), z, \bar{z})$$

where $F_{k,l,r}$ is a polynomial of bidegree (k,l) in z and $\bar{z}$ and degree r in $\Re(w)$.

A point $p \in M$ is called *umbilical* if $F_{2,2,0} = 0$. For non-umbilical points we define scalar invariants $K_{k,l,r}$ (for $k, l \geq 2, r \in \mathbb{N}$) given by $K_{k,l,r} = ||F_{k,l,r}||^2$ where $|| \ ||$ denotes the euclidean norm, i.e., the norm induced by the scalar product for which the monomials in the coordinates $\Re(w), z_i, \bar{z}_i$ constitute an orthonormal basis.

If x, y are non-umbilical points on M such that the CR-hypersurface germs (M, x) and (M, y) are isomorphic, then all these invariant $K_{k,l,r}$ must assume the same values at x and y.

For convenient application later on, we define $K_d = \sum_{k+l=d} K_{k,l,0}$ for $d \geq 4$.

4.2. Jet bundles

We recall the notion of jets (see [8]): For manifolds X and Y and points $x \in X$, $y \in Y$, the set of k-jets $J^k(X,Y)_{x,y}$ is the set of equivalence classes of map germs where two real-analytic map germs are equivalent iff their respective Taylor series developments agrees up to order k. $J^k(X,Y)$ is the disjoint union of all $J^k(X,Y)_{x,y}$ (with $x \in X$ and $y \in Y$). There is a natural manifold structure on $J^k(X,Y)$ for which we obtain a fiber bundle (*"source map"*) $\alpha : J^k(X,Y) \to X$.

4.3. Transversality

We will need the *multijet transversality theorem* ([8], thm. 4.13). Let $X^{(s)}$ denote the space of those s-tuples $(x_1,\dots,x_s) \in X^s$ where the x_i are all *distinct* elements in X. Let

$$J_s^k(X,Y) \quad = \quad \{(f_1,\dots,f_s) \quad \in \quad \left(J^k(X,Y)\right)^s \quad : \quad \alpha(f_1),\dots,\alpha(f_s) \quad \in \quad X^{(s)}\}$$

Then each $f \in C^\infty(X,Y)$ induces a map $j_s^k(f) : X^{(s)} \to J_s^k(X,Y)$ in a natural way.

Let W be a submanifold of codimension c in $J_s^k(X,Y)$.

Then the *multijet transversality theorem* implies that the function space $C^\infty(X,Y)$ contains a *residual* subset A such that $\left(j_s^k(f)\right)^{-1}(W)$ is of codimension at least c in $X^{(s)}$.

Remark. (1) In the statement on the codimension, the codimension of the empty set is to be understood as $+\infty$.

(2) A subset of a topological space V is called *residual* if it is the intersection of countably many open dense subsets. If V has the *Baire property*, then every residual subset of V is dense. The function spaces $C^\infty(X,Y)$ and $C^\omega(X,Y)$ have the Baire property (for any pair of manifolds (X,Y)).

(3) Similar results hold for the function spaces of type C^ω, i.e. real-analytic mappings, which in fact can be deduced from the transversality results for C^∞-maps, using the fact that C^ω-maps are dense in C^∞.

(4) In the real-analytic category, W does not need to be smooth, it suffices if W is a (possibly singular) real-analytic subset. As explained in [13], this can be verified using the fact that a real analytic subset W admits a stratification $W = W_0 \supset W_1 \supset W_2 \dots$ such that each $W_k \setminus W_{k+1}$ is smooth.

4.4. A proposition

Let us now assume that there is a real Lie group G acting holomorphically on X with totally real orbits. Let us furthermore assume that the action in proper. Then orbits can be separated by invariant functions. Around any given point $p \in X$, we may choose local holomorphic coordinates x_i in such a way that $x_i(p) = 0 \ \forall i$ and

$$T_p\left(G \cdot p\right) \subset T_p\left(\{x : \Re(x_i) = 0 \ \forall i\}\right).$$

It follows that for a every real homogeneous polynomial P of degree k there is a G-invariant real-analytic function f defined on some open neighbourhood of p in X such that $P(x_1, \dots, x_n) = f(x) + O(||x||^{k+1})$. As a consequence, we obtain the statement below:

Lemma 2. *Let G be a real Lie group acting holomorphically and properly on a complex manifold X with totally real orbits. Assume $\dim_{\mathbb{C}}(X) \geq 2$.*

For $x \in X$, $t \in \mathbb{R}$ let $J^k_+(X,\mathbb{R})^G_{x,t}$ denote the set of all k-jets induced by germs of G-invariant functions f for which the CR-hypersurface germ defined by $f = t$ is strictly pseudoconvex around x. Let $J^k_+(X,\mathbb{R})^G = \cup_{x\in X, t\in\mathbb{R}} J^k_+(X,\mathbb{R})^G_{x,t}$.

Then $K_4 = \ldots = K_k = 0$ defines a real-analytic subspace of codimension at least $k-3$ in $J^k_+(X,\mathbb{R})^G$.

Now we can prove the proposition given below.

Proposition 3. *Let G be a real Lie group acting holomorphically and properly on a complex manifold X with totally real orbits. Assume $\dim_{\mathbb{C}}(X) \geq 2$. Let $p \in X$.*

Then $G \cdot p$ admits an open G-invariant neighbourhood Ω such that:

(1) *The inclusion map $G \cdot p \hookrightarrow \Omega$ is a homotopy equivalence.*
(2) *The boundary $\partial\Omega$ is everywhere smooth, real-analytic and strictly pseudoconvex.*
(3) *There exists a nowhere dense real-analytic subset $\Sigma \subset \partial\Omega$ such that for every $x, y \in \partial\Omega \setminus \Sigma$ the CR-hypersurface germs $(\partial\Omega, x)$, $(\partial\Omega, y)$ are isomorphic if and only if $x = g \cdot y$ for some $g \in G$.*

Proof. Let $r = \dim_{\mathbb{R}}(X) - \dim_{\mathbb{R}}(G)$, B the open unit ball in $\mathbb{R}^r$ and $i : B \hookrightarrow X$ a real-analytic embedding with $i(0) = p$ which is everywhere transversal to the G-orbits. Then $W = G \cdot i(B)$ is an open G-invariant neighbourhood of the G-orbit $G \cdot p$. Since the G-action on X is free and proper, we may and do assume that the map $G \times B \to W$ given by $(g, x) \mapsto g \cdot i(x)$ is bijective.

We define $\rho_0 \in C^\omega(W)$ via $\rho_0(g \cdot i(v)) = ||v||^2$ for $g \in G$, $v \in B$.

An easy calculation in local coordinates shows that

$$W_\epsilon = \{x \in W : \rho_0(x) < \epsilon\}$$

is strictly pseudoconvex for all sufficiently small $\epsilon > 0$. We fix now a number $1 > \delta > 0$ such that W_δ is strictly pseudoconvex.

Then W_δ is a G-invariant open neighbourhood of $G \cdot p$ fulfilling conditions (1) and (2) of the proposition. To achieve condition (3), we have to modify the defining function $\rho_0 - \delta$ of the open domain W_δ using the theory of Chern–Moser invariants.

Every function on $B \simeq i(B)$ extends uniquely to a G-invariant function on W; this yields a bijective map

$$\zeta : C^\omega(B) \longrightarrow C^\omega(W)^G.$$

Now let Θ be an open neighborhood of $(\zeta^{-1}(\rho_0), \delta)$ in $C^\omega(B) \times \mathbb{R}$ such that the following properties hold for all $(f,t) \in \Theta$:

(1) $\{v \in B : f(v) < t\}$ is a contractible relatively compact open subset with smooth boundary in B;

(2) The domain

$$\{x \in W : \zeta(f-t)(x) < 0\} = \{g \cdot i(v) : f(v) < t, v \in B, g \in G\}$$

is everywhere strictly pseudoconvex.

Let $J_+^k(B,\mathbb{R}) = \cup_{v \in B, t \in \mathbb{R}} J_+^k(B,\mathbb{R})_{v,t}$ where $J_+^k(B,\mathbb{R})_{v,t}$ denotes the set of all k-jets induced by germs of functions $f : (B,v) \to (\mathbb{R},t)$ for which the CR-hypersurface germ

$$\{g \cdot i(x) : g \in G, x \in B, f(x) = t\}$$

is strictly pseudoconvex around $i(v)$. For $k \in \mathbb{N}$, $4 \leq d \leq k$ we define functions $\tilde{K}_d$ on $J_+^k(B,\mathbb{R})$ as follows: If j is the k-jet at $v \in B$ for some map germ $f : (B,v) \to (\mathbb{R},t)$, then $\tilde{K}_d(j)$ is defined as the scalar invariant K_d for the CR-hypersurface $\{y \in W : (\zeta(f))(y) = t\}$ at $i(v)$.

We define the "umbilical locus":

$$U_k = \{j \in J_+^k(B,\mathbb{R}) : \tilde{K}_4(j) = 0\}$$

and the "locus of coinciding scalar invariants":

$$E_k = \{(j_1,j_2) \in J_+^k(B,\mathbb{R})^2 : \tilde{K}_d(j_1) = \tilde{K}_d(j_2)\ \forall 4 \leq d \leq k\}.$$

Since $J_+^d(B,\mathbb{R})$ is an open subset in $J^d(B,\mathbb{R})$, U_k and E_k can be regarded as locally closed real-analytic subset in $J^d(B,\mathbb{R})$ resp. $J_2^d(B,\mathbb{R})$.

Fix k such that $k-3 > 2\dim_{\mathbb{R}}(B)$. Then lemma 2 implies that the codimension of E_k exceeds the dimension of $B \times B$.

The *multijet transversality theorem* implies that there is a residual set $A \subset C^\omega(B,\mathbb{R})$ such that every $f \in A$ is transversal to both U_k and E_k.

Since A is *residual*, it is dense in $C^\omega(B,\mathbb{R})$. Therefore $A \times \mathbb{R}$ intersects the open set Θ. Let $(\rho_1, t_0) \in (A \times \mathbb{R}) \cap \Theta$. Let $\Sigma_0 \subset W$ be the set of all points $x \in W$ such that the CR-hypersurface $\{y \in W : \zeta(\rho_1)(y) = \zeta(\rho_1)(x)\}$ is umbilical at x. Then transversality of ρ_1 with respect to U_k implies that Σ_0 is a nowhere dense, locally closed real-analytic subset of W. As a consequence, we can find a real number t close to t_0 such that $(\rho_1, t) \in \Theta$ and such that $\Omega = \{y \in W : \zeta(\rho_1)(y) < t\}$ has the following property:

"$\Sigma_0 \cap \Omega$ is nowhere dense in Ω."

Now $(\rho_1, t) \in \Theta$ implies that conditions (1) and (2) are fulfilled for our choice of Ω. Furthermore transversality of ρ_1 with respect to E_k (in combination with $\mathrm{codim}_{\mathbb{R}}(E_k) > \dim_{\mathbb{R}}(B)$) implies that Ω fulfills condition (3) of the proposition. This completes the proof. □

5. Privalov's theorem

We are now in position to use the classical theorem of Privalov in order to show that for every automorphism ϕ there is a sequence x_n such that both x_n and $\phi(x_n)$ converge to a point in the good part of the boundary.

Proposition 4. *Let D be a bounded domain in $\mathbb{C}^N$, $E \subset \mathbb{C}^N$, $Z \subset D$ closed analytic subsets, Ω an open subset of Z, M its boundary in Z. Assume $E \cap \Omega = \{\}$. Assume that M is everywhere smooth and that the closure of Ω in $\mathbb{C}^N$ is contained in $\Omega \cup M \cup E$. Let Ω' be the closure of Ω in Z, i.e., $\Omega' = \Omega \cup M$.*

Furthermore let $\tilde{\Omega}$ denote the universal covering of Ω and $\pi : \tilde{\Omega}' \to \Omega'$ and $\tilde{M} \to M$ the corresponding coverings.

Then for every holomorphic automorphism $\phi \in \mathrm{Aut}(\tilde{\Omega})$ there is a sequence x_n in $\tilde{\Omega}$ and points $q, \bar{q} \in \tilde{M}$ such that $\lim x_n = q$ and $\lim \phi(x_n) = \bar{q}$.

Proof. Fix $\phi \in \mathrm{Aut}(\tilde{\Omega})$. Let Δ be the unit disk in $\mathbb{C}$, $\bar{\Delta}$ its closure in $\mathbb{C}$ and $\partial\Delta$ its boundary.

We choose a C^∞ map $\zeta : \bar{\Delta} \to \tilde{\Omega}'$ such that

(1) $\zeta|_\Delta$ maps Δ holomorphically into $\tilde{\Omega}$.

(2) $\zeta^{-1}(\tilde{M})$ is a subset of positive Lebesgue measure in $\partial\Delta \simeq S^1$.

Now we consider $\eta : \Delta \to \mathbb{C}^N$ given by $\eta = \pi \circ \phi \circ \zeta$. Then η is a N-tuple of bounded holomorphic functions. It follows ([10],[12]) that the non-tangential limit exists almost everywhere on $\partial\Delta$. For $t \in \partial\Delta$, let $\lim_{n-t} \eta(t)$ denote this non-tangential limit. Evidently $\lim_{n-t} \eta(t) \in \Omega' \cup E$ wherever defined. We claim that $A = \{t : \lim_{n-t} \eta(t) \in E\}$ is a set of measure zero. Indeed $t \in A$ implies that for every holomorphic function f on $\mathbb{C}^N$ which vanishes on E, we obtain

$$\lim_{n-t} (f \circ \eta)(t) = 0.$$

If A is not a set of measure zero, it would follow from Privalov's theorem ([10]) that $f \circ \eta$ would vanish for every such f. But this would imply $\eta(\Delta) \subset E$, contradicting $\eta(\Delta) \subset \Omega$. Thus A must be a set of measure zero. It follows that there exists a point $q \in \partial\Delta \cap \zeta^{-1}(\tilde{M})$ such that the non-tangential limit for η exists at q and is not in E.

Now fix a triangle $T \subset \bar{\Delta}$ with its three edges on ∂D one of which is q (T denotes the triangle with interior, i.e., the convex hull spanned by the three edges). By the definition of the notion "non-tangential limit" we have a limit

$$\lim_{x \in T, x \to q} \eta(x) = v \in \Omega' \subset \mathbb{C}^N$$

and thus a continuous map $\bar{\eta} : T \cup \{q\} \to \Omega'$ with $\bar{\eta}|_T = \eta$. Let W be a simply-connected open neighbourhood of v in Ω', and V an open connected neighbourhood of q in $\bar{\eta}^{-1}(W)$. Observe that $\pi : \tilde{\Omega}' \to \Omega'$ is an unramified covering. Since W is simply-connected, it follows that $\pi^{-1}(W)$ is a disjoint union of connected components each of which is isomorphic to W. Connectedness of V implies that

$\phi(\zeta(V))$ is contained in one connected component of $\pi^{-1}(W)$. Together with $\lim_{x\in T, x\to q}\eta(x) = v$ this implies that there is a point $\tilde{v} \in \pi^{-1}(v)$ such that

$$\lim_{x\in T, x\to q} \phi(\zeta(x)) = \tilde{v} = \bar{q}.$$

For any sequence t_n in $int(T)$ converging to t we now obtain a sequence $x_n = \zeta(t_n)$ with convergent limits $\lim x_n = q \in \tilde{M}$, $\lim \phi(x_n) = \bar{q} \in \tilde{\Omega}'$.

Finally we note that $\bar{q}$ cannot be in $\tilde{\Omega}$: ϕ is an automorphism of $\tilde{\Omega}$ and therefore $\lim x_n \notin \tilde{\Omega}$ implies that $\phi(x_n)$ cannot converge inside of $\tilde{\Omega}$. Hence $\bar{q} \in \tilde{M}$. □

6. Extension through the boundary

We need the following well-known extension result.

Proposition 5. *Let Ω be an open subset in a Stein manifold Z. Assume that there are points $q, \bar{q} \in \partial\Omega$, an automorphism $\phi \in \mathrm{Aut}(\Omega)$, and a sequence of points $x_n \in \Omega$ with $\lim x_n = q$ and $\lim \phi(x_n) = \bar{q}$. Assume in addition that $\partial\Omega$ is real-analytic and strictly pseudoconvex near q and $\bar{q}$.*

Then there exists an open neighbourhood V of q in Z and a holomorphic map $\Phi : V \to Z$ such that $\Phi|_{\Omega\cap V} = \phi|_{\Omega\cap V}$.

Proof. First, [6] implies that ϕ can be extended to a continuous map $\bar{\phi}$ on $\bar{\Omega}$ near q. Since $\bar{\phi}$ is continuous and $\bar{\phi}|_{\Omega}$ is holomorphic, it is clear that $\bar{\phi}|_{\partial\Omega}$ is a continuous CR-map. (For a not necessarily differentiable function the notion "CR-map" is defined via regarding derivatives in the sense of distributions. Then the condition "CR" translations into the vanishing of certain integrals involving test functions – a closed condition; hence holomorphy of $\bar{\phi}|_{\Omega} = \phi$ implies that $\bar{\phi}|_{\partial\Omega}$ is a CR-map.)

Thus [3] implies that this extension is already C^∞ and finally [1] or [5] yield that there is a holomorphic extension into some open neighbourhood. □

7. Rigidity

Lemma 3. *Let Ω be a strictly pseudoconvex domain in a Stein manifold V. Let f be a holomorphic function on V such that $f(\partial\Omega) \subset \mathbb{R}$.*

Then f is constant.

Proof. By the assumption of Ω being strictly pseudoconvex it follows that for every point $p \in \Omega$ close enough to the boundary there exists a continuous map $\zeta : \bar{\Delta} \to V$ such that

(1) ζ is holomorphic on Δ,
(2) $\zeta(0) = p$,
(3) $\zeta(\partial\Delta) \subset \partial\Omega$

Now the maximum principle applied to the plurisubharmonic function $g(x) = (\Im f(x))^2$ implies that $\Im f(p) = 0$. Thus the real-analytic function $\Im f$ vanishes in some open subset of Ω and therefore (by identity principle) it vanishes everywhere. Hence f is both holomorphic and everywhere real-valued and therefore constant. □

Proposition 6. *Let Ω be an open G-invariant subset of a complex manifold Z on which G acts freely with totally real orbits. Assume that the boundary $\partial\Omega$ is a smooth CR-hypersurface.*

Let ϕ be an automorphism of Ω, $q \in \partial\Omega$ and V an open neighbourhood of q in Z such that $\phi|_{V\cap\Omega}$ extends to a holomorphic map $\bar\phi : V \to Z$.

Assume that for every $x \in V \cap \partial\Omega$ both x and $\bar\phi(x)$ are contained in the same G-orbit.

Assume furthermore that $\partial\Omega$ is strictly pseudoconvex near q.

Then there exists an element $g \in G$ such that $g \cdot x = \phi(x)$ for all $x \in \Omega$.

Proof. Let $g_0 \in G$ be such that $\bar\phi(q) = g_0 \cdot q$. We may now replace ϕ by the automorphism $x \mapsto g_0^{-1} \cdot \phi(x)$ and thereby assume that $\bar\phi(q) = q$. Now we have to show that $\phi = id_\Omega$.

Let $n = \dim_{\mathbb{C}}(\Omega)$ and $d = \dim_{\mathbb{R}}(G)$. Let $i : B_{n-d} = \{v \in \mathbb{C}^{n-d} : ||v|| < 1\} \to Z$ be an embedding such that $i(0) = q$ and that $i(B_{n-d})$ is everywhere transversal to the G-orbits. The G-action induces a real-analytic map $\psi : \mathrm{Lie}(G) \times Z \to Z$ given by $\psi(v, x) = \exp(v) \cdot x$. This extends to a holomorphic map $\psi_{\mathbb{C}} : U \to Z$ where U is an open neighbourhood of $(0, q)$ in $(\mathrm{Lie}(G) \otimes \mathbb{C}) \times Z$. By appropriately shrinking V and U we may assume that $U = N \times V$ where N is an open neighbourhood of 0 in $\mathrm{Lie}(G) \otimes \mathbb{C}$. Now we obtain a holomorphic map $\zeta : B_{n-d} \times N \to Z$ via $\zeta(w, v) = \psi_{\mathbb{C}}(v, i(w))$. Since $B_{n-d} \times N$ is an open domain in

$$\mathbb{C}^{n-d} \times \mathrm{Lie}(G) \otimes \mathbb{C} \simeq \mathbb{C}^{n-d} \times \mathbb{C}^d \simeq \mathbb{C}^n$$

this map ζ yields local holomorphic coordinates near q. In these local coordinates

$$x = (x_1, \ldots, x_n) \mapsto \bar\phi(x) - x$$

is a holomorphic map all of whose components are real-valued on $V \cap \partial\Omega$. Because $\partial\Omega$ is strictly pseudoconvex near q, it follows that this map is constant (lemma 3). Since $\bar\phi(q) = q$, constancy means that it is constant zero. Thus $\bar\phi \equiv id_V$. Finally, by identity principle it follows that $\phi(x) = x$ for all $x \in \Omega$, as desired. □

8. Reduction to the simply-connected case

Lemma 4. *Let G be a connected real Lie group, $\tilde G$ its universal covering and $\Gamma = \pi^{-1}(\{e\})$ where $\pi : \tilde G \to G$ is the natural projection map.*

Assume that there exists a simply-connected complex manifold $\tilde X$ with $\mathrm{Aut}(\tilde X) \simeq \tilde G$ such that the Γ-action on $\tilde X$ is free and properly discontinuous.

Let $X = \tilde{X}/\Gamma$. Then $\mathrm{Aut}(X) \simeq G$,

Proof. Every automorphism of X lifts to an automorphism of $\tilde{X}$, because $\tilde{X}$ is a universal covering space for X. Therefore the automorphism group of X is isomorphic to N/Γ where N denotes the group of all elements of $\mathrm{Aut}(\tilde{X})$ which normalize Γ. But Γ is the kernel of a group homomorphism, hence normal. Thus $N = \tilde{G}$ and consequently $\mathrm{Aut}(X) \simeq N/\Gamma = \tilde{G}/\Gamma \simeq G$. □

It remains to be shown that $\tilde{X}$ can be constructed in such a way that $X = \tilde{X}/\Gamma$ will be Stein and completely hyperbolic. Complete hyperbolicity is easy, since $\tilde{X}$ being completely hyperbolic implies that X is completely hyperbolic, too.

The Stein property is more involved, since for an arbitrary unramified covering $\tilde{X} \to X$, Steinness of $\tilde{X}$ does not imply that X is Stein, too.

Proposition 7. *Let G be a real Lie group, $\pi : \tilde{G} \to G$ its universal covering, $\Gamma = \pi^{-1}(e)$, and $\tilde{X}$ a complex manifold on which $\tilde{G}$ acts properly and freely with totally real orbits.*

Let $p \in \tilde{X}$.

Then there exists an open $\tilde{G}$-invariant neighbourhood $\tilde{U}$ of $\tilde{G} \cdot p$ in $\tilde{X}$ such that for every $\tilde{G}$-invariant locally Stein open submanifold $\tilde{\Omega} \subset \tilde{U}$ the complex quotient manifold $\Omega = \tilde{\Omega}/\Gamma$ is Stein.

(As usual, $\tilde{\Omega} \subset \tilde{U}$ is called locally Stein iff every point $x \in \tilde{U}$ admits an open neighbourhood V in U such that $V \cap \tilde{\Omega}$ is Stein.)

Proof. Essentially, we follow the argumentation in [16].

Let Z denote the center of $\tilde{G}$. Then there exists a discrete cocompact subgroup Λ in Z such that $\Gamma \subset \Lambda$ ([16], lemma 1). Let $G_1 = \tilde{G}/\Lambda$ and $X_1 = \tilde{X}/\Lambda$.

Let $G_{\mathbb{C}}$ be the simply-connected complex Lie group corresponding to the complex Lie algebra $\mathrm{Lie}(G) \otimes \mathbb{C}$ and $j : \tilde{G} \to G_{\mathbb{C}}$ the natural Lie group homomorphism induced by the Lie algebra embedding $\mathrm{Lie}(G) \hookrightarrow \mathrm{Lie}(G) \otimes \mathbb{C}$.

Let $\psi_0 : \mathrm{Lie}(\tilde{G}) \times \tilde{X} \to \tilde{X}$ be the map induced by the group action via $\psi_0(v, x) = \exp(v) \cdot x$. Then ψ_0 extends to a holomorphic map ψ defined on some open neighbourhood of $\mathrm{Lie}\,\tilde{G} \times \tilde{X}$ in $\mathrm{Lie}\,G_{\mathbb{C}} \times \tilde{X}$. This open neighbourhood can be chosen as product $N \times \tilde{W}$ where N is an open neighbourhood of $\mathrm{Lie}\,\tilde{G}$ in $\mathrm{Lie}\,G_{\mathbb{C}}$ and W is an open neighbourhood of p in $\tilde{X}$.

Let $n = \dim_{\mathbb{C}}(X)$ and $d = \dim_{\mathbb{R}}(G) = \dim_{\mathbb{C}}(G_{\mathbb{C}})$. Let $i : B_{n-d} = \{v \in \mathbb{C}^{n-d} : ||v|| < 1\} \to W$ be a holomorphic embedding such that $i(0) = p$ and that $i(B_{n-d})$ is everywhere transversal to the G-orbits.

We choose a small open neighbourhood $N_1 \subset N$ of 0 in $\mathrm{Lie}\,G_{\mathbb{C}}$ such that the map $\zeta : \tilde{G} \times N_1 \times B \to \tilde{X}$ given by $\zeta : (g, n, x) \mapsto g \cdot \psi(n, z)$ has the property that $\zeta(g, n, z) = \zeta(g', n', z')$ only if there is an element $v \in \mathrm{Lie}\,\tilde{G}$ such that $g' = g \cdot \exp(v)$ and $\exp(-v)\exp(n) = \exp(n')$. This is possible, because $\tilde{G}$ acts freely with totally real orbits.

For $x \in \zeta(\tilde{G} \times N_1 \times B)$ we define $\xi(x) \in Ad(G_{\mathbb{C}})$ by $Ad(g \cdot \exp(n))$ if $x = \zeta(g, n, z)$. Then ξ is a well-defined, holomorphic and $\tilde{G}$-equivariant map from an $\tilde{G}$-invariant open neighbourhood W_0 of p to $Ad(G_{\mathbb{C}})$. Moreover ξ is constant along the orbits of the center Z of $\tilde{G}$. Therefore it induces a holomorphic map $\xi_1 : W_1 \to Ad(G_{\mathbb{C}})$ where W_1 is the image of W_0 under the projection $\tilde{X} \to X_1$.

Observe that $Ad(G_{\mathbb{C}})$ is a linear complex Lie group. It follows that $Ad(G_{\mathbb{C}})$ is Stein ([11]) and hence admits a strictly plurisubharmonic exhaustion function $\rho_1 : Ad(G_{\mathbb{C}}) \to \mathbb{R}^+$.

Next we consider the real quotient map $\tau : \tilde{X} \to \tilde{X}/\tilde{G} = Y$. Let $y_1, \ldots, y_r$ be real-analytic local coordinates on Y with $y_i(\tau(p)) = 0$. Then $x \mapsto \sum_i y_i(\tau(x))^2$ defines a $\tilde{G}$-invariant real-analytic function ρ_0 on a neighbourhood of $\tilde{G} \cdot p$, which is easily verified to be strictly plurisubharmonic near $\tilde{G} \cdot p$.

By appropriately shrinking W_0 and W_1 we may assume that there is an $\epsilon > 0$ such that $W_0 = \{x : \rho_0(x) < \epsilon\}$.

We reparametrize this function via

$$\rho_0'(x) = \tan\left(\frac{\pi}{2\epsilon}\rho_0(\tau(x))\right).$$

Now $\rho_0' \to +\infty$ whenever $\rho_0 \to \epsilon$. Thus ρ_0' is an "exhaustion function modulo $\tilde{G}$", i.e. it is a $\tilde{G}$-invariant function which induces a proper continuous map from $W_0/\tilde{G}$ to $\mathbb{R}^+$.

Moreover, ρ_0' is strictly plurisubharmonic, because tan is convex and ρ_0 is strictly plurisubharmonic.

Next we recall that by lemma 2 in [16] the natural map $W_1 \to Y_1 \simeq W_1/G \times Ad(G_{\mathbb{C}})$ is proper.

Therefore $\rho_1 + \rho_0'$ is a continuous exhaustion function on W_1. On the other hand, this function is also strictly plurisubharmonic. Thus W_1 is Stein.

Let $\tilde{U} = W_0$ and $U = \tilde{U}/\Gamma$. Then U is Stein, because W_1 is Stein and we have an unramified covering $U \to W_1$.

Assume that Ω is a $\tilde{G}$-invariant open locally Stein submanifold of $\tilde{U}$. Then Ω/Γ is a G-invariant open submanifold of $U = \tilde{U}/\Gamma$ which is evidently locally Stein. But locally Stein open submanifolds of Stein manifolds are Stein. Hence Ω is Stein. □

9. Proof of the Main theorem

Here we prove our main theorem.

Proof. Let $\tilde{G}$ denote the universal covering of G, $\pi : \tilde{G} \to G$ the natural projection and $\Gamma = \pi^{-1}\{e\}$. By prop. 2 there is a quotient G_1 of $\tilde{G}$ by a central discrete subgroup Γ_1 and a G_1-action on a complex manifold Ω_1 which is free, proper and with totally real orbits. Moreover, there is a number N, a bounded domain $D \subset \mathbb{C}^N$ and closed complex analytic subsets $Z \subset D$, $E \subset \mathbb{C}^N$ and an embedding of Ω_1 as open submanifold in Z such that the closure of Ω_1 in $\mathbb{C}^N$ is contained in $Z \cup E$ and $\Omega_1 \cap E = \{\}$.

Fix $p \in \Omega_1$. We may replace Ω_1 by some appropriately choosen invariant open neighbourhood of $G_1 \cdot p$. Therefore we may and do from now on assume that $\pi_1(\Omega_1) \simeq \pi_1(G_1) \simeq \Gamma_1$. (And we keep this assumption throughout all further replacements of Ω_1 by invariant open subsets of itself.) Let $\tilde{\Omega}$ denote the universal covering of Ω_1.

From prop. 7 we deduce that, after replacing Ω_1 with some G_1-invariant open subset, we may assume that U/Γ is Stein for every open $\tilde{G}$-invariant locally Stein submanifold U of $\tilde{\Omega}$.

Next we apply prop. 3, again replacing Ω_1 by an appropriate smaller G_1-invariant open submanifold. Now Ω_1 has a smooth, real-analytic and strictly pseudoconvex boundary B in Z, and there is a nowhere dense real-analytic subset $\Sigma \subset B$ such that for every $x, y \in B \setminus \Sigma$ the CR-hypersurface germs (B, x), (B, y) are isomorphic if and only if $x = g \cdot y$ for some $g \in G_1$.

Let $\phi \in \mathrm{Aut}(\tilde{\Omega})$. Let $\tau : \tilde{\Omega} \to \Omega_1$ be the covering map. We may assume that there is a G_1-invariant open subset $X \subset Z$ such that $\bar{\Omega}_1 \cap Z \subset X$. Furthermore we may assume that the inclusion $\Omega_1 \hookrightarrow X$ induces an isomorphism of the fundamental groups. Then $\tau : \tilde{\Omega} \to \Omega_1$ extends to a covering $\tau' : \tilde{X} \to X$ with $\tilde{\Omega} \hookrightarrow \tilde{X}$. Let $\tilde{\Sigma}$ and $\tilde{B}$ denote the preimages of Σ resp. B under τ'. By prop. 4 there is a sequence of points $x_n \in \tilde{\Omega}$ and points $q, \bar{q} \in \tilde{B}$ such that $\lim x_n = q$ and $\lim \phi(x_n) = \bar{q}$. By prop. 5 it follows that ϕ extends to a holomorphic map Φ in an open neighbourhood U of q in $\tilde{X}$. Because ϕ^{-1} extends to a holomorphic map near $\bar{q}$ by the same arguments and $\phi \circ \phi^{-1} = id$, this extension Φ is locally biholomorphic and $\Phi(\tilde{B} \cap U) \subset \tilde{B}$.

Recall that $\tilde{\Sigma}$ is nowhere dense in $\tilde{B}$. Hence there is an element $q' \in (\tilde{B} \cap U) \setminus \left(\tilde{\Sigma} \cup \Phi^{-1}(\tilde{\Sigma})\right)$. Upon replacing q by q' and U by $U \setminus \left(\tilde{\Sigma} \cup \Phi^{-1}(\tilde{\Sigma})\right)$, we may from now on assume that $U \cap \Sigma$ and $\Phi(U) \cap \Sigma$ are both empty.

For every $z \in \tilde{B} \cap U$ the CR hypersurface germs $(\tilde{B}, z)$ and $(\tilde{B}, \Phi(z))$ are isomorphic and consequently there is an element $g_z \in \tilde{G}$ such that $g_z \cdot z = \Phi(z)$.

By prop. 6 it follows that there is one element $g \in \tilde{G}$ such that $\phi(z) = g \cdot z$ for all $z \in \tilde{\Omega}$. Thus $\phi \in \tilde{G}$. Since ϕ was an arbitrary automorphism of $\tilde{\Omega}$, it follows that $\mathrm{Aut}(\tilde{\Omega}) = \tilde{G}$. By lemma 4 this implies that $\mathrm{Aut}(\Omega) = G$ where $\Omega = \tilde{\Omega}/\Gamma$.

Finally let us discuss the Stein condition and hyperbolicity. Since Ω_1 injects into a bounded domain $D \subset \mathbb{C}^N$, it is hyperbolic. Because $\tilde{\Omega} \to \Omega_1$ and $\tilde{\Omega} \to \Omega$ are both unramified coverings, this implies the hyperbolicity of Ω. Moreover, by the same arguments as in [16], we may conclude that Ω is even *complete* hyperbolic.

Concerning the Stein property, let us recall application of prop. 7 further above. Our choice of Ω_1 at that time had the property that U/Γ was Stein for every locally Stein open subset U of $\tilde{\Omega}$. Subsequently we shrank Ω_1, replacing it by some open subset with strictly pseudoconvex boundary. Clearly an open subset with strictly pseudoconvex boundary is locally Stein. Therefore $\Omega = \tilde{\Omega}/\Gamma$ is Stein for our final choice of Ω_1. □

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