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On the Γ-cohomology of rings of numerical polynomials and E_∞ structures on K-theory

Andrew Baker and Birgit Richter*

Abstract. We investigate Γ-cohomology of some commutative cooperation algebras E_*E associated with certain periodic cohomology theories. For *KU* and $E(1)$, the Adams summand at a prime p, and for *KO* we show that Γ-cohomology vanishes above degree 1. As these cohomology groups are the obstruction groups in the obstruction theory developed by Alan Robinson we deduce that these spectra admit unique E_∞ structures. As a consequence we obtain an E_∞ structure for the connective Adams summand. For the Johnson–Wilson spectrum $E(n)$ with $n \geqslant 1$ we establish the existence of a unique E_∞ structure for its I_n-adic completion.

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Introduction

In homotopy theory it is often not sufficient to have homotopy ring structures on a spectrum in order to construct for instance homotopy fixed points under a group action or quotient spectra. For this, it is necessary to have ring structures which are not just given up to homotopy but where these homotopies fulfil certain coherence conditions. We will prove the existence and uniqueness of certain E_∞ structures, *i.e.*, structures on spectra which encode a coherent homotopy commutative multiplication.

In [31], [32], Alan Robinson developed a purely algebraic obstruction theory for E_∞ structures on homotopy associative and commutative ring spectra. The device for deciding whether a spectrum possesses such a structure is a cohomology theory for commutative algebras, Γ-cohomology. When applied to the cooperation algebra E_*E of a ring spectrum E, the vanishing of these cohomology groups implies the existence of an E_∞ structure on E which extends the given homotopy ring structure.

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We will apply Robinson's obstruction theory to complex K-theory KU, its p-localization $KU_{(p)}$, its Adams summand $E(1)$, with

$$KU_{(p)} \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i} E(1),$$

and to real K-theory KO. Our main topological result is

Theorem. *There are unique E_∞ structures on KU, the Adams summand $E(1)$ and KO.*

This process can then be refined to give E_∞ structures on the connective covers. For the higher Johnson–Wilson spectra $E(n)$ with $n \geqslant 1$ we will consider the I_n-adic completion.

The existence of E_∞ structures on KU and KO was already known: E_∞ structures for the connective versions ku and ko were constructed in [26, Chapter VIII], and the techniques of [17, VIII] lead to E_∞ models for KU and KO. Recently, Joachim [22] has described such a structure for KO in the context of symmetric spectra. But as far as we know the uniqueness of these structures has not previously been documented. The existence and uniqueness for $E(1)$ appears to be new. We also show that the connective Adams summand ℓ at an odd prime p admits an E_∞ structure; on the p-completion $\ell_p^\frown$ an E_∞ structure was earlier constructed by McClure and Staffeldt [27]. In subsequent work we have shown that the E_∞ structures on ku, ko, ℓ and $\ell_p^\frown$ are unique.

By [18], [29] it is known that the Lubin–Tate spectra E_n have unique E_∞ structures. In particular, $E_1 = KU_p^\frown$ has a unique E_∞ structure. The work of Hopkins and Miller [28] and Goerss and Hopkins [18] establishes A_∞ and E_∞ structures on the Lubin–Tate spectra E_n and identified the homotopy type of the space of A_∞ (resp. E_∞) maps between any two of these spectra. With the help of these results the homotopy action of the Morava-stabilizer group on E_n could be rigidified to an action of Morava-stabilizer group on E_n by A_∞ (resp. E_∞) maps.

The existence of unique E_∞ structures on $KU_p^\frown$ and $E(1)_p^\frown$ follow directly from the calculation of continuous Γ-cohomology. In [19, §1], Hovey and Strickland asked whether the I_n-adic completion of $E(n)$, $\widehat{E(n)}$, has E_∞ structures.

Theorem. *For all $n \geqslant 1$ and all primes p, $\widehat{E(n)}$ possesses a unique E_∞ structure.*

We give an elementary proof of this which relies on Robinson's obstruction theory [31]; the result for E_n then follows using ideas of [34]. However, so far we have not been able to extend this result to $E(n)$ itself since the Γ-cohomology of $E(n)_*E(n)$ appears to be very non-trivial in positive degrees for $n > 1$.

Notation and tools. All otherwise unspecified tensor products are taken over $\mathbb{Z}$ or a localization at a prime p, $\mathbb{Z}_{(p)}$. We denote by $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\widehat{\mathbb{Z}}$ the rings of p-adic integers, p-adic rationals and profinite integers respectively, while $\widehat{\mathbb{Q}} = \widehat{\mathbb{Z}} \otimes \mathbb{Q}$. For an arbitrary unital commutative ring R we denote the rationalization of R by $R_{\mathbb{Q}}$.

In the body of this paper we will repeatedly use properties of Γ-cohomology. For the reader's convenience we recall some of its crucial features here.

- Given a commutative $\Bbbk$-algebra A, Γ-cohomology of A with coefficients in an A-module M, $\mathrm{H}\Gamma^*(A|\Bbbk; M)$, is defined in [31] as the cohomology of the derived A-homomorphisms from a certain chain complex of A-modules $C^\Gamma(A|\Bbbk)_*$ to M.
- This way of defining Γ-cohomology ensures that each short exact sequence of A-modules
$$0 \to M' \longrightarrow M \longrightarrow M'' \to 0$$
leads to a long exact sequence of Γ-cohomology groups. It is also clear that in good cases there is a universal coefficient spectral sequence
$$\mathrm{E}_2^{*,*} = \mathrm{Ext}_A^{*,*}(\mathrm{H}\Gamma_*(A|\Bbbk; A), M) \Longrightarrow \mathrm{H}\Gamma^*(A|\Bbbk; M).$$
Here $\mathrm{H}\Gamma_*(A|\Bbbk; A)$ denotes the homology of $C^\Gamma(A|\Bbbk)_*$.
- In [33, 6.8], it is shown that Γ-cohomology vanishes if A is étale over $\Bbbk$, and that Γ-cohomology satisfies Flat Base Change and has a Transitivity Sequence.
- Last but not least, if A is a $\Bbbk$-algebra and $\mathbb{Q} \subseteq \Bbbk$, then the Γ-cohomology of A with coefficients in M coincides with André–Quillen cohomology $\mathrm{AQ}^*(A|\Bbbk; M)$; for details see [33, 6.4].

Convenient and concise sources for the definitions of André–Quillen homology and cohomology are [25], [36] where they are denoted $\mathrm{D}_*(\ )$ and $\mathrm{D}^*(\ )$.

The main techniques we use for the K-theoretic examples involve the passage to a continuous version of Γ-cohomology which we introduce in Section 2, and a description of the Γ-cohomology groups of the relevant cooperation algebras in terms of Γ-cohomology groups of colimits of étale algebras at each prime (which turn out to vanish) and Γ-cohomology groups of algebras which are rationally smooth and can be calculated. At the technical heart of our arguments for the K-theoretic examples is Theorem 5.1.

1. Linear compactness and cohomology

The results in this section and Section 2 will play a crucial rôle in our proof of Corollary 5.2 when we calculate the Γ-cohomology of numerical polynomials and more generally for the proof of Theorem 5.1, where Corollary 1.3 and Proposition 1.4

will help us to identify certain Γ-cohomology groups of complete algebras with inverse limits of Γ-cohomology groups of quotient algebras.

Let $\Bbbk$ be a commutative Noetherian ring and let $\mathfrak{m} \lhd \Bbbk$ be a maximal ideal. We topologize $\Bbbk$ with respect the $\mathfrak{m}$-adic topology where the open neighbourhoods of 0 are the ideals $\mathfrak{m}^k \lhd \Bbbk$ for $k \geqslant 0$. In the following we will assume that $\Bbbk$ is Hausdorff with respect to this $\mathfrak{m}$-adic topology, *i.e.*, that

$$\bigcap_{k \geqslant 0} \mathfrak{m}^k = 0. \tag{1.1}$$

For each $k \geqslant 0$, $\mathfrak{m}^k$ is a finitely-generated $\Bbbk$-module, while $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ is a finitely-generated $\Bbbk/\mathfrak{m}$-module which is therefore an Artinian $\Bbbk/\mathfrak{m}$-module.

Now let A be a commutative unital $\Bbbk$-algebra. The ideals $\mathfrak{m}^k A = A\mathfrak{m}^k \lhd A$ also generate a topology on A which is Hausdorff if

$$\bigcap_{k \geqslant 0} A\mathfrak{m}^k = 0. \tag{1.2}$$

Then the unit homomorphism $\Bbbk \longrightarrow A$ is automatically continuous and if A is augmented over $\Bbbk$ then the augmentation is also continuous. Furthermore, $(A, \Bbbk)$ is a topological algebra over the topological ring $\Bbbk$. We say that $(A, \Bbbk)$ is Hausdorff if both (1.1) and (1.2) hold.

If $(A, \Bbbk)$ is a Hausdorff topological algebra, then the $\mathfrak{m}$-*adic completion* of $(A, \Bbbk)$ is $(A^{\wedge}_{\mathfrak{m}}, \Bbbk^{\wedge}_{\mathfrak{m}})$, where

$$A^{\wedge}_{\mathfrak{m}} = \lim_k A/\mathfrak{m}^k A, \quad \Bbbk^{\wedge}_{\mathfrak{m}} = \lim_k \Bbbk/\mathfrak{m}^k.$$

We say that $(A, \Bbbk)$ is $\mathfrak{m}$-*adically complete* if $A^{\wedge}_{\mathfrak{m}} = A$ and $\Bbbk^{\wedge}_{\mathfrak{m}} = \Bbbk$. When $\mathfrak{m}$ is clear from the context we will sometimes simplify notation by writing $(\widehat{A}, \widehat{\Bbbk}) = (A^{\wedge}_{\mathfrak{m}}, \Bbbk^{\wedge}_{\mathfrak{m}})$. If $(A, \Bbbk)$ is augmented over $\Bbbk$ then $(A^{\wedge}_{\mathfrak{m}}, \Bbbk^{\wedge}_{\mathfrak{m}})$ is augmented over $\Bbbk^{\wedge}_{\mathfrak{m}}$.

Recall from [15], [21], [37] the property of a topological left module M over a topological algebra $(A, \Bbbk)$ being *linearly compact*:

- If $\{x_\lambda + N_\lambda\}_{\lambda \in \Lambda}$ is a collection of cosets of closed submodules $N_\lambda \subseteq M$ such that every finite collection of the cosets $x_\lambda + N_\lambda$ has non-trivial intersection, then
$$\bigcap_{\lambda \in \Lambda} x_\lambda + N_\lambda \neq \varnothing.$$

We will make repeated use of the following vanishing result of [21, p. 57, théorème 7.1] for the higher derived functors of the inverse limit for inverse systems of linearly compact modules.

Theorem 1.1. *Let $\{M_i\}_{i\in I}$ be an inverse system of linearly compact A-modules and continuous A-linear maps. Then for all $s > 0$ we have*

$$\lim_i{}^s M_i = 0.$$

Recall that a topological $\Bbbk$-module M is *topologically free on a countable basis* $\{b_i\}_{i\geqslant 1}$ if for each element $x \in M$ and $k \geqslant 1$, in $M/\mathfrak{m}^k M$ considered as a $\Bbbk/\mathfrak{m}^k$-module, there is a unique (finite) expansion

$$\bar{x} = \sum_{i\geqslant 1} \bar{r}_i \bar{b}_i$$

with $\bar{r}_i \in \Bbbk/\mathfrak{m}^k$ and where $\bar{b}_i \in M/\mathfrak{m}^k M$ is the residue class of b_i. As a consequence, x has a unique expansion as a limit sum

$$x = \sum_{i\geqslant 1} t_i b_i$$

where $t_i \to 0$ as $i \to \infty$; this means that for each k, there is an n_k such that for $i > n_k$ we have $t_i \in \mathfrak{m}^k$. The linear topology on M has basic open neighbourhoods of 0 of the form $\mathfrak{m}^k M$. Now the Noetherian condition on $\Bbbk$ implies that

$$\mathfrak{m}^k M = \Big\{ \sum_{i\geqslant 1} t_i b_i : t_i \in \mathfrak{m}^k,\ t_i \to 0 \text{ as } i \to \infty \Big\}. \tag{1.3}$$

For two topological left R-modules L and M over a commutative topological ring R, we let

$$\mathcal{H}om_R(L, M) \subseteq \mathrm{Hom}_R(L, M)$$

be the submodule of continuous R-module homomorphisms.

Proposition 1.2. *Suppose that M is a finitely generated $\Bbbk$-module which is complete and Hausdorff with respect to the $\mathfrak{m}$-adic topology. If L is a $\Bbbk$-module which is complete with respect to the $\mathfrak{m}$-adic topology and topologically free on a countable basis then $\mathcal{H}om_{\Bbbk}(L, M)$ is a linearly compact $\Bbbk$-module.*

Proof. Assume first that L is Hausdorff with respect to the $\mathfrak{m}$-adic topology. Note that

$$\mathcal{H}om_{\Bbbk}(L, M) = \mathcal{H}om_{\Bbbk}(L, \lim_k M/\mathfrak{m}^k M) = \lim_k \mathcal{H}om_{\Bbbk}(L, M/\mathfrak{m}^k M).$$

If $\{b_j\}_{j\geqslant 1}$ is a topological basis for L, then using the Noetherian condition on $\Bbbk$ we find that the basic neighbourhoods of 0 in L are the submodules $\mathfrak{m}^k L \subseteq L$. From this we find that

$$\begin{aligned}\mathcal{H}om_{\Bbbk}(L, M/\mathfrak{m}^k M) &= \mathcal{H}om_{\Bbbk/\mathfrak{m}^k}(L/\mathfrak{m}^k L, M/\mathfrak{m}^k M)\\ &= \mathrm{Hom}_{\Bbbk/\mathfrak{m}^k}(L/\mathfrak{m}^k L, M/\mathfrak{m}^k M)\\ &= \prod_{j\geqslant 1} \mathrm{Hom}_{\Bbbk/\mathfrak{m}^k}((\Bbbk/\mathfrak{m}^k)b_j, M/\mathfrak{m}^k M).\end{aligned}$$

But

$$\mathrm{Hom}_{\Bbbk/\mathfrak{m}^k}((\Bbbk/\mathfrak{m}^k)b_j, M/\mathfrak{m}^k M) = M/\mathfrak{m}^k M$$

and this is Artinian, hence linearly compact. This in turn implies that the final product above is also linearly compact. The claim now follows since $\lim_k \mathcal{H}om_{\Bbbk}(L, M/\mathfrak{m}^k M)$ is a closed subspace of the product

$$\prod_{j\geqslant 1} \mathrm{Hom}_{\Bbbk/\mathfrak{m}^k}((\Bbbk/\mathfrak{m}^k)b_j, M/\mathfrak{m}^k M).$$

Now we consider what happens when L is not necessarily Hausdorff. In this case, Nakayama's Lemma implies that for any $f \in \mathcal{H}om_{\Bbbk}(L, M)$ we have

$$f\Big(\bigcap_{k\geqslant 1} \mathfrak{m}^k L\Big) = 0.$$

Hence such an f factors through the quotient

$$L_0 = L\Big/\bigcap_{k\geqslant 1} \mathfrak{m}^k L,$$

so we might as well replace L by this Hausdorff quotient. Then we have

$$\mathcal{H}om_{\Bbbk}(L, M) = \mathcal{H}om_{\Bbbk}(L_0, M). \qquad \square$$

We can apply this to prove the following.

Corollary 1.3. *Suppose further that A is a topological $\Bbbk$-algebra with respect to the $\mathfrak{m}$-adic topology inherited from $\Bbbk$ and that L and M are topological A-modules. Then $\mathcal{H}om_A(L, M) \subseteq \mathcal{H}om_{\Bbbk}(L, M)$ is a closed $\Bbbk$-submodule. Hence $\mathcal{H}om_A(L, M)$ is linearly compact.*

Proof. Again we first consider the case where L is Hausdorff. The two continuous action maps

$$A \otimes_{\Bbbk} \mathcal{H}om_{\Bbbk}(L, M) \longrightarrow \mathcal{H}om_{\Bbbk}(L, M)$$

given by

$$a \otimes f \longmapsto af, \quad a \otimes f \longmapsto f(a(-))$$

are equalized on $\mathcal{H}om_A(L, M)$, so this is a closed subset of $\mathcal{H}om_{\Bbbk}(L, M)$. For L not Hausdorff we see as above that $\mathcal{H}om_A(L, M) \cong \mathcal{H}om_A(L_0, M)$. $\square$

Note that if A is not necessarily Hausdorff, then setting

$$A_0 = A \Big/ \bigcap_{k \geqslant 1} \mathfrak{m}^k A$$

we have

$$\mathcal{H}om_A(L, M) = \mathcal{H}om_{A_0}(L_0, M).$$

Proposition 1.4. *Let (C^*, δ) be a cochain complex of linearly compact and Hausdorff $\Bbbk$-modules where for each n, the coboundary $\delta^n : C^n \longrightarrow C^{n+1}$ is continuous. Then for each n, $\mathrm{H}^n(C^*, \delta)$ is linearly compact.*

Proof. Since each C^n is linearly compact and Hausdorff, the submodules $\operatorname{Im} \delta^{n-1}$ and $\operatorname{Ker} \delta^n$ of C^n are both closed. Therefore

$$\mathrm{H}^n(C^*, \delta) = \operatorname{Ker} \delta^n / \operatorname{Im} \delta^{n-1}$$

is also linearly compact. □

2. Continuous Γ-cohomology

In this section we discuss some technical issues related to our calculations of Γ-cohomology later in the paper. Continuous cohomology of profinite groups is described in [35], [36]; for analogues appearing in topology see [7], [10]; our present theory is modelled closely on the presentations in those references.

Let $\Bbbk$ be a commutative Noetherian ring and let $\mathfrak{m} \lhd \Bbbk$ be a maximal ideal. We topologize $\Bbbk$ with the $\mathfrak{m}$-adic topology. Let M be a topological left module over a topological algebra $(A, \Bbbk)$. In practise, we will usually consider the $\mathfrak{m}$-adic topology on M.

In the following we shall consider Γ-cohomology of A with coefficients in M, $\mathrm{H}\Gamma^*(A|\Bbbk; M)$. In [31], [33], Γ-cohomology is defined using a cochain complex $\mathrm{Hom}_A(C^\Gamma(A|\Bbbk)_*, M)$. Here $C^\Gamma(A|\Bbbk)_*$ is the chain complex whose degree n-part is the left A-module

$$C^\Gamma(A|\Bbbk)_n = \bigoplus_{s+r=n} \mathrm{Lie}(s+1)^* \otimes \Bbbk[\Sigma_{s+1}]^{\otimes r} \otimes_\Bbbk A^{\otimes(s+2)},$$

where $\mathrm{Lie}(s+1)^*$ is the $\Bbbk$-linear dual of the $(s+1)$-st term of the Lie operad and Σ_ℓ denotes the symmetric group on ℓ letters. In particular, $\mathrm{Lie}(s+1)^*$ and $\Bbbk[\Sigma_{s+1}]$ are finite-rank free $\Bbbk$-modules. Topologising $C^\Gamma(A|\Bbbk)_*$ with the $\mathfrak{m}$-adic topology, we can introduce the subcomplex

$$\mathcal{H}om_A(C^\Gamma(A|\Bbbk)_*, M) \subseteq \mathrm{Hom}_A(C^\Gamma(A|\Bbbk)_*, M)$$

of continuous cochains whose cohomology $\mathcal{H}\Gamma^*(A|\Bbbk; M)$ we call the *continuous Γ-cohomology* of A with coefficients in M. Note that continuous cochains can be expressed as the inverse limit

$$\mathcal{H}om_A(C^\Gamma(A|\Bbbk)_*, M) = \lim_k \operatorname{Hom}_{A/\mathfrak{m}^k A}(C^\Gamma(A/\mathfrak{m}^k A|\Bbbk/\mathfrak{m}^k)_*, M/\mathfrak{m}^k M)$$

if M is Hausdorff.

Lemma 2.1. *If $(A, \Bbbk)$ is a topological algebra as above whose completion $\widehat{A}$ is countably free on a topological basis as a $\widehat{\Bbbk}$-module and if M is an A-module such that $M/\mathfrak{m}^k M$ is Artinian and M is complete, then* $\operatorname{Hom}_A(C^\Gamma(A|\Bbbk)_n, M)$ *is linearly compact in each degree n.*

Proof. As M is complete,

$$\begin{aligned}\operatorname{Hom}_A(C^\Gamma(A|\Bbbk)_n, M) &= \operatorname{Hom}_A(C^\Gamma(A|\Bbbk)_n, \lim_k M/\mathfrak{m}^k M)\\ &= \lim_k \operatorname{Hom}_A(C^\Gamma(A|\Bbbk)_n, M/\mathfrak{m}^k M).\end{aligned} \tag{2.1}$$

For any fixed k, the homomorphisms from $C^\Gamma(A|\Bbbk)_n$ to the quotient $M/\mathfrak{m}^k M$ factor through

$$\begin{aligned}C^\Gamma(A/\mathfrak{m}^k A|\Bbbk/\mathfrak{m}^k)_n &= \bigoplus_{s+r=n} \operatorname{Lie}(s+1)^* \otimes \Bbbk/\mathfrak{m}^k[\Sigma_{s+1}]^{\otimes r} \otimes (A/\mathfrak{m}^k A)^{\otimes(s+2)}\\ &= C^\Gamma(\widehat{A}/\mathfrak{m}^k \widehat{A}|\widehat{\Bbbk}/\mathfrak{m}^k)_n.\end{aligned}$$

As we assumed that $\widehat{A}$ is countably free on a topological basis, the quotient $\widehat{A}/\mathfrak{m}^k\widehat{A}$ is free on a countable basis. Hence for any homomorphism f in $\operatorname{Hom}_A(C^\Gamma(A|\Bbbk)_n, M)$ there is a j such that f factors over the finitely-generated free submodule $C(j)_n \subseteq C^\Gamma(\widehat{A}/\mathfrak{m}^k\widehat{A}|\widehat{\Bbbk}/\mathfrak{m}^k)_n$ spanned by the first j generators of $C^\Gamma(\widehat{A}/\mathfrak{m}^k\widehat{A}|\widehat{\Bbbk}/\mathfrak{m}^k)_n$. Therefore

$$\begin{aligned}\operatorname{Hom}_A(C^\Gamma(A|\Bbbk)_n, M) &= \lim_k \operatorname{Hom}_A(C^\Gamma(A|\Bbbk)_n, M/\mathfrak{m}^k M)\\ &= \lim_k \operatorname{Hom}_{\widehat{A}}(C^\Gamma(\widehat{A}/\mathfrak{m}^k\widehat{A}|\widehat{\Bbbk}/\mathfrak{m}^k)_n, M/\mathfrak{m}^k M)\\ &= \lim_k \lim_j \operatorname{Hom}_{\widehat{A}}(C(j)_n, M/\mathfrak{m}^k M)\\ &= \lim_k \lim_j \prod_{i=1}^{j} M/\mathfrak{m}^k M.\end{aligned}$$

Therefore $\operatorname{Hom}_A(C^\Gamma(A|\Bbbk)_n, M)$ is a limit of Artinian modules and thus linearly compact. □

Notice that the above inclusion of complexes induces a forgetful homomorphism

$$\rho\colon \mathcal{H}\Gamma^*(A|\Bbbk; M) \longrightarrow \mathrm{H}\Gamma^*(A|\Bbbk; M). \tag{2.2}$$

Recall that if M is $\mathfrak{m}$-adically complete and Hausdorff, there is a short exact sequence

$$0 \to M \longrightarrow \prod_k M/\mathfrak{m}^k M \xrightarrow{\mathrm{id}-\sigma} \prod_k M/\mathfrak{m}^k M \to 0, \tag{2.3}$$

where σ is the shift-reduction map. From (2.1) and (2.3) we deduce a Milnor exact sequence relating $\mathcal{H}\Gamma^*$ to ordinary Γ-cohomology for complete coefficient modules. For a similar result see [7].

Proposition 2.2. *As above, let* $\Bbbk$ *be Noetherian with maximal ideal* $\mathfrak{m}$. *Let* M *be a complete Hausdorff topological module over* $\widehat{A}$ *which is finitely-generated over* $\widehat{\Bbbk}$. *Then for each* n *there is a short exact sequence*

$$\begin{aligned} 0 \to \lim_k{}^1 \mathrm{H}\Gamma^{n-1}(A/\mathfrak{m}^k A|\Bbbk/\mathfrak{m}^k; M/\mathfrak{m}^k M) &\longrightarrow \mathcal{H}\Gamma^n(\widehat{A}|\widehat{\Bbbk}; M) \\ &\longrightarrow \lim_k \mathrm{H}\Gamma^n(A/\mathfrak{m}^k A|\Bbbk/\mathfrak{m}^k; M/\mathfrak{m}^k M) \to 0. \end{aligned}$$

This leads to some useful calculational results, versions of which have previously appeared in [28], [29]. Notice that for any $\widehat{A}$-module M and $k \geqslant 1$, there is a natural reduction homomorphism

$$\mathrm{H}\Gamma^n(\widehat{A}|\widehat{\Bbbk}; M) \longrightarrow \mathrm{H}\Gamma^n(A|\Bbbk; M) \longrightarrow \mathrm{H}\Gamma^n(A/\mathfrak{m}^k A|\Bbbk/\mathfrak{m}^k; M/\mathfrak{m}^k M), \tag{2.4}$$

compatible with respect to different values of k. In turn there is a homomorphism

$$\mathrm{H}\Gamma^n(\widehat{A}|\widehat{\Bbbk}; M) \longrightarrow \mathrm{H}\Gamma^n(A|\Bbbk; M) \longrightarrow \lim_k \mathrm{H}\Gamma^n(A/\mathfrak{m}^k A|\Bbbk/\mathfrak{m}^k; M/\mathfrak{m}^k M). \tag{2.5}$$

The following result was inspired by [28, lemma 15.6].

Corollary 2.3. *Let* M *be an* $\widehat{A}$*-module which is complete and Hausdorff with respect to the* $\mathfrak{m}$*-adic topology and finitely-generated as a* $\widehat{\Bbbk}$*-module. Let* $\widehat{A}$ *be countably free on a topological basis. Then the natural homomorphism* ρ *induces an isomorphism*

$$\mathcal{H}\Gamma^*(\widehat{A}|\widehat{\Bbbk}; M) \cong \mathrm{H}\Gamma^*(\widehat{A}|\widehat{\Bbbk}; M).$$

In addition

$$\lim_k{}^1 \mathrm{H}\Gamma^{n-1}(A/\mathfrak{m}^k A|\Bbbk/\mathfrak{m}^k; M/\mathfrak{m}^k M) = 0$$

and the natural homomorphisms induce isomorphisms

$$\begin{aligned} \mathrm{H}\Gamma^*(A|\Bbbk; M) &\cong \mathrm{H}\Gamma^*(\widehat{A}|\widehat{\Bbbk}; M) \\ &\cong \mathcal{H}\Gamma^*(\widehat{A}|\widehat{\Bbbk}; M) \\ &\cong \lim_k \mathrm{H}\Gamma^n(A/\mathfrak{m}^k A|\Bbbk/\mathfrak{m}^k; M/\mathfrak{m}^k M). \end{aligned}$$

Proof. Using the naturality provided by (2.4) we obtain a diagram of short exact sequences from the Milnor exact sequence of Proposition 2.2 into the one for $\mathrm{H}\Gamma^n(\widehat{A}|\widehat{\Bbbk};M)$. As the homomorphisms at either end are identities, the natural map $\mathcal{H}\Gamma^*(\widehat{A}|\widehat{\Bbbk};M) \longrightarrow \mathrm{H}\Gamma^*(\widehat{A}|\widehat{\Bbbk};M)$ is an isomorphism.

Under the assumptions, the cochain complex for Γ-cohomology is linearly compact in each degree, by Lemma 2.1. Hence by Proposition 1.4, Γ-cohomology is also linearly compact in each degree. Therefore

$$\lim_k{}^1 \mathrm{H}\Gamma^{n-1}(A/\mathfrak{m}^k|\Bbbk/\mathfrak{m}^k;M/\mathfrak{m}^k M) = 0.$$

So in this case

$$\mathcal{H}\Gamma^*(\widehat{A}|\widehat{\Bbbk};M) \cong \mathrm{H}\Gamma^*(\widehat{A}|\widehat{\Bbbk};M) \cong \lim_k \mathrm{H}\Gamma^n(A/\mathfrak{m}^k A|\Bbbk/\mathfrak{m}^k;M/\mathfrak{m}^k M). \qquad \square$$

Remark 2.4. Analogous ideas apply to Hochschild cohomology for which a continuous version appears in [7].

The following result which will be used in Section 5.

Proposition 2.5. *Let* $\Bbbk$ *be Noetherian with maximal ideal* $\mathfrak{m}$ *and let* $\widehat{\Bbbk} = \lim_k \Bbbk/\mathfrak{m}^k$. *For any* $\Bbbk$*-algebra* A *with the* $\mathfrak{m}$*-adic topology for which* $\widehat{A}$ *is topologically free on a countable basis there is a long exact sequence*

$$\cdots \longrightarrow \mathrm{H}\Gamma^{n-1}(A|\Bbbk;\widehat{\Bbbk}/\Bbbk) \longrightarrow \mathrm{H}\Gamma^n(A|\Bbbk;\Bbbk) \longrightarrow \mathrm{H}\Gamma^n(A|\Bbbk;\widehat{\Bbbk})$$
$$\longrightarrow \mathrm{H}\Gamma^n(A|\Bbbk;\widehat{\Bbbk}/\Bbbk) \longrightarrow \mathrm{H}\Gamma^{n+1}(A|\Bbbk;\Bbbk) \longrightarrow \cdots.$$

Making use of the isomorphism $\mathrm{H}\Gamma^n(\widehat{A}|\widehat{\Bbbk};\widehat{\Bbbk}) \cong \mathrm{H}\Gamma^n(A|\Bbbk;\widehat{\Bbbk})$, *we also obtain an analogous exact sequence for* $\mathrm{H}\Gamma^n(\widehat{A}|\widehat{\Bbbk};\widehat{\Bbbk})$.

Proof. The short exact sequence

$$0 \to \Bbbk \longrightarrow \widehat{\Bbbk} \longrightarrow \widehat{\Bbbk}/\Bbbk \to 0$$

of coefficients together with the last isomorphism from Proposition 2.3 yields this long exact sequence. $\square$

Finally, we record a result on the Γ-(co)homology of formally étale algebras that we will repeatedly use. We call an algebra *formally étale* if it is a colimit of étale algebras.

Lemma 2.6. *If* $(A,\Bbbk)$ *is a formally étale algebra then for any* A*-module* M,

$$\mathrm{H}\Gamma_*(A|\Bbbk;M) = 0 = \mathrm{H}\Gamma^*(A|\Bbbk;M).$$

Proof. By [33, theorem 6.8 (3)], Γ-homology and cohomology vanishes for étale algebras. Also, Γ-homology commutes with colimits. Hence if $A = \operatorname{colim}_r A_r$ with A_r étale, for any A-module M we have

$$\mathrm{H}\Gamma_*(A|\mathbb{k}; M) = \operatorname*{colim}_r \mathrm{H}\Gamma_*(A_r|\mathbb{k}; M) = 0.$$

The universal coefficient spectral sequence

$$\mathrm{E}_2^{*,*} = \mathrm{Ext}_A^{*,*}(\mathrm{H}\Gamma_*(A|\mathbb{k}; A), M) \Longrightarrow \mathrm{H}\Gamma^*(A|\mathbb{k}; M)$$

has trivial E_2-term, therefore $\mathrm{H}\Gamma^*(A|\mathbb{k}; M) = 0$. □

3. Rings of numerical polynomials

We need to describe some properties of rings of numerical polynomials which appeared in a topological setting in [3], [12] and we follow these sources in our discussion. As topological motivation, we remark that A can be identified with $KU_0\,\mathbb{C}\mathrm{P}^\infty$ and A^{st} with $KU_0\,KU$ and we will calculate the Γ-cohomology of $KU_0\,KU$ later. By definition,

$$\begin{aligned} \mathrm{A} &= \{f(w) \in \mathbb{Q}[w] : \text{for all } n \in \mathbb{Z},\ f(n) \in \mathbb{Z}\}, \\ \mathrm{A}^{\mathrm{st}} &= \{f(w) \in \mathbb{Q}[w, w^{-1}] : \text{for all } n \in \mathbb{Z} - \{0\},\ f(n) \in \mathbb{Z}[1/n]\} \end{aligned}$$

are the rings of *numerical* and *stably numerical* polynomials (over $\mathbb{Z}$). If x, y are indeterminates, we can work in any of the rings $\mathrm{A}[x, y]$, $\mathrm{A}^{\mathrm{st}}[x, y]$ or $\mathbb{Q}[w, w^{-1}][x, y]$.

We will make use of the binomial coefficient functions

$$c_n(w) = \binom{w}{n} = \frac{w(w-1)\dots(w-n+1)}{n!} \in A \subseteq \mathbb{Q}[w]$$

which can be encoded in the generating function

$$(1+x)^w = \sum_{n \geqslant 0} c_n(w) x^n \in \mathrm{A}[x] \subseteq \mathbb{Q}[w][x].$$

Notice that this satisfies the formal identity

$$(1+x)^w (1+y)^w = (1 + (x + y + xy))^w. \tag{3.1}$$

Thus we have

$$c_m(w) c_n(w) = \binom{m+n}{m} c_{m+n}(w) + (\text{terms of lower degree}) \quad \text{for } m, n \geqslant 0. \tag{3.2}$$

Theorem 3.1 ([2], [12]).

(a) A *is a free* $\mathbb{Z}$*-module with a basis consisting of the* $c_n(w)$ *for* $n \geqslant 0$.

(b) A^{st} *is the localization* $\mathrm{A}^{\mathrm{st}} = \mathrm{A}[w^{-1}]$ *and it is a free* $\mathbb{Z}$*-module on a countable basis.*

Describing explicit $\mathbb{Z}$-bases for A^{st} is a non-trivial task, see [16], [23]. On the other hand, the multiplicative structure of the $\mathbb{Z}$-algebra A^{st} is in some ways more understandable. Our next result describes some generators for A^{st}.

Theorem 3.2 ([3], [12]).

(a) *The* $\mathbb{Z}$*-algebra* A *is generated by the elements* $c_m(w)$ *with* $m \geqslant 1$ *subject to the relations of* (3.2).

(b) *The* $\mathbb{Z}$*-algebra,* A^{st} *is generated by the elements* w^{-1} *and* $c_m(w)$ *with* $m \geqslant 1$.

(c) *We have*

$$\mathrm{A} \otimes \mathbb{Q} = \mathbb{Q}[w], \quad \mathrm{A}^{\mathrm{st}} \otimes \mathbb{Q} = \mathbb{Q}[w, w^{-1}].$$

For the localizations of the rings A and A^{st} at any prime p we have

$$\mathrm{A}_{(p)} = \{f(w) \in \mathbb{Q}[w] : \text{for all } u \in \mathbb{Z}_{(p)},\ f(u) \in \mathbb{Z}_{(p)}\}, \tag{3.3a}$$

$$\mathrm{A}^{\mathrm{st}}_{(p)} = \{f(w) \in \mathbb{Q}[w] : \text{for all } u \in \mathbb{Z}^{\times}_{(p)},\ f(u) \in \mathbb{Z}_{(p)}\}. \tag{3.3b}$$

Theorem 3.3 ([2], [4, proposition 2.5], [12]).

(a) $\mathrm{A}_{(p)}$ *is a free* $\mathbb{Z}_{(p)}$*-module with a basis consisting of the monomials in the binomial coefficient functions*

$$w^{r_0} c_p(w)^{r_1} c_{p^2}(w)^{r_2} \dots c_{p^\ell}(w)^{r_\ell},$$

where $r_k = 0, 1, \dots, p-1$.

(b) *The* $\mathbb{Z}_{(p)}$*-algebra* $\mathrm{A}_{(p)}$ *is generated by the elements* $c_{p^m}(w)$ *with* $m \geqslant 0$ *subject to relations of the form*

$$c_{p^m}(w)^p - c_{p^m}(w) = p\, d_{m+1}(w),$$

where $d_{m+1}(w) \in \mathrm{A}_{(p)}$ *has* $\deg d_{m+1}(w) = p^{m+1}$*. In fact the monomials*

$$w^{r_0} d_1(w)^{r_1} d_2(w)^{r_2} \dots d_\ell(w)^{r_\ell},$$

where $r_k = 0, 1, \dots, p-1$*, form a basis of* $\mathrm{A}_{(p)}$ *over* $\mathbb{Z}_{(p)}$ *and are subject to multiplicative relations of the form*

$$d_m(w)^p - d_m(w) = p\, d'_{m+1}(w),$$

where $\deg d'_{m+1}(w) = p^{m+1}$.

(c) $\mathrm{A}^{\mathrm{st}}_{(p)}$ *is the localization* $\mathrm{A}^{\mathrm{st}}_{(p)} = \mathrm{A}_{(p)}[w^{-1}]$ *and it is a free* $\mathbb{Z}_{(p)}$*-module on a countable basis.*

(d) *The* $\mathbb{Z}_{(p)}$*-algebra,* $\mathrm{A}^{\mathrm{st}}_{(p)}$ *is generated by the elements* w *and* $e_m(w) \in \mathrm{A}^{\mathrm{st}}_{(p)}$ *for* $m \geqslant 1$ *defined recursively by*

$$w^{p-1} - 1 = pe_1(w), \quad e_m(w)^p - e_m(w) = pe_{m+1}(w) \text{ for } m \geqslant 1.$$

Corollary 3.4. *Let p be a prime.*

(a) *As* $\mathbb{F}_p$*-algebras,*

$$\mathrm{A}/p\mathrm{A} = \mathbb{F}_p[c_{p^m}(w) : m \geqslant 0]/(c_{p^m}(w)^p - c_{p^m}(w) : m \geqslant 0),$$
$$\mathrm{A}^{\mathrm{st}}/p\mathrm{A}^{\mathrm{st}} = \mathbb{F}_p[w, e_m(w) : m \geqslant 0]/(w^{p-1} - 1, e_m(w)^p - e_m(w) : m \geqslant 1).$$

Hence these algebras are formally étale over $\mathbb{F}_p$*.*

(b) *For* $n \geqslant 1$, $\mathrm{A}/p^n\mathrm{A}$ *and* $\mathrm{A}^{\mathrm{st}}/p^n\mathrm{A}^{\mathrm{st}}$ *are formally étale over* $\mathbb{Z}/p^n$*.*

(c) *The p-adic completions* $\mathrm{A}_p = \lim_n \mathrm{A}/p^n\mathrm{A}$ *and* $\mathrm{A}^{\mathrm{st}}_p = \lim_n \mathrm{A}^{\mathrm{st}}/p^n\mathrm{A}^{\mathrm{st}}$ *are formally étale over* $\mathbb{Z}_p$*.*

(d) A_p *and* $\mathrm{A}^{\mathrm{st}}_p$ *are free topological* $\mathbb{Z}_p$*-modules on countable bases. Therefore they are both compact and Hausdorff.*

Proof. Parts (b) and hence (c) can be proved by induction on $n \geqslant 1$ using the *infinite-dimensional Hensel lemma* of [7, 3.9]. The case $n = 1$ is immediate from (a). Suppose that we have found a sequence of elements $s_0, s_1, \dots, s_k, \dots \in \mathrm{A}_{(p)}$ satisfying

$$s_m^p - s_m \equiv 0 \pmod{p^n} \quad \text{for } m \geqslant 0.$$

Taking $s'_m = s_m + (s_m^p - s_m)$ we find that

$$\begin{aligned} {s'_m}^p - s'_m &= (s_m + (s_m^p - s_m))^p - (s_m + (s_m^p - s_m)) \\ &\equiv s_m^p - (s_m + (s_m^p - s_m)) \pmod{p^{n+1}} \\ &= 0. \end{aligned}$$

Hence for every n we can inductively produce such elements $s_{n,m} \in \mathrm{A}_{(p)}$ for which

$$\begin{aligned} \mathrm{A}/p^n\mathrm{A} &= \mathbb{Z}/p^n[s_{n,m} : m \geqslant 0]/(s_{n,m}^p - s_{n,m} : m \geqslant 0) \\ &= \bigotimes_{m \geqslant 0} \mathbb{Z}/p^n[s_{n,m}]/(s_{n,m}^p - s_{n,m}). \end{aligned}$$

Now passing to p-adic limits we obtain elements

$$s_m = \lim_{n \to \infty} s_{n,m} \in \mathrm{A}_p$$

for which

$$s_m^p - s_m = 0.$$

In these cases we obtain for the module of Kähler differentials

$$\Omega^1_{(\mathrm{A}/p^n\mathrm{A})/\mathbb{Z}/p^n} = 0 = \Omega^1_{\mathrm{A}_p/\mathbb{Z}_p}.$$

Part (d) is related to Mahler's Theorem and a suitable exposition of this can be found in [4]. □

There are two natural choices of augmentation for A, namely evaluation at 0 or 1,

$$\varepsilon_+ \colon \mathrm{A} \longrightarrow \mathbb{Z}; \quad \varepsilon_+ f(w) = f(0),$$
$$\varepsilon_\times \colon \mathrm{A} \longrightarrow \mathbb{Z}; \quad \varepsilon_\times f(w) = f(1).$$

For our purposes, the latter augmentation will be used. Notice that there is a ring automorphism

$$\varphi \colon \mathrm{A} \longrightarrow \mathrm{A}; \quad \varphi f(w) = f(w+1) \tag{3.4}$$

for which $\varepsilon_+\varphi = \varepsilon_\times$, so these augmentations are not too dissimilar.

4. The ring of $\mathbb{Z}/(p-1)$-invariants in $\mathrm{A}^{\mathrm{st}}_{(p)}$

In this section, p always denotes an *odd* prime. The case of $p = 2$ is related to *KO* and the work of Section 7.

Since polynomial functions $\mathbb{Z}^\times_{(p)} \longrightarrow \mathbb{Q}$ are continuous with respect to the p-adic topology they extend to continuous functions $\mathbb{Z}_p^\times \longrightarrow \mathbb{Q}_p$; such functions which also map $\mathbb{Z}^\times_{(p)}$ into $\mathbb{Z}_{(p)}$ give continuous functions $\mathbb{Z}_p^\times \longrightarrow \mathbb{Z}_p$. Hence we can regard $\mathrm{A}^{\mathrm{st}}_{(p)}$ as a subring of $\mathbb{Q}_p[w, w^{-1}]$ which in turn can be viewed as a space of continuous functions on the p-adic units $\mathbb{Z}_p^\times$. For $p \geqslant 3$ there is a splitting of topological groups

$$\mathbb{Z}_p^\times \cong \mathbb{Z}/(p-1) \times (1 + p\mathbb{Z}_p),$$

where $\mathbb{Z}/(p-1)$ identifies with a subgroup generated by a primitive $(p-1)$-st root of unity ζ. There is also a bicontinuous isomorphism $1 + p\mathbb{Z}_p \cong \mathbb{Z}_p$.

For an odd prime p, the group $\langle\zeta\rangle \cong \mathbb{Z}/(p-1)$ acts continuously on $\mathbb{Q}_p[w, w^{-1}]$ by

$$\zeta \cdot f(w) = f(\zeta w)$$

and it is immediate that this action sends elements of $\mathrm{A}^{\mathrm{st}}_{(p)}$ to continuous functions $\mathbb{Z}_p^\times \longrightarrow \mathbb{Z}_p$. It then makes sense to ask for the subring of $\mathrm{A}^{\mathrm{st}}_{(p)}$ fixed by this action, ${}^\zeta\mathrm{A}^{\mathrm{st}}_{(p)}$. We will relate this subring to the algebra of cooperations of the Adams summand in Proposition 6.1.

Recall the elements $e_m(w)$ of Theorem 3.3(d). We will write $\bar{e}_m(w)$ for $w^{-1}e_m(w)$.

Proposition 4.1. *As a $\mathbb{Z}_{(p)}$-algebra, ${}^{\zeta}\mathrm{A}^{\mathrm{st}}_{(p)}$ is generated by the elements w^{p-1} and $\bar{e}_m(w)$ for $m \geqslant 1$.*

Proof. It is clear that

$$ {}^{\zeta}\mathbb{Q}[w, w^{-1}] = \mathbb{Q}[w^{p-1}, w^{-(p-1)}]. $$

Also, by construction of the $e_m(w)$,

$$ \bar{e}_m(w) \in {}^{\zeta}\mathrm{A}^{\mathrm{st}}_{(p)} \subseteq \mathbb{Q}[w^{p-1}, w^{-(p-1)}]. $$

Consider the multiplicative idempotent

$$ E_\zeta : \mathbb{Q}[w, w^{-1}] \longrightarrow \mathbb{Q}[w, w^{-1}]; \quad E_\zeta f(w) = \frac{1}{p-1} \sum_{r=1}^{p-1} f(\zeta^r w). $$

Then we have

$$ {}^{\zeta}\mathrm{A}^{\mathrm{st}}_{(p)} = E_\zeta \mathrm{A}^{\mathrm{st}}_{(p)}. $$

Each element $f(w) \in \mathbb{Q}[w, w^{-1}]$ has the form

$$ f(w) = f_0(w^{p-1}) + w f_1(w^{p-1}) + \cdots + w^{p-2} f_{p-2}(w^{p-1}), $$

where $f_k(x) \in \mathbb{Q}[x, x^{-1}]$, hence

$$ E_\zeta f(w) = f_0(w^{p-1}). $$

From this it follows that ${}^{\zeta}\mathrm{A}^{\mathrm{st}}_{(p)}$ is generated as a $\mathbb{Z}_{(p)}$-algebra by the stated elements. □

Corollary 4.2. *The following hold.*

(a) *As $\mathbb{F}_p$-algebras,*

$$ {}^{\zeta}\mathrm{A}^{\mathrm{st}}_{(p)}/p({}^{\zeta}\mathrm{A}^{\mathrm{st}}_{(p)}) = \mathbb{F}_p[w, \bar{e}_m(w) : m \geqslant 1]/(w^{p-1} - 1, \bar{e}_m(w)^p - \bar{e}_m(w) : m \geqslant 1). $$

Hence this algebra is formally étale over $\mathbb{F}_p$.

(b) *For $n \geqslant 1$, ${}^{\zeta}\mathrm{A}^{\mathrm{st}}_{(p)}/p^n({}^{\zeta}\mathrm{A}^{\mathrm{st}}_{(p)})$ is formally étale over $\mathbb{Z}/p^n$.*

(c) *The p-adic completion ${}^{\zeta}\mathrm{A}^{\mathrm{st}}_{p} = \lim_n {}^{\zeta}\mathrm{A}^{\mathrm{st}}_{(p)}/p^n({}^{\zeta}\mathrm{A}^{\mathrm{st}})_{(p)}$ is formally étale over $\mathbb{Z}_p$.*

5. The Γ-cohomology of numerical polynomials

Recall that $\widehat{\mathbb{Z}}/\mathbb{Z}$ and $\mathbb{Z}_p/\mathbb{Z}_{(p)}$ for any prime p are torsion-free divisible groups, so they are both $\mathbb{Q}$-vector spaces which have the same cardinality and (uncountable) dimensions; thus they are isomorphic. Similarly, we have $\widehat{\mathbb{Z}}/\mathbb{Z} \cong \widehat{\mathbb{Q}}/\mathbb{Q}$ and $\mathbb{Z}_p/\mathbb{Z}_{(p)} \cong \mathbb{Q}_p/\mathbb{Q}$.

In the following, we will always use the augmentations $\varepsilon_\times \colon \mathrm{A} \longrightarrow \mathbb{Z}$ and $\varepsilon_\times \colon \mathrm{A}^{\mathrm{st}} \longrightarrow \mathbb{Z}$ and their analogues for the p-localized versions. In calculating the Γ-cohomology of A, we would obtain the same result using ε_+ because of the existence of the automorphism φ of (3.4).

Theorem 5.1. *Let R be an augmented commutative $\mathbb{Z}$-algebra. Assume that, at each prime p, the p-completion R_p is topologically free on a countable basis. Suppose that for all primes p and $k \geqslant 1$, R/p^k is a formally étale algebra over $\mathbb{Z}/p^k$. Then for all $s \geqslant 0$,*

$$\mathrm{H}\Gamma^s(R|\mathbb{Z};\mathbb{Z}) \cong \mathrm{H}\Gamma^{s-1}(R_{\mathbb{Q}}|\mathbb{Q};\mathbb{Q}).$$

Proof. For each natural number n, we may write

$$n = \prod_p p^{\mathrm{ord}_p n}$$

where the product is taken over all primes p. The Chinese Remainder Theorem provides splittings

$$\mathbb{Z}/n = \prod_p \mathbb{Z}/p^{\mathrm{ord}_p n}, \tag{5.1a}$$

$$\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p. \tag{5.1b}$$

Applying the Transitivity Sequence and using that $R_{(p)}$ is étale over R, we obtain that at each prime p

$$\mathrm{H}\Gamma^*(R_{(p)}|\mathbb{Z}_{(p)};\widehat{\mathbb{Z}}) \cong \mathrm{H}\Gamma^*(R|\mathbb{Z};\widehat{\mathbb{Z}}).$$

Therefore by Corollary 2.3 we have

$$\mathrm{H}\Gamma^*(R|\mathbb{Z};\widehat{\mathbb{Z}}) = \prod_p \mathrm{H}\Gamma^*(R|\mathbb{Z};\mathbb{Z}_p) \cong \prod_p \mathrm{H}\Gamma^*(R_p|\mathbb{Z}_p;\mathbb{Z}_p).$$

For the second isomorphism, using Corollary 2.3 and the linear compactness of Γ-cohomology provided by Proposition 1.4, we can express $\mathrm{H}\Gamma^*(R|\mathbb{Z};\mathbb{Z}_p)$ as the inverse limit of the groups $\mathrm{H}\Gamma^*(R|\mathbb{Z};\mathbb{Z}/p^n)$. Here the coefficients $\mathbb{Z}/p^n$ eliminate the effect of all the p-divisible elements, therefore $\mathrm{H}\Gamma^*(R|\mathbb{Z};\mathbb{Z}/p^n)$ reduces to $\mathrm{H}\Gamma^*(R_p|\mathbb{Z}_p;\mathbb{Z}/p^n)$, where R_p denotes the p-adic completion of R.

Now for each $k \geqslant 1$, the assumption that R/p^k is formally étale over $\mathbb{Z}/p^k$ ensures that

$$\mathrm{H\Gamma}^*(R/p^k|\mathbb{Z}/p^k; \mathbb{Z}/p^k) = 0.$$

Therefore we obtain

$$\begin{aligned}\mathrm{H\Gamma}^*(R_p|\mathbb{Z}_p; \mathbb{Z}_p) &= \lim_k \mathrm{H\Gamma}^*(R/p^k|\mathbb{Z}/p^k; \mathbb{Z}/p^k) = 0 \\ &= \lim_k{}^1 \mathrm{H\Gamma}^*(R/p^k|\mathbb{Z}/p^k; \mathbb{Z}/p^k).\end{aligned}$$

For each n, Proposition 2.5 implies that

$$\begin{aligned}\mathrm{H\Gamma}^n(R_{(p)}|\mathbb{Z}_{(p)}; \mathbb{Z}_{(p)}) &= \mathrm{H\Gamma}^{n-1}(R_{(p)}|\mathbb{Z}_{(p)}; \mathbb{Z}_p/\mathbb{Z}_{(p)}), \\ \mathrm{H\Gamma}^n(R|\mathbb{Z}; \mathbb{Z}) &= \mathrm{H\Gamma}^{n-1}(R|\mathbb{Z}; \widehat{\mathbb{Z}}/\mathbb{Z}).\end{aligned}$$

As $\mathbb{Z}_p/\mathbb{Z}_{(p)}$ and $\widehat{\mathbb{Z}}/\mathbb{Z}$ are $\mathbb{Q}$-vector spaces, for all $n \neq 0$ we obtain

$$\mathrm{H\Gamma}^n(R_{(p)}|\mathbb{Z}_{(p)}; \mathbb{Z}_p/\mathbb{Z}_{(p)}) \cong \mathrm{H\Gamma}^n(R_{\mathbb{Q}}|\mathbb{Q}; \mathbb{Z}_p/\mathbb{Z}_{(p)})$$

and similarly

$$\mathrm{H\Gamma}^n(R|\mathbb{Z}; \widehat{\mathbb{Z}}/\mathbb{Z}) \cong \mathrm{H\Gamma}^n(R_{\mathbb{Q}}|\mathbb{Q}; \widehat{\mathbb{Z}}/\mathbb{Z}). \qquad \square$$

Corollary 5.2. *We have*

$$\mathrm{H\Gamma}^n(\mathrm{A}^{\mathrm{st}}|\mathbb{Z}; \mathbb{Z}) = \mathrm{H\Gamma}^n(\mathrm{A}|\mathbb{Z}; \mathbb{Z}) = \begin{cases} \widehat{\mathbb{Z}}/\mathbb{Z} & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

For each prime p,

$$\mathrm{H\Gamma}^n(\mathrm{A}^{\mathrm{st}}_{(p)}|\mathbb{Z}_{(p)}; \mathbb{Z}_{(p)}) = \mathrm{H\Gamma}^n(\mathrm{A}_{(p)}|\mathbb{Z}_{(p)}; \mathbb{Z}_{(p)}) = \begin{cases} \mathbb{Z}_p/\mathbb{Z}_{(p)} & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since $\mathrm{A}[w^{-1}]$ is étale over A, both of the Γ-cohomology groups

$$\mathrm{H\Gamma}^*(\mathrm{A}[w^{-1}]|\mathrm{A}; \mathbb{Z}) \quad \text{and} \quad \mathrm{H\Gamma}^*(\mathrm{A}_{(p)}[w^{-1}]|\mathrm{A}_{(p)}; \mathbb{Z})$$

vanish. The Transitivity Theorem [33, 3.4] implies that there are isomorphisms

$$\begin{aligned}\mathrm{H\Gamma}^*(\mathrm{A}|\mathbb{Z}; \mathbb{Z}) &\cong \mathrm{H\Gamma}^*(\mathrm{A}[w^{-1}]|\mathbb{Z}; \mathbb{Z}) = \mathrm{H\Gamma}^*(\mathrm{A}^{\mathrm{st}}|\mathbb{Z}; \mathbb{Z}), \\ \mathrm{H\Gamma}^*(\mathrm{A}_{(p)}|\mathbb{Z}_{(p)}; \mathbb{Z}_{(p)}) &\cong \mathrm{H\Gamma}^*(\mathrm{A}_{(p)}[w^{-1}]|\mathbb{Z}_{(p)}; \mathbb{Z}_{(p)}) = \mathrm{H\Gamma}^*(\mathrm{A}^{\mathrm{st}}_{(p)}|\mathbb{Z}_{(p)}; \mathbb{Z}_{(p)}),\end{aligned}$$

hence it suffices to prove the result for A and $\mathrm{A}_{(p)}$.

Corollary 3.4 ensures that $\mathrm{A}/p^k\mathrm{A}$ and therefore $\mathrm{A}_{(p)}/p^k\mathrm{A}_{(p)}$ as well is formally étale over $\mathbb{Z}/p^k$ for all $k \geqslant 1$. Now Corollaries 3.4(d) and 1.3 together guarantee that the cochains for Γ-cohomology fulfil the linear compactness requirements of Theorem 5.1. Thus we can apply this theorem and obtain the vanishing result for Γ-cohomology in dimensions different from one.

By [29, theorem 4.1] and the fact that $\mathbb{Z}_p/\mathbb{Z}_{(p)}$ and $\widehat{\mathbb{Z}}/\mathbb{Z}$ are $\mathbb{Q}$-vector spaces, we have

$$\begin{aligned}\mathrm{H}\Gamma^*(\mathrm{A}_{(p)}|\mathbb{Z}_{(p)};\mathbb{Z}_p/\mathbb{Z}_{(p)}) &= \mathrm{H}\Gamma^*(\mathrm{A}\otimes\mathbb{Q}|\mathbb{Q};\mathbb{Z}_p/\mathbb{Z}_{(p)})\\ &= \mathrm{H}\Gamma^*(\mathbb{Q}[w]|\mathbb{Q};\mathbb{Z}_p/\mathbb{Z}_{(p)}) = \mathbb{Z}_p/\mathbb{Z}_{(p)}\end{aligned}$$

and

$$\mathrm{H}\Gamma^*(\mathrm{A}|\mathbb{Z};\widehat{\mathbb{Q}}/\mathbb{Q}) = \mathrm{H}\Gamma^*(\mathrm{A}\otimes\mathbb{Q}|\mathbb{Q};\widehat{\mathbb{Z}}/\mathbb{Z}) = \mathrm{H}\Gamma^*(\mathbb{Q}[w]|\mathbb{Q};\widehat{\mathbb{Z}}/\mathbb{Z}) = \widehat{\mathbb{Z}}/\mathbb{Z}.$$

Thus we obtain

$$\mathrm{H}\Gamma^n(\mathrm{A}_{(p)}|\mathbb{Z}_{(p)};\mathbb{Z}_{(p)}) = \begin{cases}\mathbb{Z}_p/\mathbb{Z}_{(p)} & \text{if } n=1,\\ 0 & \text{otherwise,}\end{cases}$$

and

$$\mathrm{H}\Gamma^n(\mathrm{A}|\mathbb{Z};\mathbb{Z}) = \begin{cases}\widehat{\mathbb{Z}}/\mathbb{Z} & \text{if } n=1,\\ 0 & \text{otherwise,}\end{cases}$$

as claimed. □

Remark 5.3. Notice, that for the calculations of Γ-cohomology above we used the formal properties of Γ-cohomology. As André–Quillen cohomology satisfies analoguous properties, we can transfer the above results to obtain the following:

$$\mathrm{AQ}_n(\mathrm{A}^{\mathrm{st}}|\mathbb{Z};\mathbb{Z}) = \mathrm{AQ}_n(\mathrm{A}|\mathbb{Z};\mathbb{Z}) = \begin{cases}\mathbb{Q} & \text{if } n=0,\\ 0 & \text{if } n\neq 0,\end{cases}$$

$$\mathrm{AQ}^n(\mathrm{A}^{\mathrm{st}}|\mathbb{Z};\mathbb{Z}) = \mathrm{AQ}^n(\mathrm{A}|\mathbb{Z};\mathbb{Z}) = \begin{cases}\widehat{\mathbb{Q}}/\mathbb{Q} & \text{if } n=1,\\ 0 & \text{if } n\neq 1.\end{cases}$$

The results from Section 4 allow us to calculate the Γ-cohomology of ${}^{\zeta}\mathrm{A}^{\mathrm{st}}_{(p)}$ over $\mathbb{Z}_{(p)}$ directly as was done above for A^{st}. Alternatively, we may use the fact that the extension $\mathrm{A}^{\mathrm{st}}_{(p)}/{}^{\zeta}\mathrm{A}^{\mathrm{st}}_{(p)}$ is étale since it has the form B/A, where $B = A[t]/(t^{p-1}-v)$ for a unit $v \in A$, where A is a $\mathbb{Z}_{(p)}$-algebra. We can now determine the Γ-cohomology of ${}^{\zeta}\mathrm{A}^{\mathrm{st}}_{(p)}$ since the Transitivity Theorem of [33, 3.4] gives

Proposition 5.4. *For an odd prime p,*

$$\mathrm{H}\Gamma^*({}^{\zeta}\mathrm{A}^{\mathrm{st}}_{(p)}|\mathbb{Z}_{(p)};\mathbb{Z}_{(p)}) = \mathrm{H}\Gamma^*(\mathrm{A}^{\mathrm{st}}_{(p)}|\mathbb{Z}_{(p)};\mathbb{Z}_{(p)}).$$

6. Applications to E_∞ structures on K-theory

Robinson [31] has developed an obstruction theory for E_∞ structures on a homotopy commutative ring spectrum E. Provided E satisfies the following form of the Künneth and universal coefficient theorems for E_*E

$$E^*(E^{\wedge n}) \cong \mathrm{Hom}_{E_*}(E_*E^{\otimes n}, E_*),$$

then the obstructions lie in groups

$$\mathrm{H}\Gamma^{n,2-n}(E_*E|E_*; E_*),$$

while the extensions are determined by classes in

$$\mathrm{H}\Gamma^{n,1-n}(E_*E|E_*; E_*).$$

Here the bigrading (s,t) involves cohomological degree s and internal degree t. Moreover, the relevant values of n are for $n \geqslant 3$.

We want to apply this to the cases of complex KU-theory and the Adams summand $E(1)$ of $KU_{(p)}$ at a prime p. Recall that

$$KU_* = \mathbb{Z}[t, t^{-1}], \quad KU_{(p)*} = \mathbb{Z}_{(p)}[t, t^{-1}], \quad E(1)_* = \mathbb{Z}_{(p)}[u, u^{-1}],$$

where $t \in KU_2$ and $u \in E(1)_{2(p-1)}$. Our next result implies that the relevant conditions mentioned above are both satisfied for KU and $E(1)$.

Proposition 6.1. *There are isomorphisms of rings* (*in fact, of Hopf algebras*)

$$KU_0 KU \cong \mathrm{A}^{\mathrm{st}}, \quad KU_{(p)0} KU_{(p)} \cong \mathrm{A}^{\mathrm{st}}_{(p)},$$
$$E(1)_0 E(1) \cong {}^{\zeta}\mathrm{A}^{\mathrm{st}}_{(p)}.$$

Hence,

$$KU_* KU \cong KU_* \otimes \mathrm{A}^{\mathrm{st}}, \quad KU_{(p)*} KU_{(p)} \cong KU_{(p)*} \otimes \mathrm{A}^{\mathrm{st}}_{(p)},$$
$$E(1)_* E(1) \cong E(1)_* \otimes {}^{\zeta}\mathrm{A}^{\mathrm{st}}_{(p)}.$$

Proof. The isomorphisms for KU and $KU_{(p)}$ can be found in [12, p. 392].

Consider $E(1)_*E(1)$, the algebra of cooperations for $E(1)$. Since $E(1)$ is Landweber exact, we have

$$\begin{aligned} E(1)_*E(1) &\cong E(1)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} E(1)_* \\ &\cong E(1)_*[t_1, t_2, \ldots, V_1, V_1^{-1}]/(\text{relations}), \end{aligned}$$

where V_1 denotes the right unit η_r applied to v_1 and the variables t_i stem from $BP_* BP$. We also write $\overline{w} = v_1^{-1} V_1$. The relation $v_1 + pt_1 - V_1 = 0$ in $E(1)_* E(1)$ gives rise to

$$1 - v_1^{-1} V_1 = -p v_1^{-1} t_1 \in E(1)_0 E(1),$$

hence on setting $\overline{e}_1 = v_1^{-1} t_1$ we have

$$\overline{w} - 1 = p\overline{e}_1.$$

Now we may inductively define

$$\overline{e}_m = v_1^{-p^{m-1}-\cdots-p-1} t_m.$$

The higher relations

$$v_1 t_k^p - v_1^{p^k} t_k + p t_{k+1} = 0 \quad (k \geqslant 1)$$

can be used to prove the desired relations for the $\overline{e}_m$. Taking the p-th power we have

$$\overline{e}_m^p = v_1^{-p^m-\cdots-p} t_m^p.$$

Multiplying the relation

$$v_1 t_m^p - v_1^{p^m} t_m = p t_{m+1}$$

by $v_1^{-p^m-\cdots-p-1}$, we obtain

$$\overline{e}_m^p - v_1^{p^m-p^m-p^{m-1}-\cdots-p-1} t_m = p v_1^{-p^m-\cdots-p-1} t_{m+1}$$

which is precisely

$$\overline{e}_m^p - \overline{e}_m = p\overline{e}_{m+1}. \qquad \square$$

Flat base-change leads to isomorphisms

$$\mathrm{H}\Gamma^{*,*}(KU_* KU \mid KU_*; KU_*) \cong \mathrm{H}\Gamma^*(\mathrm{A}^{\mathrm{st}} | \mathbb{Z}; \mathbb{Z}) \otimes KU_*,$$
$$\mathrm{H}\Gamma^{*,*}(KU_{(p)*} KU_{(p)} | KU_{(p)*}; KU_{(p)*}) \cong \mathrm{H}\Gamma^*(\mathrm{A}^{\mathrm{st}}_{(p)} | \mathbb{Z}_{(p)}; \mathbb{Z}_{(p)}) \otimes KU_{(p)*},$$
$$\mathrm{H}\Gamma^{*,*}(E(1)_* E(1) | E(1)_*; E(1)_*) \cong \mathrm{H}\Gamma^*({}^{\zeta}\mathrm{A}^{\mathrm{st}}_{(p)} | \mathbb{Z}_{(p)}; \mathbb{Z}_{(p)}) \otimes E(1)_*.$$

With the help of Corollary 5.2 we can therefore deduce the following.

Theorem 6.2. *For a prime p and $n \geqslant 2$,*

$$\mathrm{H}\Gamma^{n,2-n}(KU_* KU \mid KU_*; KU_*) = 0$$
$$= \mathrm{H}\Gamma^{n,1-n}(KU_* KU \mid KU_*; KU_*),$$
$$\mathrm{H}\Gamma^{n,2-n}(KU_{(p)*} KU_{(p)} | KU_{(p)*}; KU_{(p)*}) = 0$$
$$= \mathrm{H}\Gamma^{n,1-n}(KU_{(p)*} KU_{(p)} | KU_{(p)*}; KU_{(p)*}),$$
$$\mathrm{H}\Gamma^{n,2-n}(E(1)_* E(1) | E(1)_*; E(1)_*) = 0$$
$$= \mathrm{H}\Gamma^{n,1-n}(E(1)_* E(1) | E(1)_*; E(1)_*).$$

Hence KU, $KU_{(p)}$, and $E(1)$ each have a unique E_∞ structure.

It is a rather old question whether the connective Adams summand, often denoted by ℓ, is an E_∞ spectrum. The E_∞ ring spectrum machinery developed in [26] yields the following general result.

Theorem 6.3. *For any E_∞ ring spectrum E, the connective cover $e \longrightarrow E$ possesses a model as an E_∞ ring spectrum.*

Proof. Proceeding as in [26, proposition VII.3.2], we first take the underlying zeroth space E_0 of the E_∞ ring spectrum E, then build a prespectrum $T(E_0)$ out of it using a bar construction which consists of suspensions and the monad for the little convex body (partial) operad. Finally we apply the spectrification functor (there called Ω^∞) to $T(E_0)$. By [26, proposition VII.3.2], this has the correct homotopy groups and is an E_∞ ring spectrum. □

Applying this result, we obtain a canonical E_∞ model for the connective cover $\ell \longrightarrow E(1)$.

Proposition 6.4. *There is at least one E_∞ structure on the connective Adams summand ℓ.*

Remark 6.5. After p-completion, we obtain an E_∞ structure on the p-completed connective Adams summand $\ell\widehat{_p}$. In subsequent work we have shown that this E_∞ structure coincides with the one constructed by McClure and Staffeldt in [27] using algebraic K-theory.

7. E_∞ structures on *KO*

The case of *KO* can be treated by similar methods but involves somewhat more delicate considerations because of the presence of 2-torsion in KO_*. Recall that

$$KO_* = \mathbb{Z}[h, y, w, w^{-1}]/(2h, h^3, hy, y^2 - 4w), \tag{7.1}$$

where $h \in KO_1$, $y \in KO_4$ and $w \in KO_8$. We will also require the graded $\mathbb{Q}$-vector space $V_* = KO_* \otimes \widehat{\mathbb{Z}}/\mathbb{Z}$.

We will prove the following algebraic result.

Theorem 7.1.

(a) *For any prime p and $k \geqslant 1$, we have*

$$\mathrm{H}\Gamma^*(KO_0\,KO/p^k|\mathbb{Z}/p^k; \mathbb{Z}/p^k) = 0,$$

and

$$\mathcal{H}\Gamma^*(KO_0\, KO\widehat{_p}\,|\mathbb{Z}_p;\mathbb{Z}_p) = \mathrm{H}\Gamma^*(KO_0\, KO\widehat{_p}\,|\mathbb{Z}_p;\mathbb{Z}_p) = 0.$$

(b) *We have*

$$\mathrm{H}\Gamma^{n,*}(KO_*\, KO \mid KO_*; KO_*) = \begin{cases} V_* & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Using this, the obstruction theory E_∞ structures and localization yield our result on E_∞ structures for *KO*.

Theorem 7.2. *KO and, for each prime p, $KO_{(p)}$ and $KO\widehat{_p}$ all have unique E_∞ structures.*

To prove Theorem 7.1, we begin with a composite result distilled from [3] and [1, p. 162].

Theorem 7.3.

(a) *$KO_*\, KO$ is a free KO_*-module on countably many generators lying in $KO_0\, KO$.*

(b) *The natural homomorphism*

$$KO_0\, KO \longrightarrow KU_0\, KU \xrightarrow{\cong} \mathrm{A}^{\mathrm{st}}$$

is a split monomorphism whose image is

$$\{f(w) \in \mathrm{A}^{\mathrm{st}} : f(-w) = f(w)\} \subseteq \mathrm{A}^{\mathrm{st}}.$$

(c) *For each prime p and $k \geqslant 1$,*

$$KO_0\, KO/p^k \longrightarrow KU_0\, KU/p^k \cong \mathrm{A}^{\mathrm{st}}/p^k$$

is a split monomorphism of $\mathbb{Z}/p^k$-modules.

Consider the short exact sequence

$$0 \to KO_* \longrightarrow KO_* \otimes \widehat{\mathbb{Z}} \longrightarrow KO_* \otimes \widehat{\mathbb{Z}}/\mathbb{Z} \to 0.$$

Since $\widehat{\mathbb{Z}}/\mathbb{Z}$ is a $\mathbb{Q}$-vector space and we have the splitting of (5.1b), we can reformulate the above exact sequence to obtain that the sequence

$$0 \to KO_* \longrightarrow \prod_p KO_* \otimes \mathbb{Z}_p \longrightarrow V_* \to 0, \tag{7.2}$$

is exact. Here V_* is defined above and we have used the fact that each group KO_n is finitely-generated. The application of Γ-cohomology of $KO_* KO$ to this sequence yields a long exact sequence which relates $H\Gamma^{*,*}(KO_* KO \mid KO_*; KO_*)$ to $H\Gamma^{*,*}(KO_* KO \mid KO_*; \prod_p KO_* \otimes \mathbb{Z}_p)$ and $H\Gamma^{*,*}(KO_* KO \mid KO_*; V_*)$.

Now, for the part with coefficients in V_* we have

$$H\Gamma^{*,*}(KO_* KO \mid KO_*; V_*) \cong H\Gamma^{*,*}(KO_* KO \otimes \mathbb{Q} \mid KO_* \otimes \mathbb{Q}; V_*),$$

and

$$KO_* \otimes \mathbb{Q} = \mathbb{Q}[y, y^{-1}], \quad KO_* KO \otimes \mathbb{Q} = \mathbb{Q}[y, y^{-1}, z, z^{-1}].$$

By [29], as in Corollary 5.2 we find that

$$H\Gamma^{n,*}(KO_* KO \mid KO_*; V_*) = \begin{cases} V_* & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{7.3}$$

For a fixed algebra, the cochain complex for Γ-cohomology commutes with limits taken over the coefficient module, therefore Γ-cohomology commutes with products of coefficient modules and the splitting $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ leads to

$$H\Gamma^{*,*}(KO_* KO \mid KO_*; KO_* \otimes \widehat{\mathbb{Z}}) \cong \prod_p H\Gamma^{*,*}(KO_* KO \mid KO_*; KO_* \otimes \mathbb{Z}_p).$$

For each prime p we obtain a short exact sequence,

$$\begin{aligned} 0 \to \lim_k{}^1\, & H\Gamma^{*-1,*}(KO_* KO \mid KO_*; KO_*/p^k) \\ & \longrightarrow H\Gamma^{*,*}(KO_* KO \mid KO_*; KO_* \otimes \mathbb{Z}_p) \\ & \qquad \longrightarrow \lim_k H\Gamma^{*,*}(KO_* KO \mid KO_*; KO_*/p^k) \to 0. \end{aligned}$$

When $p > 2$, we are reduced to considering

$$H\Gamma^{*,*}(KO_* KO \mid KO_*; KO_*/p^k) = H\Gamma^{*,*}(KO_* KO/p^k \mid KO_*/p^k; KO_*/p^k),$$

which can be determined by the methods of Section 4 using the subgroup $\{\pm 1\} \leqslant \mathbb{Z}_p^\times$ in place of the group of all $(p-1)$-st roots of unity. The result is that

$$H\Gamma^{*,*}(KO_* KO \mid KO_*; KO_*/p^k) = 0,$$

whence

$$H\Gamma^{*,*}(KO_* KO \mid KO_*; KO_* \otimes \mathbb{Z}_p) = 0.$$

The case $p = 2$ requires a more intricate analysis. First we identify $KO_0 KO_{(2)}$ and the quotients $KO_0 KO/2^k$ as rings of functions.

Theorem 7.4.

(a) *There is an isomorphism of rings*

$$KO_0\,KO_{(2)} \cong \{f(w) \in \mathbb{Q}[w, w^{-1}] : f\mathbb{Z}_{(2)}^{\times} \subseteq \mathbb{Z}_{(2)},\ f(-w) = f(w)\} \subseteq \mathrm{A}_{(2)}^{\mathrm{st}}.$$

(b) *For each* $k \geqslant 1$, *there is an isomorphism of rings*

$$KO_0\,KO/2^k \cong \mathrm{Cont}(1 + 8\mathbb{Z}_2, \mathbb{Z}/2^k),$$

where $\mathrm{Cont}(1 + 8\mathbb{Z}_2, \mathbb{Z}/2^k)$ *denotes the space of continuous maps from* $1+8\mathbb{Z}_2 \subseteq \mathbb{Z}_2^{\times} \subseteq \mathbb{Z}_2$ *with its 2-adic topology to* $\mathbb{Z}/2^k$ *with the discrete topology.*

(c) *There is an isomorphism of rings*

$$KO_0\,KO\widehat{_2} \cong \mathrm{Cont}(1 + 8\mathbb{Z}_2, \mathbb{Z}_2),$$

the space of continuous maps from $1 + 8\mathbb{Z}_2$ *to* $\mathbb{Z}_2$.

(d) *The algebras* $(KO_0\,KO/2^k, \mathbb{Z}/2^k)$ *and* $(KO_0\,KO\widehat{_2}, \mathbb{Z}_2)$ *are formally étale.*

Proof. The methods of [4] apply here, and we leave verification of the details to the reader.

The squaring map $\mathbb{Z}_2^{\times} \longrightarrow \mathbb{Z}_2^{\times}$ has image $1 + 8\mathbb{Z}_2$, hence a polynomial $f(w) \in \mathrm{A}_{(2)}^{\mathrm{st}}$ satisfying $f(-w) = f(w)$ corresponds to a continuous function $1+8\mathbb{Z}_2 \longrightarrow \mathbb{Z}_2$. By compactness of the domain, $\mathrm{Cont}(1 + 8\mathbb{Z}_2, \mathbb{Z}/2^k)$ consists of locally constant functions. If we express $x \in \mathbb{Z}_2$ in the form

$$x = x_0 + x_1 2 + x_2 2^2 + \cdots + x_n 2^n + \cdots,$$

where $x_i = 0, 1$, then the functions

$$\xi_i : 1 + 8\mathbb{Z}_2 \longrightarrow \mathbb{Z}_2; \quad \xi_i(x) = x_i$$

are locally constant and give rise to $\mathbb{Z}/2^k$-algebra generators of $\mathrm{Cont}(1+8\mathbb{Z}_2, \mathbb{Z}/2^k)$. They also satisfy the relations

$$\xi_i^2 = \xi_i,$$

and the distinct monomials

$$\xi_0^{r_0}\xi_1^{r_1}\ldots\xi_d^{r_d} \quad (r_i = 0, 1)$$

form a $\mathbb{Z}/2^k$-basis. This implies that the $\mathbb{Z}/2^k$-algebra $\mathrm{Cont}(1 + 8\mathbb{Z}_2, \mathbb{Z}/2^k)$ is formally étale. Similar considerations apply to the topological algebra $KO_0\,KO\widehat{_2}$. □

Collecting together the results of the above discussion (in particular Theorem 7.4(d)) we obtain the case $p = 2$ of Theorem 7.1(a). The proof of Theorem 7.1(b) makes use of the long exact sequence resulting from (7.2) and (7.3).

We remark that rather than working modulo powers of 2, it is also possible to consider powers of the maximal ideal $(2, h, y) \lhd KO_*$ and then we obtain

Proposition 7.5. *For $k \geqslant 1$,*

$$\mathrm{H}\Gamma^*(KO_* KO/(2,h,y)^k | KO_*/(2,h,y)^k; KO_*/(2,h,y)^k) = 0.$$

and

$$\begin{aligned}\mathcal{H}\Gamma^{*,*}(KO_* \widehat{KO_{(2,h,y)}} \,|\, \widehat{(KO_*)_{(2,h,y)}}; \widehat{(KO_*)_{(2,h,y)}})\\ = \mathrm{H}\Gamma^{*,*}(KO_* \widehat{KO_{(2,h,y)}} \,|\, \widehat{(KO_*)_{(2,h,y)}}; \widehat{(KO_*)_{(2,h,y)}}) = 0.\end{aligned}$$

8. E_∞ structures on the I_n-adic completion of $E(n)$

In this section we describe what we can prove about E_∞ structures on the I_n-adic completion of Johnson–Wilson spectrum $E(n)$ for a prime p and $n \geqslant 1$.

The coefficient ring

$$E(n)_* = \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n, v_n^{-1}]$$

is Noetherian and contains the maximal ideal

$$I_n = (p, v_1, \dots, v_{n-1}) \lhd E(n)_*.$$

Here the v_i denote the images of the Araki generators of BP_* and we sometimes write $v_0 = p$. There is a commutative ring spectrum $\widehat{E(n)}$ for which the coefficient ring $\widehat{E(n)}_*$ is the I_n-adic completion of $E(n)_*$, *i.e.*, its completion at I_n. It is known from [13], [19] that $\widehat{E(n)}$ is the $K(n)$-localization of $E(n)$. We also know from [7] that for each prime p, $\widehat{E(n)}$ possesses a unique A_∞ structure and the canonical map $\widehat{E(n)} \longrightarrow \widehat{E(n)}/I_n \simeq K(n)$ to the n-th Morava K-theory is a map of A_∞ ring spectra for any of the A_∞ structures on $K(n)$ shown to exist in [30]. Actually these results were only claimed for odd primes but the arguments also work for the prime 2.

Proposition 8.1. *Possible obstructions for an E_∞ structure on the completed Johnson–Wilson spectra $\widehat{E(n)}$ live in the continuous Γ-cohomology groups*

$$\mathcal{H}\Gamma^{*,*}(\widehat{E(n)}_*\widehat{E(n)} | \widehat{E(n)}_*; \widehat{E(n)}_*).$$

Proof. For $\widehat{E(n)}$ we have a *continuous* universal coefficient theorem, *i.e.*, possible obstructions live in the *continuous* $\widehat{E(n)}$-cohomology of $(X_m)_+ \ltimes_{\Sigma_m} E^{\wedge m}$, where X_m is a filtration quotient of an E_∞ operad as described in [31, section 5.1]. These cohomology groups can be identified with the continuous $\widehat{E(n)}_*$-homomorphisms from the corresponding $\widehat{E(n)}$-homology groups (compare [31, proposition 5.4] and [7, §1]). This proves the claim. □

For each $\ell \geqslant 0$, Proposition 2.2 yields a short exact sequence

$$\begin{aligned} 0 \to \lim_k{}^1 \, \mathrm{H}\Gamma^{\ell-1,*}(E(n)_*E(n)/I_n^k | E(n)_*/I_n^k;\, E(n)_*/I_n^k) \\ \longrightarrow \mathcal{H}\Gamma^{\ell,*}(\widehat{E(n)}_*\widehat{E(n)} | \widehat{E(n)}_*;\, \widehat{E(n)}_*) \\ \longrightarrow \lim_k \mathrm{H}\Gamma^{\ell,*}(E(n)_*E(n)/I_n^k | E(n)_*/I_n^k;\, E(n)_*/I_n^k) \to 0. \end{aligned} \tag{8.1}$$

Theorem 8.2. *The $E(n)_*/I_n^k$-algebra $E(n)_*E(n)/I_n^k$ is formally étale. Hence the Γ-cohomology of $E(n)_*E(n)/I_n^k$ over $E(n)_*/I_n^k$ is trivial,*

$$\mathrm{H}\Gamma^{*,*}(E(n)_*E(n)/I_n^k | E(n)_*/I_n^k;\, \widehat{E(n)}_*/I_n^k) = 0.$$

Proof. First we show that the algebra $E(n)_*E(n)/I_n^k$ is formally étale. In the following we use the notation chosen in [7]. As in the proof of [7, lemma 3.4], we can apply the infinite-dimensional Hensel lemma (see the proof of our Corollary 3.4) to split $E(n)_*E(n)/I_n^k$ into an infinite tensor product of $E(n)_*/I_n^k$-algebras,

$$E(n)_*E(n)/I_n^k = \bigotimes_{j \geqslant 1} E(n)_*/I_n^k[S_j]/(v_n S_j^{p^n} - v_n^{p^j} S_j).$$

We can write $E(n)_*E(n)/I_n^k$ as a colimit of finite tensor products,

$$E(n)_*E(n)/I_n^k = \operatorname{colim}_m \bigotimes_{j=1}^{m} E(n)_*/I_n^k[S_j]/(v_n S_j^{p^n} - v_n^{p^j} S_j).$$

We claim that each algebra $E(n)_*/I_n^k[S_j]/(v_n S_j^{p^n} - v_n^{p^j} S_j)$ is étale over $E(n)_*/I_n^k$. Notice that it is flat over $E(n)_*/I_n^k$ and is finitely-generated by S_j. As the ground ring $E(n)_*/I_n^k$ is Noetherian, the only thing that remains to be shown is that the module of Kähler differentials is trivial.

The Kähler differentials are generated by the symbol dS_j, but in $E(n)_*E(n)/I_n^k$ we have the relation $v_n S_j^{p^n} = v_n^{p^j} S_j$. The residue class of the element $v_n \in E(n)_*$ is a unit in the ring $E(n)_*/I_n$ and thus we can deduce

$$dS_j = v_n^{1-p^j} d(S_j^{p^n}) = p^n v_n^{1-p^j} S_j^{p^n-1} dS_j.$$

Iteration of this relation t times, where t is an integer such that $tn \geqslant k$, implies that dS_j is zero, since in the quotient $E(n)_*/I_n^k$, p^k is zero.

Now by Lemma 2.6 Γ-homology commutes with colimits, therefore

$$\begin{aligned}\mathrm{H}\Gamma_{*,*}(E(n)_*E(n)/I_n^k|E(n)_*/I_n^k;\ \widehat{E(n)}_*/I_n^k) &= 0\\ &= \mathrm{H}\Gamma^{*,*}(E(n)_*E(n)/I_n^k|E(n)_*/I_n^k;\ \widehat{E(n)}_*/I_n^k).\end{aligned}$$

This completes the proof of Theorem 8.2. □

Using (8.1) and the fact that the completion of $\widehat{E(n)}_*\widehat{E(n)}$ is free on a countable basis [9, theorem 1.1], [4], we obtain

Theorem 8.3. *For p a prime and $n \geqslant 1$, the spectrum $\widehat{E(n)}$ possesses a unique E_∞ structure.*

Using the ideas of Section 2, we can also deduce

Theorem 8.4. *For $n \geqslant 1$ and $k \geqslant 1$, we have*

$$\begin{aligned}\mathrm{AQ}_*(E(n)_*E(n)/I_n^k|E(n)_*/I_n^k;\ E(n)_*/I_n^k) &= 0\\ &= \mathrm{AQ}^*(E(n)_*E(n)/I_n^k|E(n)_*/I_n^k;\ E(n)_*/I_n^k),\\ \mathrm{AQ}_*(E(n)_*E(n)\widehat{_{I_n}}|\widehat{E(n)}_*;\ \widehat{E(n)}_*) &= 0\\ &= \mathrm{AQ}^*(E(n)_*E(n)\widehat{_{I_n}}|\widehat{E(n)}_*;\ \widehat{E(n)}_*).\end{aligned}$$

Remark 8.5. Extending Theorem 8.3 to cover $E(n)$ for $n > 1$ does not appear to be straightforward. The following two problems arise.

- There is the question of whether $E(n)_*E(n)$ is a free $E(n)_*$-module when $n > 1$. If $E(n)$ does not have a universal coefficient theorem, then the obstructions to building an E_∞ structure on $E(n)$ would live in $E(n)$-cohomology which might not then be identifiable with Γ-cohomology. In [11], the first author showed that the cooperation algebra of the I_n-localization of $E(n)$, $E(n)_{I_n}$, is free over $E(n)_{*I_n}$, so it does have a universal coefficient theorem and the above problem is overcome.
- Γ-cohomology of $E(n)_*E(n)$ is non-trivial in positive degrees. Even for $n = 2$ there are polynomial generators in $E(2)_*E(2)$ which lead to non-trivial classes in Γ-cohomology.

We aim to return to the existence of E_∞ structures on $E(n)$ and $E(n)_{I_n}$ in future work.

We end this section with some remarks on suitably completed versions of elliptic cohomology. Here $\mathcal{E}\ell\ell$ denotes the level 1 version of elliptic cohomology of Landweber, Ravenel and Stong [24] and we focus on the supersingular completions of [6]. Our above techniques together with results from [8] yield the following.

Theorem 8.6. *For each prime $p > 3$, the supersingular completions $\mathcal{E}\ell\ell^{\widehat{}}_{(p,E_{p-1})}$ and $\mathcal{E}\ell\ell^{\widehat{}}_{\mathcal{P}}$ for each maximal ideal $\mathcal{P} \lhd (\mathcal{E}\ell\ell_*)_{(p)}$ containing (p, E_{p-1}), have unique E_∞ structures.*

An analogous result applies to the $K(1)$-localization of $\mathcal{E}\ell\ell$ studied in [5] and more recently by M. Hopkins.

9. An obstruction theory for the coherence of maps

For the following, we need to work in a good category of spectra with a symmetric monoidal smash product, for example that of [17]. Where necessary, all ring spectra will be assumed to be fibrant.

Let E and F be two E_∞ ring spectra over the E_∞ operad $\mathcal{T}$ from [31, section 5.1] and let $f\colon E \longrightarrow F$ be a map of commutative ring spectra, *i.e.*, the map f commutes with the multiplication maps μ_E and μ_F up to homotopy,

$$\mu_F \circ f \wedge f \simeq f \circ \mu_E,$$

and similar coherence properties exist with respect to the homotopies for associativity and commutativity on E and F. The aim of the following discussion is to give criteria, when the map f can be made into a map which is compatible with the $\mathcal{T}$-algebra structures on E and F up to homotopy. For A_∞ structures the analogous question was addressed in [30].

From now on we will use the notation of [31]. The topological operad $\mathcal{T}$ is filtered by subspaces $\nabla^m\mathcal{T}(n) \subseteq \mathcal{T}(n)$. Let θ_E and θ_F be the action maps of the operad $\mathcal{T}$ on E and F.

Consider the sequence of topological spaces

$$\nabla^m\mathcal{T}(n) \hookrightarrow \nabla^{m+1}\mathcal{T}(n) \longrightarrow \nabla^{m+1}\mathcal{T}(n)/\nabla^m\mathcal{T}(n) \cup \partial\nabla^{m+1}\mathcal{T}(n),$$

where $\partial\nabla^{m+1}\mathcal{T}(n)$ is the part of $\mathcal{T}(n)$ which is determined by compositions in the operad of elements coming from lower filtration degrees.

Theorem 9.1.

(a) *If $\mathrm{H}\Gamma^{n,2-n}(F_*E|F_*; F_*) = 0$ for all $n \geqslant 3$, then f can be turned into a map satisfying*

$$f \circ \theta_E \simeq \theta_F \circ \underbrace{f \wedge \cdots \wedge f}_{m} \colon \mathcal{T}(m) \ltimes_{\Sigma_m} E^{\wedge m} \longrightarrow F$$

for all m.

(b) *If in addition* $\mathrm{Hom}_{F_*}(F_*E, F_*) \cong \mathrm{Hom}_{E_*}(E_*E, F_*)$, *then it suffices to prove that*

$$\mathrm{H}\Gamma^{n,2-n}(E_*E|E_*; F_*) = 0$$

for all $n \geqslant 3$.

The second condition is satisfied for instance if F is projective over E, then $F_*E \cong F_* \otimes_{E_*} E_*E$ can be used to reduce the module of F_*-linear morphisms to the module of E_*-linear morphisms.

Proof. Assume f satisfies the conditions up to filtration degree m. In order to extend f coherently over the $(m+1)$-st filtration step, we have to show that the condition of the theorem suffices to force f to fulfil

$$f \circ \theta_E|\nabla^{m+1} \simeq \theta_F|\nabla^{m+1} \circ \underbrace{f \wedge \ldots \wedge f}_{n} \colon \nabla^{m+1}\mathcal{T}(n) \ltimes_{\Sigma_n} E^{\wedge n} \longrightarrow F.$$

The map $f \circ \theta_E|\nabla^{m+1}$ corresponds to an element in $F^0(\nabla^{m+1}\mathcal{T}(n) \ltimes_{\Sigma_n} E^{\wedge n})$. Using the long exact cohomology sequence corresponding to the sequence of spaces

$$\nabla^m\mathcal{T}(n) \cup \partial\nabla^{m+1}\mathcal{T}(n) \xrightarrow{i} \nabla^{m+1}\mathcal{T}(n) \xrightarrow{j} \nabla^{m+1}\mathcal{T}(n)/\nabla^m\mathcal{T}(n) \cup \partial\nabla^{m+1}\mathcal{T}(n),$$

we find that the difference element

$$f \circ \theta_E|\nabla^{m+1} - \theta_F \circ f^{\wedge n}|\nabla^{m+1}$$

maps to zero under i^*, thus it has to be in the image of j^*. Consequently, if j^* has trivial codomain, then this difference has to be trivial as an element in F-cohomology. An argument showing that the corresponding class in

$$F^0(\nabla^{m+1}\mathcal{T}(n)/\nabla^m\mathcal{T}(n) \cup \partial\nabla^{m+1}\mathcal{T}(n) \ltimes_{\Sigma_n} E^{\wedge n})$$

has to be a cocycle in the complex for $\mathrm{H}\Gamma^*$ can be found in [31]. Therefore, if $\mathrm{H}\Gamma^{m,2-m}(F_*E|F_*; F_*)$ vanishes in all degrees $m \geqslant 3$, the potentially obstructing difference maps $f \circ \theta_E - \theta_F \circ f^{\wedge n}$ have to be nullhomotopic. □

From the triviality of $\mathrm{H}\Gamma^n$ when $n > 1$ for complex K-theory and its localization at a prime p, we can deduce the following result.

Theorem 9.2. *For each k integer prime to p, the k-th Adams operation* $\psi^k\colon KU_{(p)} \longrightarrow KU_{(p)}$ *can be refined to a coherent map with respect to the* E_∞ *structure given by the operad action of* $\mathcal{T}$ *on* $KU_{(p)}$.

Proof. The action of such an Adams operation ψ^k on $KU_{(p)_{2n}} \cong \mathbb{Z}_{(p)}$ is given by multiplication by k^n, thus it induces a different $KU_{(p)_*}KU_{(p)}$-module structure on $KU_{(p)_*}$. This corresponds to taking $E = KU_{(p)} = F$ and the map $\psi^k \colon E \longrightarrow F$, then applying Theorem 9.1(b) and using the fact that the relevant Γ-cohomology groups vanish, this being a generalization of Theorem 6.2 which is proved in a similar way (this result depends crucially on the vanishing of Γ-cohomology for formally étale extensions). □

Finally we have a result on the inclusion $j \colon E(1) \longrightarrow KU_{(p)}$ of the Adams summand into p-local K-theory which is a map of ring spectra.

Proposition 9.3. *j gives rise to a coherent map of E_∞ spectra.*

Proof. Using the Conner–Floyd isomorphism and the Landweber exactness of $E(1)$, the above argument can be adapted to prove that the relevant part of $\mathrm{H}\Gamma^{*,*}(KU_{(p)_*}(E(1))|KU_{(p)_*}; KU_{(p)_*})$ vanishes. □

Remark 9.4. With the aid of more machinery one can actually take the above arguments to obtain the existence of *strict* maps of E_∞ ring spectra. Using a comparison result of Basterra and the second author [14, theorem 2.6], we can identify Γ-cohomology groups with the obstruction groups arising in the work of Goerss and Hopkins [18]. Now the Goerss–Hopkins obstruction theory [18, §4] tells us that the vanishing of the Γ-cohomology groups $\mathrm{H}\Gamma^{*,*}(KU_{(p)_*}(E(1))|KU_{(p)_*}; KU_{(p)_*})$ and $\mathrm{H}\Gamma^{*,*}(KU_{(p)_*}(KU_{(p)})|KU_{(p)_*}; KU_{(p)_*})$ implies that the Adams operations and the map j give rise to maps of E_∞ ring spectra.

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