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## The symplectic topology of Ramanujam's surface

Paul Seidel and Ivan Smith

**Abstract.** Ramanujam's surface  $M$  is a contractible affine algebraic surface which is not homeomorphic to the affine plane. For any  $m > 1$  the product  $M^m$  is diffeomorphic to Euclidean space  $\mathbb{R}^{4m}$ . We show that, for every  $m > 0$ ,  $M^m$  cannot be symplectically embedded into a subcritical Stein manifold. This gives the first examples of exotic symplectic structures on Euclidean space which are convex at infinity. It follows that any exhausting plurisubharmonic Morse function on  $M^m$  has at least three critical points, answering a question of Eliashberg. The heart of the argument involves showing a particular Lagrangian torus  $L$  inside  $M$  cannot be displaced from itself by any Hamiltonian isotopy, via a careful study of pseudoholomorphic discs with boundary on  $L$ .

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### 1. Introduction

Ramanujam showed in [21] that the complement  $M$  of a certain singular curve in the Hirzebruch surface  $\mathbb{F}_1$  is a *contractible* algebraic surface. Algebro-geometrically,  $M$  is distinguished from the affine plane  $\mathbb{A}^2$  by being of log general type (having log Kodaira dimension 2, cf. [17]). Topologically, in spite of being contractible,  $M$  is not homeomorphic to  $\mathbb{R}^4$ , since its fundamental group at infinity is nontrivial. Now consider the  $m$ -fold product  $M^m = M \times \cdots \times M$ . This is still of log general type, in particular not isomorphic to  $\mathbb{A}^{2m}$  as an algebraic variety. However, for  $m \geq 2$  the fundamental group at infinity becomes trivial, and as a consequence  $M^m$  is diffeomorphic to  $\mathbb{R}^{4m}$ ; indeed, Dimca [unpublished] observed that any contractible affine variety of complex dimension  $d \geq 3$  is diffeomorphic to  $\mathbb{R}^{2d}$  (cf. [25, Theorem 3.2]). The conclusion is that the smooth manifolds  $\mathbb{R}^{4m}$ ,  $m \geq 2$ , admit nonstandard algebraic variety structures.

The aim of the present note is to consider this phenomenon from the symplectic perspective. We will equip  $M$  with an exhausting plurisubharmonic function  $\phi_M$ , which makes it into a Stein manifold of finite type, and consider the associated

symplectic form  $\omega_M$  (for this and related terminology, see Sections 2 and 3). On  $M^m$  we take the product structure.

**Theorem 1.** *For all  $m$ ,  $M^m$  cannot be symplectically embedded into a subcritical Stein manifold.*

In particular, for  $m \geq 2$  we obtain a symplectic structure on  $\mathbb{R}^{4m}$  which is exotic (in the usual sense, of not admitting an embedding into the standard  $\mathbb{R}^{4m}$ ). Any sufficiently large relatively compact part of  $M$  is exotic in the same sense, because of the finite type property (cf. Lemma 3 and Lemma 15 below). There is a consequence which can be stated purely in terms of Stein geometry, answering a question of Eliashberg [10, Problem 3]:

**Corollary 2.** *For all  $m$ , any exhausting plurisubharmonic Morse function on  $M^m$  must have at least 3 critical points.*

It seems appropriate to compare Theorem 1 with some other known results. There are several constructions of exotic symplectic structures on  $\mathbb{R}^{2n}$  for  $n \geq 2$ , starting with the abstract existence theorem of [15, Corollary 0.4.A<sub>2</sub>']. However, in contrast to our example, the resulting symplectic forms are not known to be convex at infinity; in fact, at least one construction [18] is explicitly designed to violate that condition. In a somewhat different direction, we should mention that Eliashberg [10] has given candidates, by an explicit Lagrangian handle decomposition, for Stein *subdomains* of  $\mathbb{C}^{2n}$ ,  $n > 2$ , which are diffeomorphic to balls, and for which he conjectures that the conclusion of Corollary 2 still holds.

The main result of [21] asserts that  $\mathbb{A}^2$  is the only algebraic surface which is contractible *and* simply-connected at infinity. The symplectic counterpart of this is the observation that (assuming the 3-dimensional Poincaré conjecture) any 4-dimensional Weinstein manifold which is contractible, simply connected at infinity, complete, and of finite type, is symplectically isomorphic to standard  $\mathbb{R}^4$ . The proof relies on the uniqueness of tight contact structures on  $S^3$  [9] and the description of Stein fillings of this structure via families of holomorphic discs [8]. Moreover, the picture changes if one drops the finite type condition: Gompf [13] has used a suitable infinite handle-body decomposition to produce Stein structures on uncountably many manifolds homeomorphic, but not diffeomorphic, to  $\mathbb{R}^4$ .

The essential ingredient in our proof of Theorem 1 is a particular Lagrangian torus  $L \subset M$ , described below. By a careful study of pseudo-holomorphic discs, and invoking a theorem of Chekanov [5], we show that  $L \subset M$  cannot be displaced from itself by a Hamiltonian isotopy. More generally, if  $i: M \rightarrow N$  is a symplectic embedding into a complete Stein manifold, then  $i(L) \subset N$  has the same non-displaceability property. This, together with the corresponding facts for products  $L^m$ , leads easily to Theorem 1.

The heart of the argument involves considering two different compactifications  $X$  and  $\bar{X} \cong \mathbb{F}_1$  of  $M$ . The complement  $\bar{X} \setminus M$  is a curve  $\bar{S} = \bar{S}' \cup \bar{S}''$  with two irreducible components, one of which  $\bar{S}''$  has a cusp singularity; the Lagrangian torus  $L$  lies in a neighbourhood of the cusp. Rather than working with a single Lagrangian torus, we consider a family  $\{L_t\}$  which as  $t \rightarrow 0$  collapses into the cusp point on  $\bar{S}''$ . Intuitively, the limit of a family of holomorphic discs  $\{(\mathbb{D}, \partial\mathbb{D}) \rightarrow (M, L_t)\}_{t \in (0,1]}$  as  $t \rightarrow 0$  is either a holomorphic sphere in  $\bar{X}$  disjoint from  $\bar{S}'$ , or is the constant map to the cusp point. The first case is excluded since  $\bar{S}'$  is ample; the second is impossible for topological reasons concerning the fundamental group  $\pi_1(\bar{V} \setminus \bar{S}'')$ , where  $\bar{V} \subset \bar{X}$  is a small neighbourhood of the cusp. The upshot is that no such families of discs can exist, enabling us to appeal to Chekanov's work.

For technical reasons, we in fact work with a blown-up compactification  $X \rightarrow \bar{X}$  of  $M$  in which the complement  $S = \bar{X} \setminus M$  is a divisor with normal crossings. The tori  $L_t$  now appear as so-called linking tori for a normal crossing point  $p$  of  $S$ . To construct them as manifestly Lagrangian tori, and to make the abovementioned limiting argument for holomorphic discs rigorous, we use a simple algebro-geometric trick. Take  $\mathbb{C}P^1 \times X$  and blow up the point  $(0, p)$ , obtaining a threefold  $Y$  with a projection  $Y \rightarrow \mathbb{C}P^1$ . The singular fibre  $Y_0$  has an irreducible component which is a  $\mathbb{C}P^2$ . We take a Clifford torus  $K_0$  in that component, and move it by parallel transport to obtain a family of Lagrangian tori  $K_t$ ,  $t \in [-1; 1]$ , in the nearby fibres  $Y_t$ . For  $t \neq 0$ , these fibres are naturally identified with  $X$ , and we define  $L_t$  to be the image of  $K_t$  under this identification, for  $t \in (-1; 0]$ . Since the total submanifold  $K = \bigcup_{t \in [-1,1]} K_t \subset Y$  is Lagrangian, Gromov compactness can be applied directly to families of discs with boundary in  $K$ . The drawback of this argument is that the  $Y_t \cong X$  carry varying Kähler forms. To take account of this, we give a careful discussion of Stein deformations in Section 2, and introduce in Section 3 the technical notion of a "Stein-essential" Lagrangian submanifold. The idea is that for any given  $E > 0$ , we can deform our Stein structure and our Lagrangian submanifold, in such a way that at the endpoint of the deformation, there are no pseudo-holomorphic discs of area less than  $E$ . Section 4 introduces Lagrangian linking tori, and describes the implications of a linking torus being Stein-essential. Only in Section 5 are these ingredients assembled to derive Theorem 1.

There are at least two possible alternative ways of analyzing the symplectic nature of  $M^m$ . One could try to use Floer homology or symplectic homology as introduced by Viterbo [24], [23] and Cieliebak–Floer–Hofer [7]. In fact, as we intend to discuss elsewhere, existence of a Stein-essential Lagrangian submanifold already implies that  $SH^*(M^m) \neq 0$ . To compute  $SH^*(M^m)$  precisely would presumably require an analysis of the Reeb flow at infinity, though since Floer homology behaves well under products [19] it would be enough to do the computation for  $M$  itself. Floer homology may also distinguish between different exotic symplectic structures on



$\mathbb{R}^{4m}$ , which falls outside the scope of the arguments used here. The other possible approach would be via a symplectic field theory decomposition argument. The aim would be to prove that if  $M^m$  is subcritical, there has to be a non-constant algebraic map  $\mathbb{A}^1 \rightarrow M$ , contradicting a property of contractible surfaces of log general type [17, Theorem 4.7.1]. Eliashberg has announced a theorem which says that if a smooth projective variety contains a smooth ample divisor with subcritical complement, then the variety has many rational curves. In examples, it appears that these rational curves are closures of maps of  $\mathbb{A}^1$  to the complement, but this is not well-understood in general. Closely related results have been obtained by Biolley [3].

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## 2. Background

We begin by reviewing the definitions and some elementary results. This follows ([11], [10], [4]) with some modifications. The proofs have been relegated to the Appendix.

Take a manifold  $M$  equipped with a symplectic form  $\omega_M$ , a one-form  $\theta_M$  such that  $d\theta_M = \omega_M$ , and an exhausting (which means proper and bounded below) smooth function  $\phi_M: M \rightarrow \mathbb{R}$ . Let  $\lambda_M$  be the Liouville vector field associated to  $\theta_M$ , so  $\omega_M(\lambda_M, \cdot) = \theta_M$ . The quadruple  $(M, \omega_M, \theta_M, \phi_M)$  is a *convex symplectic manifold* if there is a sequence  $c_1 < c_2 < \dots$  converging to  $+\infty$ , such that  $d\phi_M(\lambda_M) > 0$  on each level set  $\phi_M^{-1}(c_k)$ . We call a convex symplectic manifold *complete* if the flow of  $\lambda_M$  exists for all positive times (the corresponding statement for negative times is always true), and *of finite type* if there is a  $c_0$  such that  $d\phi_M(\lambda_M) > 0$  on  $\phi_M^{-1}([c_0; +\infty))$ . Note that if  $M$  is complete and of finite type, then the flow of  $\lambda_M$  defines a diffeomorphism  $f: [0; \infty) \times \phi_M^{-1}(c_0) \rightarrow \phi_M^{-1}([c_0; +\infty))$  satisfying  $f^*\theta_M = e^r(\theta_M|_{\phi_M^{-1}(c_0)})$ , where  $r$  is the variable in  $[0; \infty)$ . Hence  $M$  is a symplectic manifold with a conical end.

**Lemma 3.** *Let  $M, N$  be convex symplectic manifolds, with  $M$  of finite type and  $N$  complete. Take  $c_0$  such that  $d\phi_M(\lambda_M) > 0$  on  $\phi_M^{-1}([c_0; +\infty))$ . Then any embedding  $i: \phi_M^{-1}((-\infty; c_0]) \rightarrow N$  such that  $i^*\theta_N - \theta_M$  is an exact one-form, can be extended to an embedding  $M \rightarrow N$  with the same property.*

Let  $(\omega_{M,t}, \theta_{M,t}, \phi_{M,t})$ ,  $0 \leq t \leq 1$ , be a smooth family of convex symplectic structures on a fixed manifold  $M$ . We say that this is a *convex symplectic deformation* if the following two additional conditions hold: the function  $(t, x) \mapsto \phi_{M,t}(x)$  on

$[0; 1] \times M$  is proper; and for each  $t \in [0; 1]$  there is a neighbourhood  $t \in I \subset [0; 1]$  and a sequence  $c_1 < c_2 < \dots$  converging to  $+\infty$ , such that  $d\phi_{M,s}(\lambda_{M,s}) > 0$  along  $\phi_{M,s}^{-1}(c_k)$  for all  $k$  and all  $s \in I$ . A convex symplectic deformation is called *complete* if all the convex symplectic structures in it are complete, and *of finite type* if there is a  $c_0$  such that  $d\phi_{M,t}(\lambda_{M,t}) > 0$  on  $\phi_{M,t}^{-1}([c_0; +\infty))$  for all  $t$ .

**Lemma 4.** *Let  $(\omega_{M,t}, \theta_{M,t}, \phi_{M,t})$  be a complete convex symplectic deformation. For any relatively compact open subset  $U \subset M$ , there is a smooth family of embeddings  $j_t: U \rightarrow M$  starting with  $j_0 = \text{id}$ , such that  $j_t^*\theta_{M,t} - \theta_{M,0}$  are exact one-forms on  $U$ .*

**Lemma 5.** *Let  $(\omega_{M,t}, \theta_{M,t}, \phi_{M,t})$  be a complete finite type convex symplectic deformation. Then there is a smooth family of diffeomorphisms  $f_t: M \rightarrow M$ , starting with  $f_0 = \text{id}$ , such that  $f_t^*\theta_{M,t} - \theta_{M,0} = dR_t$ , where  $(t, x) \mapsto R_t(x)$  is a compactly supported function on  $[0; 1] \times M$ .*

It may be instructive to compare our definitions with some that appear elsewhere in the literature. The condition of a manifold being *Weinstein*, defined in [10], is related to but stronger than being convex symplectic; the function  $\phi_M$  is tied closely to  $\lambda_M$  by a Lyapunov condition, which is somewhat more restrictive than our requirements. Convex symplectic manifolds were introduced in [11], defined as exact symplectic manifolds  $(M, \omega_M, \theta_M)$  together with an exhaustion by relatively compact subsets  $U_1 \subset \bar{U}_1 \subset U_2 \subset \bar{U}_2 \subset U_3 \subset \dots$ , such that each  $\partial U_k$  is a smooth hypersurface and convex of contact type. This coincides with our notion, but we choose to describe the exhaustion  $U_k = \phi_M^{-1}((-\infty; c_k))$  via sublevel sets of some function. Our definition of convex symplectic deformation stays close to the same picture, since locally in the deformation parameter  $t$ , the manifolds  $(M, \omega_{M,t}, \theta_{M,t})$  have smoothly varying exhaustions  $U_{k,t} = \phi_{M,t}^{-1}((-\infty; c_k))$ . For convex symplectic manifolds which are both finite type and complete, this is the natural analogue of the notion of deformation in [10].

A *Stein manifold*  $(M, J_M, \phi_M)$  is a complex manifold  $(M, J_M)$  with an exhausting plurisubharmonic function  $\phi_M$ . Here plurisubharmonicity is always intended in the strict sense, meaning that  $-dd^c\phi_M = -d(d\phi_M \circ J_M)$  is a positive  $(1, 1)$ -form. We say that the Stein manifold is *complete* if the gradient flow of  $\phi_M$  exists for all positive times, and *of finite type* if there is a  $c_0$  such that all  $c \geq c_0$  are regular values of  $\phi_M$ . Taking  $\omega_M = -dd^c\phi_M$  and  $\theta_M = -d^c\phi_M$  then makes  $M$  into a convex symplectic manifold, whose Liouville vector field is  $\lambda_M = \nabla\phi_M$  (and which therefore satisfies  $d\phi_M(\lambda_M) > 0$  on each regular level set of  $\phi_M$ ). Completeness or finite type nature of the Stein manifold imply the corresponding convex symplectic properties.

**Lemma 6.** *Let  $(M, J_M, \phi_M)$  be a Stein manifold, and  $h: \mathbb{R} \rightarrow \mathbb{R}$  a function satisfying  $h'(c) > 0$ ,  $h''(c) \geq 0$  for all  $c$ , and such that there are  $c_0$  and  $\delta > 0$  with  $h''(c) \geq \delta h'(c)$  for all  $c \geq c_0$ . Then  $\tilde{\phi}_M = h(\phi_M)$  is again plurisubharmonic and makes*

$(M, J_M)$  into a complete Stein manifold. Denote the convex symplectic structure obtained from  $\tilde{\phi}_M$  by  $(\tilde{\omega}_M, \tilde{\theta}_M)$ . If the original Stein structure was complete and of finite type, there is a diffeomorphism  $f: M \rightarrow M$  such that  $f^*(\tilde{\theta}_M) - \theta_M = dR$  for some compactly supported function  $R$ .

**Lemma 7.** *Let  $\phi_M, \tilde{\phi}_M$  be two exhausting plurisubharmonic functions on the same complex manifold  $(M, J_M)$ , of which the second one is complete and of finite type (while the first one can be arbitrary). Then there is an embedding  $i: M \rightarrow M$  such that  $i^*(\tilde{\theta}_M) - \theta_M$  is an exact one-form.*

Let  $(J_{M,t}, \phi_{M,t})$ ,  $0 \leq t \leq 1$ , be a smooth family of Stein structures on a manifold  $M$ . We call this a *Stein deformation* if the following two additional conditions hold: the function  $(t, x) \mapsto \phi_{M,t}(x)$  on  $[0; 1] \times M$  is proper; and for each  $t \in [0; 1]$  there is a neighbourhood  $t \in I \subset [0; 1]$  and a sequence  $c_1 < c_2 < \dots$  converging to  $+\infty$ , such that  $c_k$  is a regular level set for each  $\phi_{M,s}$ ,  $s \in I$ . A Stein deformation is called *complete* if all the Stein structures in it are complete, and of *finite type* if there is a  $c_0$  such that all  $c \geq c_0$  are regular values of  $\phi_{M,t}$  for all  $t \in [0; 1]$ . Clearly, these kinds of deformations induce the corresponding convex symplectic notions.

It remains to make the connection with algebraic geometry. Let  $X$  be a smooth projective variety,  $E \rightarrow X$  an ample line bundle,  $s_E \in H^0(E)$  a nonzero holomorphic section, and  $S = s_E^{-1}(0)$  the hypersurface along which it vanishes. Ampleness means that we can put a metric  $\|\cdot\|_E$  on  $E$  such that the curvature form  $\omega_X = iF_{\nabla_E}$  of the associated connection  $\nabla_E$  is a positive  $(1, 1)$ -form. The restriction of this form to  $M = X \setminus S$  can be written as  $\omega_M = \omega_X|_M = -dd^c \phi_M$ , where  $\phi_M = -\log \|s_E\|_E$ . This is clearly an exhausting function, hence defines a Stein structure.

**Lemma 8.** *Suppose that  $S$  has only normal crossing singularities (but  $s_E$  can vanish along the irreducible components with arbitrary multiplicities). Then  $\phi_M$  is of finite type.*

There is also a version of this for deformations: in the same algebro-geometric situation, given a family  $\|\cdot\|_{E,t}$  of metrics, one gets a finite type Stein deformation  $(J_{M,t} = J_M, \phi_{M,t} = -\log \|s_E\|_{E,t})$ .

### 3. Stein-Essential Lagrangian submanifolds

Following a line of thought similar to the one in [4], we combine “soft” displacement methods for subcritical Stein manifolds with “hard” Lagrangian intersection results to derive some restrictions on embeddings of Stein manifolds.

Let  $(M, \omega_M, \theta_M)$  be any exact symplectic manifold. By a Hamiltonian isotopy of  $M$ , we will mean an isotopy  $(g_t)$ ,  $0 \leq t \leq 1$ , starting with  $g_0 = \text{id}$ , which is

induced by a smooth family of Hamiltonian functions  $H_t$ , such that  $(t, x) \mapsto H_t(x)$  has compact support in  $[0; 1] \times M$ . The *Hofer length* of  $(g_t)$  is defined as

$$\int_0^1 (\max(H_t) - \min(H_t)) dt.$$

Now let  $L \subset M$  be a Lagrangian submanifold (throughout, all such submanifolds will be assumed to be compact). Consider Hamiltonian isotopies  $(g_t)$  such that  $g_t(L) \cap L = \emptyset$ . The infimum of the Hofer lengths of all these isotopies is called the *displacement energy* of  $L$  (of course, there are cases where no such isotopy exists, and then the displacement energy is  $\infty$ ). Recall that by definition, a Lagrangian isotopy  $(L_t)$  is *exact* if the class  $[\theta_M|_{L_t}] \in H^1(L_t; \mathbb{R}) \cong H^1(L_0; \mathbb{R})$  is constant in  $t$ . These are precisely the Lagrangian isotopies which can be embedded into Hamiltonian ones, in the sense that there is a  $(g_t)$  with  $g_t(L_0) = L_t$ . Hence, the displacement energy is invariant under exact Lagrangian isotopies. Chekanov's theorem ([5], [6], [20]) says:

**Theorem 9.** *Let  $L \subset M$  be a compact Lagrangian submanifold whose displacement energy is  $E < \infty$ . Let  $J_M$  be an  $\omega_M$ -compatible almost complex structure which is convex at infinity. Then there is a non-constant  $J_M$ -holomorphic map  $u: (\mathbb{D}, \partial\mathbb{D}) \rightarrow (M, L)$ , where  $\mathbb{D} \subset \mathbb{C}$  is the closed unit disc, whose area is  $\int u^*\omega_M \leq E$ .  $\square$*

Convexity at infinity of the almost complex structure means that there is an exhausting function  $\phi_M$  such that outside a compact subset,  $-d(d\phi_M \circ J_M)$  is positive on all  $J_M$ -complex tangent planes. This holds for the given complex structure on any Stein manifold, but it also allows one to deform that structure (compatibly with the symplectic form) on a compact subset. Chekanov's theorem actually holds in somewhat greater generality, but that will not be necessary for our purpose.

Recall that a Stein manifold  $(M, J_M, \phi_M)$  is *subcritical* if  $\phi_M$  is a Morse function and has only critical points of index  $< \frac{1}{2} \dim_{\mathbb{R}} M$ . The next statement is a special case of [4, Lemma 3.2] (and technically somewhat simpler than the general result):

**Lemma 10.** *For any Lagrangian submanifold  $L$  in a complete subcritical Stein manifold  $M$ , there exists an exact Lagrangian isotopy  $(L_t)$  such that  $L_0 = L$  and  $L_1 \cap L_0 = \emptyset$ .  $\square$*

Suppose that we have a Stein manifold, containing a Lagrangian submanifold such that there are *no* non-constant holomorphic discs bounding it. Then our Stein manifold cannot be subcritical (one would use Lemma 6 to make it complete, and then combine Theorem 9 with Lemma 10). A little less obviously, this manifold cannot have an exact symplectic embedding into any subcritical Stein manifold. We will spend the rest of this section deriving a more complicated version of the latter statement.

Let  $(M, J_M, \phi_M)$  be a complete finite type Stein manifold, and take  $c_0$  as usual. On  $\phi_M^{-1}([c_0; \infty))$ , which in symplectic terms is the cone part of  $M$ , consider the splitting

$$TM = \xi_M \oplus \xi_M^\perp \quad (1)$$

where  $\xi_M = \ker(d\phi_M) \cap \ker(d^c\phi_M)$  is the contact hyperplane field on each level set  $\phi_M^{-1}(c)$ ,  $c \geq c_0$ ; and  $\xi_M^\perp = \mathbb{R}\nabla\phi_M \oplus \mathbb{R}\rho_M$  is spanned by the Liouville vector field together with the Reeb vector field on each level set, which is  $\rho_M = J_M\nabla\phi_M/\|\nabla\phi_M\|^2$ . The decomposition (1) is  $J_M$ -invariant and orthogonal with respect to  $\omega_M$ . One can therefore find an  $\omega_M$ -compatible almost complex structure  $\tilde{J}_M$  which

$$\left. \begin{array}{ll} \text{is equal to } J_M & \text{on } \phi_M^{-1}((-\infty; c_0]); \\ \text{preserves } \xi_M, \text{ and maps } \lambda_M & \text{on } \phi_M^{-1}([c_0; +\infty)); \\ \text{to a positive multiple of } \rho_M & \\ \text{is invariant under the Liouville flow} & \text{on } \phi_M^{-1}([c_0 + 1; +\infty)). \end{array} \right\} \quad (2)$$

**Lemma 11.** *Let  $\Sigma_0$  be a compact connected Riemann surface with boundary, and  $u: \Sigma_0 \rightarrow M$  a  $\tilde{J}_M$ -holomorphic map such that  $u(\partial\Sigma_0) \subset \phi_M^{-1}((-\infty; c_0])$ . Then  $u(\Sigma_0) \subset \phi_M^{-1}((-\infty; c_0])$ .*

*Proof.* The second part of (2) implies that on  $\phi_M^{-1}([c_0; +\infty))$ ,  $-d\phi_M \circ \tilde{J}_M = \eta \theta_M$  with a strictly positive function  $\eta$ . Hence

$$-d(d\phi_M \circ \tilde{J}_M) = \eta \omega_M + \frac{d\eta}{\eta} \wedge (-d\phi_M \circ \tilde{J}_M). \quad (3)$$

Suppose that we have a  $\tilde{J}_M$ -holomorphic map  $u: \Sigma_0 \rightarrow M$  such that  $u(\partial\Sigma_0) \subset \phi_M^{-1}((-\infty; c_0])$  but  $u(\Sigma_0) \not\subset \phi_M^{-1}((-\infty; c_0])$ . Then there is a  $c > c_0$  such that  $u$  intersects the level set  $\phi_M^{-1}(c)$  transversally in a nonempty set. Consider the function  $\psi = \phi_M \circ u$  on the surface  $\Sigma = u^{-1}\phi_M^{-1}([c; +\infty)) \subset \Sigma_0$ . By pulling back (3) and using the positivity of  $u^*\omega_M$  we obtain a differential inequality for  $\psi$ , which in a local holomorphic coordinate  $z = s + it$  can be written as

$$(\partial_s^2 + \partial_t^2)\psi - \sigma(s, t)\partial_s\psi - \tau(s, t)\partial_t\psi \geq 0$$

with  $\sigma = (\eta \circ u)^{-1}\partial_s(\eta \circ u)$ ,  $\tau = (\eta \circ u)^{-1}\partial_t(\eta \circ u)$ . The strong maximum principle [12, Theorem 3.5] applies to solutions of such equations, hence  $\psi \leq c$  everywhere on  $\Sigma$ , which means that  $u(\Sigma_0) \subset \phi_M^{-1}((-\infty; c])$ . Since  $c$  can be chosen arbitrarily close to  $c_0$ , the result follows.  $\square$

Because  $\tilde{J}_M$  is invariant under the Liouville flow outside a compact subset, it is tame in the sense of [1, Chapter V, Definition 4.1.1]. In particular, the monotonicity lemma [1, Chapter V, Proposition 4.3.1] applies:

**Lemma 12.** Let  $\tilde{g}_M$  be the metric associated to  $\omega_M$  and  $\tilde{J}_M$ . There is a  $\rho > 0$ , which is less than the injectivity radius of  $\tilde{g}_M$ , and an  $\epsilon > 0$ , such that the following holds. Let  $x$  be any point in  $M$ , and  $B_r(x)$  the closed ball of radius  $r \leq \rho$  around it. If  $\Sigma$  is a compact Riemann surface with boundary, and  $u: \Sigma \rightarrow B_r(x)$  a  $\tilde{J}_M$ -holomorphic map satisfying  $x \in u(\Sigma)$  and  $u(\partial\Sigma) \subset \partial B_r(x)$ , then  $\int u^* \omega_M \geq \epsilon r^2$ .  $\square$

**Lemma 13.** For every  $E > 0$  there is a  $C > 0$  with the following property. Let  $\Sigma$  be a compact connected Riemann surface with boundary, whose boundary is decomposed into two nonempty unions of circles  $\partial_- \Sigma \cup \partial_+ \Sigma$ . Let  $u: \Sigma \rightarrow M$  be a  $\tilde{J}_M$ -holomorphic map such that  $u(\partial_- \Sigma) \subset \phi_M^{-1}((-\infty; c_0])$  and  $u(\partial_+ \Sigma) \subset \phi_M^{-1}([C; \infty))$ . Then  $\int_\Sigma u^* \omega_M > E$ .

*Proof.* Consider the diffeomorphism  $f: [0; \infty) \times \phi_M^{-1}(c_0) \rightarrow \phi_M^{-1}([c_0; \infty))$  which defines the conical end structure. Since the metric  $\tilde{g}_M$  blows up on the cone, the distance between any two sets  $f(\{i\} \times \phi_M^{-1}(c_0))$ ,  $i = 0, 1, 2, \dots$  is bounded below by some  $\delta > 0$ . Take the constants  $\rho, \epsilon$  from Lemma 12. After making  $\delta$  smaller if necessary, we may assume that  $\delta/2 < \rho$ ; we then take an integer  $k$  greater than  $9\delta^{-2}\epsilon^{-1}E$ , and choose  $C$  so that  $\phi_M^{-1}([C; \infty)) \subset f([k; \infty) \times \phi_M^{-1}(c_0))$ .

Since  $\Sigma$  is connected and intersects both  $\phi_M^{-1}((-\infty; c_0])$  and  $f([k; \infty) \times \phi_M^{-1}(c_0))$  nontrivially, there are points  $z_1, \dots, z_k \in \Sigma$  such that  $x_i = u(z_i) \in f(\{i - 1/2\} \times \phi_M^{-1}(c_0))$ . The balls  $B_r(x_i)$ , for any  $r < \delta/2$ , are mutually disjoint. Choose  $r \in (\delta/3; \delta/2)$  in such a way that  $u$  is transverse to all the boundaries  $\partial B_r(x_i)$ . By Lemma 12, each  $u_i = u|_{u^{-1}(B_r(x_i))}: u^{-1}(B_r(x_i)) \rightarrow B_r(x_i)$  has area  $\geq \epsilon \delta^2/9$ . Hence, the total area of  $u$  is  $\geq k\epsilon \delta^2/9 > E$ .  $\square$

Let  $(M, J_M, \phi_M)$  be a finite type Stein manifold. We say that a compact Lagrangian submanifold  $L \subset M$  is *Stein-essential* if for each  $E > 0$  there is a finite type Stein deformation  $(J_{M,t}, \phi_{M,t})$  and a smooth family of compact submanifolds  $L_t \subset M$  ( $0 \leq t \leq 1$ ), with the following properties: at the starting point  $t = 0$ , we have the original Stein structure and Lagrangian submanifold  $L = L_0$ ; for all  $t$ ,  $L_t$  is  $\omega_{M,t}$ -Lagrangian, and the cohomology class  $[\theta_{M,t}] \in H^1(L_t; \mathbb{R}) \cong H^1(L; \mathbb{R})$  is constant in  $t$ ; and at the opposite end, every  $J_{M,1}$ -holomorphic map  $u: (\mathbb{D}, \partial\mathbb{D}) \rightarrow (M, L_1)$  with  $\int u^* \omega_{M,1} \leq E$  is constant.

**Proposition 14.** Let  $M$  be a finite type Stein manifold, which admits an embedding  $i: M \rightarrow N$  into a complete subcritical Stein manifold, such that  $i^* \theta_N - \theta_M$  is an exact one-form. Then  $M$  cannot contain any Stein-essential Lagrangian submanifolds.

*Proof.* Assume that on the contrary, there is a Stein-essential Lagrangian submanifold  $L \subset M$ . By definition, for each  $E$  we can find a finite type Stein deformation  $(J_{M,t}, \phi_{M,t})$  and family of Lagrangian submanifolds  $(L_t)$ , such that  $E$  is a strict lower bound for the area of non-constant  $J_{M,1}$ -holomorphic discs in  $(M, L_1)$ . We

may take  $E$  to be the displacement energy of  $i(L)$  inside  $N$ , which is finite by Lemma 10.

Because the deformation  $(J_{M,t}, \phi_{M,t})$  is of finite type, there is a  $c_0 > 0$  so that all  $c \geq c_0$  are regular values of  $\phi_{M,t}$  for all  $t$ . After making  $c_0$  larger if necessary, one can also assume that  $L_t \subset \phi_{M,t}^{-1}((-\infty; c_0])$ . Lemma 6 says that one can find functions  $h_t$  depending smoothly on  $t$ , such that the modified Stein structures  $(J_{M,t}, \tilde{\phi}_{M,t} = h_t(\phi_{M,t}))$  are complete. Choose these functions in such a way that  $h_t(c) = c$  for  $c \leq c_0$ , which means that the  $L_t$  remain Lagrangian for the associated modified convex symplectic structures  $(\tilde{\omega}_{M,t}, \tilde{\theta}_{M,t})$ .

By construction  $(J_{M,1}, \tilde{\phi}_{M,1})$  is complete and of finite type. Introduce a new  $\tilde{\omega}_{M,1}$ -compatible almost complex structure  $\tilde{J}_{M,1}$  as in the discussion preceding Lemma 11. More explicitly, to carry over that construction to the current situation, one should replace the notation  $J_M, \phi_M, \tilde{J}_M$  in (2) by  $J_{M,1}, \tilde{\phi}_{M,1}, \tilde{J}_{M,1}$  respectively; and similarly  $\xi_M, \lambda_M, \rho_M$  are now the contact hyperplane field, Liouville vector field, and Reeb vector field associated to  $(\tilde{\omega}_{M,1}, \tilde{\theta}_{M,1})$  and to the conical end  $[c_0; +\infty) \times \tilde{\phi}_{M,1}^{-1}(c_0) \rightarrow \tilde{\phi}_{M,1}^{-1}([c_0; +\infty))$ . We will now state some properties of the data introduced so far.

- (a) *There is an embedding  $\tilde{i}: M \rightarrow N$  with  $\tilde{i}(L) = i(L)$ , such that  $\tilde{i}^*\theta_N - \tilde{\theta}_{M,0}$  is an exact one-form.*

To obtain that, restrict  $i$  to an embedding of  $\phi_M^{-1}((-\infty; c_0])$  into  $N$ , note that  $\theta_M = \tilde{\theta}_{M,0}$  on that subset, and then extend it to the whole of  $(M, \tilde{\omega}_{M,0}, \tilde{\theta}_{M,0})$  using Lemma 3.

- (b) *There is a diffeomorphism  $\tilde{f}_1: M \rightarrow M$  such that  $\tilde{f}_1^*\tilde{\theta}_{M,1} - \tilde{\theta}_{M,0}$  is an exact one-form, and  $\tilde{f}_1(L_0)$  is exact Lagrangian isotopic to  $L_1$ .*

By definition  $(J_{M,t}, \tilde{\phi}_{M,t})$  is a complete finite type deformation, so Lemma 5 provides a family of diffeomorphisms  $\tilde{f}_t: M \rightarrow M$  such that  $\tilde{f}_t^*\tilde{\theta}_{M,t} - \tilde{\theta}_{M,0}$  are exact.  $\tilde{f}_1\tilde{f}_t^{-1}(L_t)$  is a Lagrangian isotopy between  $\tilde{f}_1(L_0)$  and  $L_1$ , and the cohomology class  $[\tilde{\theta}_{M,1}|\tilde{f}_1\tilde{f}_t^{-1}(L_t)] = [\tilde{\theta}_{M,0}|\tilde{f}_t^{-1}(L_t)] = [\tilde{\theta}_{M,t}|L_t] = [\theta_{M,t}|L_t]$  is constant in  $t$ , which means that the isotopy is exact.

- (c) *The image of any  $\tilde{J}_{M,1}$ -holomorphic disc  $\tilde{u}: (\mathbb{D}, \partial\mathbb{D}) \rightarrow (M, L_1)$  is contained in  $\tilde{\phi}_{M,1}^{-1}((-\infty; c_0])$ .*

By construction  $L_1 \subset \tilde{\phi}_{M,1}^{-1}((-\infty; c_0])$ ; therefore Lemma 11 applies and yields the desired result.

- (d) *There is a  $C > c_0$  with the following property. Let  $\Sigma$  be a compact connected Riemann surface with boundary, whose boundary is decomposed into two nonempty unions of circles  $\partial_-\Sigma \cup \partial_+\Sigma$ . Let  $\tilde{u}: \Sigma \rightarrow M$  be a  $\tilde{J}_{M,1}$ -holomorphic map such that  $\tilde{u}(\partial_-\Sigma) \subset \tilde{\phi}_{M,1}^{-1}((-\infty; c_0])$  and  $\tilde{u}(\partial_+\Sigma) \subset \tilde{\phi}_{M,1}^{-1}([C; \infty))$ . Then  $\int_\Sigma u^*\tilde{\omega}_{M,1} > E$ .*

Up to the change in notation, this is Lemma 13.

Consider the compact subset  $K = \tilde{\phi}_{M,1}^{-1}((-\infty; C+1])$ , and let  $j = \tilde{i} \circ \tilde{f}_1^{-1}|_K : K \rightarrow N$ . Clearly, one can find an  $\omega_N$ -compatible almost complex structure  $\tilde{J}_N$  with the following two properties:  $\tilde{J}_N = J_N$  outside a compact subset; and  $j^* \tilde{J}_N = \tilde{J}_{M,1}|_K$ . From (b) we know that the Lagrangian submanifold  $j(L_1)$  is exact isotopic to  $\tilde{i}(L)$ , hence its displacement energy is again  $E$ . Since  $\tilde{J}_N = J_N$  at infinity, we can apply Theorem 9, which shows that there is a non-constant  $\tilde{J}_N$ -holomorphic disc  $u : (\mathbb{D}, \partial\mathbb{D}) \rightarrow (N, j(L_1))$  with  $\int u^* \omega_N \leq E$ .

Choose a  $c \in [C; C+1]$  such that  $u$  intersects the hypersurface  $j(\tilde{\phi}_{M,1}^{-1}(c))$  transversally. Consider only the part of our  $\tilde{J}_N$ -holomorphic disc  $u$  that lies on the interior side of that hypersurface. This may have several connected components; we ignore all of them except the one which contains  $\partial\mathbb{D}$ , and compose that with  $j^{-1}$  to obtain a  $\tilde{J}_{M,1}$ -holomorphic map  $\tilde{u} : \Sigma \rightarrow K \subset M$ . By construction,  $\Sigma$  is a connected compact Riemann surface with boundary; its boundary contains one circle  $\partial_- \Sigma$  such that  $\tilde{u}(\partial_- \Sigma) \subset L_1 \subset \tilde{\phi}_{M,1}^{-1}((-\infty; c_0])$ , and if  $\partial_+ \Sigma$  is the union of all the other boundary circles, then  $\tilde{u}(\partial_+ \Sigma) \subset \tilde{\phi}_{M,1}^{-1}(c) \subset \tilde{\phi}_{M,1}^{-1}([C; \infty))$ ; finally  $\int \tilde{u}^* \tilde{\omega}_{M,1} \leq E$ . By (d) above, this is possible only if  $\partial_+ \Sigma = \emptyset$ , which means that  $\tilde{u}$  is a non-constant  $\tilde{J}_{M,1}$ -holomorphic disc in  $(M, L)$ . Applying (c) we find that the image of  $\tilde{u}$  must be contained in  $\tilde{\phi}_{M,1}^{-1}((-\infty; c_0])$ , which implies that it is in fact a  $J_{M,1}$ -holomorphic disc, with  $\int \tilde{u}^* \omega_{M,1} \leq E$ . However, given our original choice of the deformation, the existence of such a disc violates the definition of Stein-essential Lagrangian submanifold.  $\square$

In fact, the requirement that  $N$  is complete can be omitted, due to the following observation, which is similar to step (a) in the previous proof:

**Lemma 15.** *Let  $M$  be a finite type Stein manifold. If  $M$  admits an embedding  $i : M \rightarrow N$  into a subcritical Stein manifold, such that  $i^* \theta_N - \theta_M$  is an exact one-form, then it also admits an embedding into a complete subcritical Stein manifold, with the same property.*

*Proof.* Take  $c_0$  so that all  $c \geq c_0$  are regular values of  $\phi_M$ . Use Lemma 6 to find an  $h$  such that  $\tilde{\phi}_N = h(\phi_N)$  gives rise to a complete Stein structure. This is still subcritical, because the critical points and their Morse indices remain the same.  $h$  can be chosen in such a way that the new convex symplectic structure  $(\tilde{\omega}_N, \tilde{\theta}_N)$  agrees with the old one on  $i(\phi_M^{-1}((-\infty; c_0]) \subset N$ . By restricting  $i$  to  $\phi_M^{-1}((-\infty; c_0])$ , and then extending it again using Lemma 3, one gets an embedding  $j : M \rightarrow N$  such that  $j^* \tilde{\theta}_N - \theta_M$  is exact.  $\square$



#### 4. Linking tori

Throughout this section,  $X$  will be a smooth projective algebraic surface;  $S \subset X$  an algebraic curve with only normal crossing singularities; and  $p \in S$  a crossing point. Set  $M = X \setminus S$ . Take local holomorphic coordinates  $(a, b)$  centered at  $p$  in which  $S = \{ab = 0\}$ , and let  $U \subset X$  be a ball of some radius  $\rho > 0$  in those coordinates. Consider the torus  $L \subset M$  given by  $\{|a| = \mu, |b| = \nu\}$  for some  $0 < \mu, \nu < \rho/\sqrt{2}$ . We will call such an  $L$ , as well as any other torus isotopic to it inside  $U \cap M$ , a *linking torus* for  $S$  at  $p$ .

Recall that, given any algebraic curve on a smooth algebraic surface, one can resolve its singularities by blowups, until only normal crossings remain ([2], Chapter II). The linking tori constructed in this way can be viewed as lying in the complement of the original curve, since blowups leave that complement unchanged. We will now consider in more detail the simplest example of this, which is relevant for our application later on. Let  $\bar{X}$  be a smooth projective algebraic surface, and  $\bar{S} \subset \bar{X}$  a curve which has a cusp singularity at the point  $\bar{p}$ . Blow up to get a map  $q: X \rightarrow \bar{X}$ , such that  $S = q^{-1}(\bar{S})$  has only normal crossings. We assume that this resolution is the minimal one (meaning that no exceptional component of  $S$  can be blown down without violating the normal crossing condition). Take local coordinates  $(c, d)$  centered at  $\bar{p}$  in which  $\bar{S} = \{c^2 = d^3\}$ ; let  $\bar{V} \subset \bar{X}$  be a small ball in these coordinates; and set  $V = q^{-1}(\bar{V})$ . The curve  $S \cap V$  consists of a small piece of the principal component, which is the proper transform of  $\bar{S}$ , and three exceptional components of multiplicities 2, 3 and 6. Figure 1 summarizes the stages of the blowup process and the corresponding coordinate changes (the thick lines are the exceptional components, and the dots indicate the origin of the coordinate systems used).

Consider the linking torus  $L = \{|a| = \mu, |b| = \nu\}$  at the point  $(a, b) = (0, 0)$  where the principal component of  $S \cap V$  crosses the exceptional component of multiplicity 6 (this point is indicated by the small arrow in Figure 1). Its image under  $q$  is the torus  $\bar{L} = \bar{L}_{\mu, \nu}$  parametrized by

$$c = \mu^3(\nu e^{i\gamma} + 1)e^{3i\delta}, \quad d = \mu^2(\nu e^{i\gamma} + 1)e^{2i\delta}$$

for  $(\gamma, \delta) \in \mathbb{R}/2\pi\mathbb{Z}$ , and where  $\mu, \nu > 0$  are suitably small constants. By keeping  $\mu$  constant and letting  $\nu \rightarrow 0$ , one obtains a smooth family of tori in  $\bar{V} \setminus \bar{S}$ , which in the limit shrink to the loop  $(c = \mu^3 e^{3i\delta}, d = \mu^2 e^{2i\delta})$  lying on  $(\bar{S} \setminus \{(0, 0)\}) \cap \bar{V}$ .

The topological aspect of cusp singularities is well-known: the intersection  $S^3 \cap \{c^2 = d^3\}$  is a  $(2, 3)$ -torus knot, which is a trefoil  $\kappa$ . One can find a diffeomorphism  $\bar{V} \setminus \{(0, 0)\} \cong (0; 1) \times S^3$  which takes  $\bar{S} \setminus \{(0, 0)\}$  to  $(0; 1) \times \kappa$ , hence identifies  $\bar{V} \setminus \bar{S}$  with  $(0; 1) \times (S^3 \setminus \kappa)$ . From the argument given above, it follows that the loop on  $\bar{L}$  given by  $\{\gamma = \text{const.}\}$  is homotopic to a longitude of  $\kappa$ . Here, by longitude we mean a curve in  $S^3 \setminus \kappa$  which runs parallel to  $\kappa$ , for some framing which may not necessarily

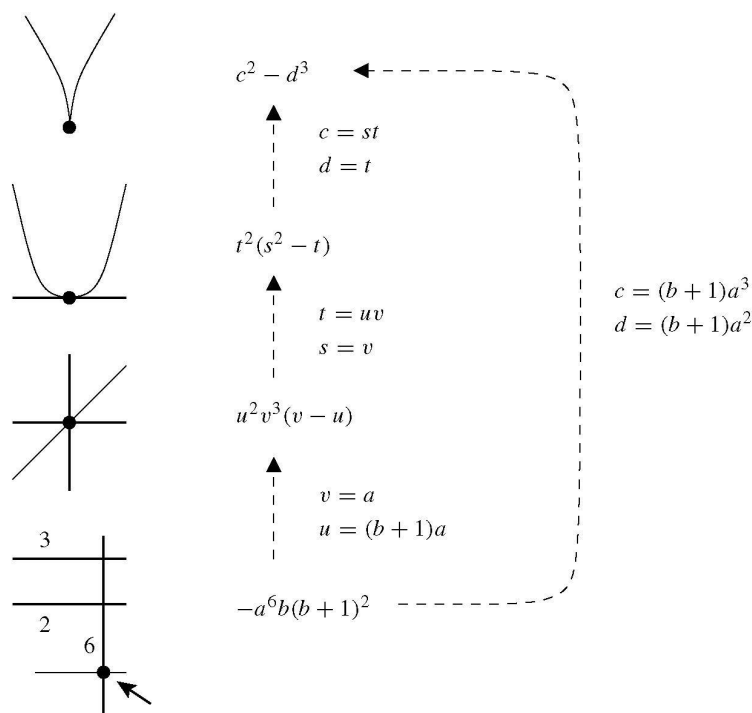


Figure 1

be the canonical one (this ambiguity could be settled by explicit computation, but it is irrelevant for our purpose). Similarly, by inspection of the limit  $v \rightarrow 0$  with fixed  $\mu$  and  $\delta$ , one sees that the other loop  $\{\delta = \text{const.}\}$  on  $\bar{L}$  is a meridian of  $\kappa$ . It is a general fact about nontrivial knots (Dehn's Lemma, see e.g. [16, Theorem 11.2]) that longitude and meridian together define an injective homomorphism  $\mathbb{Z}^2 \rightarrow \pi_1(S^3 \setminus \kappa)$ , which for us means that  $\pi_1(\bar{L}) \rightarrow \pi_1(\bar{V} \setminus \bar{S})$  is injective. Using the identification  $V \setminus S \cong \bar{V} \setminus \bar{S}$  provided by  $q$ , we arrive at this conclusion:

**Lemma 16.** *If a loop on  $L$  bounds a disc in  $V \setminus S$ , then it must be contractible on  $L$  itself.*  $\square$

Returning to the general discussion of linking tori, we now reformulate their definition using a degeneration of  $X$  to a normal crossing surface. Let  $Y$  be the variety obtained by blowing up  $(0, p) \in \mathbb{P}^1 \times X$ , and  $\pi: Y \rightarrow \mathbb{P}^1 \times X \rightarrow \mathbb{P}^1$  the projection to the first variable. The smooth fibres  $Y_t$ ,  $t \in \mathbb{P}^1 \setminus \{0\}$ , are obviously isomorphic to  $X$ . The singular fibre has two irreducible components,  $Y_0 = Z \cup P$ :  $Z$  is the blowup of  $X$

at  $p$ , and  $P = \mathbb{P}(\mathbb{C} \oplus TX_p)$  is the exceptional divisor in  $Y$ . They are joined together by a normal crossing, where one identifies the exceptional curve in  $Z$  with the line  $C_0 = \mathbb{P}(\{0\} \oplus TX_p)$  in  $P$ . Let  $T \subset Y$  be the proper transform of  $\mathbb{P}^1 \times S \subset \mathbb{P}^1 \times X$  under the blowup, and  $T_t = T \cap Y_t$ . For  $t \neq 0$  one can obviously identify  $T_t \subset Y_t$  with  $S \subset X$ ; while  $T_0$  is the union of  $T \cap Z$ , which is the proper transform of  $S$  under blowing up  $p \in X$ , and of  $T \cap P$ , which consists of the two lines  $C_1, C_2 \subset P$  obtained by projectivizing  $\mathbb{C} \times$  (tangent space to either branch of  $S$  at  $p$ ). Choose an isomorphism  $P \cong \mathbb{P}^2$  in such a way that the  $C_k$  become the coordinate lines, and let  $K_0 \subset (\mathbb{C}^*)^2 \cong P \setminus (C_0 \cup C_1 \cup C_2)$  be one of the standard Clifford tori. One can find a submanifold with boundary  $K \subset Y$ , lying in  $\pi^{-1}([-1; 1])$  and with boundary in  $\pi^{-1}(\{-1; 1\})$ , such that  $\pi|_K: K \rightarrow [-1; 1]$  is a smooth fibration, whose fibre over  $t = 0$  is the given  $K_0$ . One way to think of this is to choose a connection (a horizontal subbundle) on the open subset of  $\pi$ -regular points of  $Y$ . Since  $K_0$  lies in that subset, one can use parallel transport to move it to other fibres, and doing that in both directions along the real axis yields  $K$ . In fact, any  $K$  with the properties stated above can be obtained in this way, for some choice of connection.

**Lemma 17.** *For all sufficiently small  $t \in [-1; 1] \setminus \{0\}$ ,  $K_t = K \cap Y_t$  is a linking torus for  $S$  at the crossing point  $p$ .*

It may be appropriate to first clarify the meaning of this. As before, let  $(a, b)$  be coordinates centered at  $p$  in which  $S = \{ab = 0\}$ , and  $U$  a ball in those coordinates. By identifying  $Y_t \cong X$  for  $t \in [-1; 1] \setminus \{0\}$ , one can think of the  $K_t$  as tori inside  $X$ . A more technical formulation of the lemma is that for sufficiently small such  $t$ ,  $K_t$  lies in  $U \setminus S$ , and is isotopic inside that set to the standard linking torus  $\{|a| = \mu, |b| = \nu\}$ . (This formulation, and the following proof, are somewhat pedantic, but are engineered to adapt well to the symplectic geometry requirements to be imposed subsequently.)

*Proof.* Let  $W$  be the preimage of  $\mathbb{P}^1 \times U$  under the blowup map  $Y \rightarrow \mathbb{P}^1 \times X$ . Since  $K_0 \subset P \subset W$ , one has  $K_t \subset W \cap Y_t = U$  for sufficiently small  $t \neq 0$ . Similarly, because  $K_0 \cap T = K_0 \cap (C_1 \cup C_2) = \emptyset$ , one has  $K_t \cap S = \emptyset$  for sufficiently small  $t \neq 0$ . The next step is to show that the isotopy type of  $K_t$  inside  $U \setminus S$  is independent of the choice of  $K$ . If one thinks of that choice as given by a connection, any two connections can be deformed into each other, which gives rise to an isotopy of the associated submanifolds  $K_t$ . The previous considerations show that for small  $t$ , this isotopy will take place inside  $U \setminus S$ .

With that in mind, it is sufficient to prove the statement that the  $K_t$  are linking tori for just one choice of  $K$ . We write down the local picture near  $P \subset Y$  in coordinates:

$$\begin{aligned} Y &= \{(t, a, b, [\tau : \alpha : \beta]) \in \mathbb{C}^3 \times \mathbb{P}^2 : (t, a, b) \in [\tau : \alpha : \beta]\}, \\ \pi(t, a, b, [\tau : \alpha : \beta]) &= t, \\ Z &= \{t = 0, \tau = 0\}, \end{aligned}$$

$$P = \{t = a = b = 0\},$$

$$T = \{\alpha\beta = 0\}.$$

Here  $(t, a, b)$  should actually lie in a small neighbourhood of  $(0, 0, 0)$ , but we omit that to make the notation more transparent. By definition,  $K_0 = \{\tau = 1, |\alpha| = \mu, |\beta| = \nu\}$  for some constants  $\mu, \nu > 0$ , and one can therefore take  $K = \{t \in \mathbb{R}, |a| = \mu|t|, |b| = \nu|t|\}$ , in which case  $K_t$  ( $t \neq 0$ ) is clearly a family of linking tori, whose diameter shrinks as  $t \rightarrow 0$ .  $\square$

From this point onwards, we will make the additional assumption that there is an effective divisor  $D$  on  $X$  whose support is  $S$  (in other words,  $D$  is a sum of the irreducible components of  $S$  with positive multiplicities), which is ample. Let  $E = \mathcal{O}_X(D)$  be the associated ample line bundle. Form the tensor product  $\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes E \rightarrow \mathbb{P}^1 \times X$  and pull it back to a line bundle on  $Y$  (keeping the notation for simplicity). For  $d \gg 0$ ,  $F = (\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes E)^{\otimes d} \otimes \mathcal{O}_Y(-P) \rightarrow Y$  is again ample. We will now recall Kodaira's classical proof of this fact; for full details see e.g. [14, p. 185]. Start with a metric  $\|\cdot\|_E$  on  $E$  whose curvature (more precisely  $iF_{\nabla_E}$ , where  $\nabla_E$  is the associated connection) is a positive  $(1, 1)$ -form, denoted by  $\omega_X$ . Similarly, on  $\mathcal{O}_{\mathbb{P}^1}(1)$  we choose a metric whose curvature is a positive  $(1, 1)$ -form  $\omega_{\mathbb{P}^1}$ . Tensor them together to give a metric on  $\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes E$ , with curvature  $\omega_{\mathbb{P}^1} + \omega_X$ . By specializing to the point  $(0, p) \in \mathbb{P}^1 \times X$ , this induces a metric on the bundle  $\mathcal{O}_P(1) \rightarrow P$ , and a Fubini–Study form  $\omega_P$  on  $P$ . One can identify  $\mathcal{O}_Y(-P)|_P \cong \mathcal{O}_P(1)$ , so this gives a metric on  $\mathcal{O}_Y(-P)|_P$ , which one can extend to a small neighbourhood of  $P$ . On the complement of  $P$ ,  $\mathcal{O}_Y(-P)$  is canonically trivial, so one can take a constant metric, and patch that together with the other one using a cutoff function. The outcome is a metric on  $\mathcal{O}_Y(-P)$  whose curvature form restricts to  $\omega_P$  on  $P$ . Direct computation shows that for  $d \gg 0$ , the curvature of the resulting tensor product metric  $\|\cdot\|_F$  on  $F$  is a positive  $(1, 1)$ -form, which we denote by  $\omega_Y$ .

For our application, we suppose that  $\|\cdot\|_E$  has been chosen in such a way that the two branches of  $S$  meet orthogonally at  $p$ . This is always possible, in fact one can modify the Kähler potential to make the metric standard in any given local holomorphic coordinates; see e.g. [22, Lemma 7.2]. The advantage is that we can then identify  $P \cong \mathbb{P}^2$  in such a way that  $C_0, C_1, C_2$  become the coordinate lines, and  $\omega_P$  the standard Fubini–Study form. As a consequence, any Clifford torus  $K_0 \subset P$  is Lagrangian for  $\omega_Y|_P = \omega_P$ . The Kähler form  $\omega_Y$  also induces a symplectic connection on the set of  $\pi$ -regular points in  $Y$ . We use this connection to transport  $K_0$  into nearby fibres, as described above. The resulting  $K$  is a Lagrangian submanifold with boundary inside  $(Y, \omega_Y)$  (this is best seen in two steps: since the connection is symplectic and  $K_0$  is Lagrangian, each  $K_t$  will be Lagrangian in  $Y_t$ ; and since the horizontal subspace is defined as the  $\omega_Y$ -orthogonal complement to the fibrewise tangent spaces,  $K$  itself is Lagrangian).

**Lemma 18.** *Suppose that there is a sequence  $t_k \in (0; 1]$  with  $\lim_k t_k = 0$ , and a sequence of holomorphic discs  $u_k: (\mathbb{D}, \partial\mathbb{D}) \rightarrow (Y_k, K_k)$  whose areas  $\int u_k^* \omega_Y$  are bounded. Then, after passing to a subsequence, there is a finite collection of holomorphic maps  $v_i: \mathbb{P}^1 \rightarrow X$  with the property that (with respect to the isomorphism  $Y_k \cong X$ ) the image of  $u_k$  for  $k \gg 0$  is contained in an arbitrarily small neighbourhood of the union of the images of the  $v_i$ . Moreover, the union of the images of the  $v_i$  is connected, and contains  $p$ .*

*Proof.* Consider  $(t_k, u_k)$  as a sequence of holomorphic discs in  $Y$  with boundary on the Lagrangian submanifold  $K$ , and apply Gromov compactness to a suitable subsequence. Since the images of the discs lie in  $Y_{t_k}$ , the limiting stable disc has image in  $Y_0$ . Its components are of three kinds: holomorphic spheres  $w_i: \mathbb{P}^1 \rightarrow Z$  and  $y_i: \mathbb{P}^1 \rightarrow P$ , as well as discs  $z_i: (\mathbb{D}, \partial\mathbb{D}) \rightarrow (P, K_0)$ . A fairly weak implication of Gromov convergence is that the image of  $(t_k, u_k)$  for  $k \gg 0$  is contained in an arbitrarily small neighbourhood (in  $Y$ ) of the union of the images of the  $w_i, y_i, z_i$ . We define the  $v_i$  to be the images of all the original components under blowdown  $Y \rightarrow \mathbb{P}^1 \times X$ ; the  $y_i$  and  $z_i$  become constant, and in the latter case we replace the domain  $\mathbb{D}$  by  $\mathbb{P}^1$ . The convergence statement then holds by construction; since the original stable disc was connected, the same applies to the union of the images of the  $v_i$ ; and since there was at least one  $z_i$  component, there is at least one  $v_i$  which is the constant map with value  $p$ .  $\square$

**Lemma 19.** *Suppose that there is a union of irreducible components of  $S$ , forming a sub-curve  $S' \subset S$  with  $p \notin S'$ , and an effective nef divisor  $D'$  whose support is  $S'$ . Assume that we have  $(t_k, u_k)$  as in the previous lemma, with the additional assumption that  $u_k^{-1}(S') = \emptyset$  for all  $k$ . Then  $v_i^{-1}(S') = \emptyset$  for all  $i$ .*

*Proof.* We have  $u_k \cdot D' = 0$  because the supports are disjoint, and by looking at the Gromov limiting process,  $\sum_i v_i \cdot D' = 0$  (the fact that  $(0, p)$  is blown up in  $Y$  plays no role here, since  $p \notin S'$ ). Nefness implies that  $v_i \cdot D' = 0$  for each  $i$ , which means that the image of  $v_i$  is either contained in  $S'$  or disjoint from it. Connectedness of the Gromov limit, together with the fact that  $p \notin S'$  lies on one of the  $v_i$ , means that the first possibility is excluded.  $\square$

We now link this with the previous discussion of Stein-essential Lagrangian submanifolds.  $E$  comes with a canonical holomorphic section  $s_E$  which vanishes precisely on  $S$ . From this, the section of  $\mathcal{O}_{\mathbb{P}^1}(1)$  which vanishes exactly at  $\{\infty\}$ , and the nowhere zero meromorphic section of  $\mathcal{O}_Y(-P)$  which has a simple pole at  $P$ , one gets a section  $s_F$  of  $F$  which vanishes precisely on  $Y_\infty \cup T \cup P$ . For  $t \neq 0, \infty$ , choose the isomorphism  $F|_{Y_t} \cong E$  in such a way that  $s_F$  gets mapped to  $s_E$ . Then the restriction of  $\|\cdot\|_F$  to  $Y_t \cong X$  induces a metric  $\|\cdot\|_{E,t}$  on  $E$ , hence an exhausting plurisubharmonic function  $\phi_{M,t} = -\log \|s_E\|_{E,t}$ , which by Lemma 8 makes  $M$  into

a finite type Stein manifold. Note that the associated convex symplectic structure  $(\omega_{M,t}, \theta_{M,t})$  satisfies  $\theta_{M,t} = (d^c \log \|s_F\|_F) | (Y_t \setminus T_t)$  and  $\omega_{M,t} = \omega_Y | (Y_t \setminus T_t)$ , where we are again using the identifications  $Y_t \setminus T_t \cong X \setminus S = M$ .

By Lemma 17 one can find a  $\tau > 0$  such that for  $t \in (0; \tau] \subset \mathbb{P}^1$ ,  $K_t$  is a linking torus, and in particular disjoint from  $S$ . This yields a smooth family of submanifolds  $K_t \subset M$  which are  $\omega_{M,t}$ -Lagrangian. Moreover, the class  $[\theta_{M,t}|K_t]$  is constant in  $t$ . To see this, note that since  $K$  itself is  $\omega_Y$ -Lagrangian, the restriction of  $d^c \log \|s_F\|_F$  to  $K \cap \pi^{-1}(0; \tau]$  is a closed one-form. Its image under the restriction map  $H^1(K \cap \pi^{-1}((0; \tau]; \mathbb{R})) \rightarrow H^1(K_t; \mathbb{R})$  is  $[\theta_{M,t}|K_t]$ , which is therefore independent of  $t$  as claimed.

**Lemma 20.** *Suppose that  $K_\tau$ , as a Lagrangian submanifold of the Stein manifold  $(M, \phi_{M,\tau})$ , is not Stein-essential in the sense of Section 3. Then there is a sequence  $t_k \in (0; \tau]$  with  $\lim_k t_k = 0$ , and a sequence of non-constant holomorphic discs  $u_k: (\mathbb{D}, \partial\mathbb{D}) \rightarrow (Y_{t_k} \setminus T_{t_k}, K_{t_k})$  whose areas  $\int u_k^* \omega_Y$  are bounded.*

*Proof.* Suppose that the conclusion is false. Then as  $t \in (0; \tau]$  goes to zero, the least area of non-constant holomorphic discs bounding  $K_t$  in  $(M, \omega_{M,t})$  must go to infinity. More precisely, for each  $E > 0$  there is a  $\tau' \in (0; \tau]$  such that every holomorphic disc  $(\mathbb{D}, \partial\mathbb{D}) \rightarrow (M, K_{\tau'})$  with  $\int u^* \omega_{M,\tau'} \leq E$  is constant. The manifold  $M$ , with its given complex structure and the family of plurisubharmonic functions  $\phi_{M,t}$ ,  $t \in [\tau'; \tau]$ , is a finite type Stein deformation (see the remark following Lemma 8), and the  $K_t$ ,  $t \in [\tau'; \tau]$ , are a family of Lagrangian submanifolds such that  $[\theta_{M,t}|K_t]$  is constant. By definition, the existence of such a deformation for each  $E$  means that  $K_\tau$  is Stein-essential, contrary to our assumption.  $\square$

Slightly more generally, suppose that for some  $m \geq 1$ , the product  $K_\tau^m$  is not Stein-essential as a Lagrangian submanifold of  $M^m$  equipped with the product Stein structure (meaning the product complex structure and the plurisubharmonic function  $(x_1, \dots, x_m) \mapsto \phi_{M,\tau}(x_1) + \dots + \phi_{M,\tau}(x_m)$ , which is still of finite type). Then the same argument as before shows that one can find  $t_k$  and non-constant holomorphic discs  $(u_k^1, \dots, u_k^m): (\mathbb{D}, \partial\mathbb{D}) \rightarrow ((Y_{t_k} \setminus T_{t_k})^m, K_{t_k}^m)$  with bounded area. After choosing a non-constant component  $u_k = u_k^{i_k}$ ,  $i_k \in \{1, \dots, m\}$  of each disc, one arrives at the same conclusion as in the lemma itself.

## 5. Conclusion

We briefly recall Ramanujam's construction [21]. In  $\mathbb{P}^2$  take a smooth conic, and a cubic with a cusp singularity, which intersect each other at two points with multiplicities 1 and 5 respectively (the intersection points should also be distinct from

the cusp). Blow up the multiplicity 1 intersection point, and let  $\bar{S}'$ ,  $\bar{S}''$  be the proper transforms of the conic and the cubic, respectively, inside the blowup  $\bar{X} \cong \mathbb{F}_1$ . The union  $\bar{S} = \bar{S}' \cup \bar{S}''$  has two singular points, namely the multiplicity 5 intersection point and the cusp. Take the minimal blowup  $q: X \rightarrow \bar{X}$  such that  $S = q^{-1}(\bar{S})$  is a divisor with normal crossings. The resolution graph describing  $S$  is shown in Figure 2. The fattened vertices correspond to those components which are the proper transforms of  $\bar{S}'$  (with selfintersection  $-2$ ) and  $\bar{S}''$  (with selfintersection  $-3$ ); the other components are exceptional divisors lying above the cusp (on the left) and the multiplicity 5 intersection point (on the right). Let  $p \in S$  be the crossing point in the preimage of the cusp where the proper transform of the cubic intersects the exceptional divisor of multiplicity 6; this corresponds to the edge indicated by the arrow.

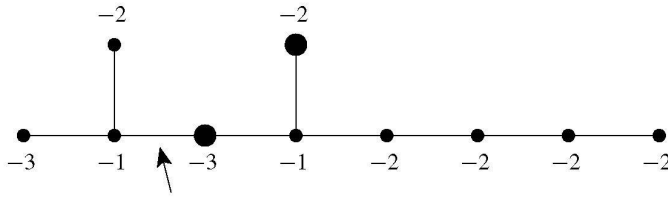


Figure 2

Because  $\bar{S}$  is ample, one can find an ample divisor  $D$  on  $X$  whose support is  $S$ . Set  $E = \mathcal{O}_X(D)$ , and carry out the construction from the previous section for the point  $p$ ; this yields a family of plurisubharmonic functions  $\phi_{M,t}$  and  $\omega_{M,t}$ -Lagrangian tori  $K_t$ . Take  $\tau$  as in the discussion before Lemma 20. Equip  $M$  with its standard complex structure  $J_M$ , the function  $\phi_M = \phi_{M,\tau}$  which makes it into a finite type Stein manifold, and the Lagrangian submanifold  $L = K_\tau$ .

Let  $S' \subset S$  be the preimage of  $\bar{S}'$ , which is its proper transform together with the exceptional curves arising from blowing up the multiplicity 5 intersection point. Since  $\bar{S}' \subset \bar{X}$  is ample, one can use Kodaira's construction to find an effective divisor  $D'$  with support  $S'$ , and a  $(1,1)$ -form representing its Poincaré dual, which is nonnegative everywhere, and positive away from the preimage of the cusp point. This means that any curve  $\Sigma$  with  $\Sigma \cdot D' \leq 0$  must lie on the preimage of the cusp point; in particular  $D'$  is nef.

*Proof of Theorem 1.* Assume that  $M^m$  has a symplectic embedding  $i$  into a subcritical Stein manifold  $N$ , which we may assume to be complete by Lemma 15. Since  $H^1(M^m; \mathbb{R}) = 0$ , the closed one-form  $i^*\theta_N - \theta_M$  is automatically exact. By Proposition 14,  $L^m$  cannot be Stein-essential. Using Lemma 20 and the remark following it, one then gets a sequence  $t_k$  and non-constant holomorphic discs  $u_k: (\mathbb{D}, \partial\mathbb{D}) \rightarrow (Y_{t_k} \setminus T_{t_k}, K_{t_k})$  with bounded energy. The isomorphism  $Y_{t_k} \cong X$

sends  $T_{t_k}$  to  $S$ , hence the image of each  $u_k$  is disjoint from  $S' \subset S$ . The limit of this, in the sense of Lemma 18, is a finite collection of holomorphic maps  $v_i : \mathbb{P}^1 \rightarrow X$ , which by Lemma 19 are disjoint from  $S'$ . Using the observation made above, it follows that the image of the  $v_i$  lies in the preimage of the cusp point. In other words, if  $\bar{V}$  is a neighbourhood of the cusp in  $\bar{X}$ , and  $V$  its preimage in  $X$ , then for  $k \gg 0$  we have a non-constant holomorphic disc  $u_k : (\mathbb{D}, \partial\mathbb{D}) \rightarrow (V \setminus S, K_{t_k})$ . Since  $K_{t_k}$  is a linking torus for a crossing point which arose from a cusp, we can apply Lemma 16 to conclude that  $u_k(\partial\mathbb{D})$  is a contractible loop on  $K_{t_k}$ . But by Stokes this implies that the area  $\int u_k^* \omega_Y$  is zero, which means that  $u_k$  is constant, hence yields a contradiction.  $\square$

*Proof of Corollary 2.* Suppose that  $M^m$  carries an exhausting plurisubharmonic Morse function  $\tilde{\phi}_M$  with just 1 critical point. In view of Lemma 6, we may assume that  $\tilde{\phi}_M$  is complete; and it is of finite type by assumption. Lemma 7 then says that there is a symplectic embedding  $i : M \rightarrow M$  such that  $i^* \tilde{\theta}_M - \theta_M$  is exact, contradicting Theorem 1.  $\square$

## Appendix

*Proof of Lemma 3.* This is a straightforward generalization of the case of cotangent bundles, discussed in [11, Proposition 1.3.A]. Without affecting the validity of the statement, we may replace  $\theta_N$  by  $\tilde{\theta}_N = \theta_N + dK$  for any compactly supported function  $K$ . A suitable choice of  $K$  ensures that  $i^* \tilde{\theta}_N = \theta_M$ , and then  $i$  takes the Liouville flow  $l_M$  to the modified Liouville flow  $\tilde{l}_N$  associated to  $\tilde{\theta}_N$ . By assumption, for any point  $x \in M$  there is a  $t \geq 0$  such that  $l_M^{-t}(x) \in \phi_M^{-1}((-\infty; c_0])$ , and on the other hand  $\tilde{l}_N^t$  is defined for all  $t \geq 0$ . Hence

$$j_t = \tilde{l}_N^t \circ i \circ l_M^{-t}, \quad t \geq 0$$

is a family of mutually compatible extensions of  $i$  to successively larger subsets, which exhaust  $M$ ; and they satisfy  $j_t^* \tilde{\theta}_N = \theta_M$ .  $\square$

*Proof of Lemma 4.* After a finite decomposition of the interval  $[0; 1]$ , we may assume that there are  $c_1 < c_2 < \dots$  converging to  $+\infty$ , such that  $d\phi_{M,t}(\lambda_{M,t}) > 0$  on  $\phi_{M,t}^{-1}(c_k)$  for all  $t \in [0; 1]$ . There is a  $k$  such that  $U \subset \phi_{M,0}^{-1}((-\infty; c_k])$ . Let  $\mu_t$  be the Moser vector field defined by  $\omega_{M,t}(\mu_t, \cdot) = -d\theta_{M,t}/dt$ . By choosing  $r \gg 0$  sufficiently large, one can achieve that  $d\phi_{M,t}(\mu_t - r\lambda_{M,t}) + \partial_t \phi_{M,t} < 0$  on  $\phi_{M,t}^{-1}(c_k)$  for all  $t \in [0; 1]$ . Hence, integrating  $\mu_t - r\lambda_{M,t}$  yields a smooth family of embeddings  $i_t : \phi_{M,0}^{-1}((-\infty; c_k]) \rightarrow \phi_{M,t}^{-1}((-\infty; c_k])$  starting with  $i_0 = \text{id}$ , such that  $i_t^* \theta_{M,t} - e^{-rt} \theta_{M,0}$  is an exact one-form. Define  $j_t$  by composing  $i_t|_U$  and the time  $rt$  flow of  $\lambda_{M,t}$ .  $\square$

*Proof of Lemma 5.* Let  $c_0$  be as in the definition of finite type deformations. Gray's Theorem on the stability of contact structures implies that there is a family of diffeomorphisms of  $\phi_{M,t}^{-1}(c_0)$  which pulls back (the restrictions of)  $\theta_{M,t}$  to  $\theta_{M,0}$ . This induces a family of diffeomorphisms which identify the cone-like ends of  $(M, \omega_{M,t}, \theta_{M,t})$  for different  $t$ . Going back from isotopies



to the generating vector fields, the outcome is that one can find vector fields  $\gamma_t$  on  $\phi_{M,t}^{-1}([c_0; \infty))$  such that  $L_{\gamma_t}\theta_{M,t} + \partial_t\theta_{M,t} = 0$  and  $[\gamma_t, \lambda_{M,t}] + \partial_t\lambda_{M,t} = 0$  (the first condition comes from Gray's argument, and the second one expresses compatibility with the conical structure). Extend these vector fields to all of  $M$  in an arbitrary way, and integrate them to get a family of diffeomorphisms  $g_t: M \rightarrow M$  such that  $g_t^*\theta_{M,t} - \theta_{M,0}$  vanishes outside a compact subset. Moser's Lemma then yields another family of diffeomorphisms  $h_t: M \rightarrow M$ , which is compactly supported and satisfies  $h_t^*g_t^*\theta_{M,t} - \theta_{M,0} = dR_t$  as desired. Set  $f_t = g_t \circ h_t$ .  $\square$

*Proof of Lemma 6.* The main statement is taken from [4, Lemma 3.1]. There, the authors observe that for any  $h$  with  $h' > 0$ ,  $h'' \geq 0$ , the modified function  $\tilde{\phi}_M = h(\phi_M)$  is again plurisubharmonic. Moreover, the Liouville vector field associated to the modified Stein structure is related to the original one by

$$\tilde{\lambda}_M = \lambda_M \cdot \frac{h'(\phi_M)}{h'(\phi_M) + h''(\phi_M)\|\nabla\phi_M\|^2},$$

with the norm taken in the original Kähler metric. This means that the modified Liouville flow has the same flow lines as the original one, but moves along them at a slower or equal rate. The additional condition  $h''(c) \geq \delta h'(c)$  for  $c \geq c_0$  implies  $d\phi_M(\tilde{\lambda}_M) \leq \delta^{-1}$  outside a compact subset, so that the flow is then complete.

To prove the last statement in the lemma, consider the family of functions  $\phi_{M,t} = h_t(\phi_M)$  with  $h_t(c) = (1-t)c + th(c)$ ,  $0 \leq t \leq 1$ . These also satisfy  $h'_t > 0$ ,  $h''_t \geq 0$ , so the flow of  $\nabla\phi_{M,t}$  (with respect to its associated Kähler metric) is slower than that of  $\nabla\phi_M$ . We are assuming the second flow is complete, hence so is the first one. Besides that, we are also assuming that  $\phi_M$  is of finite type. Hence what we get is a complete finite Stein deformation  $(M, J_M, \phi_{M,t})$ , to which Lemma 5 can be applied.  $\square$

*Proof of Lemma 7.* We imitate the argument from [11, Theorem 1.4.A], with some clarifications. We will prove the statement first in the case where  $\tilde{\phi}_M$  grows faster than  $\phi_M$ , by which we mean that the difference  $\delta = \tilde{\phi}_M - \phi_M$  is an exhausting function. Let  $U_k = \delta^{-1}((-\infty; k))$  be the associated family of exhausting subsets; after changing  $\tilde{\phi}_M$  by a constant, we may assume that  $\delta \geq 1$  everywhere, hence  $U_k = \emptyset$  for  $k \leq 1$ . Fix a smooth function  $l: \mathbb{R} \rightarrow \mathbb{R}$  such that  $l(r) = 0$  for  $r \leq -1$ ,  $l(r) = r$  for  $r \geq 1$ , and  $l''(r) \geq 0$  everywhere. For  $k = 0, 1, 2, \dots$  consider the functions  $\tilde{\phi}_{M,k} = \phi_M + l(\tilde{\phi}_M - \phi_M - k)$ . These satisfy  $\tilde{\phi}_{M,k} = \phi_M$  on  $\bar{U}_{k-1}$ ,  $\tilde{\phi}_{M,k} = \tilde{\phi}_M - k$  on  $M \setminus U_{k+1}$ , and are plurisubharmonic because

$$dd^c\tilde{\phi}_{M,k} = (1-l')dd^c\phi_M + l'dd^c\tilde{\phi}_M + l''d(\tilde{\phi}_M - \phi_M) \wedge d^c(\tilde{\phi}_M - \phi_M)$$

with  $l' = l'(\tilde{\phi}_M - \phi_M - k) \in [0; 1]$ , and  $l'' = l''(\tilde{\phi}_M - \phi_M - k) \geq 0$ . Let  $(\tilde{\omega}_{M,k}, \tilde{\theta}_{M,k})$  be the convex symplectic structure associated to  $\tilde{\phi}_{M,k}$ . By applying Moser's argument to the linear deformation between the  $k^{\text{th}}$  and  $(k+1)^{\text{st}}$  of these structures, we get a diffeomorphism  $f_k: M \rightarrow M$  which is the identity outside  $\bar{U}_{k+2} \setminus U_{k-1}$ , and such that  $f_k^*\tilde{\theta}_{M,k} - \tilde{\theta}_{M,k+1}$  is the derivative of a function  $K_k$  supported in  $\bar{U}_{k+2} \setminus U_{k-1}$ . Let  $i: M \rightarrow M$  be the infinite composition  $f_0 \circ f_1 \circ \dots$ . This is well-defined because for each  $x \in M$ , one has  $f_k(x) = x$  for all but finitely many  $k$ . The infinite composition is injective and a local diffeomorphism, hence an embedding (but not necessarily a diffeomorphism; composing the  $f_k^{-1}$  in the opposite order makes no sense). By definition  $\tilde{\theta}_{M,0} = \tilde{\theta}_M$ ; and for each relatively compact subset  $U \subset M$  there is a  $k$  such that  $\tilde{\theta}_{M,k} = \theta_M$  on  $U$ , and  $f_{k+1}|_U = f_{k+2}|_U = \dots = \text{id}_U$ . It follows that  $i^*\tilde{\omega}_M = \omega_M$ .

and moreover  $i^*\tilde{\theta}_M = \theta_M + dK$  for  $K = (f_1 \circ f_2 \circ \dots)^*K_0 + (f_2 \circ f_3 \circ \dots)^*K_1 + \dots$ , the same argument as before showing that this sum is well-defined.

We now pass to the general situation, where  $\tilde{\phi}_M$  is complete and of finite type but otherwise arbitrary. One can then find a function  $h$  as in Lemma 6 for which  $h(\tilde{\phi}_M)$  grows faster than  $\phi_M$ , and moreover this rescaling does not change the (exact) symplectic isomorphism type. One can then apply the previous argument to  $\phi_M$  and  $h(\tilde{\phi}_M)$ , and derive the desired result.  $\square$

*Proof of Lemma 8.* For simplicity, we write  $\|\cdot\| = \|\cdot\|_E$  and  $s = s_E$ . Around a point  $x \in S$ , choose local holomorphic coordinates, and a local holomorphic trivialization of our line bundle, with respect to which  $s(z) = z_1^{w_1} \dots z_n^{w_n}$ . Write  $w = w_1 + \dots + w_n$ . With respect to the trivial metric  $\|\cdot\|_0$  one has

$$\begin{aligned} |d\|s\|_0| &\gtrsim \sum_j w_j |\partial_{z_j} \|s\|_0| \gtrsim \sum_j w_j |z_1^{w_1} \dots z_j^{w_j-1} \dots z_n^{w_n}| \\ &\gtrsim \sqrt[w]{|z_1|^{w_1(w-1)} \dots |z_n|^{w_n(w-1)}} = \|s\|_0^{1-1/w}, \end{aligned}$$

where  $\gtrsim$  means greater than or equal to some small constant times the right hand side (in spite of that, we have kept the  $w_j$ , because they indicate how the inequality between arithmetic and geometric mean is applied). One also has  $|d\|s\|| + \|s\| \gtrsim |d\|s\|_0|$  and  $\|s\|_0^{1-1/w} \gtrsim \|s\|^{1-1/w}$ . After combining the inequalities, one sees that  $d\|s\|$  does not vanish at points  $z$  where  $\|s(z)\|$  is sufficiently small, hence  $x$  does not lie in the closure of the critical point set of  $\phi_M$ .  $\square$

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