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Non-vanishing for Koszul cohomology of curves

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Abstract. We study the relationship between rank $p+2$ Koszul classes and rank 2 vector bundles on a smooth curve. We show that every rank $p+2$ Koszul class is obtained from a rank 2 vector bundle and give an explicit nonvanishing theorem for Koszul classes arising in this way.

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1. Introduction

Let X be a smooth complex projective variety. The geometry of X is reflected in the behaviour of the Koszul cohomology groups $K_{p,q}(X, L)$ introduced by Green [4], more specifically the vanishing/nonvanishing of certain Koszul cohomology groups. The fundamental result in this direction is the nonvanishing theorem of Green–Lazarsfeld [5]. This theorem states that if a line bundle L admits a decomposition $L = L_1 \otimes L_2$ with $r_i = h^0(X, L_i) - 1 \geq 1$ $(i = 1, 2)$ then $K_{r_1+r_2-1,1}(X, L) \neq 0$. Voisin [9, (1.1)] has given a different proof of this result under the hypothesis that L_1 and L_2 are globally generated.

The aim of this note is to give a more geometric approach to this type of problems. The starting point is the following construction due to Voisin. Given a rank two vector bundle E on X with determinant L, Voisin [11, (2.22)] defined a homomorphism

$$\varphi\colon S^p H^0(X, E) \otimes \bigwedge^{p+2} H^0(X, E) \to \bigwedge^p H^0(X, L) \otimes H^0(X, L).$$

By [11, Lemma 5], this homomorphism produces elements of $K_{p,1}(X, L)$. If we take $E = L_1 \oplus L_2$, we get back the classes constructed by Green and Lazarsfeld. As one of the referees pointed out to us, Koh and Stillman [7] had generalised the Green–Lazarsfeld construction before from a different point of view.

Recall that the *rank* of a Koszul class $\gamma \in K_{p,1}(X, L)$ is the minimal dimension of a linear subspace $W \subset H^0(X, L)$ such that γ is represented by an element in $\bigwedge^p W \otimes H^0(X, L)$; cf. [6, Definition 2.2]. (Note that the subspace W is uniquely determined if $p \geq 2$.) By definition, the Koszul classes constructed in this paper are of rank $p+2$ if the vector bundle E is indecomposable.

Section 3 contains the main results of this paper. We first give a necessary and sufficient condition for nonvanishing of Koszul classes on smooth curves obtained from rank 2 vector bundles (Theorem 3.1). This result generalises the nonvanishing theorem of Green–Lazarsfeld in the case of curves. Our second main result, Theorem 3.4, states that every rank $p+2$ Koszul class on a smooth curve comes from a rank two vector bundle. This theorem is a generalisation of [6, Theorem 6.7].

2. Preliminaries

2.1. The method of Voisin. Let E be a rank two vector bundle on a smooth projective variety X defined over an algebraically closed field k of characteristic zero. Write $L = \det E$ and $V = H^0(X, L)$, and let

$$d\colon \bigwedge^2 H^0(X, E) \to V$$

be the determinant map. Given $t \in H^0(X, E)$, define a linear map

$$d_t\colon H^0(X, E) \to V$$

by $d_t(u) = d(t \wedge u)$, and choose a subspace $U \subset H^0(X, E)$ with $U \cap \ker(d_t) = 0$. Suppose that $\dim(U) = p+2$ with $p \geq 1$, and put $W = d_t(U) \cong U$. The restriction of d to $\bigwedge^2 U$ defines a map $\bigwedge^2 U \to V$, which we can view as an element of

$$\bigwedge^2 U^\vee \otimes V \cong \bigwedge^p U \otimes V.$$

Let

$$\gamma \in \bigwedge^p W \otimes V \subset \bigwedge^p V \otimes V$$

be the image of this element under the map d_t.

Following Voisin [11, (2.22)], we prove that γ defines a Koszul class in $K_{p,1}(X, L)$. To this end, we make the previous construction explicit using coordinates. If we choose a basis $\{e_1, \dots, e_{p+3}\}$ of $\langle t\rangle \oplus U \subset H^0(X, E)$ such that $e_1 = t$, we have

$$\begin{aligned}\gamma = \sum_{i<j} (-1)^{i+j} d(t \wedge e_2) \wedge \dots \wedge \widehat{d(t \wedge e_i)} \wedge \cdots \\ \cdots \wedge \widehat{d(t \wedge e_j)} \wedge \dots \wedge d(t \wedge e_{p+3}) \otimes d(e_i \wedge e_j).\end{aligned} \tag{1}$$

As in [11] one shows that the image of the γ by the Koszul differential

$$\delta\colon \bigwedge^p V \otimes H^0(X, L) \to \bigwedge^{p-1} V \otimes S^2 H^0(X, L)$$

equals

$$\begin{aligned}\sum_{i<j<k} &(-1)^{i+j+k} d(t\wedge e_2)\wedge\cdots\wedge \widehat{d(t\wedge e_i)}\wedge\cdots\\ &\cdots\wedge\widehat{d(t\wedge e_j)}\wedge\cdots\wedge\widehat{d(t\wedge e_k)}\wedge\cdots\wedge d(t\wedge e_{p+3})\\ &\otimes\{d(t\wedge e_i)d(e_j\wedge e_k)-d(t\wedge e_j)d(e_i\wedge e_k)+d(t\wedge e_k)d(e_i\wedge e_j)\}.\end{aligned}\tag{2}$$

Lemma 2.1 (Voisin). *Given four elements $w_1, w_2, w_3, w \in H^0(X, E)$ we have the relation*

$$d(w\wedge w_1)d(w_2\wedge w_3)-d(w\wedge w_2)d(w_1\wedge w_3)+d(w\wedge w_3)d(w_1\wedge w_2)=0$$

in $H^0(X, L^2)$.

Proof. See [11, Lemma 5]. □

The previous lemma shows that γ belongs to the kernel of the Koszul differential

$$\delta_X\colon \textstyle\bigwedge^p V\otimes H^0(X,L)\to \bigwedge^{p-1}V\otimes H^0(X,L^2).$$

Hence γ defines a Koszul class $[\gamma]=\gamma(U,t)\in K_{p,1}(X,L,W)\subseteq K_{p,1}(X,L)$.

Remark 2.2. If $U'\subset\langle t\rangle\oplus U\subset d_t^{-1}(W)$ is another lifting of W, then $\gamma(U,t)=\gamma(U',t)$. In particular, if $\ker(d_t)=\mathbb{C}.t$ the given class only depends on t and W; we write $[\gamma]=\gamma(W,t)$ in this case.

2.2. The method of Green–Lazarsfeld. Let L_1, L_2 be two line bundles on a smooth projective variety X such that $r_i = h^0(X, L_i)-1\geq 1$ $(i=1,2)$. Write $L_i = M_i+F_i$ with M_i the mobile part and F_i the fixed part. Let B be the divisorial part of $F_1\cap F_2$. It is possible to choose $s_i\in H^0(X,L_i)$ such that $V(s_1,s_2)=B\cup Z$ with $\operatorname{codim}(Z)\geq 2$. Set $L=L_1\otimes L_2$, and put $t=(s_1,s_2)\in H^0(X,L_1\oplus L_2)$, $W=\operatorname{im}(d_t)\subset H^0(X,L(-B))$. By construction $h^0(X,\mathcal{O}_X(B))=1$, hence $\ker(d_t)=\mathbb{C}.t$ and $\dim W=r_1+r_2+1$. By the previous discussion, we obtain a Koszul class $\gamma(W,t)\in K_{r_1+r_2-1,1}(X,L)$. We call such classes *Green–Lazarsfeld classes*.

Note that the rank of a Green–Lazarsfeld class is either $p+1$ or $p+2$. Classes of rank $p+1$ are of scrollar type; see e.g. [8] or [6, Corollary 5.2].

Definition 2.3. Given a nonnegative integer $k\geq 0$, let $K_{k,1}(X,L)_{\mathrm{GL}}\subseteq K_{k,1}(X,L)$ be the subspace generated by Green–Lazarsfeld classes for all decompositions $L=L_1\otimes L_2$ with $k=r_1+r_2-1$, $(r_1\geq 1, r_2\geq 1)$.

2.3. The method of Koh–Stillman. Voisin's method produces syzygies of rank $\leq p+2$. As we have seen in the previous subsection, rank $p+1$ syzygies are Green–Lazarsfeld syzygies of scrollar type. Rank $p+2$ syzygies can be obtained in the following way. Suppose that L is a globally generated line bundle on a projective variety X, and let $[\gamma] \in K_{p,1}(X, L)$ be a nonzero class represented by an element $\gamma \in \bigwedge^p W \otimes V$ with $\dim W = p+2$. We view γ as an element in $\bigwedge^2 W^\vee \otimes V \cong \operatorname{Hom}\big(\bigwedge^2 W, V\big)$. Following [6, Proof of Theorem 6.1] we consider the map

$$\gamma' \colon \textstyle\bigwedge^2(\mathbb{C} \oplus W) = W \oplus \bigwedge^2 W \to V$$

defined by taking the direct sum of γ and the inclusion $W \hookrightarrow V$. If we choose a generator e_1 for the first summand and a basis $\{e_2, \dots, e_{p+3}\}$ for W, we obtain a skew-symmetric $(p+3) \times (p+3)$ matrix A by setting

$$a_{ij} = \gamma'(e_i \wedge e_j).$$

By construction, the inclusion $W \to V$ corresponds to the map $\gamma'(e_1 \wedge -)$. This allows us to identify a_{1j} and e_j, $2 \leq j \leq p+3$. Let α be the image of γ under the Koszul differential

$$\delta \colon \textstyle\bigwedge^p V \otimes V \to \bigwedge^{p-1} V \otimes S^2 V.$$

Writing this out, we obtain

$$\alpha = \sum_{i<j<k} (-1)^{i+j+k} a_{12} \wedge \dots \wedge \widehat{a_{1,i}} \wedge \dots \wedge \widehat{a_{1,j}} \wedge \dots \wedge \widehat{a_{1,k}} \wedge \dots \wedge a_{1,p+3} \otimes \mathrm{Pf}_{1ijk}(A). \tag{3}$$

As the elements $\{a_{12}, \dots, a_{1,p+3}\} = \{e_2, \dots, e_{p+3}\}$ are linearly independent, this expression is nonzero if and only if at least one of the Pfaffians $\mathrm{Pf}_{1ijk}(A)$ is nonzero. Furthermore, since α maps to zero in $\bigwedge^{p-1} V \otimes H^0(X, L^2)$ the Pfaffians $\mathrm{Pf}_{1ijk}(A)$ have to vanish on the image of X.

The preceding discussion shows that every rank $p+2$ syzygy arises from a skew-symmetric $(p+3) \times (p+3)$ matrix A such that

(i) the elements $\{a_{12}, \dots, a_{1,p+3}\}$ are linearly independent;

(ii) there exists a nonzero Pfaffian $\mathrm{Pf}_{1ijk}(A)$;

(iii) the Pfaffians $\mathrm{Pf}_{1ijk}(A)$ vanish on the image of X in $\mathbb{P}(V^\vee)$.

This is exactly the method used by Koh and Stillman to produce syzygies; see [7, Lemma 1.3].

Remark 2.4. In the geometric setting of Section 2.1, let Y be the image of X in $\mathbb{P}(V^\vee)$. The expression (2) shows that the canonical isomorphism

$$K_{p,1}(X, L) \cong K_{p-1,2}(\mathbb{P}^r, \mathcal{I}_Y, \mathcal{O}_{\mathbb{P}}(1))$$

maps the class $[\gamma]$ to the element α defined in (3). Moreover, if d does not vanish on decomposable elements then $[\gamma] \neq 0$. Indeed, this condition is satisfied if and only if the matrix A has no generalised zero; cf. [7, Definition (1.1)]. One then applies [loc. cit., Remark p. 122].

3. Main results

Theorem 3.1. *Let X be a smooth curve, let L be a base-point free line bundle on X and let $W \subset H^0(X, L)$ be a linear subspace. Put $B = \mathrm{Bs}(W)$, and let t be a section of $H^0(X, \mathcal{O}_X(B))$ vanishing on B. Consider an extension*

$$0 \to \mathcal{O}_X(B) \to E \to L(-B) \to 0 \tag{4}$$

such that

$$W \subset (\ker H^0(X, L(-B)) \xrightarrow{\delta} H^1(X, \mathcal{O}_X(B))).$$

Then the Koszul classes $\gamma(U, t)$ defined in Section 2.1 *are nonzero for all liftings U of W if and only if the extension* (4) *is non-split.*

Proof. The proof proceeds in several steps. We use the notation of Section 2.1.

"Only if". Suppose that the extension (4) splits, hence $W \subset H^0(X, E)$ canonically. We then put $U = W$. In this case, one readily verifies that d vanishes identically on $\bigwedge^2 U$. The formula (1) then shows that $\gamma(U, t) = 0$.

"If". Suppose there exists U such that $\gamma(U, t) = 0$. We proceed in several steps.

Step 1. There exists a linear map $h \colon U \to \mathbb{C}$ such that

$$d(u_1 \wedge u_2) = h(u_2)d_t(u_1) - h(u_1)d_t(u_2) \tag{5}$$

for all $u_1, u_2 \in U$.

Indeed, suppose that there exists a nonzero element $\tilde{\gamma} \in \bigwedge^{p+1} W \cong W^\vee$ such that γ is the image of $\tilde{\gamma}$ under the Koszul differential. Then γ coincides with the composition of maps

$$\bigwedge^2 W \xrightarrow{\delta} W \otimes W \xrightarrow{\tilde{\gamma} \otimes \mathrm{id}} W \hookrightarrow V.$$

Since

$$\begin{aligned} d(u_1 \wedge u_2) &= \gamma(d_t(u_1) \wedge d_t(u_2)) \\ &= \tilde{\gamma}(d_t(u_2))d_t(u_1) - \tilde{\gamma}(d_t(u_1))d_t(u_2)), \end{aligned}$$

condition (5) is satisfied with $h = \tilde{\gamma} \circ d_t \colon U \to \mathbb{C}$.

Step 2. Let $u_1, u_2 \in \langle t\rangle \oplus U$ be two sections such that $d_t(u_1)$ and $d_t(u_2)$ generate $L(-B)$. If $d(u_1 \wedge u_2) = 0$, the extension (4) splits.

To prove this assertion, put $s_i = d_t(u_i)$ $(i = 1, 2)$ and consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_X(B) \arrow[r] & E \arrow[r] & L(-B) \arrow[r] & 0 \\ & 0 \arrow[r] & \langle u_1, u_2\rangle \otimes \mathcal{O}_X \arrow[u, "\mathrm{ev}_2"] \arrow[r, "\sim"] & \langle s_1, s_2\rangle \otimes \mathcal{O}_X \arrow[u, "\mathrm{ev}_1"] \arrow[r] & 0. \end{tikzcd}$$

Put $M = \ker(\mathrm{ev}_1)$, and note that $\ker(\mathrm{ev}_2) \cong L^{-1}(B)$ since ev_2 is surjective. By the Snake Lemma we obtain an exact sequence

$$0 \to M \to L^{-1}(B) \to \mathcal{O}_X(B) \to \mathrm{coker}\,(\mathrm{ev}_1) \to 0.$$

Note that

$$d(u_1 \wedge u_2) = 0 \iff \mathrm{rank}\ \mathrm{im}(\langle u_1, u_2\rangle \otimes \mathcal{O}_X \to E) = 1 \iff \mathrm{rank}\ M = 1$$

where the first equivalence follows from [10, p. 380]. If $d(u_1 \wedge u_2) = 0$ the above exact sequence shows that $M \cong L^{-1}(B)$, hence the isomorphism $\langle u_1, u_2\rangle \otimes \mathcal{O}_X \xrightarrow{\sim} \langle s_1, s_2\rangle \otimes \mathcal{O}_X$ induces an isomorphism $\mathrm{im}(\mathrm{ev}_1) \cong L(-B)$. The inverse of this isomorphism provides a splitting of the extension (4).

Step 3. By Step 1, there exists a linear map $h\colon U \to \mathbb{C}$ verifying the relation (5). If h is identically zero, then we can apply Step 1 and Step 2 to conclude. Suppose $h \not\equiv 0$. Consider the morphism

$$\pi : X \to \mathbb{P}(W^\vee)$$

defined by the base-point free linear system $W \subset H^0(X, L(-B))$, and choose a linear subspace $\Lambda \subset \mathbb{P}(W^\vee)$ of codimension two such that $\Lambda \cap \pi(X) = \emptyset$. The hyperplane $\ker(h) \subset W$ corresponds to a point $p \in \mathbb{P}(W^\vee)$. Put $H_1 = \langle \Lambda, p\rangle$ and choose a hyperplane $H_2 \subset \mathbb{P}(W^\vee)$ containing Λ such that $p \notin H_2$. Let u_1, u_2 be the sections corresponding to H_1, H_2. Then $d_t(u_1)$ and $d_t(u_2)$ generate $L(-B)$ and $u_1 \in \ker(h)$, $u_2 \notin \ker(h)$. Equation (5) yields the identity

$$d(u_1 \wedge u_2) = h(u_2)d_t(u_1).$$

Rewriting this identity, we obtain $d(u_1 \wedge (u_2 + h(u_2)t)) = 0$. Since the pair $\{d_t(u_1), d_t(u_2 + h(u_2)t)\} = \{d_t(u_1), d_t(u_2)\}$ generates $L(-B)$, Step 2 implies that the extension (4) splits. □

Remark 3.2. In the statement of Theorem 3.1 it is not necessary to suppose that L is globally generated, since $K_{p,1}(X, L(-\mathrm{Bs}(L))) \cong K_{p,1}(X, L)$.

Theorem 3.1 yields a short, geometric proof of the Green–Lazarsfeld nonvanishing theorem for curves.

Theorem 3.3 (Green–Lazarsfeld). *Let X be a smooth curve, and let L be a line bundle on X that admits a decomposition $L = L_1 \otimes L_2$ with $r_i = \dim |L_i| \geq 1$ for $i = 1, 2$. Then $K_{r_1+r_2-1,1}(X, L) \neq 0$.*

Proof. We define s_1, s_2, t, W, B and $\gamma(W, t)$ as in Section 2.2. Let C be the base locus of W, seen as a subspace of $H^0(X, L(-B))$. We prove that $\gamma(W, t) \neq 0$. Suppose that $\gamma(W, t) = 0$. Consider the extension

$$0 \to \mathcal{O}_X(B) \to L_1 \oplus L_2 \to L(-B) \to 0.$$

Pulling back this extension along the injective homomorphism $L(-B-C) \to L(-B)$, we obtain an induced extension

$$0 \to \mathcal{O}_X(B) \to E \to L(-B-C) \to 0.$$

Applying Theorem 3.1 to the line bundle $L(-C)$, we find that this extension splits. Hence there exists an injective homomorphism

$$\mathcal{O}_X(B) \oplus L(-B-C) \to L_1 \oplus L_2.$$

In particular there exists $i \in \{1, 2\}$ such that $\operatorname{Hom}(L(-B-C), L_i) \neq 0$. This implies that

$$r_i + 1 = h^0(X, L_i) \geq h^0(X, L(-B-C)) \geq \dim W = r_1 + r_2 + 1,$$

and this is impossible since $r_1 \geq 1$ and $r_2 \geq 1$. □

Theorem 3.4. *Let X be a smooth curve, and let $\alpha \neq 0 \in K_{p,1}(X, L)$ be a Koszul class of rank $p+2$ represented by an element of $\bigwedge^p W \otimes H^0(X, L)$ with $\dim W = p+2$. There exist a rank 2 vector bundle E on X, a section $t \in H^0(X, E)$ and a subspace $W \cong U \subset H^0(X, E)$ such that $\alpha = \gamma(U, t)$.*

Proof. Put $T = \mathbb{C} \oplus W$, and choose a basis $\{e_1, \dots, e_{p+3}\}$ of T such that $t = e_1$ is the generator of the first summand. Writing $z_{ij} = e_i \wedge e_j$, we obtain a skew-symmetric matrix $Z = (z_{ij})$ and coordinates $(z_{ij})_{1 \leq i < j \leq p+3}$ on $\mathbb{P}(\bigwedge^2 T^\vee)$. Consider the Grassmannian $G = G(2, T)$ of 2-dimensional quotients of T. The ideal of G under the Plücker embedding $G \subset \mathbb{P}(\bigwedge^2 T^\vee)$ is generated by the 4×4 Pfaffians $\mathrm{Pf}_{ijkl}(Z)$ of the matrix Z. Taking exterior powers in the exact sequence

$$0 \to \langle t \rangle \to T \to W \to 0$$

we obtain an exact sequence

$$0 \to \langle t\rangle \otimes W \to \textstyle\bigwedge^2 T \to \bigwedge^2 W \to 0.$$

The linear subspace $\mathbb{P}(\bigwedge^2 W^\vee) \subset \mathbb{P}(\bigwedge^2 T^\vee)$ is defined by the vanishing of the linear forms z_{1j}, $j = 2, \dots, p+3$. A straightforward computation then shows that the ideal of the union

$$G(2,T) \cup \mathbb{P}(\textstyle\bigwedge^2 W^\vee) \subset \mathbb{P}(\bigwedge^2 T^\vee)$$

is generated by the Pfaffians $\mathrm{Pf}_{1ijk}(Z)$. The tautological exact sequence

$$0 \to S \to T \otimes \mathcal{O}_G \to Q \to 0$$

induces an isomorphism $T \cong H^0(G, Q)$. Under this isomorphism, we have $G(2, W) = V(t)$.

As in Section 2.3 we associate to the Koszul class α a matrix $A = (a_{ij})$ of linear forms such that

(a) the linear forms in the first row of A span W;

(b) there exists a nonzero 4×4 Pfaffian of A involving the first row and column;

(c) the 4×4 Pfaffians involving the first row and column of A vanish on the image of X in $\mathbb{P}H^0(X, L)^\vee$.

Let C be the base locus of the image of A. Replacing L by $L(-C)$ if necessary (W is obviously contained in the image of A) we can suppose that C is empty, hence the matrix A defines a morphism

$$\psi\colon X \to \mathbb{P}(\textstyle\bigwedge^2 T^\vee).$$

Condition (c) implies that the image $Y = \psi(X)$ is contained in the union $G(2,T) \cup \mathbb{P}(\bigwedge^2 W^\vee)$, and condition (a) shows that Y is not contained in $\mathbb{P}(\bigwedge^2 W^\vee)$. As Y is irreducible, this implies that Y is contained in $G(2,T)$.

Put $E = \psi^* Q$. Twisting the exact sequence

$$0 \to \mathcal{I}_Y \to \mathcal{O}_G \to \psi_*\mathcal{O}_X \to 0$$

by the universal quotient bundle Q and taking global sections, we obtain an exact sequence

$$0 \to H^0(G, Q \otimes \mathcal{I}_Y) \to H^0(G, Q) \xrightarrow{\psi^*} H^0(G, \psi_*\mathcal{O}_X \otimes Q) \cong H^0(X, E).$$

Condition (a) implies that Y is not contained in $G(2, W) = G(2,T) \cap \mathbb{P}(\bigwedge^2 W^\vee)$, hence t does not vanish identically on X and defines a global section of E. The zero locus of this section is given by the equations $a_{12} = \dots = a_{1,p+3} = 0$, hence it coincides with the base locus B of the sublinear system of $|L|$ induced by W. Consequently the line bundle E is given by an extension

$$0 \to \mathcal{O}_X(B) \to E \to L(-B) \to 0. \tag{6}$$

Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
H^0(G, \mathcal{O}_G) \arrow[r] \arrow[d, ".t"] & H^0(X, \mathcal{O}_X(B)) \arrow[d, ".\psi^*(t)"] \\
H^0(G, Q) \arrow[r, "\psi^*"] \arrow[d] & H^0(X, E) \arrow[d, "d_t"] \\
W \arrow[r, "i"] & H^0(X, L(-B)).
\end{tikzcd}$$

Note that $\ker i = W \cap H^0(G, \mathcal{O}_G(1) \otimes \mathcal{I}_Y) = 0$ by condition (a). As the map $H^0(G, Q) \to W$ is surjective, we find that W is contained in the image of the map $d_t \colon H^0(X, E) \to H^0(X, L(-B))$. The embedding $W \subset H^0(G, Q) = \langle t \rangle \oplus W$ composed with ψ^* is a section of d_t. Put $U = \psi^*(W)$. By construction we obtain $\gamma = \gamma(U, t)$. □

Remark 3.5. The union $G(2, T) \cup \mathbb{P}(\bigwedge^2 W^\vee)$ is a generic syzygy scheme; see [6, Theorem 6.1]. In [loc. cit., Theorem 6.7] it was shown that a rank $p+2$ syzygy gives rise to a rank 2 vector bundle if L is very ample and the ideal of X is generated by quadrics.

The condition of Theorem 3.1 can be reinterpreted in terms of surjectivity of a natural multiplication map.

Proposition 3.6. *Let X be a smooth curve, and let $W \subset H^0(X, L)$ be a linear subspace. We put $B = \mathrm{Bs}(W)$ and view W as a base-point free linear subspace of $H^0(X, L(-B))$. Let*

$$\mu \colon W \otimes H^0(X, K_X(-B)) \to H^0(K_X \otimes L(-2B))$$

be the multiplication map. The following conditions are equivalent.

(i) *The map μ is not surjective.*

(ii) *There exists a non-split extension $0 \to \mathcal{O}_X(B) \to E \to L(-B) \to 0$ such that W is contained in the kernel of the map $\delta \colon H^0(X, L(-B)) \to H^1(X, \mathcal{O}_X(B))$.*

Proof. We first show that (i) implies (ii). Since μ is not surjective, there exists a hyperplane $H \subset H^0(X, K_X \otimes L(-2B))$ that contains $\mathrm{im}(\mu)$. Let η be a linear functional defining H. Put $0 \neq \xi = \eta^\vee \in H^1(X, L^{-1}(2B))$, and let

$$0 \to \mathcal{O}_X(B) \to E \to L(-B) \to 0$$

be the corresponding non-split extension. Given $w \in W$ and $v \in H^0(X, K_X(-B))$, the formula

$$\delta(w)(v) = (\eta \circ \mu)(w \otimes v) \tag{7}$$

shows that W is contained in the kernel of δ.

For the converse, note that formula (7) implies that $\eta|_{\mathrm{im}\,\mu} \equiv 0$. □

Remark 3.7. If B is a fixed divisor, the result of the previous Proposition follows from Green's duality theorem [4, Corollary (2.c.10)]. Indeed,

$$\operatorname{coker} \mu \cong K_{0,1}(X, K_X(-B), L(-B), W) \cong K_{p,1}(X, B, L(-B), W)^\vee \tag{8}$$

and since $h^0(X, \mathcal{O}_X(B)) = 1$ we have an injection

$$K_{p,1}(X, B, L(-B), W) \hookrightarrow K_{p,1}(X, L).$$

Theorem 3.4 shows that Voisin's method may produce nontrivial Koszul classes that are not contained in the space $K_{p,1}(X, L)_{\mathrm{GL}}$ spanned by Green–Lazarsfeld classes.

Example 3.8. By [2, Theorem 3.6 and Theorem 4.3] there exists a smooth curve of genus 14 and Clifford index 5 whose Clifford index is computed by a unique line bundle L such that $L^2 = K_X$. The line bundle L embeds X in $\mathbb{P}^4$ as a projectively normal curve of degree 13 which is not contained in any quadric of rank ≤ 4, and the ideal of X is generated by the 4×4 Pfaffians of a skew-symmetric matrix $(a_{ij})_{1 \leq i,j \leq 5}$ with

$$\deg(a_{ij}) = \begin{cases} 2 \text{ if } i = 1 \text{ or } j = 1 \\ 1 \text{ if } i \geq 2 \text{ and } j \geq 2 \end{cases}$$

such that the quadric $Q = a_{23}a_{45} - a_{24}a_{35} + a_{25}a_{34}$ has rank 5.

By [loc.cit.] the group $K_{1,1}(X, L)$ is generated by $[Q]$, hence I_X contains no quadrics of rank ≤ 4. If $K_{1,1}(X, L)$ contains a Green–Lazarsfeld class this class would be of scrollar type, since it necessarily comes from two pencils $|L_1|$, $|L_2|$. This is impossible, since classes of scrollar type give rise to quadrics of rank ≤ 4.

The Koszul class $[Q] \in K_{1,1}(X, L)$ has rank 3, since it is represented by the linear subspace $W = \langle a_{23}, a_{24}, a_{25} \rangle$. Hence $[Q]$ comes from Voisin's method by Theorem 3.4.

Remark 3.9. A more geometric description of a subspace W representing $[Q]$ is the following. A smooth intersection of the quadric $V(Q) \subset \mathbb{P}H^0(X, L)^\vee$ with one of the cubic Pfaffians is a $K3$ surface in $\mathbb{P}H^0(X, L)^\vee$ containing a line ℓ which is disjoint from X by [2, Proposition 4.1]. The line ℓ corresponds to a 3-dimensional linear subspace $W \subset H^0(X, L)$, which is base-point-free since ℓ does not meet X.

One could ask whether the syzygies constructed in Section 2.1 span $K_{p,1}(X, L)$. In principle it may be possible to obtain higher rank syzygies as linear combinations of rank $p + 2$ syzygies. However, if $K_{p,1}(X, L)$ is spanned by a single syzygy of rank $\geq p + 3$ this is not possible.

Example 3.10 (Eusen–Schreyer). Eusen and Schreyer [3, Theorem 1.7 (a)] have constructed a smooth curve $X \subset \mathbb{P}^5$ of genus 7 and Clifford index 3 embedded by the linear system $|K_X(-x)|$ such that $K_{2,1}(X, K_X(-x)) \cong \mathbb{C}$ is spanned by a syzygy s_0. The explicit expression for s_0 given on p. 8 of [loc. cit.] shows that s_0 is a rank 5 syzygy. Hence s_0 cannot be obtained by the Green–Lazarsfeld construction or the method of Section 2.1.

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