The M-ellipsoid, symplectic capacities and volume

Shiri Artstein-Avidan, Vitali Milman* and Yaron Ostrover

Abstract. In this work we bring together tools and ideology from two different fields, symplectic geometry and asymptotic geometric analysis, to arrive at some new results. Our main result is a dimension-independent bound for the symplectic capacity of a convex body by its volume radius.

Mathematics Subject Classification (2000). 53D05, 53C15, 46B07, 52A20, 46B20.

Keywords. Symplectic capacities, convex bodies, volume, M-ellipsoid.

1. Introduction and main result

In this work we bring together tools and ideology from two different fields, symplectic geometry and asymptotic geometric analysis, to arrive at some new results. Our main result is a dimension-independent bound for the symplectic capacity of a convex body by its volume radius. This type of inequality was first suggested by C. Viterbo, who conjectured that among all convex bodies in $\mathbb{R}^{2n}$ with a given volume, the Euclidean ball has maximal symplectic capacity. In order to state our results we proceed with a more formal presentation.

Consider the $2n$-dimensional Euclidean space $\mathbb{R}^{2n}$ with the standard linear co-ordinates $(x_1, y_1, \ldots, x_n, y_n)$. One equips this space with the standard symplectic structure $\omega_{\text{st}} = \sum_{j=1}^{n} dx_j \wedge dy_j$, and with the standard inner product $g_{\text{st}} = \langle \cdot, \cdot \rangle$. Note that under the identification between $\mathbb{R}^{2n}$ and $\mathbb{C}^n$ these two structures are the real and the imaginary parts of the standard Hermitian inner product in $\mathbb{C}^n$, and $\omega(v, iv) = \langle v, v \rangle$.

In [20], Viterbo investigated the relation between the classical Riemannian way of measuring the size of sets using the canonical volume and the symplectic way using symplectic capacities. Before we proceed let us recall the definition of symplectic capacities and their basic properties.

Definition 1.1. A symplectic capacity on $(\mathbb{R}^{2n}, \omega_{st})$ associates to each subset $U \subset \mathbb{R}^{2n}$ a number $c(U) \in [0, \infty]$ such that the following three properties hold:

(P1) $c(U) \leq c(V)$ for $U \subseteq V$ (monotonicity);

(P2) $c(\psi(U)) = |\alpha|\, c(U)$ for $\psi \in \mathrm{Diff}(\mathbb{R}^{2n})$ such that $\psi^* \omega_{st} = \alpha\, \omega_{st}$ (conformality);

(P3) $c(B^{2n}(r)) = c(B^2(r) \times \mathbb{C}^{n-1}) = \pi r^2$ (nontriviality and normalization), where $B^{2k}(r)$ is the open $2k$-dimensional ball of radius r.

Note that the third property disqualifies any volume-related invariant, while the first two properties imply that every two sets $U, V \subset \mathbb{R}^{2n}$ such that there exists a symplectomorphism sending U onto V, will have the same capacity. Recall that a *symplectomorphism* of $\mathbb{R}^{2n}$ is a diffeomorphism which preserves the symplectic structure i.e., $\psi \in \mathrm{Diff}(\mathbb{R}^{2n})$ such that $\psi^* \omega_{st} = \omega_{st}$. We will denote by $\mathrm{Symp}(\mathbb{R}^{2n}) = \mathrm{Symp}(\mathbb{R}^{2n}, \omega_{st})$ the group of all symplectomorphisms of $(\mathbb{R}^{2n}, \omega_{st})$.

A priori, it is not clear that symplectic capacities exist. The celebrated non-squeezing theorem of Gromov [5] shows that for $R > r$ the ball $B^{2n}(R)$ does not admit a symplectic embedding into the symplectic cylinder $Z^{2n}(r) := B^2(r) \times \mathbb{C}^{n-1}$. This theorem led to the following definitions:

Definition 1.2. The symplectic radius of a non-empty set $U \subset \mathbb{R}^{2n}$ is

$$c_B(U) := \sup\left\{\pi r^2 \mid \text{there exists } \psi \in \mathrm{Symp}(\mathbb{R}^{2n}) \text{ with } \psi(B^{2n}(r)) \subset U\right\}.$$

The cylindrical capacity of U is

$$c^Z(U) := \inf\left\{\pi r^2 \mid \text{there exists } \psi \in \mathrm{Symp}(\mathbb{R}^{2n}) \text{ with } \psi(U) \subset Z^{2n}(r)\right\}.$$

Note that both the symplectic radius and the cylindrical capacity satisfy the axioms of Definition 1.1 by the non-squeezing theorem. Moreover, it follows from Definition 1.1 that for every symplectic capacity c and every open set $U \subset \mathbb{R}^{2n}$ we have $c_B(U) \leq c(U) \leq c^Z(U)$.

The above axiomatic definition of symplectic capacities is originally due to Ekeland and Hofer [3]. Nowadays, a variety of symplectic capacities can be constructed in different ways. For several of the detailed discussions on symplectic capacities we refer the reader to [2], [7], [8], [10], [12] and [21].

We may now proceed with the description of Viterbo's Conjecture and the previous results leading to this paper. We will be interested in an inequality relating the symplectic capacity of a convex body in $\mathbb{R}^{2n}$ and its volume. As mentioned above, Viterbo [20] conjectured that among all convex bodies in $\mathbb{R}^{2n}$ with a given volume, the symplectic capacity is maximal for the Euclidean ball. More precisely, denoting by $\mathrm{Vol}(K)$ the volume of K and abbreviating B^{2n} for the open Euclidean unit ball in $\mathbb{R}^{2n}$, Viterbo conjectured that:

Conjecture 1.3. For any symplectic capacity c and for any convex body $K \subset \mathbb{R}^{2n}$

$$\frac{c(K)}{c(B^{2n})} \leq \left(\frac{\mathrm{Vol}(K)}{\mathrm{Vol}(B^{2n})}\right)^{1/n},$$

with equality achieved only for symplectic images of the Euclidean ball.

Note that this conjecture trivially holds for the symplectic radius c_B. The first result in the direction of the above conjecture is due to Viterbo [20]. Using linear methods, namely John's ellipsoid, he proved:

Theorem 1.4 (Viterbo). *For a convex body $K \subset \mathbb{R}^{2n}$ and a symplectic capacity c one has*

$$\frac{c(K)}{c(B^{2n})} \leq \gamma_n \left(\frac{\mathrm{Vol}(K)}{\mathrm{Vol}(B^{2n})}\right)^{1/n},$$

where $\gamma_n = 2n$ if K is centrally symmetric and $\gamma_n = 32n$ for general convex bodies.

In [6], Hermann constructed starshaped domains in $\mathbb{R}^{2n}$, for $n > 1$, with arbitrarily small volume and fixed cylindrical capacity. Therefore, in the category of starshaped domains the above theorem with *any* constant γ_n independent of the body K must fail. In addition, Hermann proved the above conjecture for a special class of convex bodies which admit many symmetries, called convex Reinhardt domains (for definitions see [6]).

In [1], the first and third named authors used methods from asymptotic geometric analysis to reduce the order of the above mentioned constant γ_n. They showed:

Theorem 1.5. *There exists a universal constant A_1 such that for every even dimension $2n$, any convex body $K \subset \mathbb{R}^{2n}$, and any symplectic capacity c, one has*

$$\frac{c(K)}{c(B^{2n})} \leq A_1 (\log 2n)^2 \left(\frac{\mathrm{Vol}(K)}{\mathrm{Vol}(B^{2n})}\right)^{1/n}$$

(so, $\gamma_n \leq A_1 (\log 2n)^2$).

They also showed that for many classes of convex bodies, the logarithmic term is not needed. Among these classes are all the ℓ_p^n-balls for $1 \leq p \leq \infty$, all zonoids (bodies that can be approximated by Minkowski sums of segments) and other classes of convex bodies, see [1].

In this work we use some more advanced methods from asymptotic geometric analysis to eliminate the logarithmic factor from the above theorem and prove an upper bound for γ_n that is independent of the dimension. This bound, which is the constant A_0 in Theorem 1.6 below is a universal constant and not difficult to compute

(as was the constant A_1 in Theorem 1.5 above). However, we avoid this computation here, since what is of interest for us is the fact that this constant is independent of the dimension. The question whether it equals 1, i.e., Conjecture 1.3 remains open, and cannot follow from our method. Finding dimension independent estimates is a frequent goal in asymptotic geometric analysis, where surprising phenomena such as concentration of measure (see e.g. [17] and a more recent survey [4]) imply the existence of order and structures in high dimension, despite the huge complexity it involves. It is encouraging to see that such phenomena also exist in symplectic geometry, and although this is just a first example, we hope more will follow.

We emphasize that, as in [1], we work exclusively in the category of linear symplectic geometry. That is, the tools we use are purely linear and the reader should not expect any difficult symplectic analysis. It turns out that even in this limited category of linear symplectic transformations, there are tools which are powerful enough to obtain a dimension independent estimate for γ_n in Theorem 1.4. While this fits with the philosophy of asymptotic geometric analysis, this is less expected from the point of view of symplectic geometry where for strong results one expects to need highly nonlinear objects.

More precisely, let $\mathrm{Sp}(\mathbb{R}^{2n}) = \mathrm{Sp}(\mathbb{R}^{2n}, \omega_{\mathrm{st}})$ denote the group of linear symplectic transformations of $\mathbb{R}^{2n}$. We consider a more restricted notion of linearized cylindrical capacity, which is similar to c^Z but where the transformation ψ is taken only in $\mathrm{Sp}(\mathbb{R}^{2n})$ namely

$$c_{\mathrm{lin}}^Z(U) := \inf \left\{ \pi r^2 \mid \text{there exists } \psi \in \mathrm{Sp}(\mathbb{R}^{2n}) \text{ with } \psi(U) \subset Z^{2n}(r) \right\}.$$

(Note that it is no longer a symplectic capacity.) Of course, it is always true that for every symplectic capacity c we have $c \leq c^Z \leq c_{\mathrm{lin}}^Z$.

Our main result is that for some universal constant A_0 one has $\gamma_n \leq A_0$ for all n. This follows from the following theorem, which we prove in Section 3.

Theorem 1.6. *There exists a universal constant A_0 such that for every even dimension $2n$ and any convex body $K \subset \mathbb{R}^{2n}$ one has*

$$\frac{c_{\mathrm{lin}}^Z(K)}{c(B^{2n})} \leq A_0 \left(\frac{\mathrm{Vol}(K)}{\mathrm{Vol}(B^{2n})} \right)^{1/n}.$$

Notation. In this paper the letters A_0, A_1, A_2, A_3 and C are used to denote universal positive constants which do not depend on the dimension nor on the body involved. We denote by B^{2n} the Euclidean unit ball in $\mathbb{R}^{2n}$. In what follows we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ by associating $z = x + iy$, where $x, y \in \mathbb{R}^n$, to the vector $(x_1, y_1, \ldots, x_n, y_n)$, and consider the standard complex structure given by complex multiplication by i, i.e. $i(x_1, y_1, \ldots, x_n, y_n) = (-y_1, x_1, \ldots, -y_n, x_n)$. We denote by $\langle \cdot, \cdot \rangle$ the standard Euclidean inner product on $\mathbb{R}^{2n}$. We shall denote by $e^{i\theta}$ the standard action of S^1 on $\mathbb{C}^n$

which rotates each coordinate by angle θ, i.e., $e^{i\theta}(z_1, \ldots, z_n) = (e^{i\theta}z_1, \ldots, e^{i\theta}z_n)$. By $x^{\perp}$ we denote the hyperplane orthogonal to x with respect to the Euclidean inner product. For two sets A, B in $\mathbb{R}^{2n}$, we denote their Minkowski sum by $A + B = \{a + b : a \in A, b \in B\}$. By a convex body we shall mean a convex bounded set in $\mathbb{R}^{2n}$ with non-empty interior. Finally, since affine translations in $\mathbb{R}^{2n}$ are symplectic maps, we shall assume throughout the text that any convex body K has the origin in its interior.

Structure of the paper. The paper is organized as follows. In the next section we describe the central ingredient in the proof, called the M-ellipsoid, which is coming from asymptotic geometric analysis. In Section 3 we prove our main result, and in the last section we show an additional result about convex bodies, generalizing a result of Rogers and Shephard.

2. Asymptotic geometric analysis background: M-position

In this section we work in $\mathbb{R}^n$ with the Euclidean structure, without a symplectic or complex structure. We review some well-known theorems from asymptotic geometric analysis which we will use in later sections. In what follows, we use the notion "a position of a body K" to denote the image of the body K under a volume preserving linear transformation. A position of a convex body is in fact equivalent to a choice of a Euclidean structure, or, in other words, a choice of some ellipsoid as the Euclidean unit ball. A fundamental object in asymptotic geometric analysis, which was discovered by the second named author in relation with the reverse Brunn–Minkowski inequality, is a special ellipsoid now called the Milman ellipsoid, abbreviated M-ellipsoid. This ellipsoid has several essentially equivalent definitions, the simplest of which may be the following:

Definition 2.1. An ellipsoid $\mathcal{E}_K$ is called an M-ellipsoid (with constant C) of K if $\text{Vol}(\mathcal{E}_K) = \text{Vol}(K)$ and it satisfies

$$\text{Vol}(K + \mathcal{E}_K)^{1/n} \leq C\,\text{Vol}(K)^{1/n}, \quad \text{and} \quad \text{Vol}(K \cap \mathcal{E}_K)^{1/n} \geq C^{-1}\,\text{Vol}(K)^{1/n}.$$

The fact that there exists a universal C_0 such that every convex body K has an M-ellipsoid (with constant C_0) was proved in [14] for a symmetric body K. The fact that the body K need not be symmetric, for the existence of an M-ellipsoid with the properties which we use in the proof of our main result, was proved in [15] (see

Theorem 1.5 there). A complete extension of all M-ellipsoid properties in the non-symmetric case was performed in [16], where it was shown that the right choice of the origin (translation) in the case of a general convex body is the barycenter (center of mass) of the body.

This ellipsoid was invented in order to study the "reverse Brunn–Minkowski inequality" which is proved in [14], and we begin by recalling this inequality, which we will strongly use in the proof of our main theorem. We then describe some further properties of this ellipsoid. Recall that the classical Brunn–Minkowski inequality states that if A and B are non-empty compact subsets of $\mathbb{R}^n$, then

$$\mathrm{Vol}(A + B)^{1/n} \geq \mathrm{Vol}(A)^{1/n} + \mathrm{Vol}(B)^{1/n}.$$

Although at first sight it seems that one cannot expect any inequality in the reverse direction (consider, for example, two very long and thin ellipsoids pointing in orthogonal directions in $\mathbb{R}^2$), if one allows for an extra choice of "position", a reverse inequality is possible, which we now describe.

It was discovered in [14] that one can reverse the Brunn–Minkowski inequality, up to a universal constant factor, as follows: for every convex body K there exists a linear transformation T_K, which is volume preserving, such that for any two bodies K_1 and K_2, the bodies $T_{K_1} K_1$ and $T_{K_2} K_2$ satisfy an inverse Brunn–Minkowski inequality up to some universal constant. A volume preserving linear image of a convex body is called a "position" of the body. It turns out that the right choice of T_K is such that the ellipsoid $r T_K^{-1} B^n$ (for the right choice of r) is an M-ellipsoid of K, which we denote as before by $\mathcal{E}_K$. We then say that the body $T_K K$ is in M-position (or that it is an M-position of K). Thus, a body is in M-position if a multiple of the Euclidean ball B^n is an M-ellipsoid for K. In particular, if K is in M-position then so is any rotation/reflection of K. We remark that an M-ellipsoid of a body is far from being unique, and a body can have many different such ellipsoids. The important fact is, as mentioned after Definition 2.1, that there is some C_0 such that every convex body has an M-ellipsoid with constant C_0. For a detailed account about M-ellipsoids we refer the readers to [15] and [18], where they will also find proofs of the theorems below. The property of M-position which we use in this paper for the proof of Theorem 1.6 is the following

Theorem 2.2. *There exists a universal constant C such that if $\widetilde{K}_1$, $\widetilde{K}_2 \subset \mathbb{R}^n$ are two convex bodies in M-position then*

$$\mathrm{Vol}(\widetilde{K}_1 + \widetilde{K}_2)^{1/n} \leq C\big(\mathrm{Vol}(\widetilde{K}_1)^{1/n} + \mathrm{Vol}(\widetilde{K}_2)^{1/n}\big). \tag{1}$$

In particular this theorem implies that for a convex body K there exists a transformation T_K, which depends solely on K, such that for any two convex bodies K_1 and K_2, denoting $\widetilde{K}_1 = T_{K_1}(K_1)$, $\widetilde{K}_2 = T_{K_2}(K_2)$, we have that (1) is satisfied. The transformation T_K is the transformation which takes the ellipsoid $\mathcal{E}_K$ to a multiple of B^n.

Therefore, it is clear that any composition of T_K with an orthogonal transformation from the left will also satisfy this property.

This ellipsoid $\mathcal{E}_K$ has many more well-known intriguing properties. We recall one of them, which we will use in Section 4:

Theorem 2.3. *There exists a universal constant C such that for any convex body K, the ellipsoid $\mathcal{E}_K$ satisfies the following: for every convex body P one has that*

$$C^{-1} \operatorname{Vol}(P + \mathcal{E}_K)^{1/n} \leq \operatorname{Vol}(P + K)^{1/n} \leq C \operatorname{Vol}(P + \mathcal{E}_K)^{1/n}. \tag{2}$$

3. Proof of the main result

We return to $\mathbb{R}^{2n}$ equipped with the standard symplectic structure and the standard Euclidean inner product. We first present the main ingredient needed for the proof of the main theorem. Using the notion of M-position, we show that every convex body K has a linear symplectic image $K' = SK$ such that the couple K' and iK' satisfy the inverse Brunn–Minkowski inequality. For this we need to recall a well-known fact about the relation between a symplectic form and a positive definite quadratic form. The following theorem by Williamson [22] (see also [8] and [13]) concerns simultaneous normalization of a symplectic form and an inner product.

Williamson's theorem. *For any positive definite symmetric matrix A there exists an element $S \in \operatorname{Sp}(2n)$ and a diagonal matrix with positive entries D with the property $iD = Di$ (complex linear), such that $A = S^T D S$.*

An immediate corollary (for a proof see [1]) is

Corollary 3.1. *Let T be a volume preserving $2n$-dimensional real matrix. Then there exists a linear symplectic matrix $S \in \operatorname{Sp}(\mathbb{R}^{2n})$, an orthogonal transformation $W \in O(2n)$ and a diagonal complex linear matrix D with positive entries such that*

$$T = WDS.$$

This decomposition, together with Theorem 2.2, implies the following (in the sequel we will only use the special case $\theta = \pi/2$, i.e., multiplication by i)

Theorem 3.2. *Every convex body K in $\mathbb{R}^{2n}$ has a symplectic image $K' = SK$, where $S \in \operatorname{Sp}(2n)$, such that for any $0 \leq \theta \leq 2\pi$*

$$\operatorname{Vol}(K)^{1/2n} \leq \operatorname{Vol}(K' + e^{i\theta} K')^{1/2n} \leq A_2 \operatorname{Vol}(K)^{1/2n},$$

where A_2 is a universal constant.

Proof. The first inequality holds trivially for any $K' = SK$ since $K' \subset K' + e^{i\theta}K'$ and S is volume preserving. Next, let K be a convex body in $\mathbb{R}^{2n}$. Set $K_1 = TK$, where T is a volume-preserving linear transformation which takes the body K to an M-position. It follows from Corollary 3.1 that $T = WDS$ where W is orthogonal, S is symplectic, and D is a complex linear transformation. We set $K' = SK$. The remark after Theorem 2.2 implies that we can assume $K_1 = DSK$ where D and S are as above, since an orthogonal image of a body in M-position is also in M-position. Note that the rotated body $e^{i\theta}K_1$ is in M-position as well, since multiplication by a complex number of module 1 is a unitary transformation. Next, it follows from Theorem 2.2 that

$$\mathrm{Vol}(K_1 + e^{i\theta}K_1)^{1/2n} \leq C\left(\mathrm{Vol}(K_1)^{1/2n} + \mathrm{Vol}(e^{i\theta}K_1)^{1/2n}\right) = 2C\,\mathrm{Vol}(K)^{1/2n},$$

where $C > 0$ is a universal constant. Since D is complex linear it commutes with multiplication by $e^{i\theta}$, and using also the fact that it is volume preserving we conclude that

$$\mathrm{Vol}(K' + e^{i\theta}K')^{1/2n} = \mathrm{Vol}(K_1 + e^{i\theta}K_1)^{1/2n} \leq 2C\,\mathrm{Vol}(K)^{1/2n}.$$

The proof is now complete. $\qquad\square$

In order to complete the proof of the main theorem, we shall need two more ingredients. The first is the following easy observation

Lemma 3.3. *Let K be a symmetric convex body satisfying $K = iK$, and let $rB^{2n} \subset K$ be the largest multiple of the Euclidean ball contained in K. Then*

$$c_{\mathrm{lin}}^Z(K) \leq 2\pi r^2.$$

Proof. Since the body K is assumed to be symmetric there are at least two contact points x and $-x$ which belong to ∂K, the boundary of K, and to rS^{2n-1}, the boundary of rB^{2n}. Note that the supporting hyperplanes to K at these points must be $\pm x + x^{\perp}$ since they are also supporting hyperplanes of rB^{2n} at the tangency points. Thus, the body K lies between the hyperplanes $-x + x^{\perp}$ and $x + x^{\perp}$. However, since K is invariant under multiplication by i, the points $\pm ix$ are contact points for ∂K and rS^{2n-1} as well. Thus, the body K lies also between $-ix + ix^{\perp}$ and $ix + ix^{\perp}$. Note that the length of the vectors x and ix is r. We conclude that the projection of K onto the plane spanned by x and ix is contained in a square of edge length $2r$, which in turn is contained in a disc of radius $\sqrt{2}r$. Therefore K is contained in a cylinder of radius $\sqrt{2}r$ with base spanned by x and ix. Since this cylinder is a unitary image of the standard symplectic cylinder $Z^{2n}(\sqrt{2}r)$, the lemma follows. $\qquad\square$

Remark. The factor 2π above can be replaced by 4 if we replace c_{lin}^Z by c^Z. For this we need only to take a small step out of the linear category and use a non-linear symplectomorphism which is essentially two-dimensional.

The last tool we need is a famous result of Rogers and Shephard [19]. This result, which we generalize in some sense in Section 4 below, states that for a convex body $K \subset \mathbb{R}^n$ the volume of the so called "difference body" $K - K$ is not much larger than the volume of the original body. They showed that one has

$$\mathrm{Vol}(K - K) \leq 4^n \, \mathrm{Vol}(K). \tag{3}$$

We are now in a position to prove our main result:

Proof of Theorem 1.6. Let K be a convex body in $\mathbb{R}^{2n}$ and set $K_1 = K - K$. Note that K_1 is symmetric and by (3) we have $\mathrm{Vol}(K_1) \leq 4^{2n} \, \mathrm{Vol}(K)$. It follows from Theorem 3.2 that there exists a symplectic map $S \in \mathrm{Sp}(\mathbb{R}^{2n})$ for which $\mathrm{Vol}(SK_1 + iSK_1) \leq A_2^{2n} \, \mathrm{Vol}(K_1)$. Denote $K_2 = SK_1$, $K_3 = K_2 + iK_2$. Thus $\mathrm{Vol}(K_2) = \mathrm{Vol}(K_1)$ and $\mathrm{Vol}(K_3) \leq A_2^{2n} \, \mathrm{Vol}(K_2)$. Let $r > 0$ be the largest radius such that $rB^{2n} \subset K_3$. We thus have

$$r^{2n} \, \mathrm{Vol}(B^{2n}) \leq \mathrm{Vol}(K_3) \leq A_2^{2n} \, \mathrm{Vol}(K_2) = A_2^{2n} \, \mathrm{Vol}(K_1) \leq (4A_2)^{2n} \, \mathrm{Vol}(K).$$

On the other hand, since $K_3 = iK_3$, it follows from the monotonicity property of symplectic capacities and from Lemma 3.3 that

$$c_{\mathrm{lin}}^Z(K) \leq c_{\mathrm{lin}}^Z(K_1) = c_{\mathrm{lin}}^Z(K_2) \leq c_{\mathrm{lin}}^Z(K_3) \leq 2\pi r^2.$$

Joining these two together we conclude

$$\frac{c_{\mathrm{lin}}^Z(K)}{c(B^{2n})} \leq 2(4A_2)^2 \left(\frac{\mathrm{Vol}(K)}{\mathrm{Vol}(B^{2n})} \right)^{1/n},$$

and the proof of the theorem is complete. $\qquad\square$

4. Generalized Rogers–Shephard

In this section we again work in $\mathbb{R}^n$ equipped only with the Euclidean structure. The above type of reasoning led us to the following simple generalization of the theorem of Rogers and Shephard (3) above. In this generalization, instead of considering the Minkowski sum and the Minkowski difference of a body and itself, we consider the sum and the difference of two different bodies, and show with the use of M-ellipsoid that both have the same volume radius up to a universal constant.

Theorem 4.1. *There exists a universal constant A_3 such that for any two convex bodies $A, B \subset \mathbb{R}^n$ one has*

$$\mathrm{Vol}(A + B)^{1/n} \leq A_3 \, \mathrm{Vol}(A - B)^{1/n}.$$

Proof. In the case where one of the bodies is centrally symmetric the statement is trivial. In the case where both of them are not symmetric, we will use the property of the M-ellipsoid described in Theorem 2.3 above. Let $\mathcal{E}_B$ be the M-ellipsoid of B, which is of course centrally symmetric. We see that

$$\mathrm{Vol}(A+B)^{1/n} \leq C\,\mathrm{Vol}(A+\mathcal{E}_B)^{1/n} = C\,\mathrm{Vol}(A-\mathcal{E}_B)^{1/n} \leq C^2\,\mathrm{Vol}(A-B)^{1/n}. \quad \square$$

We remark that the constant in (3) is equal to 2 (if we put it in the setting of the theorem above) whereas the constant A_3 in the theorem above, although universal, may a-priori be much worse. However, A. Litvak recently found a simpler argument for Theorem 4.1 which gives the constant $A_3 = 4$.

References

[1] S. Artstein-Avidan, Y. Ostrover, On Symplectic Capacities and Volume Radius. Preprint, 2006; math.SG/0603411. arXiv:math.SG/0603411

[2] K. Cieliebak, H. Hofer, J. Latschev, F. Schlenk, Quantitative symplectic geometry. Preprint, 2005; arXiv:math.SG/0506191

[3] I. Ekeland, H. Hofer, Symplectic topology and Hamiltonian dynamics. *Math. Z.* **200** (3) (1989), 355–378. Zbl 0641.53035 MR 0978597

[4] A. A. Giannopoulos, V. D. Milman, Euclidean structure in finite dimensional normed spaces. In *Handbook of the geometry of Banach spaces*, Vol. I, North-Holland, Amsterdam 2001, 707–779.

[5] M. Gromov, Pseudoholomorphic curves in symplectic manifolds. *Invent. Math.* **82** (2) (1985), 307–347. Zbl 0592.53025 MR 0809718

[6] D. Hermann, Non-equivalence of symplectic capacities for open sets with restricted contact type boundary. Prépublication d'Orsay numéro 32 (29/4/1998). http://www.math.u-psud.fr/ biblio/pub/1998/abs/ppo1998_32.html

[7] H. Hofer, Symplectic capacities. In *Geometry of low-dimensional manifolds. 2* (Durham, 1989), London Math. Soc. Lecture Note Ser. 151, Cambridge University Press, Cambridge 1990, 15–34. Zbl 0729.58026 MR 1171906

[8] H. Hofer, E. Zehnder, *Symplectic Invariants and Hamiltonian Dynamics*. Birkhäuser Adv. Texts/Basler Lehrbücher, Birkhäuser, Basel 1994. Zbl 0805.58003 MR 1306732

[9] F. John, Extremum problems with inequalities as subsidiary conditions. In *Courant Anniversary Volume*, Interscience, New York 1948, 187–204. Zbl 0034.10503 MR 0030135

[10] F. Lalonde, Energy and capacities in symplectic topology. In *Geometric topology* (Athens, GA, 1993), , AMS/IP Stud. Adv. Math. 2.1, Amer. Math. Soc., Providence, RI, 1997, 328–374. Zbl 0939.53002 MR 1470736

[11] F. Lalonde, D. McDuff, The geometry of symplectic energy. *Ann. of Math.* **141** (1995), 349–371. Zbl 0829.53025 MR 1324138

[12] D. McDuff, Symplectic topology and capacities. In *Prospects in mathematics* (Princeton, NJ, 1996), Amer. Math. Soc., Providence, RI, 1999, 69–81. Zbl 0947.53043 MR 1660473

[13] D. McDuff, D. Salamon, *Introduction to Symplectic Topology*. 2nd edition, Oxford Math. Monogr., Oxford University Press, Oxford 1998. Zbl 1066.53137 MR 1698616

[14] V. D. Milman, Inégalité de Brunn-Minkowski inverse et applications à la théorie locale des espaces normés. *C. R. Acad. Sci. Paris Sér. I Math.* **302** (1) (1986), 25–28. Zbl 0604.52003 MR 0827101

[15] V. D. Milman, Isomorphic symmetrizations and geometric inequalities. In *Geometric aspects of functional analysis* (1986/87), Lecture Notes in Math. 1317, Springer-Verlag, Berlin 1988, 107–131. Zbl 0652.52008 MR 0950978

[16] V. D. Milman, A. Pajor, Entropy and asymptotic geometry of non-symmetric convex bodies. *Adv. Math.* **152** (2) (2000), 314–335. Zbl 0974.52004 MR 1764107

[17] V. D. Milman, G. Schechtman, *Asymptotic theory of finite dimensional normed spaces.* Lecture Notes in Math. 1200, Springer-Verlag, Berlin 1986. Zbl 0606.46013 MR 0856576

[18] G. Pisier, *The volume of convex bodies and Banach space geometry.* Cambridge Tracts in Math. 94, Cambridge University Press, Cambridge 1989. Zbl 0698.46008 MR 1036275

[19] C. A. Rogers, C. Shephard, The difference body of a convex body. *Arch. Math.* **8** (1957), 220–233. Zbl 0082.15703 MR 0092172

[20] C. Viterbo, Metric and isoperimetric problems in symplectic geometry. *J. Amer. Math. Soc.* (2) 13 (2000), 411–431. Zbl 0964.53050 MR 1750956

[21] C. Viterbo, Capacités symplectiques et applications (d'après Ekeland-Hofer, Gromov). Séminaire Bourbaki, Vol. 1988/89; *Astérisque* **177–178** (1989), 345–362. Zbl 0698.53019 MR 1040580

[22] J. Williamson, On the algebraic problem concerning the normal forms of linear dynamical systems. *Amer. J. Math.* **58** (1963), 141–163. Zbl 0013.28401 MR 1507138

Shiri Artstein-Avidan, Vitali Milman, School of Mathematical Science, Tel Aviv University, Ramat Aviv, Tel Aviv, 69978, Israel

E-mail: shiri@post.tau.ac.il; milman@post.tau.ac.il

Yaron Ostrover, Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, U.S.A.

E-mail: ostrover@math.mit.edu