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Autor(en): **Kucharz, Wojciech**

Objektyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **84 (2009)**

PDF erstellt am: **10.08.2024**

Persistenter Link: <https://doi.org/10.5169/seals-99112>

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Transcendental submanifolds of projective space

Wojciech Kucharz*

Abstract. Given integers m and c satisfying $m-2 \geq c \geq 2$, we explicitly construct a nonsingular m -dimensional algebraic subset of $\mathbb{P}^{m+c}(\mathbb{R})$ that is not isotopic to the set of real points of any nonsingular complex algebraic subset of $\mathbb{P}^{m+c}(\mathbb{C})$ defined over \mathbb{R} . The first examples of this type were obtained by Akbulut and King in a more complicated and nonconstructive way, and only for certain large integers m and c .

Mathematics Subject Classification (2000). 57R55, 14P25.

Keywords. Smooth manifold, algebraic set, isotopy.

1. Introduction

Denote by $\mathbb{P}^n(\mathbb{R})$ and $\mathbb{P}^n(\mathbb{C})$ real and complex projective n -spaces. We regard $\mathbb{P}^n(\mathbb{R})$ as a subset of $\mathbb{P}^n(\mathbb{C})$. A smooth (of class \mathcal{C}^∞) submanifold M of $\mathbb{P}^n(\mathbb{R})$ is said to be of *algebraic type* if it is isotopic in $\mathbb{P}^n(\mathbb{R})$ to the set of real points of a nonsingular complex algebraic subset of $\mathbb{P}^n(\mathbb{C})$ defined over \mathbb{R} ; otherwise M is said to be *transcendental*. It is not at all obvious that transcendental submanifolds exist. However, Akbulut and King [2] proved the existence of transcendental submanifolds M of $\mathbb{P}^n(\mathbb{R})$ which can even be realized as nonsingular algebraic subsets of $\mathbb{P}^n(\mathbb{R})$. Their examples are obtained in a nonconstructive way, by a method which requires both $m = \dim M$ and $n - m$ to be large integers satisfying $2m - n \geq 2$. In the present paper we explicitly construct such examples, assuming only $n - m \geq 2$ and $2m - n \geq 2$. Moreover, we verify that M is a transcendental submanifold of $\mathbb{P}^n(\mathbb{R})$ using only the Barth–Larsen theorem [6, Corollary 6.5] and completely avoiding all results of [1], [2]. More precisely, denote by S^k the unit k -sphere,

$$S^k = \{(y_1, \dots, y_{k+1}) \in \mathbb{R}^{k+1} \mid y_1^2 + \dots + y_{k+1}^2 = 1\}.$$

In Section 3 we prove the following:

*The paper was completed at the Max-Planck-Institut für Mathematik in Bonn, whose support and hospitality are gratefully acknowledged.

Theorem 1.1. *Let m and n be positive integers satisfying $n - m \geq 2$ and $2m - n \geq 2$. Let*

$$\varphi: \mathbb{P}^2(\mathbb{R}) \times S^{m-2} \longrightarrow \mathbb{P}^n(\mathbb{R})$$

be defined by

$$\begin{aligned} & \varphi((x_1 : x_2 : x_3), (y_1, \dots, y_{m-1})) \\ &= (x_1^2 + x_2^2 + x_3^2 : x_1x_2 : x_1x_3 : x_2x_3 : \sigma y_1 : \dots : \sigma y_{m-1} : 0 : \dots : 0), \end{aligned}$$

where 0 is repeated $n - m - 2$ times and $\sigma = x_1^2 + 2x_2^2 + 3x_3^2$. Then:

- (i) *The image $M = \varphi(\mathbb{P}^2(\mathbb{R}) \times S^{m-2})$ is an m -dimensional nonsingular algebraic subset of $\mathbb{P}^n(\mathbb{R})$.*
- (ii) *$\varphi: \mathbb{P}^2(\mathbb{R}) \times S^{m-2} \rightarrow M$ is a biregular isomorphism.*
- (iii) *M is a transcendental submanifold of $\mathbb{P}^n(\mathbb{R})$.*

It follows directly from Theorem 1.1 that for any integers m and c satisfying $m - 2 \geq c \geq 2$, there is a nonsingular algebraic set M in $\mathbb{P}^{m+c}(\mathbb{R})$ such that $\dim M = m$ and M is a transcendental submanifold. In particular, there are transcendental submanifolds of arbitrary dimension $m \geq 4$. The existence of transcendental submanifolds of dimension 2 or 3 remains unsettled at this time. There are no transcendental submanifolds of dimension 1 or of codimension 1. The last assertion is a special case of the following well known fact.

Remark 1.2. Let M be a smooth m -dimensional submanifold of $\mathbb{P}^n(\mathbb{R})$. If either $n - m = 1$ or $2m + 1 \leq n$, then there exists a smooth embedding $e: M \rightarrow \mathbb{P}^n(\mathbb{R})$, arbitrarily close in the \mathcal{C}^∞ topology to the inclusion map $M \hookrightarrow \mathbb{P}^n(\mathbb{R})$, such that $e(M)$ is the set of real points of a nonsingular complex algebraic subset of $\mathbb{P}^n(\mathbb{C})$ defined over \mathbb{R} .

If $n - m = 1$, the claim is explicitly established for example in [3, Theorem 7.1].

For the second case, consider $\mathbb{P}^n(\mathbb{R})$ as a subset of $\mathbb{P}^k(\mathbb{R})$, where k is a large integer. By [8], there exists a smooth embedding $j: M \rightarrow \mathbb{P}^k(\mathbb{R})$, arbitrarily close in the \mathcal{C}^∞ topology to the inclusion map $M \hookrightarrow \mathbb{P}^k(\mathbb{R})$, such that $j(M)$ is a nonsingular algebraic subset of $\mathbb{P}^k(\mathbb{R})$. Increasing k if necessary and making use of Hironaka's resolution of singularities theorem [7], we may assume that the Zariski complex closure of $j(M)$ in $\mathbb{P}^k(\mathbb{C})$ is nonsingular. If $2m + 1 \leq n$, we obtain an embedding $e: M \rightarrow \mathbb{P}^n(\mathbb{R})$ with the required properties by composing j with an appropriate generic projection onto $\mathbb{P}^n(\mathbb{R})$.

2. A criterion for transcendence

First we need some results related to the Picard group. Following the current custom, we state them in the language of schemes.

Let V be a smooth projective scheme over \mathbb{R} . Assume that the set $V(\mathbb{R})$ of \mathbb{R} -rational points of V is nonempty. We regard $V(\mathbb{R})$ as a compact smooth manifold. Every invertible sheaf \mathcal{L} on V determines a real line bundle on $V(\mathbb{R})$, denoted $\mathcal{L}(\mathbb{R})$. The correspondence which assigns to each invertible sheaf \mathcal{L} on V the first Stiefel–Whitney class $w_1(\mathcal{L}(\mathbb{R}))$ of $\mathcal{L}(\mathbb{R})$ gives rise to a canonical homomorphism

$$w_1 : \text{Pic}(V) \longrightarrow H^1(V(\mathbb{R}), \mathbb{Z}/2),$$

defined on the Picard group $\text{Pic}(V)$ of isomorphism classes of invertible sheaves on V . We set

$$H_{\text{alg}}^1(V(\mathbb{R}), \mathbb{Z}/2) = w_1(\text{Pic}(V)).$$

It will be convenient to recall another description of $\text{Pic}(V)$. Consider the scheme $V_{\mathbb{C}} = V \times_{\mathbb{R}} \mathbb{C}$ over \mathbb{C} and its Picard group $\text{Pic}(V_{\mathbb{C}})$. The Galois group $G = \text{Gal}(\mathbb{C}/\mathbb{R})$ of \mathbb{C} over \mathbb{R} acts on $\text{Pic}(V_{\mathbb{C}})$. We denote by $\text{Pic}(V_{\mathbb{C}})^G$ the subgroup of $\text{Pic}(V_{\mathbb{C}})$ consisting of the elements fixed by G . Given an invertible sheaf \mathcal{L} on V , we write $\mathcal{L}_{\mathbb{C}}$ for the corresponding sheaf on $V_{\mathbb{C}}$. The correspondence $\mathcal{L} \rightarrow \mathcal{L}_{\mathbb{C}}$ defines a canonical group homomorphism

$$\alpha : \text{Pic}(V) \longrightarrow \text{Pic}(V_{\mathbb{C}})^G.$$

It follows from the general theory of descent [4] that α is an isomorphism (a simple treatment of the case under consideration can also be found in [5]).

As usual, we set $\mathbb{P}_{\mathbb{R}}^n = \text{Proj}(\mathbb{R}[T_0, \dots, T_n])$ and identify $\mathbb{P}_{\mathbb{R}}^n(\mathbb{R})$ with $\mathbb{P}^n(\mathbb{R})$. Thus if V is a subscheme of $\mathbb{P}_{\mathbb{R}}^n$, then $V(\mathbb{R})$ is a subset of $\mathbb{P}^n(\mathbb{R})$.

Proposition 2.1. *Let V be a closed smooth m -dimensional subscheme of $\mathbb{P}_{\mathbb{R}}^n$. If $2m - n \geq 2$, then*

$$H_{\text{alg}}^1(V(\mathbb{R}), \mathbb{Z}/2) = i^*(H^1(\mathbb{P}^n(\mathbb{R}), \mathbb{Z}/2)),$$

where $i : V(\mathbb{R}) \hookrightarrow \mathbb{P}^n(\mathbb{R})$ is the inclusion map.

Proof. Let $j : V \hookrightarrow \mathbb{P}_{\mathbb{R}}^n$ and $j_{\mathbb{C}} : V_{\mathbb{C}} \hookrightarrow \mathbb{P}_{\mathbb{C}}^n = \mathbb{P}_{\mathbb{R}}^n \times_{\mathbb{R}} \mathbb{C}$ be the inclusion morphisms. By the Barth–Larsen theorem [6, Corollary 6.5], the induced homomorphism

$$j_{\mathbb{C}}^* : \text{Pic}(\mathbb{P}_{\mathbb{C}}^n) \longrightarrow \text{Pic}(V_{\mathbb{C}})$$

is an isomorphism. Since $j_{\mathbb{C}}^*$ is G -equivariant, the restriction

$$j_{\mathbb{C}}^* : \text{Pic}(\mathbb{P}_{\mathbb{C}}^n)^G \longrightarrow \text{Pic}(V_{\mathbb{C}})^G$$

is an isomorphism. We have the following commutative diagram:

$$\begin{array}{ccc}
 \text{Pic}(\mathbb{P}_{\mathbb{C}}^n)^G & \xrightarrow{j_{\mathbb{C}}^*} & \text{Pic}(V_{\mathbb{C}})^G \\
 \alpha \uparrow & & \uparrow \alpha \\
 \text{Pic}(\mathbb{P}_{\mathbb{R}}^n) & \xrightarrow{j^*} & \text{Pic}(V) \\
 w_1 \downarrow & & \downarrow w_1 \\
 H^1(\mathbb{P}^n(\mathbb{R}), \mathbb{Z}/2) & \xrightarrow{i^*} & H^1(V(\mathbb{R}), \mathbb{Z}/2).
 \end{array}$$

Since the homomorphisms α are isomorphisms and

$$H^1(\mathbb{P}^n(\mathbb{R}), \mathbb{Z}/2) = H_{\text{alg}}^1(\mathbb{P}^n(\mathbb{R}), \mathbb{Z}/2),$$

it follows that

$$H_{\text{alg}}^1(V(\mathbb{R}), \mathbb{Z}/2) = i^*(H^1(\mathbb{P}^n(\mathbb{R}), \mathbb{Z}/2)),$$

as required. \square

Note that a smooth submanifold of $\mathbb{P}^n(\mathbb{R})$ is of algebraic type if and only if it is isotopic in $\mathbb{P}^n(\mathbb{R})$ to $V(\mathbb{R})$ for some closed smooth subscheme V of $\mathbb{P}_{\mathbb{R}}^n$. Hence Proposition 2.1 yields the following criterion for transcendence.

Proposition 2.2. *Let M be a compact smooth m -dimensional submanifold of $\mathbb{P}^n(\mathbb{R})$. Assume that the inclusion map $e: M \hookrightarrow \mathbb{P}^n(\mathbb{R})$ induces a trivial homomorphism*

$$e^*: H^1(\mathbb{P}^n(\mathbb{R}), \mathbb{Z}/2) \longrightarrow H^1(M, \mathbb{Z}/2),$$

that is, $e^ = 0$. If M is nonorientable and $2m - n \geq 2$, then M is a transcendental submanifold of $\mathbb{P}^n(\mathbb{R})$.*

Proof. Suppose to the contrary that M is of algebraic type. Let V be a closed smooth subscheme of $\mathbb{P}_{\mathbb{R}}^n$ with $V(\mathbb{R})$ isotopic to M in $\mathbb{P}^n(\mathbb{R})$. Then the homomorphism

$$i^*: H^1(\mathbb{P}^n(\mathbb{R}), \mathbb{Z}/2) \longrightarrow H^1(V(\mathbb{R}), \mathbb{Z}/2),$$

induced by the inclusion map $i: V(\mathbb{R}) \hookrightarrow \mathbb{P}^n(\mathbb{R})$, is trivial. Since $\dim V = m$ and $2m - n \geq 2$, Proposition 2.1 implies

$$H_{\text{alg}}^1(V(\mathbb{R}), \mathbb{Z}/2) = 0.$$

On the other hand, the first Stiefel–Whitney class $w_1(V(\mathbb{R}))$ of $V(\mathbb{R})$ is nonzero, $V(\mathbb{R})$ being a nonorientable manifold. Moreover, $w_1(V(\mathbb{R})) = w_1(\mathcal{K}(\mathbb{R}))$, where \mathcal{K} is the canonical invertible sheaf of V , and hence, $w_1(V(\mathbb{R}))$ is in $H_{\text{alg}}^1(V(\mathbb{R}), \mathbb{Z}/2)$. In view of this contradiction, the proof is complete. \square

3. Transcendental submanifolds

We begin with some preliminary observations. Identify \mathbb{R}^n with its image under the map

$$\mathbb{R}^n \longrightarrow \mathbb{P}^n(\mathbb{R}), \quad (x_1, \dots, x_n) \longmapsto (1 : x_1 : \dots : x_n);$$

thus $\mathbb{R}^n \subset \mathbb{P}^n(\mathbb{R})$. An algebraic subset X of \mathbb{R}^n is said to be *projectively closed* if X is also an algebraic subset of $\mathbb{P}^n(\mathbb{R})$. One readily checks that X is projectively closed if and only if it can be defined by a real polynomial equation

$$f(x_1, \dots, x_n) = 0,$$

where the homogeneous form of top degree in f vanishes only at 0 in \mathbb{R}^n .

Lemma 3.1. *Let X be an algebraic subset of \mathbb{R}^k contained in the open half-space*

$$H = \{(x_1, \dots, x_k) \in \mathbb{R}^k \mid x_k > 0\}.$$

Then the map $\psi : X \times S^\ell \rightarrow \mathbb{R}^{k+\ell}$ defined by

$$\psi((x_1, \dots, x_k), (y_1, \dots, y_{\ell+1})) = (x_1, \dots, x_{k-1}, x_k y_1, \dots, x_k y_{\ell+1})$$

is an algebraic embedding, that is, the image $Y = \psi(X \times S^\ell)$ is an algebraic subset of $\mathbb{R}^{k+\ell}$ and $\psi : X \times S^\ell \rightarrow Y$ is a biregular isomorphism. Moreover, if X is projectively closed in \mathbb{R}^k , then Y is projectively closed in $\mathbb{R}^{k+\ell}$.

Proof. Let

$$f(u, v) = 0$$

be a real polynomial equation defining X , where $u = (x_1, \dots, x_{k-1})$ and $v = x_k$. Since

$$X \subset H, \tag{1}$$

the subset Y of $\mathbb{R}^{k+\ell}$ is defined by the equation

$$f(u, \rho) = 0, \tag{2}$$

where

$$\rho = (x_k^2 + x_{k+1}^2 + \dots + x_{k+\ell}^2)^{\frac{1}{2}}.$$

We will now show that (2) can be replaced by a polynomial equation in $x_1, \dots, x_{k-1}, x_k, \dots, x_{k+\ell}$. To this end we write

$$f(u, v) = g(u, v^2) + v h(u, v^2), \tag{3}$$

where g and h are real polynomials in (u, v) . Then (2) is equivalent to

$$g(u, \rho^2) + \rho h(u, \rho^2) = 0, \quad (4)$$

and in view of (1) also to

$$(g(u, \rho^2))^2 - \rho^2 (h(u, \rho^2))^2 = 0, \quad (5)$$

which is a polynomial equation, as required. Consequently, Y is an algebraic subset of $\mathbb{R}^{k+\ell}$.

It is clear that ψ is injective and $\theta: Y \rightarrow X$,

$$\theta(x_1, \dots, x_{k-1}, x_k, \dots, x_{k+\ell}) = \left(x_1, \dots, x_{k-1}, \frac{x_k}{\rho}, \dots, \frac{x_{k+\ell}}{\rho} \right),$$

is the inverse of $\psi: X \rightarrow Y$. By (4),

$$\rho = -\frac{g(x_1, \dots, x_{k-1}, x_k^2 + \dots + x_{k+\ell}^2)}{h(x_1, \dots, x_{k-1}, x_k^2 + \dots + x_{k+\ell}^2)}$$

for $(x_1, \dots, x_{k-1}, x_k, \dots, x_{k+\ell})$ in Y , and hence θ is a regular map. Thus $\psi: X \rightarrow Y$ is a biregular isomorphism.

Assume now that X is projectively closed in \mathbb{R}^k . We may also assume that the homogeneous form of top degree in f , denoted F , vanishes only at 0 in \mathbb{R}^k . Note that $F(u, \rho^2)F(u, -\rho^2)$ is the homogeneous form of top degree in equation (5). This form vanishes only at 0 in $\mathbb{R}^{k+\ell}$, and hence Y is projectively closed in $\mathbb{R}^{k+\ell}$. \square

Lemma 3.2. *The map $g: \mathbb{P}^2(\mathbb{C}) \rightarrow \mathbb{P}^4(\mathbb{C})$,*

$$g((x_1 : x_2 : x_3)) = (x_1^2 + x_2^2 + x_3^2 : x_1x_2 : x_1x_3 : x_2x_3 : x_1^2 + 2x_2^2 + 3x_3^2),$$

is an algebraic embedding. In particular, the restriction $f: \mathbb{P}^2(\mathbb{R}) \rightarrow \mathbb{P}^4(\mathbb{R})$ of g is an algebraic embedding.

Proof. One readily checks that g is injective. Moreover, the (complex) differential of g at each point of $\mathbb{P}^2(\mathbb{C})$ is of rank 2. It follows that g is an algebraic embedding, and hence f is an algebraic embedding. \square

Proof of Theorem 1.1. Let $f: \mathbb{P}^2(\mathbb{R}) \rightarrow \mathbb{P}^4(\mathbb{R})$ be the algebraic embedding of Lemma 3.2. Note that the image $X = f(\mathbb{P}^2(\mathbb{R}))$ is a projectively closed algebraic subset of $\mathbb{R}^4 \subset \mathbb{P}^4(\mathbb{R})$, contained in the open half-space

$$\{(u_1, u_2, u_3, u_4) \in \mathbb{R}^4 \mid u_4 > 0\}.$$

Let

$$\psi: X \times S^{m-2} \longrightarrow \mathbb{R}^{4+(m-2)} = \mathbb{R}^{m+2} \subset \mathbb{P}^{m+2}(\mathbb{R})$$

be the algebraic embedding of Lemma 3.1 (with $k = 4$ and $\ell = m - 2$). Note that $\psi(X \times S^{m-2})$ is projectively closed in \mathbb{R}^{m+2} , and hence is an algebraic subset of $\mathbb{P}^{m+2}(\mathbb{R})$.

Clearly, if $i : S^{m-2} \rightarrow S^{m-2}$ is the identity map, then

$$f \times i : \mathbb{P}^2(\mathbb{R}) \times S^{m-2} \longrightarrow X \times S^{m-2}$$

is a biregular isomorphism. Denoting by $j : \mathbb{P}^{m+2}(\mathbb{R}) \rightarrow \mathbb{P}^n(\mathbb{R})$ the standard embedding,

$$j((v_0 : \dots : v_{m+2})) = (v_0 : \dots : v_{m+2} : 0 : \dots : 0),$$

we obtain

$$\varphi = j \circ \psi \circ (f \times i),$$

which implies that φ is an algebraic embedding. In other words, conditions (i) and (ii) are satisfied. Moreover, $M \subset \mathbb{R}^n \subseteq \mathbb{P}^n(\mathbb{R})$. Since M is nonorientable and $2m - n \geq 2$, condition (iii) follows from Proposition 2.2. \square

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Received February 28, 2007

Wojciech Kucharz, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87131-0001, U.S.A.

E-mail: kucharz@math.unm.edu