String topology for spheres

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String topology for spheres

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With an appendix by Gerald Gaudens and Luc Menichi

Dedicated to Jean-Claude Thomas, on the occasion of his 60th birthday

Abstract. Let M be a compact oriented d-dimensional smooth manifold. Chas and Sullivan have defined a structure of Batalin–Vilkovisky algebra on $\mathbb{H}_*(LM)$. Extending work of Cohen, Jones and Yan, we compute this Batalin–Vilkovisky algebra structure when M is a sphere S^d, $d \geq 1$. In particular, we show that $\mathbb{H}_*(LS^2;\mathbb{F}_2)$ and the Hochschild cohomology $HH^*(H^*(S^2);H^*(S^2))$ are surprisingly not isomorphic as Batalin–Vilkovisky algebras, although we prove that, as expected, the underlying Gerstenhaber algebras are isomorphic. The proof requires the knowledge of the Batalin–Vilkovisky algebra $H_*(\Omega^2 S^3;\mathbb{F}_2)$ that we compute in the Appendix.

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1. Introduction

Let M be a compact oriented d-dimensional smooth manifold. Denote by $LM := \mathrm{map}(S^1, M)$ the free loop space on M. In 1999, Chas and Sullivan [2] have shown that the shifted free loop homology $\mathbb{H}_*(LM) := H_{*+d}(LM)$ has a structure of Batalin–Vilkovisky algebra (Definition 5). In particular, they showed that $\mathbb{H}_*(LM)$ is a Gerstenhaber algebra (Definition 8). This Batalin–Vilkovisky algebra has been computed when M is a complex Stiefel manifold [25] and very recently over $\mathbb{Q}$ when M is a $K(\pi,1)$ [28]. In this paper, we compute the Batalin–Vilkovisky algebra $\mathbb{H}_*(LM;\mathbb{k})$ when M is a sphere S^n, $n \geq 1$ over any commutative ring $\mathbb{k}$ (Theorems 10, 16, 17, 24 and 25).

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In fact, few calculations of this Batalin–Vilkovisky algebra structure or even of the underlying Gerstenhaber algebra structure have been done because the following conjecture has not yet been proved.

Conjecture 1 (due to [2, "dictionary" p. 5] or [7]?). If M is simply connected then there is an isomorphism of Gerstenhaber algebras $\mathbb{H}_*(LM) \cong HH^*(S^*(M); S^*(M))$ between the free loop space homology and the Hochschild cohomology of the algebra of singular cochains on M.

In [7], [5], Cohen and Jones proved that there is an isomorphism of graded algebras over any field

$$\mathbb{H}_*(LM) \cong HH^*(S^*(M); S^*(M)).$$

Over the reals or over the rationals, two proofs of this isomorphism of graded algebras have been given by Merkulov [23] and Félix, Thomas, Vigué-Poirrier [11]. Motivated by this conjecture, Westerland [30] has computed the Gerstenhaber algebra $HH^*(S^*(M;\mathbb{F}_2); S^*(M;\mathbb{F}_2))$ when M is a sphere or a projective space.

What about the Batalin–Vilkovisky algebra structure?

Suppose that M is formal over a field, then since the Gerstenhaber algebra structure on Hochschild cohomology is preserves by quasi-isomorphism of algebras [10, Theorem 3], we obtain an isomorphism of Gerstenhaber algebras

$$HH^*(S^*(M); S^*(M)) \cong HH^*(H^*(M); H^*(M)). \tag{2}$$

Poincaré duality induces an isomorphism of $H^*(M)$-modules

$$\Theta\colon H^*(M) \to H^*(M)^\vee.$$

Therefore, we obtain the isomorphism

$$HH^*(H^*(M); H^*(M)) \cong HH^*(H^*(M); H^*(M)^\vee)$$

and the Gerstenhaber algebra structure on $HH^*(H^*(M); H^*(M))$ extends to a Batalin–Vilkovisky algebra [26], [22], [19] (See above Proposition 20 for details). This Batalin–Vilkovisky algebra structure is further extended in [27], [9], [20], [21] to a richer algebraic structure. It is natural to conjecture that this Batalin–Vilkovisky algebra on $HH^*(H^*(M); H^*(M))$ is isomorphic to the Batalin–Vilkovisky algebra $\mathbb{H}_*(LM)$. We show (Corollary 30) that this is not the case over $\mathbb{F}_2$ when M is the sphere S^2. See [6, Comments 2, Chapter 1] or the papers of Tradler and Zeinalian [26], [27] for a related conjecture when M is not assumed to be necessarily formal. On the contrary, we prove (Corollary 23) that the above conjecture is satisfied for $M = S^2$ over $\mathbb{F}_2$.

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2. The Batalin–Vilkovisky algebra structure on $\mathbb{H}_*(LM)$

We recall here the definition of the Batalin–Vilkovisky algebra on $\mathbb{H}_*(LM;\Bbbk)$ given by Chas and Sullivan [2] over any commutative ring $\Bbbk$ and deduce that this Batalin–Vilkovisky algebra $\mathbb{H}_*(LM;\Bbbk)$ behaves well with respect to change of rings.

We first recall the definition of the loop product following Cohen and Jones [7], [6]. Let M be a closed oriented smooth manifold of dimension d. The inclusion $e\colon \mathrm{map}(S^1 \vee S^1, M) \hookrightarrow LM \times LM$ can be viewed as a codimension d embedding between infinite dimension manifolds [24, Proposition 5.3]. Denote by ν its normal bundle. Let $\tau_e\colon LM \times LM \twoheadrightarrow \mathrm{map}(S^1 \vee S^1, M)^\nu$ its Thom–Pontryagin collapse map. Recall that the umkehr (Gysin) map $e_!$ is the composite of τ_e and the Thom isomorphism:

$$
\begin{aligned}
H_*(LM \times LM;\Bbbk) &\xrightarrow{H_*(\tau_e;\Bbbk)} H_*(\mathrm{map}(S^1 \vee S^1, M)^\nu;\Bbbk)\\
&\xrightarrow[\cong]{\cap u_\Bbbk} H_{*-d}(\mathrm{map}(S^1 \vee S^1, M);\Bbbk).
\end{aligned}
$$

The Thom isomorphism is given by taking a relative cap product $\cap$ with a Thom class for ν, $u_\Bbbk \in H^d(\mathrm{map}(S^1 \vee S^1, M)^\nu;\Bbbk)$. A Thom class with coefficients in $\mathbb{Z}$, $u_\mathbb{Z}$, gives rise to a Thom class $u_\Bbbk$ with coefficients in $\Bbbk$, under the morphism

$$
H^d(\mathrm{map}(S^1 \vee S^1, M);\mathbb{Z}) \to H^d(\mathrm{map}(S^1 \vee S^1, M);\Bbbk)
$$

induced by the ring homomorphism $\mathbb{Z} \to \Bbbk$ [16, p. 441]. So we have the commutative diagram

$$
\begin{array}{ccc}
H_*(LM \times LM;\mathbb{Z}) & \xrightarrow{e_!} & H_{*-d}(\mathrm{map}(S^1 \vee S^1, M);\mathbb{Z})\\
\downarrow & & \downarrow\\
H_*(LM \times LM;\Bbbk) & \xrightarrow{e_!} & H_{*-d}(\mathrm{map}(S^1 \vee S^1, M);\Bbbk).
\end{array}
$$

Let $\gamma\colon \mathrm{map}(S^1 \vee S^1, M) \to LM$ be the map obtained by composing loops. The loop product is the composite

$$
\begin{aligned}
&H_*(LM;\Bbbk) \otimes H_*(LM;\Bbbk) \to H_*(LM \times LM;\Bbbk)\\
&\quad\xrightarrow{e_!} H_{*-d}(\mathrm{map}(S^1 \vee S^1, M);\Bbbk) \xrightarrow{H_{*-d}(\gamma;\Bbbk)} H_{*-d}(LM;\Bbbk).
\end{aligned}
$$

So clearly, we have proved

Lemma 3. *The morphism of abelian groups $\mathbb{H}_*(LM;\mathbb{Z}) \to \mathbb{H}_*(LM;\Bbbk)$ induced by $\mathbb{Z} \to \Bbbk$ is a morphism of graded rings.*

Suppose that the circle S^1 acts on a topological space X. Then we have an action of the algebra $H_*(S^1)$ on $H_*(X)$,

$$H_*(S^1) \otimes H_*(X) \to H_*(X).$$

Denote by $[S^1]$ the fundamental class of the circle. Then we define an operator of degree 1, $\Delta\colon H_*(X;\mathbb{k}) \to H_{*+1}(X;\mathbb{k})$, which sends x to the image of $[S^1] \otimes x$ under the action. Since $[S^1]^2 = 0$, $\Delta \circ \Delta = 0$. The following lemma is obvious.

Lemma 4. *Let X be a S^1-space. We have the commutative diagram*

$$\begin{array}{ccc} H_*(X;\mathbb{Z}) & \xrightarrow{\Delta} & H_{*+1}(X;\mathbb{Z}) \\ \downarrow & & \downarrow \\ H_*(X;\mathbb{k}) & \xrightarrow{\Delta} & H_{*+1}(X;\mathbb{k}), \end{array}$$

where the vertical maps are induced by the ring homomorphism $\mathbb{Z} \to \mathbb{k}$.

The circle S^1 acts on the free loop space on M by rotating the loops. Therefore we have a operator Δ on $\mathbb{H}_*(LM)$. Chas and Sullivan [2] have shown that $\mathbb{H}_*(LM)$ equipped with the loop product and the Δ-operator, is a Batalin–Vilkovisky algebra.

Definition 5. A *Batalin–Vilkovisky algebra* is a commutative graded algebra A equipped with an operator $\Delta\colon A \to A$ of degree 1 such that $\Delta \circ \Delta = 0$ and

$$\begin{aligned}\Delta(abc) = \Delta(ab)c &+ (-1)^{|a|}a\Delta(bc) + (-1)^{(|a|-1)|b|}b\Delta(ac) \\ &- (\Delta a)bc - (-1)^{|a|}a(\Delta b)c - (-1)^{|a|+|b|}ab(\Delta c).\end{aligned} \tag{6}$$

Consider the bracket $\{\ ,\ \}$ of degree $+1$ defined by

$$\{a,b\} = (-1)^{|a|}\big(\Delta(ab) - (\Delta a)b - (-1)^{|a|}a(\Delta b)\big)$$

for any $a, b \in A$. (6) is equivalent to the following relation called the *Poisson relation*:

$$\{a,bc\} = \{a,b\}c + (-1)^{(|a|+1)|b|}b\{a,c\}. \tag{7}$$

Getzler [14, Proposition 1.2] has shown that $\{\,,\}$ is a Lie bracket and therefore that a Batalin–Vilkovisky algebra is a Gerstenhaber algebra.

Definition 8. A *Gerstenhaber algebra* is a commutative graded algebra A equipped with a linear map $\{-,-\}\colon A \otimes AG \to A$ of degree 1 such that:

a) the bracket $\{-,-\}$ gives to A a structure of a graded Lie algebra of degree 1. This means that for each a, b and $c \in A$,

$$\{a,b\} = -(-1)^{(|a|+1)(|b|+1)}\{b,a\},$$

and

$$\{a,\{b,c\}\} = \{\{a,b\},c\} + (-1)^{(|a|+1)(|b|+1)}\{b,\{a,c\}\}.$$

b) The product and the Lie bracket satisfy the Poisson relation (7).

Using Lemma 3 and Lemma 4, we deduce

Proposition 9. *The $\mathbb{k}$-linear map*

$$\mathbb{H}_*(LM;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{k} \hookrightarrow \mathbb{H}_*(LM;\mathbb{k})$$

is an inclusion of Batalin–Vilkovisky algebras.

In particular, by the universal coefficient theorem,

$$\mathbb{H}_*(LM;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{H}_*(LM;\mathbb{Q}).$$

More generally, this proposition tells us that if $\mathrm{Tor}^{\mathbb{Z}}(\mathbb{H}_*(LM;\mathbb{Z}),\mathbb{k}) = 0$ then the Batalin–Vilkovisky algebra $\mathbb{H}_*(LM;\mathbb{Z})$ determines the Batalin–Vilkovisky algebra $\mathbb{H}_*(LM;\mathbb{k})$.

3. The circle and an useful lemma

In this section, we compute the structure of the Batalin–Vilkovisky algebra on the homology of the free loop space on the circle S^1 using a lemma which gives information on the image of Δ on elements of lower degree in $H_*(LM)$.

Theorem 10. *As a Batalin–Vilkovisky algebra, the homology of the free loop space on the circle is given by*

$$\mathbb{H}_*(LS^1;\mathbb{k}) \cong \mathbb{k}[\mathbb{Z}] \otimes \Lambda a_{-1}.$$

Denote by x a generator of $\mathbb{Z}$. The operator Δ is

$$\Delta(x^i \otimes a_{-1}) = i(x^i \otimes 1), \quad \Delta(x^i \otimes 1) = 0$$

for all $i \in \mathbb{Z}$.

Let X be a pointed topological space. Consider the free loop fibration $\Omega X \overset{j}{\hookrightarrow} LX \overset{\mathrm{ev}}{\twoheadrightarrow} X$. Denote by $\mathrm{hur}_X \colon \pi_n(X) \to H_n(X)$ the Hurewicz map.

Lemma 11. *Let $n \in \mathbb{N}$. Let $f \in \pi_{n+1}(X)$. Denote by $\tilde{f} \in \pi_n(\Omega X)$ the adjoint of f. Then*

$$(H_*(\mathrm{ev}) \circ \Delta \circ H_*(j) \circ \mathrm{hur}_{\Omega X})(\tilde{f}) = \mathrm{hur}_X(f).$$

Proof. Take in homology the image of $[S^1] \otimes [S^n]$ in the following commutative diagram:

$$\begin{array}{ccccc} S^1 \times \Omega X & \xrightarrow{S^1 \times j} & S^1 \times LX & \xrightarrow{\mathrm{act}_{LX}} & LX \\ {\scriptstyle S^1 \times \tilde{f}} \uparrow & & & & \downarrow {\scriptstyle \mathrm{ev}} \\ S^1 \times S^n & \longrightarrow & S^1 \wedge S^n & \xrightarrow[f]{} & X, \end{array}$$

where $\mathrm{act}_{LX} \colon S^1 \times LX \to LX$ is the action of the circle on LX. □

Proof of Theorem 10. More generally, let G be a compact Lie group. Consider the homeomorphism $\Theta_G \colon \Omega G \times G \xrightarrow{\cong} LG$ which sends the couple (w, g) to the free loop $t \mapsto w(t)g$. In fact, Θ_G is an isomorphism of fiberwise monoids. Therefore by [15, Part 2 of Theorem 8.2],

$$\mathbb{H}_*(\Theta_G) \colon H_*(\Omega G) \otimes \mathbb{H}_*(G) \to \mathbb{H}_*(LG)$$

is a morphism of graded algebras. Since $H_*(S^1)$ has no torsion,

$$\mathbb{H}_*(\Theta_{S^1}) \colon H_*(\Omega S^1) \otimes \mathbb{H}_*(S^1) \cong \mathbb{H}_*(LS^1)$$

is an isomorphism of algebras. Since Δ preserves path-connected components,

$$\Delta(x^i \otimes a_{-1}) = \alpha(x^i \otimes 1)$$

where $\alpha \in \mathbb{k}$. Denote by $\varepsilon_{\mathbb{k}[\mathbb{Z}]}$ the canonical augmentation of the group ring $\mathbb{k}[\mathbb{Z}]$. Since $H_*(\mathrm{ev} \circ \Theta_{S^1}) = \varepsilon_{\mathbb{k}[\mathbb{Z}]} \otimes H_*(S^1)$,

$$(H_*(\mathrm{ev}) \circ \Delta)(x^i \otimes a_{-1}) = \alpha 1.$$

On the other hand, applying Lemma 11 to the degree i map $S^1 \to S^1$, we obtain that $(H_*(\mathrm{ev}) \circ \Delta \circ H_*(j))(x^i) = i1$. Therefore $\alpha = i$. □

4. Computations using Hochschild homology

In this section, we compute the Batalin–Vilkovisky algebra $\mathbb{H}_*(LS^n)$, $n \geq 2$, using the following elementary technique:

The algebra structure has been computed by Cohen, Jones and Yan using the Serre spectral sequence [8]. On the other hand, the action of $H_*(S^1)$ on $H_*(LS^n)$ can be computed using Hochschild homology. Using the compatibility between the product and Δ, we determine the Batalin–Vilkovisky algebra $\mathbb{H}_*(LS^n)$ up to isomorphism. This elementary technique will fail for $\mathbb{H}_*(LS^2)$.

Let A be an augmented differential graded algebra. Denote by $s\bar{A}$ the suspension of the augmentation ideal $\bar{A}$, $(s\bar{A})_i = \bar{A}_{i-1}$. Let d_1 be the differential on the tensor product of complexes $A \otimes T(s\bar{A})$. The (normalized) Hochschild chain complex, denoted $\mathcal{C}_*(A; A)$, is the complex $(A \otimes T(s\bar{A}), d_1 + d_2)$ where

$$\begin{aligned} d_2 a[sa_1|\dots|sa_k] =& (-1)^{|a|} aa_1[sa_2|\dots|sa_k] \\ &+ \sum_{i=1}^{k-1} (-1)^{\varepsilon_i} a[sa_1|\dots|sa_i a_{i+1}|\dots|sa_k] \\ &- (-1)^{|sa_k|\varepsilon_{k-1}} a_k a[sa_1|\dots|sa_{k-1}]. \end{aligned}$$

Here $\varepsilon_i = |a| + |sa_1| + \dots + |sa_i|$.

Connes' boundary map B is the map of degree $+1$

$$B\colon A \otimes (s\bar{A})^{\otimes p} \to A \otimes (s\bar{A})^{\otimes p+1}$$

defined by

$$B(a_o[sa_1|\dots|sa_p]) = \sum_{i=0}^{p} (-1)^{|sa_0 \dots sa_{i-1}||sa_i \dots sa_p|} [sa_i|\dots|sa_p|sa_0|\dots|sa_{p-1}].$$

Up to the isomorphism $s^p(A^{\otimes(p+1)}) \to A \otimes (sA)^{\otimes p}$, $s^p(a_0[a_1|\dots|a_p]) \mapsto (-1)^{p|a_0|+(p-1)|a_1|+\dots+|a_{p-1}|} a_0[sa_1|\dots|sa_p]$, our signs coincides with those of [29].

The Hochschild homology of A (with coefficient in A) is the homology of the Hochschild chain complex:

$$HH_*(A; A) := H_*(\mathcal{C}_*(A; A)).$$

The Hochschild cohomology of A (with coefficient in $A^\vee$) is the homology of the dual of the Hochschild chain complex:

$$HH^*(A; A^\vee) := H_*(\mathcal{C}_*(A; A)^\vee).$$

Consider the dual of Connes' boundary map, $B^\vee(\varphi) = (-1)^{|\varphi|}\varphi \circ B$. On $HH^*(A; A^\vee)$, $B^\vee$ defines an action of $H_*(S^1)$.

Example 12. Let $n \geq 2$. Let $\Bbbk$ be any commutative ring. Let $A := H^*(S^n) = \Lambda x_{-n}$ be the exterior algebra on a generator of lower degree $-n$. Denote by $[sx]^k := 1[sx|\dots|sx]$ and $x[sx]^k := x[sx|\dots|sx]$ the elements of $\mathcal{C}_*(A; A)$ where the term sx appears k times. These elements form a basis of $\mathcal{C}_*(A; A)$. Denote by $[sx]^{k\vee}$, $x[sx]^{k\vee}$, $k \geq 0$, the dual basis. The differential $d^\vee$ on $\mathcal{C}_*(A; A)^\vee$ is given by

$d^\vee([sx]^{k\vee}) = 0$ and $d^\vee(x[sx]^{k\vee}) = \pm\left(1-(-1)^{k(n+1)}\right)[sx]^{(k+1)\vee}$. The dual of Connes' boundary map $B^\vee$ is given by

$$B^\vee([sx]^{k\vee}) = \begin{cases} (-1)^{n+1}k\ x[sx]^{(k-1)\vee} & \text{if } (k+1)(n+1) \text{ is even,} \\ 0 & \text{if } (k+1)(n+1) \text{ is odd,} \end{cases}$$

and $B^\vee(x[sx]^{k\vee}) = 0$. We remark that $[sx]^{k\vee}$ is of (lower) degree $k(n-1)$ and $x[sx]^{k\vee}$ of degree $n + k(n-1)$.

Theorem 13 ([17]). *Let X be a simply connected space such that $H_*(X;\mathbb{k})$ is of finite type in each degree. Then there is a natural isomorphism of $H_*(S^1)$-modules between the homology of the free loop space on X and the Hochschild cohomology of the algebra of singular cochain $S^*(X;\mathbb{k})$:*

$$H_*(LX) \cong HH^*(S^*(X;\mathbb{k}); S^*(X;\mathbb{k})^\vee). \tag{14}$$

In this paper, when we will apply this theorem, $H_*(X;\mathbb{k})$ is assumed to be $\mathbb{k}$-free of finite type in each degree and X will be always $\mathbb{k}$-formal: the algebra $S^*(X;\mathbb{k})$ will be linked by quasi-isomorphisms of cochain algebras to $H_*(X;\mathbb{k})$. Therefore

$$HH^*(S^*(X;\mathbb{k}); S^*(X;\mathbb{k})^\vee) \cong HH^*(H^*(X;\mathbb{k}); H^*(X;\mathbb{k})^\vee). \tag{15}$$

Theorem 16. *For $n > 1$ odd, as a Batalin–Vilkovisky algebra,*

$$\mathbb{H}_*(LS^n;\mathbb{k}) = \mathbb{k}[u_{n-1}] \otimes \Lambda a_{-n},$$

$$\Delta(u_{n-1}^i \otimes a_{-n}) = i(u_{n-1}^{i-1} \otimes 1),$$

$$\Delta(u_{n-1}^i \otimes 1) = 0.$$

Proof. For the algebra structure, Cohen, Jones and Yan [8] proved that $\mathbb{H}_*(LS^n;\mathbb{Z}) = \mathbb{k}[u_{n-1}] \otimes \Lambda a_{-n}$ when $\mathbb{k} = \mathbb{Z}$. Their proof works over any $\mathbb{k}$ (alternatively, using Proposition 9, we could assume that $\mathbb{k} = \mathbb{Z}$). Computing Connes' boundary map on $HH^*(H^*(S^n); H_*(S^n))$ (Example 12), we see that Δ on $\mathbb{H}_*(LS^n;\mathbb{k})$ is null in even degree and in degree $-n$, and is an isomorphism in degree -1. Therefore $\Delta(u_{n-1}^i \otimes 1) = 0$, $\Delta(1 \otimes a_{-n}) = 0$ and $\Delta(u_{n-1} \otimes a_{-n}) = \alpha 1$ where α is invertible in $\mathbb{k}$. Replacing a_{-n} by $\frac{1}{\alpha}a_{-n}$ or u_{n-1} by $\frac{1}{\alpha}u_{n-1}$, we can assume up to isomorphisms that $\Delta(u_{n-1} \otimes a_{-n}) = 1$. Therefore $\{u_{n-1}, a_{-n}\} = 1$. Using the Poisson relation (7), $\{u_{n-1}^i, a_{-n}\} = iu_{n-1}^{i-1}$. Therefore $\Delta(u_{n-1}^i \otimes a_{-n}) = i(u_{n-1}^{i-1} \otimes 1)$. $\square$

Theorem 17. *For $n \geq 2$ even, there exists a constant $\varepsilon_0 \in \mathbb{F}_2$ such that as a Batalin–Vilkovisky algebra,*

$$\begin{aligned}\mathbb{H}_*(LS^n;\mathbb{Z}) &= \Lambda b \otimes \frac{\mathbb{Z}[a,v]}{(a^2,ab,2av)}\\ &= \bigoplus_{k=0}^{+\infty} \mathbb{Z}v_{2(n-1)}^k \oplus \bigoplus_{k=0}^{+\infty} \mathbb{Z}b_{-1}v^k \oplus \mathbb{Z}a_{-n} \oplus \bigoplus_{k=1}^{+\infty} \frac{\mathbb{Z}}{2\mathbb{Z}}av^k,\end{aligned}$$

for all $k \geq 0$, $\Delta(v^k) = 0$, $\Delta(av^k) = 0$ and

$$\Delta(bv^k) = \begin{cases} (2k+1)v^k + \varepsilon_0 av^{k+1} & \text{if } n = 2,\\ (2k+1)v^k & \text{if } n \geq 4.\end{cases}$$

Proof. For the algebra structure, Cohen, Jones and Yan [8] proved the equality. Computing Connes' boundary map on $HH^*(H^*(S^n); H_*(S^n))$ (Example 12), we see that Δ on $\mathbb{H}_*(LS^n;\mathbb{k})$ is null in even degree and is injective in odd degree.

Case $n \neq 2$. This case is simple, since all the generators of $\mathbb{H}_*(LS^n)$, v^k, bv^k and av^k, $k \geq 0$, have different degrees. Using Example 12, we also see that for all $k \geq 0$,

$$\Delta\colon \mathbb{H}_{-1+2k(n-1)} = \mathbb{Z}b_{-1}v^k \hookrightarrow \mathbb{H}_{2k(n-1)} = \mathbb{Z}v^k$$

has cokernel isomorphic to $\frac{\mathbb{Z}}{(2k+1)\mathbb{Z}}$. Therefore $\Delta(bv^k) = \pm(2k+1)v^k$. By replacing b_{-1} by $-b_{-1}$, we can assume up to isomorphims that $\Delta(b) = 1$. Let $k \geq 1$. Let $\alpha_k \in \{-2k-1, 2k+1\}$ such that $\Delta(bv^k) = \alpha_k v^k$. Using formula (6), we obtain that $\Delta(bv^k v^k) = (2\alpha_k - 1)v^{2k}$. We know that $\Delta(bv^{2k}) = \pm(4k+1)v^{2k}$. Therefore α_k must be equal to $2k+1$.

Case $n = 2$. This case is complicated, since for $k \geq 0$, v^k and av^{k+1} have the same degree. Using Example 12, we also see that

$$\Delta\colon \mathbb{H}_{-1+2k} = \mathbb{Z}b_{-1}v^k \hookrightarrow \mathbb{H}_{2k} = \mathbb{Z}v^k \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}av^{k+1}$$

has cokernel, denoted $\operatorname{Coker}\Delta$, isomorphic to $\frac{\mathbb{Z}}{(2k+1)\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}$. There exists unique $\alpha_k \in \mathbb{Z}^*$ and $\varepsilon_k \in \frac{\mathbb{Z}}{2\mathbb{Z}}$ such that $\Delta(bv^k) = \alpha_k v^k + \varepsilon_k av^{k+1}$. The injective map Δ

fits into the commutative diagram of short exact sequences (Noether's Lemma)

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{H}_{-1+2k} & \xrightarrow{\mathrm{id}} & \mathbb{H}_{-1+2k} & \longrightarrow & 0 & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\times 2} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{H}_{-1+2k} & \xrightarrow{\Delta} & \mathbb{H}_{2k} & \longrightarrow & \mathrm{Coker}\Delta & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\cong} & & \\
0 & \longrightarrow & \frac{\mathbb{Z}}{2\mathbb{Z}} & \xrightarrow{\overline{\Delta}} & \frac{\mathbb{Z}}{2\alpha_k\mathbb{Z}}\oplus\frac{\mathbb{Z}}{2\mathbb{Z}} & \longrightarrow & \mathrm{Coker}\overline{\Delta} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}
$$

The cokernel of $\overline{\Delta}$, denoted $\mathrm{Coker}\overline{\Delta}$ is of cardinal $2|\alpha_k|$. So $|\alpha_k| = 2k+1$. Therefore $\Delta(bv^k) = \pm(2k+1)v^k + \varepsilon_k av^{k+1}$.

By replacing b_{-1} by $-b_{-1}$, we can assume up to isomorphims that $\Delta(b) = 1 + \varepsilon_0 av$. Using formula (6), we obtain that

$$\Delta(bv^k v^l) = (\alpha_k + \alpha_l - 1)v^{k+l} + (\varepsilon_k + \varepsilon_l - \varepsilon_0)av^{k+l+1}.$$

Therefore

$$\Delta(bv^k v^k) = (2\alpha_k - 1)v^{2k} + \varepsilon_0 av^{2k+1} = \pm(4k+1)v^{2k} + \varepsilon_{2k}av^{2k+1}.$$

So $\alpha_k = 2k+1$, $\varepsilon_{2k} = \varepsilon_0$ and $\varepsilon_{2k+1} = \varepsilon_{2k} + \varepsilon_1 - \varepsilon_0 = \varepsilon_1$.

The map $\Theta\colon \mathbb{H}_*(LS^2) \to \mathbb{H}_*(LS^2)$ given by $\Theta(b_{-1}v^k) = b_{-1}v^k$, $\Theta(v^k) = v^k + kav^{k+1}$, $\Theta(av^k) = av^k$, $k \geq 0$ is an involutive isomorphism of algebras. Therefore, by replacing v by $v + av^2$, we can assume that $\varepsilon_1 = \varepsilon_0$. So we have proved

$$\Delta(bv^k) = (2k+1)v^k + \varepsilon_0 av^{k+1}, \quad k \geq 0. \qquad \square$$

These two cases $\varepsilon_0 = 0$ and $\varepsilon_0 = 1$ correspond to two non-isomorphic Batalin–Vilkovisky algebras whose underlying Gerstenhaber algebras are the same. Therefore even if we have not yet computed the Batalin–Vilkovisky algebra $\mathbb{H}_*(LS^2;\mathbb{Z})$, we have computed its underlying Gerstenhaber algebra. Using the definition of the bracket, straightforward computations give the following corollary.

Corollary 18. *For $n \geq 2$ even, as Gerstenhaber algebra*

$$\mathbb{H}_*(LS^n;\mathbb{Z}) = \Lambda b_{-1} \otimes \frac{\mathbb{Z}[a_{-n}, v_{2(n-1)}]}{(a^2, ab, 2av)}$$

with $\{v^k, v^l\} = 0$, $\{bv^k, v^l\} = -2lv^{k+l}$, $\{bv^k, bv^l\} = 2(k-l)bv^{k+l}$, $\{a, v^l\} = 0$, $\{av^k, bv^l\} = -(2l+1)av^{k+l}$ *and* $\{av^k, av^l\} = 0$ *for all* $k, l \geq 0$.

5. When Hochschild cohomology is a Batalin–Vilkovisky algebra

In this section, we recall the structure of Gerstenhaber algebra on the Hochschild cohomology of an algebra whose degrees are bounded. We recall from [26], [22], [27], [19] the Batalin–Vilkovisky algebra on the Hochschild cohomology of the cohomology $H^*(M)$ of a closed oriented manifold M. We compute this Batalin–Vilkovisky algebra $HH^*(H^*(M); H^*(M))$ when M is a sphere.

Throughout in this section we will work over the prime field $\mathbb{F}_2$. Let A be an augmented graded algebra such that the augmentation ideal $\bar{A}$ is concentrated in degree ≤ -2 and bounded below (or concentrated in degree ≥ 0 and bounded above). Then the (normalized) Hochschild cochain complex, denoted $\mathcal{C}^*(A, A)$, is the complex

$$\mathrm{Hom}(Ts\bar{A}, A) \cong \bigoplus_{p\geq 0} \mathrm{Hom}((s\bar{A})^{\otimes p}, A)$$

with a differential d_2. For an element f in $\mathrm{Hom}((s\bar{A})^{\otimes p}, A)$, the differential $d_2 f$ in $\mathrm{Hom}((s\bar{A})^{\otimes p+1}, A)$ is given by

$$\begin{aligned}(d_2 f)([sa_1|\dots|sa_{p+1}]) &:= a_1 f([sa_2|\dots|sa_{p+1}]) \\ &\quad + \sum_{i=1}^{p} f([sa_1|\dots|s(a_i a_{i+1})|\dots|sa_{p+1}]) + f([sa_1|\dots|sa_p])a_p.\end{aligned}$$

The Hochschild cohomology of A with coefficient in A is the homology of the Hochschild cochain complex:

$$HH^*(A; A) := H_*(\mathcal{C}^*(A; A)).$$

We remark that $HH^*(A; A)$ is bigraded. Our degree is sometimes called the total degree: sum of the external degree and the internal degree. The Hochschild cochain complex $\mathcal{C}^*(A, A)$ is a differential graded algebra. For $f \in \mathrm{Hom}((s\bar{A})^{\otimes p}, A)$ and $g \in \mathrm{Hom}((s\bar{A})^{\otimes q}, A)$, the (cup) product of f and g, $f \cup g \in \mathrm{Hom}((s\bar{A})^{\otimes p+q}, A)$ is defined by

$$(f \cup g)([sa_1|\dots|sa_{p+q}]) := f([sa_1|\dots|sa_p])g([sa_{p+1}|\dots|sa_{p+q}]).$$

The Hochschild cochain complex $\mathcal{C}^*(A, A)$ has also a Lie bracket of (lower) degree $+1$.

$$\begin{aligned}(f \mathbin{\bar{\circ}} g)&([sa_1|\dots|sa_{p+q-1}]) \\ &:= \sum_{i=1}^{p} f\left([sa_1|\dots|sa_{i-1}|sg([sa_i|\dots|sa_{i+q-1}])|sa_{i+q}|\dots|sa_{p+q-1}]\right).\end{aligned}$$

$\{f, g\} = f \bar{\circ} g - g \bar{\circ} f$. Our formulas are the same as in the non-graded case [13]. We remark that if A is not assumed to be bounded, the formulas are more complicated. Gerstenhaber has shown that $HH^*(A; A)$ equipped with the cup product and the Lie bracket is a Gerstenhaber algebra.

Let M be a closed d-dimensional smooth manifold. Poincaré duality induces an isomorphism of $H^*(M; \mathbb{F}_2)$-modules of (lower) degree d:

$$\Theta\colon H^*(M; \mathbb{F}_2) \xrightarrow{\cap[M]} H_*(M; \mathbb{F}_2) \cong H^*(M; \mathbb{F}_2)^\vee. \tag{19}$$

More generally, let A be a graded algebra equipped with an isomorphism of A-bimodules of degree d, $\Theta\colon A \xrightarrow{\cong} A^\vee$. Then we have the isomorphism

$$HH^*(A, \Theta)\colon HH^*(A, A) \xrightarrow{\cong} HH^*(A, A^\vee).$$

Therefore on $HH^*(A, A)$, we have both a Gerstenhaber algebra structure and an operator Δ given by the dual of Connes' boundary map B. Motivated by the Batalin–Vilkovisky algebra structure of Chas–Sullivan on $\mathbb{H}_*(LM)$, Thomas Tradler [26] proved that $HH^*(A, A)$ is a Batalin–Vilkovisky algebra. See [22, Theorem 1.6] for an explicit proof. In [19] or [27, Corollary 3.4] or [9, Section 1.4] or [20, Theorem B] or [21, Section 11.6], this Batalin–Vilkovisky algebra structure on $HH^*(A, A)$ extends to a structure of algebra on the Hochschild cochain complex $\mathcal{C}^*(A, A)$ over various operads or PROPs: the so-called cyclic Deligne conjecture. Let us compute this Batalin–Vilkovisky algebra structure when M is a sphere.

Proposition 20 ([30] and [31, Corollary 4.2]). *Let $d \geq 2$. As Batalin–Vilkovisky algebra, for the Hochschild cohomology of $H^*(S^d; \mathbb{F}_2) = \Lambda x_{-d}$, we have*

$$HH^*(H^*(S^d; \mathbb{F}_2); H^*(S^d; \mathbb{F}_2)) \cong \Lambda g_{-d} \otimes \mathbb{F}_2[f_{d-1}]$$

with $\Delta(g_{-d} \otimes f_{d-1}^k) = k(1 \otimes f_{d-1}^{k-1})$ and $\Delta(1 \otimes f_{d-1}^k) = 0, k \geq 0$. In particular, the underlying Gerstenhaber algebra is given by $\{f^k, f^l\} = 0$, $\{gf^k, f^l\} = lf^{k+l-1}$ and $\{gf^k, gf^l\} = (k-l)gf^{k+l-1}$ for $k, l \geq 0$.

Proof. Denote by $A := H^*(S^d; \mathbb{F}_2)$. The differential on $\mathcal{C}^*(A; A)$ is null. Let $f \in \mathrm{Hom}(s\bar{A}, A) \subset \mathcal{C}^*(A; A)$ such that $f([sx]) = 1$. Let $g \in \mathrm{Hom}(\mathbb{F}_2, A) = \mathrm{Hom}((s\bar{A})^{\otimes 0}, A) \subset \mathcal{C}^*(A; A)$ such that $g([]) = x$. The k-th power of f is the map $f^k \in \mathrm{Hom}((s\bar{A})^{\otimes k}, A)$ such that $f^k([sx|\dots|sx]) = 1$. The cup product $g \cup f^k \in \mathrm{Hom}((s\bar{A})^{\otimes k}, A)$ sends $[sx|\dots|sx]$ to x. So we have proved that $\mathcal{C}^*(A; A)$ is isomorphic to the tensor product of graded algebras $\Lambda g_{-d} \otimes \mathbb{F}_2[f_{d-1}]$.

The unit 1 and x_{-d} form a linear basis of $H^*(S^d)$. Denote by $1^\vee$ and $x^\vee$ the dual basis of $A^\vee = H^*(S^d)^\vee$. Poincaré duality induces the isomorphism $\Theta\colon H^*(S^d) \xrightarrow{\cong} H^*(S^d)^\vee$, $1 \mapsto x^\vee$ and $x \mapsto 1^\vee$. The two families of elements of the form $1[sx|\dots|sx]$ and of the form $x[sx|\dots|sx]$ form a basis of $\mathcal{C}_*(A;A)$. Denote by $1[sx|\dots|sx]^\vee$ and $x[sx|\dots|sx]^\vee$ the dual basis in $\mathcal{C}_*(A;A)^\vee$. The isomorphism Θ induces an isomorphism of complexes of degree d, $\widehat{\Theta}\colon \mathcal{C}^*(A;A) \xrightarrow[\cong]{\mathcal{C}^*(A;\Theta)} \mathcal{C}^*(A;A^\vee) \xrightarrow{\cong} \mathcal{C}_*(A;A)^\vee$. Explicitly [22, Section 4] this isomorphism sends $f \in \operatorname{Hom}((s\bar{A})^{\otimes p}, A)$ to the linear map $\widehat{\Theta}(f) \in (A \otimes (s\bar{A})^{\otimes p})^\vee \subset \mathcal{C}_*(A;A)^\vee$ defined by

$$\widehat{\Theta}(f)(a_0[sa_1|\dots|sa_p]) = \big((\Theta \circ f)[sa_1|\dots|sa_p]\big)(a_0).$$

Here with $A = \Lambda x$, $\widehat{\Theta}(f^k) = x[sx|\dots|sx]^\vee$ and $\widehat{\Theta}(g \cup f^k) = 1[sx|\dots|sx]^\vee$. Computing Connes' boundary map $B^\vee$ on $\mathcal{C}_*(A;A)^\vee$ (Example 12) and using that $\widehat{\Theta} \circ \Delta = B^\vee \circ \widehat{\Theta}$ by definition of Δ, we obtain the desired formula for Δ. □

6. The Gerstenhaber algebra $\mathbb{H}_*(LS^2;\mathbb{F}_2)$

Using the same Hochschild homology technique as in Section 4, we compute, up to an indeterminacy, the Batalin–Vilkovisky algebra $\mathbb{H}_*(LS^2;\mathbb{F}_2)$. Nevertheless, this will give the complete description of the underlying Gerstenhaber algebra on $\mathbb{H}_*(LS^2;\mathbb{F}_2)$.

Lemma 21. *There exists a constant $\varepsilon \in \{0,1\}$ such that as a Batalin–Vilkovisky algebra, the homology of the free space loop on the sphere S^2 is*

$$\mathbb{H}_*(LS^2;\mathbb{F}_2) = \Lambda a_{-2} \otimes \mathbb{F}_2[u_1],$$

$$\Delta(a_{-2} \otimes u_1^k) = k(1 \otimes u_1^{k-1} + \varepsilon a_{-2} \otimes u_1^{k+1}) \quad \textit{and} \quad \Delta(1 \otimes u_1^k) = 0, \quad k \geq 0.$$

Proof. In [8], Cohen, Jones and Yan proved that the Serre spectral sequence for the free loop fibration $\Omega M \overset{j}{\hookrightarrow} LM \overset{\mathrm{ev}}{\twoheadrightarrow} M$ is a spectral sequence of algebras converging toward the algebra $\mathbb{H}_*(LM)$. Using Hochschild homology, we see that there is an isomorphism of vector spaces $\mathbb{H}_*(LS^2;\mathbb{F}_2) \cong \mathbb{H}_*(S^2;\mathbb{F}_2) \otimes H_*(\Omega S^2;\mathbb{F}_2)$. Therefore the Serre spectral sequence collapses. Since there is no extension problem, we have the isomorphism of algebras

$$\mathbb{H}_*(LS^2;\mathbb{F}_2) \cong \mathbb{H}_*(S^2;\mathbb{F}_2) \otimes H_*(\Omega S^2;\mathbb{F}_2) = \Lambda(a_{-2}) \otimes \mathbb{F}_2[u_1].$$

Computing Connes' boundary map on $HH^*(H^*(S^2;\mathbb{F}_2); H_*(S^2;\mathbb{F}_2))$ (see Example 12), we see that Δ on $\mathbb{H}_*(LS^2;\mathbb{F}_2)$ is null in even degree and that

$$\Delta\colon \mathbb{H}_{2k-1} \to \mathbb{H}_{2k}$$

is a linear map of rank 1, $k \geq 0$. In particular Δ is injective in degree -1.

Applying Lemma 11 to the identity map id: $S^2 \to S^2$, we see that the composite

$$H_1(\Omega S^2;\mathbb{F}_2) \xrightarrow{H_1(j;\mathbb{F}_2)} H_1(LS^2;\mathbb{F}_2) \xrightarrow{\Delta} H_2(LS^2;\mathbb{F}_2) \xrightarrow{H_2(\mathrm{ev};\mathbb{F}_2)} H_2(S^2;\mathbb{F}_2)$$

is non-zero. Since $\mathbb{H}_*(\mathrm{ev})$ is a morphism of algebras, $\mathbb{H}_0(ev)(a_{-2}u_1^2) = 0$. And so $\Delta(a_{-2}u_1) = 1 + \varepsilon a_{-2}u_1^2$ with $\varepsilon \in \mathbb{F}_2$.

We remark that when $b = c$, formula (6) takes the simple form

$$\Delta(ab^2) = \Delta(a)b^2 + a\Delta(b^2). \tag{22}$$

Using this formula, we obtain that

$$\Delta(a_{-2}u_1^{2k+1}) = \Delta((a_{-2}u_1)(u_1^k)^2) = u_1^{2k} + \varepsilon a_{-2}u_1^{2k+2}, \quad k \geq 0.$$

Since $\Delta\colon \mathbb{H}_1 = \mathbb{F}_2 a_{-2}u_1^3 \oplus \mathbb{F}_2 u_1 \to \mathbb{H}_2$ is of rank 1 and $\Delta(a_{-2}u_1^3) \neq 0$, $\Delta(u_1) = \lambda\Delta(a_{-2}u_1^3)$ with $\lambda = 0$ or $\lambda = 1$. Using again formula (22), we have that

$$\Delta(u_1^{2k+1}) = \Delta(u_1(u_1^k)^2) = \lambda\Delta(a_{-2}u_1^3)u_1^{2k} = \lambda\Delta(a_{-2}u_1^{2k+3}), \quad k \geq 0.$$

So finally

$$\Delta(a_{-2}u_1^k) = ku_1^{k-1} + \varepsilon k a_{-2}u_1^{k+1} \text{ and } \Delta(u_1^k) = \lambda\Delta(a_{-2}u_1^{k+2}), \quad k \geq 0.$$

The cases $\lambda = 0$ and $\lambda = 1$ correspond to isomorphic Batalin–Vilkovisky algebras: Let $\Theta\colon \mathbb{H}_*(LS^2;\mathbb{F}_2) \to \mathbb{H}_*(LS^2;\mathbb{F}_2)$ be an automorphism of algebras which is not the identity. Since $\Theta(a_{-2}) \neq 0$, $\Theta(a_{-2}) = a_{-2}$. Since $\Theta(a_{-2})$ and $\Theta(u_1)$ must generate the algebra $\Lambda a_{-2} \otimes \mathbb{F}_2[u_1]$, $\Theta(u_1) \neq a_{-2}u_1^3$. Since $\Theta(u_1) \neq u_1$, $\Theta(u_1) = u_1 + a_{-2}u_1^3$. Therefore there is an unique automorphism of algebras $\Theta\colon \mathbb{H}_*(LS^2;\mathbb{F}_2) \to \mathbb{H}_*(LS^2;\mathbb{F}_2)$ which is not the identity. Explicitly, Θ is given by $\Theta(u_1^k) = u_1^k + ka_{-2}u_1^{k+2}$, $\Theta(a_{-2}u_1^k) = a_{-2}u_1^k$, $k \geq 0$. One can check that Θ is an involutive isomorphism of Batalin–Vilkovisky algebras who transforms the cases $\lambda = 0$ into the cases $\lambda = 1$ without changing ε. Therefore, by replacing u_1 by $u_1 + a_{-2}u_1^3$, we can assume that $\lambda = 0$. □

Consider the Batalin–Vilkovisky algebras $\Lambda a_{-2} \otimes \mathbb{F}_2[u_1]$ with $\Delta(a_{-2} \otimes u_1^k) = k(1 \otimes u_1^{k-1} + \varepsilon a_{-2} \otimes u_1^{k+1})$, $\Delta(1 \otimes u_1^k) = \lambda\Delta(a_{-2}u_1^{k+2})$, $k \geq 0$, given by the different values of ε, $\lambda \in \{0, 1\}$. These four Batalin–Vilkovisky algebras have only two underlying Gerstenhaber algebras given by $\{u_1^k, u_1^l\} = 0$, $\{a_{-2}u_1^k, u_1^l\} = lu^{k+l-1} + l(\varepsilon - \lambda)a_{-2}u^{k+l+1}$ and $\{a_{-2}u_1^k, a_{-2}u_1^l\} = (k-l)a_{-2}u^{k+l-1}$ for $k, l \geq 0$. Via the above isomorphism Θ, these two Gerstenhaber algebras are isomorphic.

Corollary 23. *The free loop space modulo 2 homology $\mathbb{H}_*(LS^2;\mathbb{F}_2)$ is isomorphic as Gerstenhaber algebra to the Hochschild cohomology of $H^*(S^2;\mathbb{F}_2)$,*

$$HH^*(H^*(S^2;\mathbb{F}_2); H^*(S^2;\mathbb{F}_2)).$$

7. The Batalin–Vilkovisky algebra $\mathbb{H}_*(LS^2)$

In this section, we complete the calculations of the Batalin–Vilkovisky algebras $\mathbb{H}_*(LS^2;\mathbb{F}_2)$ and $\mathbb{H}_*(LS^2;\mathbb{Z})$ started respectively in Sections 6 and 4, using a purely homotopic method.

Theorem 24. *As a Batalin–Vilkovisky algebra, the homology of the free loop space on the sphere S^2 with mod 2 coefficients is*

$$\mathbb{H}_*(LS^2;\mathbb{F}_2) = \Lambda a_{-2} \otimes \mathbb{F}_2[u_1],$$

$$\Delta(a_{-2} \otimes u_1^k) = k(1 \otimes u_1^{k-1} + a_{-2} \otimes u_1^{k+1}) \quad \textit{and} \quad \Delta(1 \otimes u_1^k) = 0, \quad k \geq 0.$$

Theorem 25. *With integer coefficients, as a Batalin–Vilkovisky algebra,*

$$\begin{aligned}\mathbb{H}_*(LS^2;\mathbb{Z}) &= \Lambda b \otimes \frac{\mathbb{Z}[a,v]}{(a^2, ab, 2av)} \\ &= \bigoplus_{k=0}^{+\infty} \mathbb{Z}v_2^k \oplus \bigoplus_{k=0}^{+\infty} \mathbb{Z}b_{-1}v^k \oplus \mathbb{Z}a_{-2} \oplus \bigoplus_{k=1}^{+\infty} \frac{\mathbb{Z}}{2\mathbb{Z}} av^k\end{aligned}$$

for all $k \geq 0$, $\Delta(v^k) = 0$, $\Delta(av^k) = 0$ and $\Delta(bv^k) = (2k+1)v^k + av^{k+1}$.

Denote by $s\colon X \hookrightarrow LX$ the trivial section of the evaluation map $\mathrm{ev}\colon LX \twoheadrightarrow X$.

Lemma 26. *The image of $\Delta\colon H_1(LS^2;\mathbb{F}_2) \to H_2(LS^2;\mathbb{F}_2)$ is not contained in the image of $H_2(s;\mathbb{F}_2)\colon H_2(S^2;\mathbb{F}_2) \hookrightarrow H_2(LS^2;\mathbb{F}_2)$.*

Lemma 27. *The image of $\Delta\colon H_1(LS^2;\mathbb{Z}) \to H_2(LS^2;\mathbb{Z})$ is not contained in the image of $H_2(s;\mathbb{Z})\colon H_2(S^2;\mathbb{Z}) \hookrightarrow H_2(LS^2;\mathbb{Z})$.*

Proof of Lemma 27 *assuming Lemma* 26. Consider the commutative diagram

$$\begin{array}{ccc}
H_1(LS^2;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_2 & \xrightarrow{\cong} & H_1(LS^2;\mathbb{F}_2) \\
\downarrow{\scriptstyle \Delta\otimes_{\mathbb{Z}}\mathbb{F}_2} & & \downarrow{\scriptstyle \Delta} \\
H_2(LS^2;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_2 & \xrightarrow{\cong} & H_2(LS^2;\mathbb{F}_2) \\
\uparrow{\scriptstyle H_2(s;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{F}_2} & & \uparrow{\scriptstyle H_2(s;\mathbb{F}_2)} \\
H_2(S^2;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_2 & \xrightarrow{\cong} & H_2(S^2;\mathbb{F}_2).
\end{array}$$

Since $H_1(LS^2;\mathbb{Z}) \cong H_0(LS^2;\mathbb{Z}) \cong \mathbb{Z}$, the horizontal arrows are isomorphisms by the universal coefficient theorem. The top rectangle commutes according to Lemma 4.

Suppose that the image of $\Delta\colon H_1(LS^2;\mathbb{Z}) \to H_2(LS^2;\mathbb{Z})$ is included in the image of $H_2(s;\mathbb{Z})$. Then the image of $\Delta \otimes_{\mathbb{Z}} \mathbb{F}_2$ is included in the image of $H_2(s;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_2$. Using the above diagram, the image of $\Delta\colon H_1(LS^2;\mathbb{F}_2) \to H_2(LS^2;\mathbb{F}_2)$ is included in the image of $H_2(s;\mathbb{F}_2)$. This contradicts Lemma 26. □

Proof of Theorem 24 *assuming Lemma* 26. It suffices to show that the constant ε in Lemma 21 is not zero. Suppose that $\varepsilon = 0$. Then by Lemma 21, $\Delta(a_{-2} \otimes u_1) = 1$.

It is well known that $\mathbb{H}_*(s)\colon \mathbb{H}_*(M) \to \mathbb{H}_*(LM)$ is a morphism of algebras. In particular, let $[S^2]$ be the fundamental class of S^2, $H_2(s)([S^2])$ is the unit of $\mathbb{H}_*(LS^2)$. So $\Delta(a_{-2} \otimes u_1) = H_2(s)([S^2])$. This contradicts Lemma 26. □

The proof of Theorem 25 assuming Lemma 27 is the same. To complete the computation of this Batalin–Vilkovisky algebra on the homology of the free loop space of a manifold, we will relate it to another structure of a Batalin–Vilkovisky algebra that arises in algebraic topology: the homology of the double loop space.

Let X be a pointed topological space. The circle S^1 acts on the sphere S^2 by "rotating the earth". Hence the circle also acts on $\Omega^2 X = \text{map}\big((S^2, \text{North pole}), (X, *)\big)$. So we have an induced operator $\Delta\colon H_*(\Omega^2 X) \to H_{*+1}(\Omega^2 X)$. With Theorem 32 and the following proposition, we will able to prove Lemma 26.

Proposition 28. *Let X be a pointed topological space. There is a natural morphism $r\colon L\Omega X \to \text{map}_*(S^2, X)$ of S^1-spaces between the free loop space on the pointed loops of X and the double pointed loop space of X such that:*

- *If we identify S^2 and $S^1 \wedge S^1$, r is a retract up to homotopy of the inclusion $j : \Omega(\Omega X) \hookrightarrow L(\Omega X)$.*
- *The composite $r \circ s\colon \Omega X \hookrightarrow L(\Omega X) \to \text{map}_*(S^2, X)$ is homotopically trivial.*

Proof. Let $\sigma\colon S^2 \twoheadrightarrow \frac{S^1 \times S^1}{S^1 \times *} = S^1_+ \wedge S^1$ be the quotient map that identifies the North pole and the South pole on the earth S^2. The circle S^1 acts without moving the based point on $S^1_+ \wedge S^1$ by multiplication on the first factor. On the torus $S^1 \times S^1$, the circle can act by multiplication on both factors. But when you pinch a circle to a point in the torus, the circle can act only on one factor. If we make a picture, we easily see that $\sigma\colon S^2 \twoheadrightarrow S^1_+ \wedge S^1$ is compatible with the actions of S^1. Therefore $r\colon = \text{map}_*(\sigma, X)\colon L\Omega X \to \text{map}_*(S^2, X)$ is a morphism of S^1-spaces.

- Let $\pi : S^1_+ \wedge S^1 \twoheadrightarrow S^1 \wedge S^1 = \frac{S^1_+ \wedge S^1}{* \times S^1}$ be the quotient map. The inclusion map $j : \Omega(\Omega X) \to L(\Omega X)$ is $\text{map}_*(\pi, X)$. The composite $\pi \circ \sigma : S^2 \twoheadrightarrow S^1 \wedge S^1$ is the quotient map obtained by identifying a meridian with a point in the sphere S^2. The composite $\pi \circ \sigma$ can also be viewed as the quotient map from the non-reduced suspension of S^1 to the reduced suspension of S^1. So the composite $\pi \circ \sigma : S^2 \twoheadrightarrow S^1 \wedge S^1$ is a homotopy equivalence. Let $\Theta\colon S^1 \wedge S^1 \xrightarrow{\cong} S^2$ be any given homeomorphism.

The composite $\Theta \circ \pi \circ \sigma\colon S^2 \to S^2$ is of degree ± 1. The reflection through the equatorial plane is a morphism of S^1-spaces. By replacing eventually σ by its composite with the previous reflection, we can suppose that $\Theta \circ \pi \circ \sigma\colon S^2 \to S^2$ is homotopic to the identity map of S^2, i.e. $\sigma \circ \Theta$ is a section of π up to homotopy. Therefore $\mathrm{map}_*(\sigma \circ \Theta, X) = \mathrm{map}_*(\Theta, X) \circ r$ is a retract of j up to homotopy.

• Let $\rho\colon S^1_+ \wedge S^1 = \frac{S^1 \times S^1}{S^1 \times *} \twoheadrightarrow S^1$ be the map induced by the projection on the second factor. Since $\pi_2(S^1) = *$, the composite $\rho \circ \sigma$ is homotopically trivial. Therefore $r \circ s$, the composite of $r = \mathrm{map}_*(\sigma, X)$ and $s = \mathrm{map}_*(\rho, X)\colon \Omega X \to L(\Omega X)$ is also homotopically trivial. □

Proof of Lemma 26. Denote by $\mathrm{ad}_{S^n}\colon S^n \to \Omega S^{n+1}$ the adjoint of the identity map $\mathrm{id}\colon S^{n+1} \to S^{n+1}$. The map $L(\mathrm{ad}_{S^2})\colon LS^2 \to L\Omega S^3$ is obviously a morphism of S^1-spaces. Therefore using Proposition 28, the composite $r \circ L(\mathrm{ad}_{S^2})\colon LS^2 \to L\Omega S^3 \to \Omega^2 S^3$ is also a morphism of S^1-spaces. Therefore $H_*(r \circ L(\mathrm{ad}_{S^2}))$ commutes with the corresponding operators Δ in $H_*(LS^2)$ and $H_*(\Omega^2 S^3)$.

Consider the commutative diagram up to homotopy:

$$\begin{array}{ccccc} \Omega S^2 & \xrightarrow{j} & LS^2 & \xleftarrow{s} & S^2 \\ \downarrow{\scriptstyle \Omega(\mathrm{ad}_{S^2})} & & \downarrow{\scriptstyle L(\mathrm{ad}_{S^2})} & & \downarrow{\scriptstyle \mathrm{ad}_{S^2}} \\ \Omega^2 S^3 & \xrightarrow{j} & L\Omega S^3 & \xleftarrow{s} & \Omega S^3 \\ & {\scriptstyle \mathrm{id}}\searrow & \downarrow{\scriptstyle r} & \swarrow{\scriptstyle *} & \\ & & \Omega^2 S^3. & & \end{array} \tag{29}$$

Using the left part of this diagram, we see that $\pi_1(r \circ L(\mathrm{ad}))$ maps the generator of $\pi_1(LS^2) = \mathbb{Z}(j \circ \mathrm{ad}_{S^1})$ to the composite $\Omega(\mathrm{ad}_{S^2}) \circ \mathrm{ad}_{S^1}\colon S^1 \to \Omega S^2 \to \Omega^2 S^3$ which is the generator of $\pi_1(\Omega^2 S^3) \cong \mathbb{Z}$. Therefore $\pi_1(r \circ L(\mathrm{ad}))$ is an isomorphism.

So we have the commutative diagram

$$\begin{array}{ccccc} \pi_1(LS^2) \otimes \mathbb{F}_2 & \xrightarrow[\cong]{\mathrm{hur}} & H_1(LS^2; \mathbb{F}_2) & \xrightarrow{\Delta} & H_2(LS^2; \mathbb{F}_2) \\ \downarrow{\scriptstyle \pi_1(r \circ L(\mathrm{ad}_{S^2})) \otimes \mathbb{F}_2}\ \cong & & \downarrow{\scriptstyle H_1(r \circ L(\mathrm{ad}_{S^2}); \mathbb{F}_2)} & & \downarrow{\scriptstyle H_2(r \circ L(\mathrm{ad}_{S^2}); \mathbb{F}_2)} \\ \pi_1(\Omega^2 S^3) \otimes \mathbb{F}_2 & \xrightarrow[\cong]{\mathrm{hur}} & H_1(\Omega^2 S^3; \mathbb{F}_2) & \xrightarrow{\Delta} & H_2(\Omega^2 S^3; \mathbb{F}_2). \end{array}$$

By Theorem 32, $\Delta\colon H_1(\Omega^2 S^3; \mathbb{F}_2) \to H_2(\Omega^2 S^3; \mathbb{F}_2)$ is non-zero. Therefore using the above diagram, the composite $H_2(r \circ L(\mathrm{ad}_{S^2})) \circ \Delta$ is also non-zero. On the other hand, using the right part of diagram (29), we have that the composite $H_2(r \circ L(\mathrm{ad}_{S^2})) \circ H_2(s)$ is null. □

Corollary 30. *The free loop space modulo 2 homology* $\mathbb{H}_*(LS^2;\mathbb{F}_2)$ *is not isomorphic as Batalin–Vilkovisky algebra to the Hochschild cohomology of* $H^*(S^2;\mathbb{F}_2)$, $HH^*(H^*(S^2;\mathbb{F}_2);H^*(S^2;\mathbb{F}_2))$.

This means exactly that there exists no isomorphism between $\mathbb{H}_*(LS^2;\mathbb{F}_2)$ and $HH^*(H^*(S^2;\mathbb{F}_2);H^*(S^2;\mathbb{F}_2))$ which at the same time

- is an isomorphism of algebras and
- commutes with the Δ-operators,

although separately

- there exists (Corollary 23) an isomorphism of algebras between $\mathbb{H}_*(LS^2;\mathbb{F}_2)$ and $HH^*(H^*(S^2;\mathbb{F}_2);H^*(S^2;\mathbb{F}_2))$ and
- there exists also an isomorphism commuting with the Δ-operators between them.

Proof. By Proposition 20, $HH^*(H^*(S^2);H^*(S^2))$ is the Batalin–Vilkovisky algebra given by $\varepsilon = 0$ in Lemma 21. On the contrary, by Theorem 24, $\mathbb{H}_*(LS^2;\mathbb{F}_2)$ is the Batalin–Vilkovisky algebra given by $\varepsilon = 1$. At the end of the proof of Lemma 21, we saw that the two cases $\varepsilon = 0$ and $\varepsilon = 1$ correspond to two non-isomorphic Batalin–Vilkovisky algebras. □

More generally, we believe that for any prime p, the free loop space modulo p of the complex projective space $\mathbb{H}_*(L\mathbb{CP}^{p-1};\mathbb{F}_p)$[1] is not isomorphic as Batalin–Vilkovisky algebra to the Hochschild cohomology

$$HH^*(H^*(\mathbb{CP}^{p-1};\mathbb{F}_p);H^*(\mathbb{CP}^{p-1};\mathbb{F}_p)).$$

Such phenomena for formal manifolds should not appear over a field of characteristic 0.

Recall that by Poincaré duality, we have the isomorphism (cf. Equation (19))

$$\Theta\colon H^*(S^2) \xrightarrow{\cong} H^*(S^2)^\vee.$$

Therefore we have the isomorphism

$$HH^*(H^*(S^2);\Theta)\colon HH^*(H^*(S^2);H^*(S^2)) \xrightarrow{\cong} HH^*(H^*(S^2);H^*(S^2)^\vee).$$

Consider any isomorphism of graded algebras

$$\mathbb{H}_*(LS^2) \cong HH^*(S^*(S^2);S^*(S^2)). \tag{31}$$

[1] Bökstedt and Ottosen [1] have recently announced the computation of the Batalin–Vilkovisky algebra $\mathbb{H}_*(L\mathbb{CP}^n;\mathbb{F}_p)$.

By Corollary 23, such isomorphism exists. Cohen and Jones ([7, Theorem 3] and [5]) proved that such isomorphism exists for any manifold M. Since S^2 is formal, we have the isomorphism of algebras (cf. Equation (2))

$$HH^*(S^*(S^2); S^*(S^2)) \xrightarrow{\cong} HH^*(H^*(S^2); H^*(S^2)).$$

By [17], we have the isomorphisms of $H_*(S^1)$-modules

$$H_*(LS^2) \overset{(14)}{\cong} HH^*(S^*(S^2); S^*(S^2)^\vee) \overset{(15)}{\cong} HH^*(H^*(S^2); H^*(S^2)^\vee).$$

Corollary 30 implies that the following diagram does not commute over $\mathbb{F}_2$:

$$\begin{array}{ccccc} & & HH^*(S^*(S^2); S^*(S^2)^\vee) & \xrightarrow{(15)} & HH^*(H^*(S^2); H^*(S^2)^\vee) \\ & \overset{(14)}{\nearrow} & & & \uparrow{\scriptstyle HH^*(H^*(S^2);\Theta)} \\ H_*(LS^2) & & & & \\ & \underset{(31)}{\searrow} & & & \\ & & HH^*(S^*(S^2); S^*(S^2)) & \xrightarrow{(2)} & HH^*(H^*(S^2); H^*(S^2)). \end{array}$$

This is surprising because as explained by Cohen and Jones [7, p. 792], the composite of the isomorphism (14) given by Jones in [17] and an isomorphism induced by Poincaré duality should give an isomorphism of algebras between $\mathbb{H}_*(LS^2)$ and $HH^*(S^*(S^2); S^*(S^2))$.

8. Appendix by Gerald Gaudens and Luc Menichi

Let X be a pointed topological space. Recall that the circle S^1 acts on the double loop space $\Omega^2 X$. Consider the induced operator $\Delta\colon H_*(\Omega^2 X) \to H_{*+1}(\Omega^2 X)$. Getzler [14] has shown that $H_*(\Omega^2 X)$ equipped with the Pontryagin product and this operator Δ forms a Batalin–Vilkovisky algebra. In [12], Gerald Gaudens and the author have determined this Batalin–Vilkovisky algebra $H_*(\Omega^2 S^3; \mathbb{F}_2)$. The key was the following theorem. In [18, Proposition 7.46], answering to a question of Gerald Gaudens, Sadok Kallel and Paolo Salvatore give another proof of this theorem.

Theorem 32 ([12]). *The operator* $\Delta\colon H_1(\Omega^2 S^3; \mathbb{F}_2) \to H_2(\Omega^2 S^3; \mathbb{F}_2)$ *is non-trivial.*

Both proofs [12] and [18, Proposition 7.46] are unpublished and publicly unavailable yet. So the goal of this section is to give a proof of this theorem which is as simple as possible.

Denote by $*$ the Pontryagin product in $H_*(\Omega^2 X)$ and by $\circ$ the map induced in homology by the composition map $\Omega^2 X \times \Omega^2 S^2 \to \Omega^2 X$. Denote by $\Omega^2_n S^2$, the path-connected component of the degree n maps. Denote by v_1 the generator of $H_1(\Omega^2_0 S^2; \mathbb{F}_2)$ and by [1] the generator of $H_0(\Omega^2_1 S^2; \mathbb{F}_2)$.

Lemma 33. *For $x \in H_*(\Omega^2 X; \mathbb{F}_2)$, $\Delta x = x \circ (v_1 * [1])$.*

Proof. The circle S^1 acts on the sphere S^2. Therefore we have a morphism of topological monoids $\Theta\colon (S^1, 1) \to (\Omega^2_1 S^2, \mathrm{id}_{S^2})$. The action of S^1 on $\Omega^2 X$ is the composite $S^1 \times \Omega^2 X \xrightarrow{\Theta \times \Omega^2 X} \Omega^2_1 S^2 \times \Omega^2 X \xrightarrow{\circ} \Omega^2 X$. Therefore for $x \in H_*(\Omega^2 X; \mathbb{F}_2)$, $\Delta x = x \circ (H_1(\Theta)[S^1])$.

Suppose that $H_1(\Theta)[S^1] = 0$. Then for any topological space X, the operator Δ on $H_*(\Omega^2 X; \mathbb{F}_2)$ is null. Therefore, for any x and $y \in H_*(\Omega^2 X; \mathbb{F}_2)$, $\{x, y\} = \Delta(xy) - (\Delta x)y - x(\Delta y) = 0$. That is the modulo 2 Browder brackets on any double loop space are null. This is obviously false. For example, Cohen in [3] explains that the Gerstenhaber algebra $H_*(\Omega^2 \Sigma^2 Y)$ has in general many non-trivial Browder brackets. So the assumption $H_1(\Theta)[S^1] = 0$ is false.

Since the loop multiplication by id_{S^2} in the H-group $\Omega^2 S^2$ is a homotopy equivalence, the Pontryagin product by [1], $*[1]\colon H_*(\Omega^2_0 S^2) \xrightarrow{\cong} H_*(\Omega^2_1 S^2)$ is an isomorphism. Therefore $v_1 * [1]$ is a generator of $H_1(\Omega^2_1 S^2)$, hence $H_1(\Theta)[S^1] = v_1 * [1]$. So finally

$$\Delta x = x \circ (H_1(\Theta)[S^1]) = x \circ (v_1 * [1]). \qquad \square$$

Recall that v_1 denotes the generator of $H_1(\Omega^2_0 S^2; \mathbb{F}_2)$.

Lemma 34. *In the Batalin–Vilkovisky algebra $H_*(\Omega^2 S^2; \mathbb{F}_2)$, $\Delta(v_1) = v_1 * v_1$.*

Proof. Recall that [1] is the generator of $H_0(\Omega^2_1 S^2)$. By Lemma 33,

$$\Delta[1] = [1] \circ (v_1 * [1]) = (v_1 * [1]).$$

Denote by $Q\colon H_q(\Omega^2_n S^2) \to H_{2q+1}(\Omega^2_{2n} S^2)$ the Dyer–Lashof operation. It is well known that $Q[1] = v_1 * [2]$. So by [4, p. 218, Theorem 1.3 (4)]

$$\{v_1 * [2], [1]\} = \{Q[1], [1]\} = \{[1], \{[1], [1]\}\}.$$

By [4, p. 215, Theorem 1.2 (3)], $\{[1], [1]\} = 0$. Therefore on one hand, $\{v_1 * [2], [1]\}$ is null. And on the other hand, using the Poisson relation (7), since $\{[2], [1]\} = \{[1] * [1], [1]\} = 2\{[1], [1]\} * [1] = 0$,

$$\{v_1 * [2], [1]\} = \{v_1, [1]\} * [2] + v_1 * \{[2], [1]\} = \{v_1, [1]\} * [2].$$

Since $*[1]\colon H_*(\Omega^2 S^2) \xrightarrow{\cong} H_*(\Omega^2 S^2)$ is an isomorphism, we obtain that the Browder bracket $\{v_1, [1]\}$ is null. Therefore,

$$\Delta(v_1 * [1]) = (\Delta v_1) * [1] + v_1 * (\Delta[1]) = ((\Delta v_1) - v_1 * v_1) * [1].$$

But $\Delta(v_1 * [1]) = (\Delta \circ \Delta)([1]) = 0$. Therefore (Δv_1) must be equal to $v_1 * v_1$. □

Proof of Theorem 32. We remark that since Δ preserves path-connected components and since the loop multiplication of two homotopically trivial loops is a homotopically trivial loop, $H_*(\Omega^2_0 S^2)$ is a sub Batalin–Vilkovisky algebra of $H_*(\Omega^2 S^2)$.

Let $S^1 \hookrightarrow S^3 \overset{\eta}{\twoheadrightarrow} S^2$ be the Hopf fibration. After double looping, the Hopf fibration gives the fibration $\Omega^2 S^1 \hookrightarrow \Omega^2 S^3 \overset{\Omega^2\eta}{\twoheadrightarrow} \Omega^2_0 S^2$ with contractile fiber $\Omega^2 S^1$ and path-connected base $\Omega^2_0 S^2$. Therefore $\Omega^2\eta\colon \Omega^2 S^3 \xrightarrow{\simeq} \Omega^2_0 S^2$ is a homotopy equivalence. And so $H_*(\Omega^2\eta)\colon H_*(\Omega^2 S^3) \xrightarrow{\simeq} H_*(\Omega^2_0 S^2)$ is an isomorphism of Batalin–Vilkovisky algebras.

Let u_1 be the generator of $H_1(\Omega^2 S^3)$. Lemma 34 implies that $\Delta(u_1) = u_1 * u_1$. Since $u_1 * u_1$ is non-zero in $H_*(\Omega^2 S^3; \mathbb{F}_2)$, $\Delta(u_1)$ is non-trivial. □

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