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Bounding the symbol length in the Galois cohomology of function fields of p -adic curves

Venapally Suresh

Dedicated to my teacher Professor R. Parimala on her 60th birthday

Abstract. Let K be a function field of a p -adic curve and l a prime not equal to p . Assume that K contains a primitive l^{th} root of unity. We show that every element in the l -torsion subgroup of the Brauer group of K is a tensor product of two cyclic algebras over K .

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Introduction

Let k be a field and l a prime number not equal to the characteristic of k . Let μ_l be the group of l^{th} roots of unity and $\mu_l(m)$ the tensor product of m copies of μ_l . For $n \geq 0$, let $H^n(k, \mu_l)$ denote the n^{th} Galois cohomology group with coefficients in μ_l . Let $k^* = k \setminus \{0\}$. We have an isomorphism $k^*/k^{*l} \rightarrow H^1(k, \mu_l)$. For $a \in k^*$, let (a) denote its image in $H^1(k, \mu_l)$. For $a_1, \dots, a_m \in k^*$, the cup product gives an element $(a_1) \cdot (a_2) \dots (a_m) \in H^n(k, \mu_l(m))$, which we call a *symbol*.

Assume that k contains a primitive l^{th} root of unity. Fix a primitive l^{th} root of unity $\zeta \in k$. Then we have isomorphisms $\mu_l \rightarrow \mu_l(m)$ of Galois groups. Hence we have isomorphisms $H^n(k, \mu_l(m)) \rightarrow H^n(k, \mu_l)$. A *symbol* in $H^n(k, \mu_l)$ is simply the image of a symbol under this map.

A classical theorem of Merkurjev ([M]) asserts that every element in $H^2(k, \mu_2)$ is a sum of symbols. A deep result of Merkurjev and Suslin ([MS]) says that every element in $H^2(k, \mu_l)$ is a sum of symbols. By a theorem of Voevodsky ([V]), every element in $H^n(k, \mu_2)$ is a sum of symbols. Suppose that k is a p -adic field. Local class field theory tells us that every element in $H^2(k, \mu_l)$ is a symbol and $H^n(k, \mu_l) = 0$ for $n \geq 3$. If k is a global field, then the global class field theory asserts that every element in $H^n(k, \mu_l)$ is a symbol.

Question 1. Do there exist integers $N_l(n)(k)$ such that every element in $H^n(k, \mu_l)$ is a sum of at most $N_l(n)(k)$ symbols?

Of course, the answer to the above question is negative in general. It can be shown that for $K = k(X_1, \dots, X_n, \dots)$, there is no such $N_l(n)(K)$ for $n \geq 2$. However we can restrict to some special fields. It is well-known that if $N_l(n)(k)$ exist for k , then $N_l(n)(k((t)))$ exist. We ask the following

Question 2. Suppose that $N_l(n)(k)$ exist for some field k . Do they exist for $k(t)$?

This is an open question. However we can restrict to fields of arithmetic interest. For example we consider the p -adic fields. The most important result in this direction is the following

Theorem (Saltman, [S1], (cf. [S2])). *Let k be a p -adic field and $K/k(t)$ be a finite extension. Suppose that $l \neq p$. If A is a central simple algebra over K representing an element in $H^2(K, \mu_l)$, then $\text{ind}(A)$ divides l^2 .*

Let K be as in the above theorem. Suppose $p \neq 2$. Let $\alpha \in H^2(K, \mu_2)$ and A a central simple algebra over K representing α . Then by the above theorem, we have $\text{ind}(A) = 1, 2, 4$. If $\text{ind}(A) = 1$, then α is a trivial element. If $\text{ind}(A) = 2$, then it is well known that α is a symbol. Assume that $\text{ind}(A) = 4$. By a classical theorem of Albert ([A]), α is a sum of two symbols. For $H^3(K, \mu_l)$, we have the following

Theorem ([PS2], 3.5, (cf. [PS1], 3.9)). *Let k be a p -adic field and $K/k(t)$ be a finite extension. Suppose that $l \neq p$. Every element in $H^3(K, \mu_l)$ is a symbol.*

Let k and K be as above. The field K is of cohomological dimension 3 and $H^n(K, \mu_l) = 0$ for $n \geq 4$. By the above theorem, $N_l(3)(K) = 1$ and the only case where $N_l(n)(K)$ is to be determined is for $n = 2$. It is known that $N_l(2)(K) \geq 2$ (cf. [S1], Appendix). In this article we prove the following

Theorem. *Let k be a p -adic field and $K/k(t)$ be a finite extension. Suppose that $l \neq p$. Every element in $H^2(K, \mu_l)$ is a sum of at most two symbols; in other words, $N_l(2)(K) = 2$.*

1. Some preliminaries

In this section we recall a few basic facts about Galois cohomology groups and divisors on arithmetic surfaces. We refer the reader to ([C]), ([Li1]), ([Li2]) and ([Se]).

Let k be a field and l a prime number not equal to the characteristic of k . Assume that k contains a primitive l^{th} root of unity. Let $\zeta \in k$ be a primitive l^{th} root of unity. Let μ_l be the group of l^{th} roots of unity. Since k contains a primitive l^{th} root of unity, the absolute Galois group of k acts trivially on μ_l . For $m \geq 1$, let $\mu_l(m)$ denote the tensor product of m copies of μ_l . By fixing a primitive l^{th} root of unity ζ in k , we have isomorphisms of Galois modules $\mu_l(m) \rightarrow \mu_l$. Throughout this paper we fix a primitive l^{th} root of unity and identify $\mu_l(m)$ with μ_l .

Let $H^n(k, A)$ be the n^{th} Galois cohomology group of the absolute Galois group Γ of k with values in a discrete Γ -module A . The identification of $\mu_l(m)$ with μ_l gives an identification of $H^n(k, \mu_l(m))$ with $H^n(k, \mu_l)$. In the rest of this paper we use this identification.

Let $k^* = k \setminus \{0\}$. For $a, b, c \in k^*$ we have the following relations in $H^2(k, \mu_l)$.

- (1) $(a) \cdot (bc) = (a) \cdot (b) + (a) \cdot (c)$;
- (2) $(a) \cdot (b) = -((b) \cdot (a))$;
- (3) $(a) \cdot (b^l) = 0$;
- (4) $(a) \cdot (-a) = 0$.

If $l \geq 3$, we have $(a) \cdot (a) = (a) \cdot ((-1)^l a) = (a) \cdot (-a) = 0$.

Let K be a field and l a prime number not equal to the characteristic of K . Let v be a discrete valuation of K . The residue field of v is denoted by $\kappa(v)$. Suppose $\text{char}(\kappa(v)) \neq l$. Then there is a residue homomorphism $\partial_v: H^n(K, \mu_l(m)) \rightarrow H^{n-1}(\kappa(v), \mu_l(m-1))$. Let $\alpha \in H^n(K, \mu_l(m))$. We say that α is *unramified* at v if $\partial_v(\alpha) = 0$; otherwise it is said to be *ramified* at v .

Let \mathcal{X} be a regular integral scheme of dimension d , with function field K . Let \mathcal{X}^1 be the set of points of \mathcal{X} of codimension 1. A point $x \in \mathcal{X}^1$ gives rise to a discrete valuation v_x on K . The residue field of this discrete valuation ring is denoted by $\kappa(x)$. The corresponding residue homomorphism is denoted by ∂_x . We say that an element $\zeta \in H^n(K, \mu_l(m))$ is *unramified* at x if $\partial_x(\zeta) = 0$; otherwise it is said to be *ramified* at x . We define the ramification divisor $\text{ram}_{\mathcal{X}}(\zeta) = \sum x$ as x runs over points in \mathcal{X}^1 where ζ is ramified. Suppose C is an irreducible subscheme of \mathcal{X} of codimension 1. Then the generic point x of C belongs to \mathcal{X}^1 and we set $\partial_C = \partial_x$. If $\alpha \in H^n(K, \mu_l(m))$ is unramified at x , then we say that α is *unramified* at C . We say that α is *unramified* on \mathcal{X} if it is unramified at every point of \mathcal{X}^1 .

Let k be a p -adic field and K the function field of a smooth projective geometrically integral curve X over k . By the resolution of singularities for surfaces (cf. [Li1] and [Li2]), there exists a regular projective model \mathcal{X} of X over the ring of integers \mathcal{O}_k of k . We call such an \mathcal{X} a *regular projective model* of X . Since the generic fibre X of \mathcal{X} is geometrically integral, it follows that the special fibre $\overline{\mathcal{X}}$ is connected. Further if D is a divisor on \mathcal{X} , there exists a proper birational morphism $\mathcal{X}' \rightarrow \mathcal{X}$ such that the total transform of D on \mathcal{X}' is a divisor with normal crossings (cf. [Sh], Theorem, p. 38 and Remark 2, p. 43). We use this result throughout this paper without further

reference. If $P \in \mathcal{X}$ is a closed point and $f \in K$ is a unit at P , then we denote the image of f in the residue field at P by $f(P)$.

Let k be a p -adic field and K the function field of a smooth projective geometrically integral curve over k . Let l be a prime number not equal to p . Assume that k contains a primitive l^{th} root of unity. Let $\alpha \in H^2(K, \mu_l)$. Let \mathcal{X} be a regular projective model of K such that $\text{ram}_{\mathcal{X}}(\alpha) = C + E$, where C and E are regular curves with normal crossings. We have the following

Theorem 1.1 (Saltman [S1]). *Let $K, \alpha, \mathcal{X}, C$ and E be as above and $P \in C \cup E$. Let R be the local ring at P . Let $\pi, \delta \in R$ be local equations of C and E respectively at P .*

- (1) *If $P \in C \setminus E$ (or $E \setminus C$), then $\alpha = \alpha' + (\pi) \cdot (u)$ (or $\alpha = \alpha' + (\delta) \cdot (u)$) for some unit $u \in R$, $\alpha' \in H^2(K, \mu_l)$ unramified on R .*
- (2) *If $P \in C \cap E$, then either $\alpha = \alpha' + (\pi) \cdot (u) + (\delta) \cdot (v)$ or $\alpha = \alpha' + (\pi) \cdot (u\delta^i)$ for some units $u, v \in R$, $\alpha' \in H^2(K, \mu_l)$ unramified on R .*

Let $P \in C \cap E$. Suppose that $\alpha = \alpha' + (\pi) \cdot (u) + (\delta) \cdot (v)$ for some units $u, v \in R$, $\alpha' \in H^2(K, \mu_l)$ unramified on R and π, δ are local equations of C and E respectively. Then $u(P) = \partial_C(\alpha)(P)$ and $v(P) = \partial_E(\alpha)(P)$. Note that $u(P)$ and $v(P)$ are uniquely defined modulo l^{th} powers. Following Saltman ([S3], §2), we say that P is a *hot point* of α if $u(P)$ and $v(P)$ do not generate the same subgroup of $\kappa(P)^*/\kappa(P)^{*l}$.

We have the following

Theorem 1.2 (Saltman ([S3], 5.2)). *Let $k, K, \alpha, \mathcal{X}$ be as above. Then α is a symbol if and only if there are no hot points of α .*

2. The main theorem

Let k be a p -adic field and $K/k(t)$ be a finite extension. Let $l \geq 3$ be a prime number not equal to p . Assume that k contains a primitive l^{th} root of unity. Let $\beta \in H^2(K, \mu_l)$ and \mathcal{X} a regular proper model of K . Let $\phi: \mathcal{X}' \rightarrow \mathcal{X}$ be a blow-up such that \mathcal{X}' is a regular proper model of K and $\text{ram}_{\mathcal{X}'}(\beta) = C' + E'$, where C' and E' are two regular curves with normal crossings (cf. §1 or [S1], Proof of 2.1). Let $Q \in C' \cap E'$. Let $C'_1 \subset C'$ and $E'_1 \subset E'$ be the irreducible curves containing Q . Let $R' = \mathcal{O}_{\mathcal{X}', Q}$ be the regular local ring at Q and m_Q its maximal ideal. We have $m_Q = (\pi', \delta')$, where π' and δ' are local equations of C'_1 and E'_1 at Q respectively. Let $v_{C'_1}$ and $v_{E'_1}$ be the discrete valuations on K at C'_1 and E'_1 respectively.

Let $P = \phi(Q)$. Let R be the regular local ring at P and m_P its maximal ideal. We have the induced homomorphism $\phi^*: R \rightarrow R'$ of local rings, which is injective. Let $\pi, \delta \in R$ be such that $m_P = (\pi, \delta)$.

Lemma 2.1. *Suppose that $\beta = \beta' + (f') \cdot (g')$ for some $f', g' \in K$ and β' unramified on R' . Then Q is not a hot point of β .*

Proof. Since β' is unramified on R' , the ramification data of β on R' is same as that of $(f') \cdot (g')$. Since $(f') \cdot (g')$, being a symbol, has no hot points ([S3], cf. 1.2), Q is not a hot point of β . \square

Lemma 2.2. *Suppose that $\beta = \beta' + (\delta) \cdot (gv) + (f) \cdot (g)$, where β' is unramified on R' , $f \in R$ is not divisible by δ and $v, g \in R$ are units with $g(P) = v(P)$. Then Q is not a hot point of β .*

Proof. We have $m_Q = (\pi', \delta')$ and β has ramification on R' only at π' and δ' . Since $R/m_P \hookrightarrow R'/m_Q$, we have $g(Q) = g(P) = v(P) = v(Q)$. If C'_1 (or E'_1) is the strict transform of a curve on \mathcal{X} , then either δ is a local equation of C'_1 or $v_{C'_1}(\delta) = 0$. In fact, if C'_1 is the strict transform of C_1 on \mathcal{X} , then $v_{C_1}(\delta) = v_{C'_1}(\delta)$ and δ itself being a prime in R , the assertion follows.

Suppose that C'_1 and E'_1 are strict transforms of two irreducible curves on \mathcal{X} . If δ is not a local equation for either C'_1 or E'_1 , we claim that $(\delta) \cdot (gv)$ is unramified on R' . In fact, since g and v are units in R , $(\delta) \cdot (gv)$ is unramified on R except possibly at δ . Since f is not divisible by δ , $(f) \cdot (g)$ is unramified at δ . Since β is ramified on R' only at π' and δ' and δ is not one of them, $(\delta) \cdot (gv)$ is unramified on R' . By (2.1), Q is not a hot point of β . Assume that δ is a local equation for one of them, say C'_1 . Since δ does not divide f , we have $\partial_{C'_1}(\beta) = \bar{v}g$ and $\partial_{E'_1}(\beta) = \tilde{g}^{v_{E'_1}(f)}$, where $\bar{}$ denotes the image in the residue field of C'_1 and $\tilde{}$ denotes the image in the residue field of E'_1 . Since β is ramified at E'_1 , $v_{E'_1}(f)$ is not a multiple of l . We have $\partial_{C'_1}(\beta)(Q) = v(Q)g(Q) = g(Q)^2$ and $\partial_{E'_1}(\beta)(Q) = g(Q)^{v_{E'_1}(f)}$. Since $l \neq 2$ and $v_{E'_1}(f)$ is not a multiple of l , $g(Q)^2$ and $g(Q)^{v_{E'_1}(f)}$ generate the same subgroup modulo l^{th} powers. Hence Q is not a hot point of β .

Suppose that C'_1 is a strict transform of an irreducible curve on \mathcal{X} and E'_1 is an exceptional curve on \mathcal{X}' . We have $\partial_{E'_1}(\beta) = (\tilde{g}v)^{v_{E'_1}(\delta)} \tilde{g}^{v_{E'_1}(f)}$. Since E'_1 is an exceptional fibre in \mathcal{X}' , the residue field of R is contained in the residue field at E'_1 . Hence $\partial_{E'_1}(\beta) = (g(P)v(P))^{v_{E'_1}(\delta)} g(P)^{v_{E'_1}(f)} = g(P)^{2v_{E'_1}(\delta) + v_{E'_1}(f)}$. Since β is ramified at E'_1 , $2v_{E'_1}(\delta) + v_{E'_1}(f)$ is not a multiple of l . Suppose δ is a local equation of C'_1 at Q . Since δ does not divide f and $v_{C'_1}(\delta) = 1$, we have $\partial_{C'_1}(\beta) = \bar{g}v$. Thus $\partial_{C'_1}(\beta)(Q) = g(P)v(P) = g(P)^2$. Since $l \neq 2$ and $2v_{E'_1}(\delta) + v_{E'_1}(f)$ is not a multiple of l , the subgroups generated by $g(Q)^2$ and $g(P)^{2v_{E'_1}(\delta) + v_{E'_1}(f)}$ are equal modulo l^{th} powers. Hence Q is not a hot point of β . Suppose δ is not a local equation of C'_1 at Q . We have $\partial_{C'_1}(\beta) = \tilde{g}^{v_{C'_1}(f)}$. Since β is ramified at C'_1 , $v_{C'_1}(f)$ is not a multiple of l . Thus as above Q is not a hot point of β .

The case E'_1 is a strict transform of an irreducible curve on \mathcal{X} and C'_1 is an exceptional curve in \mathcal{X}' follows on similar lines.

Suppose that both C'_1 and E'_1 are exceptional curves in \mathcal{X}' . Then as above we have $\partial_{C'_1}(\beta) = g(P)^{2\nu_{C'_1}(\delta) + \nu_{C'_1}(f)}$ and $\partial_{E'_1}(\beta) = g(P)^{2\nu_{E'_1}(\delta) + \nu_{E'_1}(f)}$. Since β is ramified at C'_1 and E'_1 , $2\nu_{C'_1}(\delta) + \nu_{C'_1}(f)$ and $2\nu_{E'_1}(\delta) + \nu_{E'_1}(f)$ are not multiples of l . In particular, the subgroups generated by $g(P)^{2\nu_{C'_1}(\delta) + \nu_{C'_1}(f)}$ and $g(P)^{2\nu_{E'_1}(\delta) + \nu_{E'_1}(f)}$ are equal modulo the l^{th} powers. Thus Q is not a hot point of β . \square

Lemma 2.3. *Suppose that $\beta = \beta' + (\pi) \cdot (u) + (\delta) \cdot (v)$, where β' unramified on R' and $u, v \in R$ units with $u(P) = v(P)$. Then Q is not a hot point of β .*

Proof. Since β is ramified at C'_1 , either $\nu_{C'_1}(\pi)$ or $\nu_{C'_1}(\delta)$ is not divisible by l . In particular their sum $\nu_{C'_1}(\pi\delta)$ is non-zero. We have

$$\partial_{C'_1}(\beta)(Q) = u(P)^{\nu_{C'_1}(\pi)} v(P)^{\nu_{C'_1}(\delta)} = u(P)^{\nu_{C'_1}(\pi\delta)}.$$

Suppose that $\nu_{C'_1}(\pi\delta)$ is a multiple of l . Since $\nu_{C'_1}(\pi\delta)$ is non-zero, C'_1 is an exceptional curve. As in the proof of (2.2), we see that $\partial_{C'_1}(\beta) = u(P)^{\nu_{C'_1}(\pi\delta)} = 1$. Which is a contradiction, as β is ramified at C'_1 . Hence $\nu_{C'_1}(\pi\delta)$ is not a multiple of l . Similarly, we have $\partial_{E'_1}(\beta)(Q) = u(P)^{\nu_{E'_1}(\pi\delta)}$ and $\nu_{E'_1}(\pi\delta)$ is not a multiple of l . Hence $u(P)^{\nu_{C'_1}(\pi\delta)}$ and $u(P)^{\nu_{E'_1}(\pi\delta)}$ generate the same subgroup of $\kappa(P)^*$ modulo $\kappa(P)^{*l}$ and Q is not a hot point of β . \square

Theorem 2.4. *Let k be a p -adic field and $K/k(t)$ be a finite extension. Let l be a prime number not equal to p . Suppose that k contains a primitive l^{th} root of unity. Then every element in $H^2(K, \mu_l)$ is a sum of at most two symbols.*

Proof. If $l = 2$, then, as we mentioned before, by ([A]), α is a sum of at most two symbols. Assume that $l \geq 3$. Let $\alpha \in H^2(K, \mu_l)$. Let \mathcal{X} be a regular proper model of K such that $\text{ram}_{\mathcal{X}}(\alpha) = C + E$, where C and E are regular curves with normal crossings.

Let $P \in C \cup E$ be a closed point of \mathcal{X} . Let R_P be the regular local ring at P on \mathcal{X} and m_P be its maximal ideal.

Let T be a finite set of closed points of \mathcal{X} containing $C \cap E$ and at least one closed point from each irreducible curve in C and E . Let A be the semi-local ring at T on \mathcal{X} . Let $\pi_1, \dots, \pi_r, \delta_1, \dots, \delta_s \in A$ be prime elements corresponding to irreducible curves in C and E respectively. Let $f_1 = \pi_1 \dots \pi_r \delta_1 \dots \delta_s \in A$. Let $P \in C \cap E$. Then $P \in C_i \cap E_j$ for unique irreducible curves C_i in C and E_j in E . Then $\pi = \pi_i$ and $\delta = \delta_j$ are local equations of C and E at P . We have $\alpha = \alpha' + (\pi) \cdot (u_P) + (\delta) \cdot (v_P)$

or $\alpha = \alpha' + (\pi) \cdot (u_P \delta^i)$ for some units, $u_P, v_P \in R$ and α' unramified on R ([S1], cf. 1.1). By the choice of f_1 , we have $f_1 = \pi \delta w_P$ for some $w_P \in A$ which is a unit at P . Let $u \in A$ be such that $u(P) = w_P(P)^{-1} u_P(P)$ for all $P \in C \cap E$. Let $f = f_1 u \in A$. Then, we have $(f) = C + E + F$, where F is a divisor on \mathcal{X} which avoids C, E and all the points of $C \cap E$. Further, for each $P \in C_i \cap E_j$, we have $f = \pi_i \delta_j w_{ij}$ for some $w_{ij} \in A$ such that $w_{ij}(P) = u_P(P)$.

By a similar argument, choose $g \in K$ satisfying

- (1) $(g) = C + G$, where G is a divisor on \mathcal{X} which avoids C, E, F and also avoids the points of $C \cap E, C \cap F, E \cap F$;
- (2) if $P \in E \cap F$ and $\alpha = \alpha' + (\delta) \cdot (v)$ for some unit $v \in R_P$ and α' is unramified at P , then $g(P) = v(P)$.

Since $C \cap E \cap F = \emptyset$, such a g exists.

We claim that $\beta = \alpha + (f) \cdot (g)$ is a symbol.

Let $\phi: \mathcal{X}' \rightarrow \mathcal{X}$ be a blow up of \mathcal{X} such that \mathcal{X}' is a regular proper model of K and $\text{ram}_{\mathcal{X}'}(\beta) = C' + E'$, where C' and E' are regular curves with normal crossings.

To show that β is a symbol, it is enough to show that β has no hot points ([S3], cf. 1.2). Let $Q \in C' \cap E'$. Let $P = \phi(Q)$. Then P is a closed point of \mathcal{X} , $R = \mathcal{O}_{\mathcal{X},P} \subset \mathcal{O}_{\mathcal{X}',Q} = R'$ and the maximal ideal m_P of R is contained in the maximal ideal m_Q of R' . Let $m_Q = (\pi', \delta')$, with π' and δ' be local equations of C' and E' at Q respectively.

Suppose that $P \notin C \cup E$. Then α is unramified at P and hence unramified at Q . By (2.1), Q is not a hot point of β .

Assume that $P \in C \cup E$.

Suppose that $P \in C \cap E$. Let π and δ be local equations of C and E at P respectively. Then $m_P = (\pi, \delta)$. By the choice of f and g , we have $f = \pi \delta w_1$ and $g = \pi w_2$ for some units $w_1, w_2 \in R$. In particular, β is ramified on R only at π and δ . Suppose that $\alpha = \alpha' + (\pi) \cdot (u) + (\delta) \cdot (v)$ for some units $u, v \in R$ and α' unramified on R . We have

$$\begin{aligned} \beta &= \alpha + (f) \cdot (g) \\ &= \alpha' + (\pi) \cdot (u) + (\delta) \cdot (v) + (\pi \delta w_1) \cdot (\pi w_2) \\ &= \alpha' + (\pi) \cdot (u) + (\delta w_1) \cdot (v) + (w_1^{-1}) \cdot (v) + (\pi) \cdot (\pi w_2) + (\delta w_1) \cdot (\pi w_2) \\ &= \alpha' + (w_1^{-1}) \cdot (v) + (\pi) \cdot (u \pi w_2) + (\delta w_1) \cdot (\pi w_2 v) \\ &= \alpha' + (w_1^{-1}) \cdot (v) + (\pi) \cdot (u w_2) + (\delta w_1) \cdot (\pi w_2 v) \\ &= \alpha' + (w_1^{-1}) \cdot (v) + (\pi w_2 v) \cdot (u w_2) + (w_2^{-1} v^{-1}) \cdot (u w_2) + (\delta w_1) \cdot (\pi w_2 v) \\ &= \alpha' + (w_1^{-1}) \cdot (v) + (w_2^{-1} v^{-1}) \cdot (u w_2) + (\pi w_2 v) \cdot (u w_2 \delta^{-1} w_1^{-1}). \end{aligned}$$

Since $\alpha' + (w_1^{-1}) \cdot (v) + (w_2^{-1} v^{-1}) \cdot (u w_2)$ is unramified on R , by (2.1), Q is not a hot point of β .

Suppose that $\alpha = \alpha' + (\pi) \cdot (u\delta^i)$ for some units, $u, v \in R$ and α' unramified on R . Then we have

$$\begin{aligned}\beta &= \alpha + (f) \cdot (g) \\ &= \alpha' + (\pi) \cdot (u\delta^i) + (\pi\delta w_1) \cdot (\pi w_2) \\ &= \alpha' + (\pi) \cdot (u\delta^i) + (\delta w_1 w_2^{-1}) \cdot (\pi w_2) \\ &= \alpha' + (\pi) \cdot (u\delta^i (\delta w_1 w_2^{-1})^{-1}) + (\delta w_1 w_2^{-1}) \cdot (w_2) \\ &= \alpha' + (\pi) \cdot (\delta^{i-1} u w_1^{-1} w_2) + (\delta w_1 w_2^{-1}) \cdot (w_2).\end{aligned}$$

If $i = 1$, then $\beta = \alpha' + (\pi) \cdot (u w_1^{-1} w_2) + (\delta w_1 w_2^{-1}) \cdot (w_2)$. Since, by the choice of f , $u(P) = w_1(P)$, by (2.3), Q is not a hot point of β . Assume that $i > 1$. Then $1 \leq i - 1 < l - 1$. Let i' be the inverse of $1 - i$ modulo l . We have

$$\begin{aligned}\beta &= \alpha' + (\pi) \cdot (\delta^{i-1} u w_1^{-1} w_2) + (\delta w_1 w_2^{-1}) \cdot (w_2) \\ &= \alpha' + (\delta^{1-i} u^{-1} w_1 w_2^{-1}) \cdot (\pi) + (\delta w_1 w_2^{-1}) \cdot (w_2) \\ &= \alpha' + ((\delta(u^{-1} w_1 w_2^{-1})^{i'})^{1-i}) \cdot (\pi) + (\delta(u^{-1} w_1 w_2^{-1})^{i'}) \cdot (w_2) \\ &\quad + ((u^{-1} w_1 w_2^{-1})^{-i'}) \cdot (w_2) + (w_1 w_2^{-1}) \cdot (w_2) \\ &= \alpha' + ((u^{-1} w_1 w_2^{-1})^{-i'}) \cdot (w_2) + (w_1 w_2^{-1}) \cdot (w_2) \\ &\quad + ((\delta(u^{-1} w_1 w_2^{-1})^{i'})^{1-i}) \cdot (\pi^{1-i} w_2).\end{aligned}$$

Since $\alpha' + ((u^{-1} w_1 w_2^{-1})^{-i'}) \cdot (w_2) + (w_1 w_2^{-1}) \cdot (w_2)$ is unramified on R , by (2.1), Q is not a hot point of β .

Suppose that $P \in C \setminus E$. We have $\alpha = \alpha' + (\pi) \cdot (u)$ for some unit u in R and α' unramified on R . We also have $f = \pi f_1$ for some $f_1 \in R$ which is not divisible by π . We have

$$\begin{aligned}\beta &= \alpha + (f) \cdot (g) \\ &= \alpha' + (\pi) \cdot (u) + (\pi f_1) \cdot (g) \\ &= \alpha' + (f_1^{-1}) \cdot (u) + (\pi f_1) \cdot (u) + (\pi f_1) \cdot (g) \\ &= \alpha' + (f_1^{-1}) \cdot (u) + (\pi f_1) \cdot (gu).\end{aligned}$$

If f_1 is a unit in R , then $\alpha' + (f_1^{-1}) \cdot (u)$ is unramified on R , by (2.1), Q is not a hot point of β . Assume that f_1 is not a unit in R . Then $P \in C \cap F$ and $g = \pi g_1$ for some unit $g_1 \in R$. We have

$$\begin{aligned}\beta &= \alpha + (f) \cdot (g) \\ &= \alpha' + (\pi) \cdot (u) + (\pi f_1) \cdot (\pi g_1) \\ &= \alpha' + (\pi g_1) \cdot (u) + (g_1^{-1}) \cdot (u) + (\pi f_1) \cdot (\pi g_1) \\ &= \alpha' + (g_1^{-1}) \cdot (u) + (\pi g_1) \cdot (u(\pi f_1)^{-1}).\end{aligned}$$

Since $\alpha' + (g_1^{-1}) \cdot (u)$ is unramified on R , by (2.1), Q is not a hot point of β .

Suppose that $P \in E \setminus C$. Then $\alpha = \alpha' + (\delta) \cdot (v)$ for some unit $v \in R$ and $f = \delta f_1$ for some $f_1 \in R$ which is not divisible by δ . Suppose that f_1 is a unit in R . Then, as above, Q is not a hot point of β . Assume that f_1 is not a unit in R . Then $P \in E \cap F$ and g is a unit in R . We have

$$\begin{aligned}\beta &= \alpha + (f) \cdot (g) \\ &= \alpha' + (\delta) \cdot (v) + (\delta f_1) \cdot (g) \\ &= \alpha' + (\delta) \cdot (vg) + (f_1) \cdot (g).\end{aligned}$$

Since α' is unramified on R and by the choice of g , $g(P) = v(P)$, by (2.2), Q is not a hot point of β .

By ([S3], cf. 1.2), β is symbol. Thus $\alpha = (f) \cdot (g) - \beta$ is a sum of at most two symbols. \square

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