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Bounding the regularity of subschemes invariant under Pfaff fields on projective spaces

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Abstract. A Pfaff field on $\mathbb{P}^n_k$ is a map $\eta\colon \Omega^s_{\mathbb{P}^n_k} \to \mathcal{L}$ from the sheaf of differential s-forms to an invertible sheaf. The interesting ones are those arising from a Pfaff system, as they give rise to a distribution away from their singular locus. A subscheme $X \subseteq \mathbb{P}^n_k$ is said to be invariant under η if η induces a Pfaff field $\Omega^s_X \to \mathcal{L}|_X$. We give bounds for the Castelnuovo–Mumford regularity of invariant complete intersection subschemes (more generally, arithmetically Cohen–Macaulay subschemes) of dimension s, depending on how singular these schemes are, thus bounding the degrees of the hypersurfaces that cut them out.

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1. Introduction

In 1891, Poincaré [Po], p. 161, posed the problem of bounding a priori the degree of the first integral of a polynomial vector field on the complex plane, when the integral is algebraic. The importance of such a bound is that it allows us to decide whether the integral is algebraic or not by making purely algebraic computations.

Poincaré himself produced bounds in special cases. But no bounds have been found in general. Actually, many obstructions to finding such bounds have been discovered: For instance, Lins Neto [Ln] produced examples to show that a bound cannot depend only on the degree m of the vector field and on the analytic type of its singularities in the plane or at infinity.

The current interest in Poincaré's problem was revived exactly a hundred years later by Lins Neto and Cerveau [CeLn], who showed that an algebraic curve invariant under the vector field has degree at most $m + 2$ if the singularities of the curve are ordinary double points, the bound achieved only if the curve is reducible; see loc. cit., Theorem 1, p. 891. Since then many papers have concentrated on this related problem,

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of bounding the degrees of algebraic curves invariant under the vector field. This has often been called the *Poincaré problem*. Works on this problem, allowing for more singular curves, are [CmCr], [Cr], [EK13], [dPW] and [Pe], to cite a few.

The problem has also been considered for higher dimensional spaces. One of the first to do so was Soares [S]. In today's language, and in great generality, let $\mathbb{P}^n_k$ denote the n-dimensional projective space over an algebraically closed field k, and consider a *Pfaff field*, a map $\eta\colon \Omega^s_{\mathbb{P}^n_k} \to \mathcal{L}$ from the sheaf of s-forms $\Omega^s_{\mathbb{P}^n_k} := \bigwedge^s \Omega^1_{\mathbb{P}^n_k}$, for an integer s between 1 and $n-1$ called the *rank* of η, to an invertible sheaf $\mathcal{L}$. Besides its rank, the unique other numerical global invariant under deformations of η is $m := \deg(\mathcal{L}) + s$, the *degree* of η. The *singular locus* of η is its degeneracy scheme $\mathcal{S}$, supported on the set of points where η is not surjective. A closed subscheme $X \subseteq \mathbb{P}^n_k$ is said to be *invariant* under η if η induces a Pfaff field $\Omega^s_X \to \mathcal{L}|_X$ on X. The above terminology is taken from [EK12], Section 3, to where the reader is directed for more details.

The Pfaff field η may arise by taking determinants from a *Pfaff system*, which, as defined by Jouanolou [J], pp. 136–138, is a map $\Omega^1_{\mathbb{P}^n_k} \to \mathcal{E}$ to a locally free sheaf $\mathcal{E}$ of rank s. (This is automatic for $s = 1$ but a strong condition for $s > 1$.) A Pfaff system may be seen as a "singular distribution," as it gives rise to an actual distribution on $\mathbb{P}^n_k - \mathcal{S}$. Then subschemes of pure dimension s that are invariant under η are solutions of the corresponding Pfaff system; see [EK12], Proposition 3.2, p. 3782, for a precise statement. Also, the degree m can be given a geometric interpretation in this case; see Section 4.

If $s = 1$ then η is the homogenization of a polynomial vector field on $\mathbb{C}^n$. If $s = n-1$, through the perfect pairing $\Omega^s_{\mathbb{P}^n_k} \otimes \Omega^{n-s}_{\mathbb{P}^n_k} \to \Omega^n_{\mathbb{P}^n_k}$, we may view η as the homogenization of a polynomial differential 1-form on $\mathbb{C}^n$. In both cases, η arises from a distribution away from $\mathcal{S}$.

Some of the statements in the literature, and all of the statements in the present article, work in positive characteristic, under suitable assumptions. However, to simplify the ongoing discussion, assume that k has characteristic zero.

For $s = n-1$ one may search for bounds on the degrees of hypersurfaces invariant under η. For instance, under the harmless assumption that $\dim(\mathcal{S}) \leq n-2$, Brunella and Mendes [BMe] showed that an invariant reduced hypersurface with at most normal-crossings singularities has degree at most $m+2$, generalizing the theorem by Cerveau and Lins Neto mentioned above; see loc. cit., p. 594, for a more general statement.

For $s = 1$ many inequalities have been produced for the degree and the genus of (reduced, equidimensional) curves invariant under η, for instance in [CmCrG] and [EK11]. However, in the spirit of Poincaré's original problem, one should look for bounds on global invariants that could reduce to purely algebraic computations the question of whether η has an invariant curve or not. The (Castelnuovo–Mumford) regularity is such an invariant, as it is well-known that a subscheme $X \subseteq \mathbb{P}_k^n$ is cut out by hypersurfaces with degree at most its regularity, $\operatorname{reg}(X)$.

Though a good measure of the complexity of a subscheme $X \subseteq \mathbb{P}_k^n$, by the reason explained above, the regularity is a concept of a rather arithmetic nature. At any rate, if X is arithmetically Cohen–Macaulay (a.C.M.), for example a complete intersection (see Subsection 2.2), the regularity acquires a more geometric meaning: cut X by as many general hyperplanes as its dimension to obtain a set Γ of points; then the regularity of X is the smallest integer r such that for each $P \in \Gamma$ there is a hypersurface of degree $r-1$ passing through all the points of Γ but P. So the regularity of Γ is higher the more special the position of the points of Γ is. For instance, if X is generically reduced, whence Γ is reduced by Bertini Theorem, and all the points of Γ are on a line, then it follows from Bezout Theorem that r is the number of points of Γ, the degree of X.

In [E] the second author shows that an invariant a.C.M. curve C, with at most ordinary double points for singularities, such that $\mathcal{S} \cap C$ is finite has regularity at most $m+2$, with equality only if the curve is reducible; see loc. cit., Theorem 1, p. 3. Since complete intersections are a.C.M., and since the regularity of a hypersurface is its degree, the statement is another generalization of Cerveau's and Lins Neto's result.

Later, the second author and Kleiman showed that the inequality $\operatorname{reg}(X) \leq m+2$ for an invariant a.C.M. curve (for $s=1$) or invariant reduced hypersurface $X \subseteq \mathbb{P}_k^n$ (for $s=n-1$) with normal-crossings singularities was a consequence of the fact that $h^s(\Omega_X^s(1)) = 0$, and that the same holds for intermediate s. More precisely, for any s, an invariant, reduced, a.C.M. subscheme $X \subseteq \mathbb{P}_k^n$ of pure dimension s whose irreducible components are not contained in $\mathcal{S}$ has regularity bounded by $m+2$ if $h^s(\Omega_X^s(1)) = 0$, and bounded by $m+1$ if $h^s(\Omega_X^s) = 1$; see [EK12], Corollary 4.5, p. 3790, and Remarks 4.6 and 4.7, p. 3791, from which the assertion can be extracted.

However, no further conditions for when $h^1(\Omega_X^1) = 1$ or $h^1(\Omega_X^1(1)) = 0$ are given in [EK12]. These appear later in [EK13], by the same authors, but only for $n=2$. There a (reduced) plane curve C of degree d is considered, and it is shown that if the singular locus of C has regularity σ bounded by $d-2$ then $h^1(\Omega_C^1) = 1$; and hence $d \leq m+1$ if C is invariant. The highly singular case is handled as well, being shown that if C is invariant and $\rho := \sigma - d + 2$ is positive, then $d \leq m+1+\rho$, with equality if $d \geq 2m+2$ and $\mathcal{S}$ is finite; see loc. cit., Theorem 2.5, p. 61.

In the present article, we extend the results of [EK13] for $n > 2$ and any s. More precisely, our Theorem 3.1 states that a connected, reduced subscheme $X \subseteq \mathbb{P}^n$ of pure dimension $s > 0$ satisfies $h^s(\Omega_X^s) = 1$ if X is a.C.M. and subcanonical, for instance a complete intersection, and if its singular locus has regularity σ bounded by $r-2$, where r is the regularity of X. From it follows Theorem 4.1, stating that $r \leq m+1$ if in addition X is invariant and $\dim(\mathcal{S} \cap X) < s$. Furthermore, by our Theorem 4.3, if X is simply a.C.M., and is invariant with $\dim(\mathcal{S} \cap X) < s$, then

$r \leq m + 1 + \rho$, where $\rho := \max(1, \sigma - r + 2)$. Finally, Theorem 5.3 says that $r = m + 1 + \rho$ if all the following conditions hold: $s = 1$ and $\mathcal{S}$ is finite; X is a.C.M., subcanonical and invariant; $r \geq 5$ if $m = 1$ or $r \geq mn - n + 4$ if $m > 1$.

Since complete intersections are a.C.M., subcanonical subschemes, we obtain as a corollary that, if $X \subseteq \mathbb{P}^n_k$ is a reduced complete intersection of hypersurfaces of degrees $d_1, \dots, d_{n-s}$, and is invariant under η with $\dim(\mathcal{S} \cap X) < s$, then

$$d_1 + \cdots + d_{n-s} \leq \begin{cases} m + n - s & \text{if } \rho \leq 0, \\ m + n - s + \rho & \text{if } \rho > 0, \end{cases}$$

where $\rho := \sigma + n - s + 1 - d_1 - \cdots - d_{n-s}$, with σ denoting the regularity of the singular locus of X; see Corollary 4.4.

The techniques we use are quite simple: basically, a detailed analysis of the long exact sequences in cohomology of several short exact sequences of sheaves associated to the problem.

The pervasive hypothesis of arithmetic Cohen–Macaulayness is necessary, as the example of a sequence of smooth curves in $\mathbb{P}^3$ of increasing regularity but invariant under degree-1 rank-1 Pfaff fields, presented in [E], Remark 21, p. 14, shows. What is not clearly necessary is the hypothesis of subcanonicalness.

The possibility that $r = m + 1 + \rho$ is investigated only for $s = 1$, because then $\mathcal{S}$ is easier to understand. Then, if $\mathcal{S}$ has dimension 0, which is the expected dimension and the case when η is general, the regularity of $\mathcal{S}$ is 1 if $m = 1$ and $mn - n + 2$ if $m > 1$; see Proposition 5.1 and the remark thereafter. This regularity gives the bound above which r must be for the equality $r = m + 1 + \rho$ to hold. On the other hand, for $s \geq 2$, those η having an invariant reduced subscheme of pure dimension s have large singular locus; indeed, $\dim(\mathcal{S}) \geq s - 1$ by [EKl2], Corollary 4.5, p. 3790. In particular, $\mathcal{S}$ does no have the expected dimension.

Section 2 collects a few results on the Castelnuovo–Mumford regularity and on arithmetically Cohen–Macaulay subschemes. In Section 3 we give conditions for when a subscheme $X \subseteq \mathbb{P}^n_k$ of pure dimension s satisfies $h^s(\Omega^s_X) = 1$. In Section 4 we prove our bounds on the regularity of closed subschemes invariant under Pfaff fields. Finally, in Section 5 we prove that these bounds are attained, if the regularity is large enough, in the case of rank-1 Pfaff fields.

2. Arithmetically Cohen–Macaulay subschemes

2.1. The Castelnuovo–Mumford regularity. Fix a positive integer n. Given $m \in \mathbb{Z}$, we say that a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_k$ is *m-regular* if $H^i(\mathcal{F}(m-i)) = 0$ for each integer $i > 0$.

Let $X \subseteq \mathbb{P}^n_k$ be a closed subscheme. If $X \neq \mathbb{P}^n_k$ then the *Castelnuovo–Mumford regularity* of X, or simply *regularity*, is the smallest integer m for which its sheaf of ideals is m-regular. By definition, the regularity of $\mathbb{P}^n_k$ is 1. Denote the regularity of X by $\operatorname{reg}(X)$.

The regularity is well-defined. In fact, let $\mathcal{I}_X$ denote the sheaf of ideals of X, and consider the natural exact sequence:

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{O}_{\mathbb{P}^n_k} \longrightarrow \mathcal{O}_X \longrightarrow 0. \tag{1}$$

Twisting it by $m-n$ and taking cohomology we get the following exact sequence:

$$H^n(\mathcal{I}_X(m-n)) \longrightarrow H^n(\mathcal{O}_{\mathbb{P}^n_k}(m-n)) \longrightarrow H^n(\mathcal{O}_X(m-n)).$$

The middle group is zero if and only if $m \geq 0$. If $X \neq \mathbb{P}^n_k$ then the last group is zero, and hence $H^n(\mathcal{I}_X(m-n)) = 0$ only if $m \geq 0$.

The above reasoning shows that $\operatorname{reg}(X) \geq 0$. Furthermore, $\operatorname{reg}(X) = 0$ if and only if $X = \emptyset$. Indeed, if X is empty, $\mathcal{I}_X = \mathcal{O}_{\mathbb{P}^n_k}$, which is 0-regular by Serre computation. On the other hand, if $\mathcal{I}_X$ is 0-regular then $\mathcal{I}_X$ is globally generated, by [Mu], p. 99. Since $\operatorname{reg}(X) \neq 1$, we have that $X \neq \mathbb{P}^n_k$, and hence $\mathcal{I}_X \neq 0$. So $H^0(\mathcal{I}_X) \neq 0$, which implies that $\mathcal{I}_X = \mathcal{O}_{\mathbb{P}^n_k}$, and thus $X = \emptyset$.

Also, $\operatorname{reg}(X) = 1$ if and only if X is a linear subspace of $\mathbb{P}^n_k$. Indeed, if $\operatorname{reg}(X) = 1$ then $\mathcal{I}_X(1)$ is globally generated, which implies that X is cut out by a system of hyperplanes. Conversely, suppose X is a linear subspace of $\mathbb{P}^n_k$. Twisting (1) by $1-i$ and taking cohomology, we get the following exact sequence:

$$H^{i-1}(\mathcal{O}_{\mathbb{P}^n_k}(1-i)) \longrightarrow H^{i-1}(\mathcal{O}_X(1-i)) \longrightarrow H^i(\mathcal{I}_X(1-i)) \longrightarrow H^i(\mathcal{O}_{\mathbb{P}^n_k}(1-i)).$$

If $i > 1$, the second and last groups are zero, by Serre computation, and thus $H^i(\mathcal{I}_X(1-i)) = 0$ for $i > 1$. For $i = 1$ the last group is zero, and the first map is an isomorphism. Thus $H^1(\mathcal{I}_X) = 0$. So $\operatorname{reg}(X) \leq 1$. Since $X \neq \emptyset$, it follows that $\operatorname{reg}(X) = 1$.

Proposition 2.1. *Let $X \subseteq \mathbb{P}^n_k$ be a closed subscheme. If $\dim(X) = 0$ then $\operatorname{reg}(X)$ is the smallest nonnegative integer r such that $H^1(\mathcal{I}_X(r-1)) = 0$, where $\mathcal{I}_X$ is the sheaf of ideals of X.*

Proof. Clearly, $H^i(\mathcal{I}_X(m)) = 0$ for every $i > n$ and every $m \in \mathbb{Z}$. Thus the assertion follows from the definition of regularity if $n = 1$.

Suppose now that $n > 1$. We need only show that $H^i(\mathcal{I}_X(r-i)) = 0$ for each $i = 2, \ldots, n$ and each $r \geq 0$. Let $m \in \mathbb{Z}$. Since X has dimension zero, $H^i(\mathcal{O}_X(m)) = 0$ for every $i \geq 1$. On the other hand, from Serre computation, $H^i(\mathcal{O}_{\mathbb{P}^n_k}(m)) = 0$ for each $i = 1, \ldots, n-1$. Twisting the natural exact sequence

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{O}_{\mathbb{P}^n_k} \longrightarrow \mathcal{O}_X \longrightarrow 0 \tag{2}$$

by m, and taking cohomology, we get, for each $i = 2, \dots, n$, the exact sequence

$$H^{i-1}(\mathcal{O}_X(m)) \to H^i(\mathcal{I}_X(m)) \to H^i(\mathcal{O}_{\mathbb{P}^n_k}(m)) \to H^i(\mathcal{O}_X(m)).$$

If $i = 2, \dots, n-1$ then one has $H^{i-1}(\mathcal{O}_X(m)) = H^i(\mathcal{O}_{\mathbb{P}^n_k}(m)) = 0$, and therefore $H^i(\mathcal{I}_X(m)) = 0$. If $i = n$, since $H^{n-1}(\mathcal{O}_X(m)) = H^n(\mathcal{O}_X(m)) = 0$ because $n \geq 2$, we have

$$H^n(\mathcal{I}_X(m)) \cong H^n(\mathcal{O}_{\mathbb{P}^n_k}(m)).$$

But, from Serre computation, $H^n(\mathcal{O}_{\mathbb{P}^n_k}(m)) = 0$ if $m \geq -n$. Thus $H^n(\mathcal{I}_X(r-n)) = 0$ for each $r \geq 0$. □

2.2. Arithmetically Cohen–Macaulay subschemes. An equidimensional closed subscheme $X \subseteq \mathbb{P}^n_k$ is said to be *arithmetically Cohen–Macaulay* (or simply a.C.M.) if its coordinate ring is Cohen–Macaulay. Alternatively, if X has positive dimension, X is a.C.M. if the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^n_k}(m)) \longrightarrow H^0(\mathcal{O}_X(m))$$

is surjective and $H^j(\mathcal{O}_X(m)) = 0$ for each $m \in \mathbb{Z}$ and $j = 1, \dots, \dim(X) - 1$. Or, equivalently, X is a.C.M. if $H^j(\mathcal{I}_X(m)) = 0$ for each $m \in \mathbb{Z}$ and $j = 1, \dots, \dim(X)$, where $\mathcal{I}_X$ is the sheaf of ideals of X. Notice that it follows that $h^0(\mathcal{O}_X) = 1$, and hence that X is connected.

Complete intersections are the simplest examples of a.C.M. subschemes.

Proposition 2.2. *Let $X \subseteq \mathbb{P}^n_k$ be a closed subscheme of pure dimension $s > 0$. If X is arithmetically Cohen–Macaulay then* $\operatorname{reg}(X)$ *is the smallest nonnegative integer r such that $H^s(\mathcal{O}_X(r-s-1)) = 0$.*

Proof. Suppose first that $s = n$, that is, $X = \mathbb{P}^n_k$. By definition, the regularity of $\mathbb{P}^n_k$ is 1. On the other hand, by Serre computation, $H^n(\mathcal{O}_{\mathbb{P}^n_k}(r-n-1)) = 0$ if and only if $r \geq 1$. So, the proposition holds for $s = n$.

Now, assume $s < n$. Let $\mathcal{I}_X$ denote the sheaf of ideals of X. Since X is a.C.M., $H^i(\mathcal{I}_X(r-i)) = 0$ for every $r \in \mathbb{Z}$ and each $i = 1, \dots, s$. On the other hand, twisting the natural short exact sequence

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{O}_{\mathbb{P}^n_k} \longrightarrow \mathcal{O}_X \longrightarrow 0 \tag{3}$$

by $r - i$, and taking cohomology, we get the following exact sequence, for each integer $i > 0$:

$$H^{i-1}(\mathcal{O}_{\mathbb{P}^n_k}(r-i)) \longrightarrow H^{i-1}(\mathcal{O}_X(r-i)) \longrightarrow H^i(\mathcal{I}_X(r-i)) \longrightarrow H^i(\mathcal{O}_{\mathbb{P}^n_k}(r-i)). \tag{4}$$

For $i = s+2, \ldots, n-1$, since

$$H^{i-1}(\mathcal{O}_X(r-i)) = H^i(\mathcal{O}_{\mathbb{P}_k^n}(r-i)) = 0,$$

we have that $H^i(\mathcal{I}_X(r-i)) = 0$ for every $r \in \mathbb{Z}$. Also, since $H^n(\mathcal{O}_{\mathbb{P}_k^n}(r-n)) = 0$ for $r \geq 0$, and $H^{n-1}(\mathcal{O}_X(r-n)) = 0$ if $s < n-1$, it follows that $H^n(\mathcal{I}_X(r-n)) = 0$ for every $r \geq 0$ if $s < n-1$.

So, $\operatorname{reg}(X)$ is the smallest nonnegative integer r such that $H^{s+1}(\mathcal{I}_X(r-s-1)) = 0$. But, if $r \geq 0$ then

$$H^s(\mathcal{O}_{\mathbb{P}_k^n}(r-s-1)) = H^{s+1}(\mathcal{O}_{\mathbb{P}_k^n}(r-s-1)) = 0,$$

because $0 < s < n$, by Serre computation. So, by the exactness of (4) for $i = s+1$,

$$H^s(\mathcal{O}_X(r-s-1)) \cong H^{s+1}(\mathcal{I}_X(r-s-1))$$

for every integer $r \geq 0$. □

2.3. Subcanonical subschemes. Let $X \subseteq \mathbb{P}_k^n$ be a closed subscheme. Let ω_X be the dualizing sheaf of X, that is,

$$\omega_X := \mathcal{E}xt^{n-s}_{\mathcal{O}_{\mathbb{P}_k^n}}(\mathcal{O}_X, \mathcal{O}_{\mathbb{P}_k^n}(-1-n)),$$

where $s := \dim(X)$. If there is $a \in \mathbb{Z}$ such that $\omega_X \cong \mathcal{O}_X(a)$, then we say that X is *a-subcanonical* (or simply *subcanonical*). If $\dim(X) > 0$ then a is unique.

Proposition 2.3. *Let $X \subseteq \mathbb{P}_k^n$ be an arithmetically Cohen–Macaulay a-subcanonical subscheme of pure dimension $s > 0$. Then $a \geq -s-1$ and* $\operatorname{reg}(X) = a+s+2$.

Proof. Observe first that $H^0(\mathcal{O}_X(i)) = 0$ if and only if $i < 0$. Indeed, since $\mathcal{O}_X(1)$ is very ample, $h^0(\mathcal{O}_X(i)) > 0$ for $i \geq 0$. On the other hand, if there were a nonzero global section of $\mathcal{O}_X(i)$ for a certain $i < 0$, multiplying it by $-iH$, for a sufficiently general hyperplane section $H \subset X$, we would obtain a nonzero global section of $\mathcal{O}_X$ vanishing at H, which is absurd.

By duality, since $\mathcal{O}_X(a)$ is the dualizing sheaf of X,

$$h^s(\mathcal{O}_X(r-s-1)) = h^0(\mathcal{O}_X(a-r+s+1)).$$

So, $H^s(\mathcal{O}_X(r-s-1)) = 0$ if and only if $a-r+s+1 < 0$, that is, if and only if $r \geq a+s+2$. It follows now from Proposition 2.2 that $\operatorname{reg}(X) = \max(a+s+2, 0)$. However, since $\operatorname{reg}(X) > 0$, we have that $a+s+2 > 0$ and $\operatorname{reg}(X) = a+s+2$. □

3. The singular locus of a k-scheme

Let X be an algebraic k-scheme. For each integer $s \geq 0$, denote by Ω^s_X the sheaf of Kähler s-forms of X, that is, $\Omega^s_X := \bigwedge^s \Omega^1_X$, where Ω^1_X is the sheaf of Kähler differentials of X.

Assume X is reduced, projective and of pure dimension $s > 0$. Let ω_X be its dualizing sheaf and $\gamma_X \colon \Omega^s_X \to \omega_X$ the canonical map. The map γ_X is constructed as follows. Let $X_1, \dots, X_m$ be the irreducible components of X with their reduced induced subscheme structures. For each $i = 1, \dots, m$ there is a natural map $\gamma_i : \Omega^s_{X_i} \to \widetilde{\omega}_{X_i}$, where $\widetilde{\omega}_{X_i}$ is Kunz's sheaf of regular differential forms of X_i. Also, the map is an isomorphism on the smooth locus of X_i; see [Ku], pp. 103–105. Furthermore, by [Ku], Satz 2.2, p. 95, or [Lp], Theorem 0.2B, p. 15, the sheaf $\widetilde{\omega}_{X_i}$ is dualizing, in a natural way; so there is a natural isomorphism $\xi_i : \widetilde{\omega}_{X_i} \to \omega_{X_i}$.

The restriction map $\tau \colon \mathcal{O}_X \to \mathcal{O}_{X_1} \oplus \cdots \oplus \mathcal{O}_{X_m}$ induces a map $\tau' \colon \omega_{X_1} \oplus \cdots \oplus \omega_{X_m} \to \omega_X$. As τ is an isomorphism on the smooth locus of X, so is τ'. Then γ_X is, by definition, the composition

$$\Omega^s_X \longrightarrow \bigoplus_{i=1}^m \Omega^s_{X_i} \xrightarrow{(\gamma_1,\dots,\gamma_m)} \bigoplus_{i=1}^m \widetilde{\omega}_{X_i} \xrightarrow{(\xi_1,\dots,\xi_m)} \bigoplus_{i=1}^m \omega_{X_i} \xrightarrow{\tau'} \omega_X,$$

where the first map is induced by restriction. All the above maps are isomorphisms on the smooth locus of X, and thus so is γ_X.

Let Σ_X be the scheme-theoretic support of the cokernel of γ_X. We call Σ_X the *singular locus* of X. Since X is reduced, whence generically smooth, γ_X is generically an isomorphism, and hence $\dim(\Sigma_X) < s$.

The sheaf ω_X is torsion-free, rank-1. Indeed, it is generically isomorphic to Ω^s_X, whence has rank 1. Its torsion subsheaf $\mathcal{T}(\omega_X)$ is supported on a subscheme of dimension less than s, and hence $H^s(\mathcal{T}(\omega_X)) = 0$. On the other hand, the injection $\mathcal{T}(\omega_X) \to \omega_X$ corresponds by duality to a map $H^s(\mathcal{T}(\omega_X)) \to k$. Since this map is zero, so is the injection, that is, $\mathcal{T}(\omega_X) = 0$.

Since ω_X is torsion-free, and γ_X is generically an isomorphism, the kernel of γ_X is the torsion subsheaf $\mathcal{T}(\Omega^s_X) \subseteq \Omega^s_X$. Thus, we get an injection

$$\mathcal{I}_{\Sigma_X, X}\, \omega_X \hookrightarrow \frac{\Omega^s_X}{\mathcal{T}(\Omega^s_X)}. \tag{5}$$

If X is Gorenstein then ω_X is invertible, and hence (5) is an isomorphism.

Theorem 3.1. *Let $X \subseteq \mathbb{P}^n_k$ be a connected, reduced, arithmetically Cohen–Macaulay subcanonical subscheme of pure dimension $s > 0$. Let Σ_X be its singular locus. Let $r := \operatorname{reg}(X)$ and $\sigma := \operatorname{reg}(\Sigma_X)$. If $\sigma = 0$ or $\sigma \leq r - 2$ then $H^s(\Omega^s_X) \cong k$.*

Proof. The assertion follows from Serre computation if $s = n$. Assume $s < n$. Consider the injection

$$\mathcal{I}_{\Sigma_X,X}\,\omega_X \hookrightarrow \frac{\Omega^s_X}{\mathcal{T}(\Omega^s_X)}.$$

Since X is reduced, both the source and target of this injection are of rank 1. So the injection is generically an isomorphism. Since the torsion subsheaf $\mathcal{T}(\Omega^s_X)$ is supported in dimension at most $s-1$, it follows that

$$H^s(\Omega^s_X) \cong H^s(\mathcal{I}_{\Sigma_X,X}\,\omega_X).$$

Since $\omega_X \cong \mathcal{O}_X(r-s-2)$ by Proposition 2.3, we must show that

$$H^s(\mathcal{I}_{\Sigma_X,X}(r-s-2)) \cong k.$$

Set $a := r-s-2$. Let $\mathcal{I}_{\Sigma_X}$ and $\mathcal{I}_X$ be the sheaves of ideals of Σ_X and X in $\mathbb{P}^n_k$. We claim that

$$H^{s+1}(\mathcal{I}_X(a)) \cong k. \tag{6}$$

Indeed, twisting the natural exact sequence

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{O}_{\mathbb{P}^n_k} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

by a and taking cohomology, we get the exact sequence

$$H^s(\mathcal{O}_{\mathbb{P}^n_k}(a)) \longrightarrow H^s(\mathcal{O}_X(a)) \longrightarrow H^{s+1}(\mathcal{I}_X(a)) \longrightarrow H^{s+1}(\mathcal{O}_{\mathbb{P}^n_k}(a)).$$

The first and last groups above are zero because $s < n$ and $r > 0$, respectively. Thus

$$H^{s+1}(\mathcal{I}_X(a)) \cong H^s(\mathcal{O}_X(a)).$$

But $\omega_X \cong \mathcal{O}_X(a)$. So, by Serre Duality,

$$H^s(\mathcal{O}_X(a)) \cong H^0(\mathcal{O}_X) \cong k,$$

where the last isomorphism follows from the connectedness of X.

Now, twisting the natural exact sequence

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{I}_{\Sigma_X} \longrightarrow \mathcal{I}_{\Sigma_X,X} \longrightarrow 0$$

by a, and taking cohomology, we get the exact sequence

$$\begin{aligned} H^s(\mathcal{I}_X(a)) \longrightarrow H^s(\mathcal{I}_{\Sigma_X}(a)) \longrightarrow H^s(\mathcal{I}_{\Sigma_X,X}(a)) \\ \longrightarrow H^{s+1}(\mathcal{I}_X(a)) \longrightarrow H^{s+1}(\mathcal{I}_{\Sigma_X}(a)). \end{aligned} \tag{7}$$

Since X is a.C.M. of dimension s, the first group is zero. The last group is also zero. Indeed, twisting the natural exact sequence

$$0 \longrightarrow \mathcal{I}_{\Sigma_X} \longrightarrow \mathcal{O}_{\mathbb{P}^n_k} \longrightarrow \mathcal{O}_{\Sigma_X} \longrightarrow 0$$

by a and taking cohomology, we get the exact sequence

$$H^s(\mathcal{O}_{\Sigma_X}(a)) \longrightarrow H^{s+1}(\mathcal{I}_{\Sigma_X}(a)) \longrightarrow H^{s+1}(\mathcal{O}_{\mathbb{P}^n_k}(a)).$$

The first and last groups above are zero because $\dim(\Sigma_X) < s$ and $r > 0$, respectively. Thus $H^{s+1}(\mathcal{I}_{\Sigma_X}(a)) = 0$.

So the boundary map in (7) is surjective. Furthermore, since (6) holds, we have that $H^s(\mathcal{I}_{\Sigma_X,X}(a)) \cong k$ if and only if the boundary map is injective, which is the case if and only if $H^s(\mathcal{I}_{\Sigma_X}(a)) = 0$. But, if $\sigma = 0$ then $\Sigma_X = \emptyset$, and hence $\mathcal{I}_{\Sigma_X} = \mathcal{O}_{\mathbb{P}^n_k}$; then $H^s(\mathcal{I}_{\Sigma_X}(a)) = 0$ because $s < n$. And if $r - 2 \geq \sigma$, then $a \geq \sigma - s$, and thus $H^s(\mathcal{I}_{\Sigma_X}(a)) = 0$. $\square$

Remark 3.2. The above proof establishes an equivalence:

$$H^s(\Omega^s_X) \cong k \quad \text{if and only if} \quad H^s(\mathcal{I}_{\Sigma_X}(r-s-2)) = 0.$$

If $s = 1$ then Σ_X is finite. If X is a line then $\sigma = 0$. Otherwise, $r \geq 2$, and it follows from Proposition 2.1 that $H^1(\mathcal{I}_{\Sigma_X}(r-3)) = 0$ only if $\sigma \leq r-2$. In other words, the converse to Theorem 3.1 holds if $s = 1$.

4. Pfaff fields

Let V be an algebraic k-scheme. By definition, a *Pfaff field* on V is a map $\eta\colon \Omega^s_V \to \mathcal{L}$ of $\mathcal{O}_V$-modules, where $\mathcal{L}$ is an invertible sheaf on V and s is a positive integer. We call s the *rank* of η. Define the *singular locus* of η to be the closed subscheme $S \subseteq V$ defined by the sheaf of ideals $\mathrm{Im}(\eta \otimes \mathcal{L}^{-1})$.

A closed subscheme $X \subseteq V$ is said to be *invariant* under η if there is a Pfaff field $\varphi\colon \Omega^s_X \to \mathcal{L}|_X$ making the following diagram commute:

$$\begin{tikzcd} \Omega^s_V \arrow[r, "\eta"] \arrow[d] & \mathcal{L} \arrow[d] \\ \Omega^s_X \arrow[r, "\varphi"] & \mathcal{L}|_X \end{tikzcd}$$

where the vertical maps are the natural restrictions.

If $X \subseteq V$ is reduced and invariant by η, then any union Y of components of X, with its reduced induced subscheme structure, is also invariant by η. Indeed, in this situation, the restriction $\Omega^s_X|_Y \to \Omega^s_Y$ is surjective with generically zero kernel, and thus any map $\Omega^s_X|_Y \to \mathcal{L}|_Y$ factors through the restriction.

Assume now that $V = \mathbb{P}^n_k$ and $\eta\colon \Omega^s_{\mathbb{P}^n_k} \to \mathcal{L}$ is a nonzero Pfaff field on $\mathbb{P}^n_k$ of rank $s < n$. Then $m \geq 0$, where $m := \deg(\mathcal{L}) + s$. Indeed, since $\mathbb{P}^n_k$ is smooth of dimension n, the field η corresponds to a nonzero element of

$$H^0(\Omega^{n-s}_{\mathbb{P}^n_k} \otimes \mathcal{L} \otimes (\Omega^n_{\mathbb{P}^n_k})^{-1}).$$

So $H^0(\Omega^{n-s}_{\mathbb{P}^n_k}(m+n+1-s)) \neq 0$. By [D], Theorem 1.1, p. 40, this is only possible if $m+n+1-s > n-s$, that is, if $m \geq 0$.

We say that m is the *degree* of η. If η arises from a Pfaff system, that is, if $\eta = \bigwedge^s \eta'$ for a map $\eta' \colon \Omega^1_{\mathbb{P}^n_k} \to \mathcal{E}$ to a locally free sheaf $\mathcal{E}$ of rank s, then the degree has a geometric interpretation: Given a general linear subspace H of $\mathbb{P}^n_k$ of codimension s, the degree m is the degree of the "critical" hypersurface $Y \subset H$ consisting of the points $P \in H$ for which the tangent space $T_{H,P}$ of H at P and the subspace of $T_{\mathbb{P}^n_k,P}$ given by the image of $(\eta'|_P)^*$ do not generate $T_{\mathbb{P}^n_k,P}$. More precisely, Y is the degeneration scheme of the map of locally free sheaves

$$\Omega^1_{\mathbb{P}^n_k}|_H \xrightarrow{(\eta'|_H,\beta)} \mathcal{E}|_H \oplus \Omega^1_H,$$

where β is the natural restriction. If H is general then Y is a hypersurface. That its degree is indeed m follows by taking determinants, noticing that $\det \Omega^1_{\mathbb{P}^n_k} \cong \mathcal{O}_{\mathbb{P}^n_k}(-n-1)$ and $\det \Omega^1_H \cong \mathcal{O}_H(-n+s-1)$.

Theorem 4.1. *Let $X \subseteq \mathbb{P}^n_k$ be a connected, reduced, arithmetically Cohen–Macaulay subcanonical subscheme of pure dimension $s > 0$ and degree d. Let Σ_X be the singular locus of X. Assume the characteristic of k is 0 or does not divide d. Assume X is invariant under a Pfaff field $\eta\colon \Omega^s_{\mathbb{P}^n_k} \to \mathcal{L}$ of rank s in such a way that no irreducible component of X is contained in the singular locus of η. Set*

$$\sigma := \mathrm{reg}(\Sigma_X), \quad r := \mathrm{reg}(X), \quad m := \deg(\mathcal{L}) + s.$$

If $\sigma = 0$ or $\sigma \leq r-2$ then $r \leq m+1$.

Proof. By Theorem 3.1, we have $h^s(\Omega^s_X) = 1$. So, by [EK12], Corollary 4.5, p. 3790, since X is a.C.M., $r \leq s + \deg(\mathcal{L}) + 1$, as claimed. □

Lemma 4.2. *Let X be an equidimensional, reduced, projective k-scheme. Let Y be a union of irreducible components of X, with its reduced induced subscheme structure. Let Σ_X and Σ_Y be the singular loci of X and Y. Then $\Sigma_Y \subseteq \Sigma_X$.*

Proof. Let $\mathcal{H}$ be the cokernel of γ_X and $\mathcal{G}$ that of γ_Y. It is enough to observe that $\mathcal{G}$ is a subsheaf of a quotient of $\mathcal{H}|_Y$. If $Y = X$ the assertion is trivial. So assume $Y \neq X$. Let $Z := \overline{X - Y}$, again with the reduced induced subscheme structure. From the way γ_X is defined, we see that γ_X decomposes as

$$\Omega^s_X \longrightarrow \Omega^s_Y \oplus \Omega^s_Z \xrightarrow{(\gamma_Y,\gamma_Z)} \omega_Y \oplus \omega_Z \xrightarrow{\lambda} \omega_X, \tag{8}$$

where the first map is induced by restriction of forms, and the last map, λ, is induced from the natural restriction map $\mathcal{O}_X \to \mathcal{O}_Y \oplus \mathcal{O}_Z$. Let $\mathcal{T}(\omega_X|_Y)$ be the torsion subsheaf of $\omega_X|_Y$, and denote by $\omega_{X,Y}$ the quotient. Restricting (8) to Y and removing torsion, we get the following composition:

$$\Omega_X^s|_Y \xrightarrow{\beta} \Omega_Y^s \xrightarrow{\gamma_Y} \omega_Y \xrightarrow{\iota} \omega_{X,Y},$$

where β is the restriction map of s-forms, and ι is the composition of the canonical injection $\omega_Y \to \omega_Y \oplus \omega_Z$ with λ and the quotient map $\omega_X \to \omega_{X,Y}$. Since λ is generically an isomorphism and ω_Y is torsion-free, ι is injective. Since β is surjective and ι is injective, we get an injective map from $\mathcal{G}$ to

$$\frac{\omega_X|_Y}{\operatorname{Im}(\gamma_X|_Y) + \mathcal{T}(\omega_X|_Y)},$$

which is a quotient of $\mathcal{H}|_Y$. □

Theorem 4.3. *Let $X \subseteq \mathbb{P}_k^n$ be a reduced, arithmetically Cohen–Macaulay subscheme of pure dimension $s > 0$. Let Σ_X be the singular locus of X. Assume X is invariant under a Pfaff field $\eta\colon \Omega_{\mathbb{P}_k^n}^s \to \mathcal{L}$ of rank s in such a way that no irreducible component of X is contained in the singular locus of η. Set*

$$\sigma := \operatorname{reg}(\Sigma_X), \quad r := \operatorname{reg}(X), \quad m := \deg(\mathcal{L}) + s.$$

Then $r \leq m + 1 + \rho$, where $\rho := \max(1, \sigma - r + 2)$.

Proof. Set $a := r - s - 2$. Let ℓ be any integer such that $\ell \geq \rho$. Since $\rho \geq 1$, we have $a + \ell \geq r - s - 1$. Let $H \subset \mathbb{P}_k^n$ be a general hyperplane. Multiplication by $(a + \ell - r + s + 1)H$ induces an injection $\mathcal{O}_X(r - s - 1) \to \mathcal{O}_X(a + \ell)$. Since $H^s(\mathcal{O}_X(r - s - 1)) = 0$ by Proposition 2.2, and since the cokernel of the injection is supported in dimension at most $s - 1$, it follows that

$$H^s(\mathcal{O}_X(a + \ell)) = 0. \tag{9}$$

Let $\mathcal{I}_X$ and $\mathcal{I}_{\Sigma_X}$ be the sheaves of ideals of X and Σ_X in $\mathbb{P}_k^n$. Twisting the natural short exact sequence

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{O}_{\mathbb{P}_k^n} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

by $a + \ell$ and taking cohomology we get the exact sequence

$$H^s(\mathcal{O}_X(a + \ell)) \longrightarrow H^{s+1}(\mathcal{I}_X(a + \ell)) \longrightarrow H^{s+1}(\mathcal{O}_{\mathbb{P}_k^n}(a + \ell)).$$

Since $a + \ell \geq r - s - 1 > -s - 1 \geq -n - 1$, the last group is zero by Serre computation, and thus, using (9), we get

$$H^{s+1}(\mathcal{I}_X(a + \ell)) = 0. \tag{10}$$

On the other hand, since $\sigma = \mathrm{reg}(\Sigma_X)$ and $a + \rho \geq \sigma - s$, we have

$$H^s(\mathcal{I}_{\Sigma_X}(a + \ell)) = 0. \tag{11}$$

If Y is a union of irreducible components of X, with its reduced induced subscheme structure, then, since $\mathcal{I}_X \subset \mathcal{I}_Y$ with quotient supported in dimension at most s, Equation (10) implies that

$$H^{s+1}(\mathcal{I}_Y(a + \ell)) = 0. \tag{12}$$

Similarly, since $\mathcal{I}_{\Sigma_X} \subset \mathcal{I}_{\Sigma_Y}$ by Lemma 4.2, and the quotient is supported in dimension at most $s - 1$, Equation (11) implies

$$H^s(\mathcal{I}_{\Sigma_Y}(a + \ell)) = 0. \tag{13}$$

Twisting the short exact sequence

$$0 \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{I}_{\Sigma_Y} \longrightarrow \mathcal{I}_{\Sigma_Y,Y} \longrightarrow 0 \tag{14}$$

by $a + \ell$, and taking cohomology, we get the exact sequence

$$H^s(\mathcal{I}_{\Sigma_Y}(a + \ell)) \longrightarrow H^s(\mathcal{I}_{\Sigma_Y,Y}(a + \ell)) \longrightarrow H^{s+1}(\mathcal{I}_Y(a + \ell)).$$

Using (12) and (13) we get that

$$H^s(\mathcal{I}_{\Sigma_Y,Y}(a + \ell)) = 0. \tag{15}$$

Now, it follows from Proposition 2.2 that $H^s(\mathcal{O}_X(a)) \neq 0$. Thus, by Serre Duality, there is a nonzero map $\tau\colon \mathcal{O}_X(a) \to \omega_X$. If X is subcanonical, this map is an isomorphism. At any rate, since both $\mathcal{O}_X(a)$ and ω_X are torsion-free, there is a union Y of irreducible components of X, with its reduced induced subscheme structure, such that τ factors though an injection $\mathcal{O}_Y(a) \to \omega_X$. This map factors through the natural map $\omega_Y \to \omega_X$, yielding an injection $\mathcal{O}_Y(a) \to \omega_Y$. Of course, this injection induces one from $\mathcal{I}_{\Sigma_Y,Y}(a)$ to $\mathcal{I}_{\Sigma_Y,Y}\omega_Y$, which can be composed with the injection $\mathcal{I}_{\Sigma_Y,Y}\omega_Y \to \widetilde{\Omega}^s_Y$, where

$$\widetilde{\Omega}^s_Y := \frac{\Omega^s_Y}{\mathcal{T}(\Omega^s_Y)},$$

with $\mathcal{T}(\Omega^s_Y)$ denoting the torsion subsheaf of Ω^s_Y. Since $\mathcal{I}_{\Sigma_Y,Y}(a)$ and $\widetilde{\Omega}^s_Y$ are rank-1, the cokernel of the composition $\mathcal{I}_{\Sigma_Y,Y}(a) \to \widetilde{\Omega}^s_Y$ is supported in dimension at most $s - 1$. Thus, it follows from (15) that

$$H^s(\widetilde{\Omega}^s_Y(\ell)) = 0. \tag{16}$$

Notice that $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n_k}(m-s)$. Since X is invariant under η, so is Y. So there is a Pfaff field $\varphi\colon \Omega^s_Y \to \mathcal{O}_Y(m-s)$ making the following diagram commute:

$$\begin{tikzcd} \Omega^s_{\mathbb{P}^n_k} \arrow[r, "\eta"] \arrow[d] & \mathcal{O}_{\mathbb{P}^n_k}(m-s) \arrow[d] \\ \Omega^s_Y \arrow[r, "\varphi"] & \mathcal{O}_Y(m-s) \end{tikzcd}$$

where the vertical maps are the natural restrictions. The image of η is by definition $\mathcal{I}_{\mathcal{S},\mathbb{P}^n_k}(m-s)$, where $\mathcal{S}$ is the singular locus of η. So, since the vertical maps are surjective, the image of φ is $\mathcal{I}_{\mathcal{S}\cap Y,Y}(m-s)$.

Now, since $\dim(\mathcal{S}\cap Y) < s$, the map φ is generically surjective, and hence, since Y is generically smooth, generically injective. In this case, the kernel of φ is the torsion subsheaf $\mathcal{T}(\Omega^1_Y)$. So $\widetilde{\Omega}^s_Y \cong \mathcal{I}_{\mathcal{S}\cap Y,Y}(m-s)$, and hence (16) implies that

$$H^s(\mathcal{I}_{\mathcal{S}\cap Y,Y}(m-s+\ell)) = 0. \tag{17}$$

Twisting the natural exact sequence

$$0 \longrightarrow \mathcal{I}_{\mathcal{S}\cap Y,Y} \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_{\mathcal{S}\cap Y} \longrightarrow 0$$

by $m-s+\ell$ and taking cohomology, we get the exact sequence

$$H^s(\mathcal{I}_{\mathcal{S}\cap Y,Y}(m-s+\ell)) \longrightarrow H^s(\mathcal{O}_Y(m-s+\ell)) \longrightarrow H^s(\mathcal{O}_{\mathcal{S}\cap Y}(m-s+\ell)).$$

Since $\dim(\mathcal{S}\cap Y) < s$, the last group is zero. So, it follows from (17) that

$$H^s(\mathcal{O}_Y(m-s+\ell)) = 0. \tag{18}$$

However, since there is an injection $\mathcal{O}_Y(a) \to \omega_Y$, we have that $H^s(\mathcal{O}_Y(a)) \neq 0$. Since (18) holds for each $\ell \geq \rho$, we have $a \leq m-s+\rho-1$, from which follows the stated inequality. □

Corollary 4.4. *Let $X \subseteq \mathbb{P}^n_k$ be a reduced complete intersection of hypersurfaces of degrees $d_1,\dots,d_{n-s}$ for a certain positive integer s. Let Σ_X be the singular locus of X. Set $\sigma := \operatorname{reg}(\Sigma_X)$ and put*

$$\rho := \sigma + n - s + 1 - d_1 - \cdots - d_{n-s}.$$

Assume the characteristic of k is 0 or does not divide any of the d_i. Assume X is invariant under a Pfaff field $\eta\colon \Omega^s_{\mathbb{P}^n_k} \to \mathcal{L}$ of rank s. Set $m := \deg(\mathcal{L}) + s$. If $\dim(\mathcal{S}\cap X) < s$, where $\mathcal{S}$ is the singular locus of η, then

$$d_1 + \cdots + d_{n-s} \leq \begin{cases} m+n-s & \text{if } \rho \leq 0,\\ m+n-s+\rho & \text{if } \rho > 0.\end{cases}$$

Proof. Since X is a complete intersection, and of positive dimension, X is a.C.M. and connected. Also, the conormal sheaf $\mathcal{C}$ of X satisfies

$$\mathcal{C} \cong \mathcal{O}_X(d_1) \oplus \cdots \oplus \mathcal{O}_X(d_{n-s}).$$

Thus

$$\omega_X \cong \bigwedge^{n} \Omega^1_{\mathbb{P}^n_k}|_X \otimes \Big(\bigwedge^{n-s} \mathcal{C}\Big)^\vee \cong \mathcal{O}_X(d_1 + \cdots + d_{n-s} - n - 1).$$

Hence, by Proposition 2.3,

$$r = d_1 + \cdots + d_{n-s} - n + s + 1.$$

Apply Theorems 4.1 and 4.3 now. □

5. Rank-1 Pfaff fields

Proposition 5.1. *Let $\eta\colon \Omega^1_{\mathbb{P}^n_k} \to \mathcal{O}_{\mathbb{P}^n_k}(m-1)$ be a rank-1 Pfaff field on $\mathbb{P}^n_k$, for $n \geq 2$. If $m \geq 1$ and the singular locus $\mathcal{S}$ of η is finite then*

$$\operatorname{reg}(\mathcal{S}) = nm - n + 2.$$

Proof. Let $\mathcal{I}_{\mathcal{S}}$ be the sheaf of ideals of $\mathcal{S}$ and $\eta' := \eta(1-m)$. Then $\eta'\colon \Omega^1_{\mathbb{P}^n_k}(1-m) \to \mathcal{O}_{\mathbb{P}^n_k}$ has image $\mathcal{I}_{\mathcal{S}}$, or degeneration scheme $\mathcal{S}$. Consider the Koszul complex of η':

$$0 \longrightarrow \Omega^n_{\mathbb{P}^n_k}(n - nm) \xrightarrow{d_n} \cdots \xrightarrow{d_2} \Omega^1_{\mathbb{P}^n_k}(1-m) \xrightarrow{d_1} 0 \qquad (19)$$

where $d_1 := \eta'$. Since $\mathcal{S}$ is finite, $\mathcal{S}$ is of the expected codimension. Since $\mathbb{P}^n_k$ is Cohen–Macaulay, the dual to η' is a regular section, and hence the complex above is exact at positive level.

Let $\mathcal{I}_j := \operatorname{Im}(d_j)$ for $j = 1, \ldots, n$. Then $\mathcal{I}_1 = \mathcal{I}_{\mathcal{S}}$ and $\mathcal{I}_n \cong \Omega^n_{\mathbb{P}^n_k}(n - nm)$. Also, we can break (19) in the following short exact sequences:

$$0 \longrightarrow \mathcal{I}_{j+1} \longrightarrow \Omega^j_{\mathbb{P}^n_k}(j - jm) \longrightarrow \mathcal{I}_j \longrightarrow 0, \quad j = 1, \ldots, n-1. \qquad (20)$$

Twisting these sequences by $r-1$, and taking cohomology, we get the exact sequences

$$\begin{aligned} &H^j(\Omega^j_{\mathbb{P}^n_k}(\ell_{j,r})) \longrightarrow H^j(\mathcal{I}_j(r-1)) \\ &\qquad \longrightarrow H^{j+1}(\mathcal{I}_{j+1}(r-1)) \longrightarrow H^{j+1}(\Omega^j_{\mathbb{P}^n_k}(\ell_{j,r})) \end{aligned} \qquad (21)$$

for $j = 1, \ldots, n-1$, where $\ell_{j,r} := j - jm + r - 1$.

Set $b := nm - n + 2$. Notice that, since $m \geq 1$,

$$r - m = \ell_{1,r} \geq \ell_{2,r} \geq \cdots \geq \ell_{n-1,r} = r + m - b.$$

So, if $r \geq b-1$ then $\ell_{j,r} \geq m-1$ for $j = 1, \dots, n-1$, with equality only if $r = b-1$. In particular, $\ell_{j,r} \geq 0$ for $r \geq b-1$, with equality only if $r = b-1$. So, from [D], Theorem 1.1, p. 40, it follows that $H^j(\Omega^j_{\mathbb{P}^n_k}(\ell_{j,r})) = 0$ for $r \geq b$, while $H^{j+1}(\Omega^j_{\mathbb{P}^n_k}(\ell_{j,r})) = 0$ for $r \geq b-1$, for $j = 1, \dots, n-1$. Then, from the exact sequences (21) we get surjections

$$\begin{aligned} &H^1(\mathcal{I}_1(r-1)) \longrightarrow H^2(\mathcal{I}_2(r-1)) \longrightarrow \cdots \\ &\qquad \cdots \longrightarrow H^{n-1}(\mathcal{I}_{n-1}(r-1)) \longrightarrow H^n(\mathcal{I}_n(r-1)) \end{aligned}$$

for $r \geq b-1$, which are all isomorphisms for $r \geq b$. Now, $\mathcal{I}_n(r-1) \cong \Omega^n_{\mathbb{P}^n_k}(n - nm + r - 1)$. So, again by [D], Theorem 1.1, p. 40, we have that $h^n(\mathcal{I}_n(r-1)) \neq 0$ if $r \leq b-1$, whereas $h^n(\mathcal{I}_n(r-1)) = 0$ if $r \geq b$. Then $\operatorname{reg}(\mathcal{S}) = b$ by Proposition 2.1. □

Remark 5.2. If $m = 0$ and $\eta \neq 0$ then $\mathcal{S}$ consists of a point, and thus $\operatorname{reg}(\mathcal{S}) = 1$.

Theorem 5.3. *Let $C \subseteq \mathbb{P}^n_k$ be a reduced, arithmetically Cohen–Macaulay, subcanonical subscheme of dimension 1. Let Σ_C be the singular locus of C. Assume C is invariant under a rank-1 Pfaff field $\eta\colon \Omega^1_{\mathbb{P}^n_k} \to \mathcal{L}$ of degree $m \geq 1$. Set*

$$\sigma := \operatorname{reg}(\Sigma_C) \quad \text{and} \quad r := \operatorname{reg}(C).$$

Assume that $r \geq 5$ if $m = 1$ or $r \geq mn - n + 4$ if $m > 1$. If the singular locus of η is finite, then $r = m + 1 + \rho$, where $\rho := \sigma - r + 2$.

Proof. Since $r \geq 4$, we have $n \geq 2$. Then $r \geq m + 4$. Indeed,

$$n(m-1) + 4 \geq 2(m-1) + 4 = m + (m-2) + 4 \geq m + 4$$

if $m > 1$. So $\rho \geq 3$ and $r \leq m + 1 + \rho$ by Theorem 4.3. In particular, $\sigma > 0$. We need only prove that $r \geq m + 1 + \rho$.

Let $\mathcal{S}$ denote the singular locus of η. Let $\mathcal{I}_{\mathcal{S}}$ and $\mathcal{I}_C$ be the sheaves of ideals of $\mathcal{S}$ and C, and $\mathcal{I}_{\mathcal{S} \cap C}$ that of $\mathcal{S} \cap C$ in $\mathbb{P}^n_k$. Set $j := m + \rho - 2$. Twisting the natural short exact sequence

$$0 \longrightarrow \mathcal{I}_{\mathcal{S}} \longrightarrow \mathcal{I}_{\mathcal{S} \cap C} \longrightarrow \mathcal{I}_{\mathcal{S} \cap C, \mathcal{S}} \longrightarrow 0$$

by j, and taking cohomology, we obtain the exact sequence

$$H^1(\mathcal{I}_{\mathcal{S}}(j)) \longrightarrow H^1(\mathcal{I}_{\mathcal{S} \cap C}(j)) \longrightarrow H^1(\mathcal{I}_{\mathcal{S} \cap C, \mathcal{S}}(j)). \tag{22}$$

Since $\mathcal{S}$ is finite, the last group is zero. Furthermore, since $r \leq m + 1 + \rho$, we have

$$j + 1 \geq r - 2 \geq mn - n + 2.$$

Since $\text{reg}(\mathcal{S}) = mn - n + 2$ by Proposition 5.1, also $H^1(\mathcal{I}_{\mathcal{S}}(j)) = 0$. Thus

$$H^1(\mathcal{I}_{\mathcal{S}\cap C}(j)) = 0.$$

Now, twist the natural short exact sequence

$$0 \longrightarrow \mathcal{I}_C \longrightarrow \mathcal{I}_{\mathcal{S}\cap C} \longrightarrow \mathcal{I}_{\mathcal{S}\cap C, C} \longrightarrow 0$$

by j, and take cohomology to get the exact sequence

$$H^1(\mathcal{I}_{\mathcal{S}\cap C}(j)) \longrightarrow H^1(\mathcal{I}_{\mathcal{S}\cap C,C}(j)) \longrightarrow H^2(\mathcal{I}_C(j)). \tag{23}$$

Since $H^1(\mathcal{I}_{\mathcal{S}\cap C}(j)) = 0$, if we show that $H^1(\mathcal{I}_{\mathcal{S}\cap C,C}(j)) \neq 0$, then it follows from the exactness of (23) that $H^2(\mathcal{I}_C(j)) \neq 0$, and hence that $r \geq j + 3$.

Since $j + 3 = m + \rho + 1$, we need only show that $H^1(\mathcal{I}_{\mathcal{S}\cap C,C}(j)) \neq 0$. Since C is invariant under η, and $\mathcal{S}$ is finite, we have that $\mathcal{I}_{\mathcal{S}\cap C,C}(m-1) \cong \widetilde{\Omega}_C^1$, where

$$\widetilde{\Omega}_C^1 := \frac{\Omega_C^1}{\mathcal{T}(\Omega_C^1)},$$

with $\mathcal{T}(\Omega_C^1)$ denoting the torsion subsheaf of Ω_C^1. Since C is subcanonical, $\omega_C \cong \mathcal{O}_C(r-3)$ by Proposition 2.3. Furthermore, C is Gorenstein, whence $\mathcal{I}_{\Sigma_C,C}\,\omega_C \cong \widetilde{\Omega}_C^1$. So, since $j = m+\rho-2$ and $r+\rho-2 = \sigma$, it follows that $H^1(\mathcal{I}_{\mathcal{S}\cap C,C}(j)) \neq 0$ is equivalent to

$$H^1(\mathcal{I}_{\Sigma_C,C}(\sigma - 2)) \neq 0. \tag{24}$$

Let $\mathcal{I}_{\Sigma_C}$ be the sheaf of ideals of Σ_C in $\mathbb{P}_k^n$. Twisting the natural exact sequence

$$0 \longrightarrow \mathcal{I}_C \longrightarrow \mathcal{I}_{\Sigma_C} \longrightarrow \mathcal{I}_{\Sigma_C,C} \longrightarrow 0$$

by $\sigma - 2$, and taking cohomology, we get the exact sequence

$$H^1(\mathcal{I}_C(\sigma-2)) \longrightarrow H^1(\mathcal{I}_{\Sigma_C}(\sigma-2)) \longrightarrow H^1(\mathcal{I}_{\Sigma_C,C}(\sigma-2)). \tag{25}$$

Since $r \geq m + 4$, we have

$$r \leq m + \rho + 1 = m + 3 + \sigma - r \leq m + 3 + \sigma - m - 4 = \sigma - 1.$$

So, since $r = \text{reg}(C)$, we have

$$H^1(\mathcal{I}_C(\sigma - 2)) = 0.$$

On the other hand, since Σ_C is finite and nonempty, $H^1(\mathcal{I}_{\Sigma_C}(\sigma - 2)) \neq 0$ by Proposition 2.1. So, from the exactness of (25) we get (24). $\square$

References

[BMe] M. Brunella and L. G. Mendes, Bounding the degree of solutions to Pfaff equations, *Publ. Mat.* **44** (2000), 593–604. Zbl 0986.32015 MR 1800822

[CmCr] A. Campillo and M. Carnicer, Proximity inequalities and bounds for the degree of invariant curves by foliations of $\mathbb{P}^2_{\mathbb{C}}$. *Trans. Amer. Math. Soc.* **349** (1997), 2211–2228. Zbl 0873.32030 MR 1407696

[CmCrG] A. Campillo and M. Carnicer and J. García de la Fuente, Invariant curves by vector fields on algebraic varieties. *J. London Math. Soc.* (2) **62** (2000), 56–70. Zbl 1040.32027 MR 1771850

[Cr] M. Carnicer, The Poincaré problem in the nondicritical case. *Ann. of Math.* **140** (1994) 289–294. Zbl 0821.32026 MR 1298714

[CeLn] D. Cerveau and A. Lins Neto, Holomorphic foliations in $CP(2)$ having an invariant algebraic curve. *Ann. Inst. Fourier* **41** (1991), 883–903. Zbl 0734.34007 MR 1150571

[D] P. Deligne, Cohomologie des intersections complètes. In *Groupes de monodromie en géométrie algébrique*. II (Séminaire de Géométrie Algébrique du Bois-Marie 1967-–1969), Lecture Notes in Math. 340, Springer-Verlag, Berlin 1973, 39–61. Zbl 0265.14007 MR 0354657

[E] E. Esteves, The Castelnuovo–Mumford regularity of an integral variety of a vector field on projective space. *Math. Res. Letters* **9** (2002), 1–15. Zbl 1037.14022 MR 1892310

[EKl1] E. Esteves and S. Kleiman, Bounds on leaves of one-dimensional foliations. *Bull. Braz. Math. Soc.* **34** (2003), 145–169. Zbl 1048.37049 MR 1993042

[EKl2] E. Esteves and S. Kleiman, Bounding solutions of Pfaff equations. *Comm. Algebra* **31** (2003), 3771–3793. Zbl 1081.32020 MR 2007384

[EKl3] E. Esteves and S. Kleiman, Bounds on leaves of foliations of the plane. *Contemp. Math.* **354** (2004), 57–67. Zbl 1065.37035 MR 2087804

[J] J. P. Jouanolou, *Equations de Pfaff algébriques*. Lecture Notes in Math. 708, Springer-Verlag, Berlin 1979. Zbl 0477.58002 MR 0537038

[Ku] E. Kunz, Holomorphe Differentialformen auf algebraischen Varietäten mit Singularitäten. *Manuscripta Math.* **25** (1975), 91–108. Zbl 0299.14013 MR 0384802

[Ln] A. Lins Neto, Some examples for Poincaré and Painlevé problems. *Ann. Sci. École Norm. Sup.* (4) **35** (2002), 231–266. Zbl 1130.34301 MR 1914932

[Lp] J. Lipman, Dualizing sheaves, differentials and residues on algebraic varieties. *Astérisque* **117** (1984). Zbl 0562.14003 MR 0759943

[dPW] A. A. du Plessis and C. T. C. Wall, Application of the theory of the discriminant to highly singular plane curves. *Math. Proc. Cambridge Philos. Soc.* **126** (1999), 259–266. Zbl 0926.14012 MR 1670229

[Mu] D. Mumford, *Lectures on curves on an algebraic surface*. Ann. of Math. Stud. 59, Princeton University Press, Princeton, NJ, 1966. Zbl 0187.42701 MR 0209285

[Pe] J. V. Pereira, On the Poincaré problem for foliations of general type. *Math. Ann.* **323** (2002), 217–226. Zbl 1016.32013 MR 1913040

[Po] H. Poincaré, Sur l'intégration algébrique des équations differentielles du premier ordre et du premier degré. *Rend. Circ. Mat. Palermo* **5** (1891), 161–191. JFM 23.0319.01

[S] M. Soares, The Poincaré problem for hypersurfaces invariant by one-dimensional foliations. *Invent. Math.* **128** (1997), 495–500. Zbl 0923.32025 MR 1452431

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