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Anna Erschler

Abstract. We study the Poisson–Furstenberg boundary of random walks on $C = A \wr B$, where $A = \mathbb{Z}^d$ and B is a finitely generated group having at least 2 elements. We show that for $d \geq 5$, for any measure on C such that its third moment is finite and the support of the measure generates C as a group, the Poisson boundary can be identified with the limit "lamplighter" configurations on A. This provides a partial answer to a question of Kaimanovich and Vershik [44]. Also, for free metabelian groups $S_{d,2}$ on d generators, $d \geq 5$, we answer a question of Vershik [56] and give a complete description of the Poisson–Furstenberg boundary for any non-degenerate random walk on $S_{d,2}$ having finite third moment. Finally, we give various examples of slowly decaying measures on wreath products with non-standard boundaries.

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1. Introduction and formulation of main results

Let G be a finitely generated group and μ be a probability measure on G. Consider the random walk on G with transition probabilities $p(x, y) = \mu(x^{-1}y)$, starting at the identity. We say that the random walk (G, μ) is *non-degenerate*, if the support of μ generates G as a semi-group. We say that the random walk is *adapted*, if the support of μ generates G as a group. It is clear that if the measure is symmetric, then it is non-degenerate if and only if it is adapted. Also, these two notion coincide for the class of centered measures on $G = \mathbb{Z}^d$ (we recall that a measure μ on $\mathbb{Z}^d$ is called centered, if $\sum_{g \in \mathbb{Z}^d} g\mu(g) = 0$). Given a probability measure μ on G we denote by $\check{\mu}$ the measure defined by $\check{\mu}(g) = \mu(g^{-1})$ for all $g \in G$.

Given a finite generating set $\mathcal{S}$ in G, we denote by $l_{G,\mathcal{S}}$ the word length, associated to $\mathcal{S}$. In the sequel we will usually omit the index $\mathcal{S}$ and we will write l_G for a word length $l_{G,\mathcal{S}}$ with respect to some finite generating set $\mathcal{S}$. We say that a random walk on G has finite i-th first moment, if the i-th moment with respect to some word length $l_{G,\mathcal{S}}$ is finite, that is, $\sum_{g\in G} l^i_{G,\mathcal{S}}(g)\mu(g) < \infty$. It is clear that the finiteness of the moment does not depend on the choice of $\mathcal{S}$.

The space of onesided infinite trajectories G^∞ of the random walk (G,μ) is equipped with the probability measure $\mathbb{P}$ which is the image of the infinite product measure of μ under the following map from G^∞ to G^∞:

$$(x_1, x_2, x_3, \dots) \to (x_1, x_1x_2, x_1x_2x_3, \dots).$$

Throughout this paper, the words "with probability 1" refer to this probability on the space of infinite trajectories. The words "with probability close to 1" mean that for any $\epsilon > 0$ the event concerning the n-th step of the random walk happens with probability at least $1-\epsilon$ for any sufficiently large given n.

Poisson–Furstenberg boundary. Consider two infinite trajectories X and Y. We say that they are equivalent if there exist N, K such that $X_i = Y_{i+K}$ for all $i > N$. Consider the measurable hull of this equivalence relation in the space of infinite trajectories. The quotient by the obtained equivalence relation is called *Poisson–Furstenberg boundary*.

Equivalently, the Poisson–Furstenberg boundary is the space of ergodic components of the time shift in the path space G^∞ (for an overview of basic facts about boundaries see [44], [38]). The Poisson–Furstenberg boundary is often also called *Poisson boundary*, and its σ-field is also called *invariant σ-field*.

We recall that a function $F: G \to \mathbb{R}$ is called *μ-harmonic*, if for all $g \in G$ it holds $F(g) = \sum_{h\in G} F(gh)\mu(h)$. It is known that the group G admits nonconstant positive harmonic functions with respect to some adapted measure μ if and only if the Poisson–Furstenberg boundary of the random walk is non-trivial. The boundary can be equivalently defined in terms of bounded harmonic functions (see [24], [26], [44]).

It is known that if the support of μ generates a non-amenable group, then the Poisson–Furstenberg boundary is non-trivial and that any amenable group G admits a symmetric measure with the support generating G such that the boundary of random walk is trivial [43], [44], [53]. First examples of symmetric random walks on amenable groups with nontrivial Poisson boundary were constructed in [44], where it was shown that a simple random walk on wreath products of $\mathbb{Z}^d$, $d \geq 3$, with a finite group (with at least two elements) has non-trivial Poisson boundary.

We recall that the *wreath product* of the groups A and B is a semidirect product of A and $\sum_A B$, where A acts on $\sum_A B$ by shifts: if $a \in A$, $f: A \to B$, $f \in \sum_A B$, then $f^a(x) = f(a^{-1}x)$, $x \in A$. Let $A \wr B$ denote the wreath product.

By definition, any element of the wreath product is a pair (a, f), $a \in A$, $f: A \to B$ is such that for all but finite number of a we have $f(a) \neq e_B$, where e_B is the neutral element of B. We say that the support of f, denoted by supp f, is the set of elements a such that $f(a) \neq e_B$.

Kaimanovich and Vershik have shown in [44] that for a simple random walk on $\mathbb{Z}^d$, $d \geq 3$, the value of the configuration at any given point of the base $A = \mathbb{Z}^d$ stabilizes along infinite trajectories (since the random walk on A is transient), and this implies non-triviality of the exit boundary. In [37] and [39] (Theorem 3.6.6) Kaimanovich has shown that a similar argument works also for measures with finite first moment. In that papers the statement is formulated under some assumption on A and B that are not used in the proof. For the convenience of the reader we provide in Section 4 the proof of the lemma below. A version of this statement, when A is a free group, appears in [47].

Lemma 1.1. *Let $C = A \wr B$, μ be a measure on C having finite first moment. Suppose that the projection of μ on A defines a transient random walk. Then for all $a_* \in A$ and $\mathbb{P}$-almost every trajectory $X_i = (Y_i, f_i)$ $(i = 1, 2, \dots)$, the values $f_i(a_*)$ stabilize from some time onwards, that is, $f_i(a_*) = f_{i+1}(a_*)$ for any sufficiently large i.*

Lemma 1.1 shows that with probability 1 we can assign to each trajectory $X_1 = (Y_1, f_1), X_2 = (Y_2, f_2), \dots$ on C the limit configuration $f: A \to B$, for all a_* putting $f(a_*) = \lim f_i(a_*)$.

Observe that the limit configuration is the same for any two infinite trajectories that coincide after some instant (that is, for trajectories X_i, X_i' are such that $X_i = X_{i+K}'$ for some constant K and all sufficiently large i). Note that the space of limit configurations carries a measure which is a projection of the measure $\mathbb{P}$ on the space of infinite trajectories. Note also that C acts on the space of limit configurations by shifts, and that this action commutes with the action of C on infinite trajectories. A space $\mathcal{B}$ with such property is called *a μ-boundary* of the random walk (C, μ). It is known that every μ-boundary is a quotient of the Poisson boundary. We denote by $\pi_{\mathcal{B}}$ the corresponding mapping from the space of infinite trajectories G^∞ to $\mathcal{B}$.

If we assume additionally that μ is adapted, then this lemma implies that the Poisson boundary of the random walk (C, μ) is non-trivial. Indeed, assume the contrary. Then there is a configuration F such that the limit configuration is equal to F with probability 1, for any trajectory of the random walk starting from the identity element. This implies that for any $x \in C$ all trajectories starting from x have the same limit configuration F_x with probability 1. Note that for any $y \in \operatorname{supp} \mu$ we have $F_x = yF_{xy}$. If support μ generates C, this cannot happen.

It is known, moreover, that the boundary is non-trivial for every finite entropy measure μ as above such that the projection of the random walk on A is transient (see [20], where it is proved under the assumption that μ is non-degenerate).

The following theorem states that for $C = A \wr B$, under suitable assumptions on A and μ, the μ-boundary of the limit configuration for the random walk (C, μ) is equal to the Poisson–Furstenberg boundary of (C, μ).

Theorem 1. *Let μ be an adapted measure on $C = A \wr B$, $A = \mathbb{Z}^d$, $d \geq 5$, $\#B \geq 2$. Assume that the third moment of μ is finite and that the projection of μ to $\mathbb{Z}^d$ is centered. Under this assumption the Poisson boundary is equal to the space of limit configurations.*

Theorem 1 holds in general, without the assumption that the projection of μ to $\mathbb{Z}^d$ is centered. If this projection is not centered, than the projected random walk on $\mathbb{Z}^d$ has positive drift. For measures such that the projection has positive drift, the result is due to Kaimanovich, [39], Example 3.6.7 after Theorem 3.6.6.

Theorem 1 gives an answer, for $d \geq 5$, to the question of Vershik and Kaimanovich about the boundary of simple random walks on $\mathbb{Z}^d \wr B$ which goes back to [43], see also [44]. Until now there are no known results about a complete description of the Poisson–Furstenberg boundary of simple random walks on wreath products $\mathbb{Z}^d \wr B$ or for any other symmetric random walks on these groups. There were some more results, however, about the non-reversible case. James and Peres have shown in [32] that the number of visits of points of the base provides a complete description of the Poisson–Furstenberg boundary of a certain measure on $\mathbb{Z}^d \wr \mathbb{Z}^+$. See more on this in Subsection 4.1.

The complete description of the Poisson–Furstenberg boundary has been known for the following finitely generated groups (under certain conditions on the decay of the probability measure defining the random walk): discrete subgroups in semi-simple Lie group (Furstenberg [27] for a particular case of an infinitely supported measure, "Furstenberg approximation", Ledrappier [50] for the case of discrete subgroups of $\mathrm{SL}(d, \mathbb{R})$, Kaimanovich [40] for a general class of measures), free groups (Dynkin, Malyutov [18] for simple random walk on standard generators, Derriennic [14] for measures with finite support), more generally for hyperbolic groups (Ancona [1] for measures with finite support, Kaimanovich [40] for measures of finite entropy and with finite logarithmic moments; see also [5]), groups with infinitely many ends (Woess [58] for finitely supported measures, [40] for more general class of measures), the mapping class group (Kaimanovich, Masur [42]), braid groups (Farb, Masur [23]), for wreath products of free groups with finite groups (Karlsson, Woess [47]), Coxeter groups (follows from Karlsson, Margulis [46], see Theorem 6.1 in [45] for an explanation). Sometimes it is easier to identify the boundary for certain non-symmetric random walks, rather than for symmetric ones. It was done for random walks on the wreath product $\mathbb{Z}^d \wr B$ with a non-zero drift of the projection on $\mathbb{Z}^d$ [40], for random walks on solvable Baumslag–Solitar groups with a non-zero drift of the projection on $\mathbb{Z}$ [37], and, more generally, for such random walks on the group of rational affinities [10]. Note that in the last two examples simple random walks have trivial boundary.

The idea of the proof of Theorem 1 can be applied not only to wreath products, but also to some other solvable groups and group extensions. In particular, we use it in this paper to give the complete description of the Poisson boundary for free metabelian groups on d generators, $d \geq 5$.

Consider a group A, and assume that $A = F_m/H$, where F_m is a the free group on m generators and H is a normal subgroup of F_m. Let $\mathcal{S}_m$ be the free generating set of F_m. Let C_A be $F_m/[H, H]$, where $[H, H]$ is the subgroup of F_m generated by all commutators $[h_1, h_2]$, $h_1, h_2 \in H$. Let J be the image of H in C_A. Observe that J is a normal subgroup of C_A and that C_A/J is equal to A. In fact, it is easy to see that C_A depends on A only.

Elements of C_A can be identified with pairs $((a, f))$, where $a \in A$ and f is a finitely supported map from the edges of the oriented Cayley graph of $(A, \mathcal{S}_m)$ to the integers with the following condition: f is an integer unit flow from e to a on the oriented Cayley graph of $(A, \mathcal{S}_m)$, when $a \neq e$, and a unit flow without source and sink when $a = e$. This means that for any $a_* \in A$ such that a_* is distinct from a and e, or $a_* = a = e$ (in the case $a = e$) the sum of the values of f on all incoming edges (in a_*) is equal to the sum of values of f on outgoing edges. If $e \neq a$ the first sum is equal the second sum plus 1 for a and the first sum is equal to the second sum minus 1 for $e = a$ (for more on this see e.g. [56]).

The following lemma is analogous to Lemma 1.1.

Lemma 1.2. *Let A be a finitely generated group and let μ be a measure on C_A with a finite first moment and such that its projection on A defines a transient random walk. Then for all edges E_* and $\mathbb{P}$-almost every trajectory $X_i = (Y_i, f_i)$ $(i = 1, 2, \dots)$, the values $f_i(E_*)$ stabilize from some time onwards, that is, $f_i(E_*) = f_{i+1}(E_*)$ for any sufficiently large i.*

For the case of finitely supported μ the statement of the lemma is proved in [56]. We prove this lemma in Section 5.

If $A = \mathbb{Z}^d$, then the group C_A is called a free metabelian group on d generators. In this case, following [52], we denote C_A by $S_{d,2}$. For background on free metabelian groups see [52] and references therein.

Theorem 2. *Let $S_{d,2}$ be the free metabelian group on d generators, $d \geq 5$, and μ be an adapted measure with finite third moment on $S_{d,2}$, such that it is projection to $A = \mathbb{Z}^d$ is centered. Then the Poisson boundary of $(S_{d,2}, \mu)$ is equal to the space of the $\mathbb{Z}$-valued limit configurations on the edges of the Cayley graph $(A, \mathcal{S}_m)$.*

If the projection of μ to $A = \mathbb{Z}^d$ is not centered, then the claim of Theorem 2 is true for any finite first moment μ, and the proof is analogous to that of Theorem 3.6.6 in [39].

The paper has the following structure. In the next section we describe the idea of the proof of Theorems 1 and 2. The proof is based on the Ray Criterion of Kaimanovich. An essential step of the proof is the construction of the ray approximation, required by this criterion. This is done in Section 2: For each n we construct a measurable mapping ϕ_n (and ϕ_n^{CE}) from the limit configurations space to C (and $S_{d,2}$).

In Section 3 we prove auxiliary facts about random walks on the base group A.

In Section 4 we give the proof of Theorem 1 (about wreath products). We prove that the map ϕ_n, constructed in Section 2, satisfies the assumption of Proposition 1 (see Section 2 below), a version of Kaimanovich's Ray Criterion.

In Section 5 we show that the ray approximation for free metabelian groups ϕ_n^{CE} satisfies the assumption of Proposition 1 and we prove Theorem 2.

In Section 6 we discuss measures with slow decay and provide examples of random walks on wreath products, where the Poisson–Furstenberg boundary is non-trivial, and there is no non-trivial partition of the boundary which can be defined in terms of finite configurations.

2. Idea of the proofs, construction of ray approximations

Idea of the proof of Theorem 1. Informally speaking, we will proceed as follows. First, for a fixed boundary point $\boldsymbol{b}$ we want to recover some information about the trajectory that converges to $\boldsymbol{b}$ and about the points of the group A, visited by the projection to A of this trajectory. This set of points cannot be recovered exactly. We will recover it "approximately", by constructing "growing neighborhoods" of the support of $\boldsymbol{b}$ (set $U_*(\boldsymbol{b}) \subset A$, a definition will be given below). This set will be in a sense close to our trajectory. If this set is not connected, we show that we can find a finite neighborhood $U(\boldsymbol{b})$ which is connected. Then, given a point $a \in A$ we want to "guess" at what instant the projection to A of the infinite trajectory, converging to $\boldsymbol{b}$, passed not too far from a. This is easier to guess if the point a corresponds to a center of a "cut ball", that is, a ball with center at a of a certain radius, depending on $l_A(a)$, that cuts our set $U(\boldsymbol{b})$ in such a way that e_A lies in a bounded connected component (in other words, this cut ball "separates the identity in A from infinity"). We will show that desired "cut balls" do exist, and that moreover they can be chosen satisfying certain additional properties. The advantage of considering the centers a of such "cut balls" is as follows. If we assume that at some instant t the projection of the trajectory has visited a (or some other point not far from a), we could expect that almost all time before this instant t the projected trajectory was in the connected component of the identity, and soon after this instant t the projected trajectory will stay forever outside of this connected component. If this is the case, we want to "guess" the instant t granted $\boldsymbol{b}$ and the constructed sets (depending on the support of $\boldsymbol{b}$). We do this by counting the cardinality of the intersection of the support of $\boldsymbol{b}$ with the connected component of the identity (and normalizing this cardinality by the constant C_{lamp}, defined in Lemma 2.1). Finally, after "guessing" the instant t we can "guess" the element of the wreath product, visited at the time t: we consider the element (a, f), where a is a center of a certain "cut ball", described above, and f is the configuration which coincides with $\boldsymbol{b}$ in the connected component of the identity, while it is trivial (identically e_B) outside this component.

The idea to consider cut balls is reminiscent of the work of James and Peres, who used cut points of the trajectory in order to describe the exchangeability boundary of the random walk on $\mathbb{Z}^d$ [32] (for more on this see Section 4.1 of our paper). In their situation it was sufficient to observe that if we know the number of visits of all points in $\mathbb{Z}^d$, and if some cut-point of the trajectory is visited exactly once, then we know the instant of the visit of this cut point (summing up the number of visits of all points visited before). The difficulty in our case is that we do not know these number of visits points and we do not know even the set of visited points. Moreover, we have already mentioned that this set (and the number of visits) cannot be recovered in terms of $\boldsymbol{b}$, but we need an "approximate construction", described above.

Ray Criterion of Kaimanovich. Below we provide a version of the Ray Criterion from [40]. Recall that a *μ-boundary* of a random walk is a quotient of the Poisson boundary with respect to a G-invariant measurable partition.

Proposition 1 (Kaimanovich). *Consider a random walk (G, μ) and assume that $\mathcal{B}$ is a μ-boundary of this random walk. Suppose that for any $n \geq 1$ there is a measurable mapping $\phi_n \colon \mathcal{B} \to G$ such that the following holds. Take a trajectory $X = (X_1, X_2, \dots)$ of the random walk (G, μ) and let $\boldsymbol{b} = \pi_{\mathcal{B}}(X)$, that is, $\boldsymbol{b}$ is the corresponding point of the μ-boundary $\mathcal{B}$. Then, under each of the following two assumptions* A) *and* B)*, the Poisson boundary of (G, μ) is equal to $\mathcal{B}$.*

A) *For any $\epsilon > 0$ there exists N such that for any fixed n, $n > N$, with probability at least $1 - \epsilon$ it holds $\mathrm{dist}_G(X_n, \phi_n(\boldsymbol{b})) \leq \epsilon n$.*

B) *There exists a family of subsets of G, $Q(g, n, \epsilon)$, such that for some $C_1 > 0$ and any $g \in G$, n and $\epsilon > 0$ such that $\#Q(g, n, \epsilon) \leq \exp(C_1 \epsilon) n$. We assume that for any $\epsilon > 0$ there exists N such that for any fixed n, $n > N$, the element X_n belongs to $Q(\phi_n(\boldsymbol{b}), n, \epsilon)$ with probability at least $1 - \epsilon$.*

In [40] it was assumed that the conditions in the statement of the proposition hold with probability 1, but the same argument proves the statement under the weaker condition above. Note that the assumption A) of the proposition implies that there exists a sequence n_i such that $\mathrm{dist}_G(X_{n_i}, \phi_{n_i}(\boldsymbol{b}))/n_i$ tends to 0 with probability 1 in the first case, and, analogously, under the assumption B) there exists a sequence n_i such that for instants defined by this sequence the points belong to corresponding Q-sets with probability 1.

A) is a particular case of B), since one can consider $Q(g, n, \epsilon)$ to be equal to the ball with the center at g and of radius ϵn.

The mapping ϕ_n as in Proposition 1 is called ray approximation. We will prove Theorem 1 and Theorem 2 by constructing an appropriate ray approximation.

Construction of the ray approximation for wreath products. First we recall a simple fact about random walks on wreath products.

Lemma 2.1. *Let A and B be arbitrary finitely generated groups. Consider a random walk on $C = A \wr B$ defined by a measure with a finite first moment, and a trajectory $X_i = (Y_i, f_i)$, $i = 1, 2, \dots$, of this random walk. Then*

$$\lim \frac{\#\mathrm{supp}\, f_i}{i} = C_{\mathrm{lamp}}.$$

Here # denotes the cardinality of the set.

The constant C_{lamp} is studied by Gilch in [28].

Proof. Consider $X_{i,m} = \big(Y_{i,m}, f_{i,m}\big)$ such that $X_{i+m} = X_i \times X_{i,m}$. Note that for all $i, m \geq 0$,

$$\#\mathrm{supp}\, f_{i+m} \leq \#\mathrm{supp}\, f_i + \#\mathrm{supp}\, f^{im}.$$

Note also that $\#\mathrm{supp}\, f_i \geq 0$ and $E\#\mathrm{supp}\, f_1$ is finite, since the first moment of the measure is finite. We see that the cardinality of the support is non-negative and subadditive. Therefore, the statement of the lemma follows from Kingman's subadditive theorem [48]. □

Now fix non-decreasing positive valued functions $r_1(l)$ and $r_2(l)$ with $r_1(l) < r_2(l) < l$. Let $\boldsymbol{b}$ be a limit configuration for a random walk on the wreath product $C = A \wr B$. Define

$$U_*(\boldsymbol{b}) = U_*^{r_1}(\boldsymbol{b}) = \bigcup_{a \,\in\, \mathrm{supp}\, \boldsymbol{b} \text{ or } a \,=\, e_A} B(a, r_1(l_A(a))).$$

Here and in the sequel l_A denotes a word length in A with respect to some finite generating set of A.

Consider a minimal positive integer $C_{\boldsymbol{b}}$ such that the $C_{\boldsymbol{b}}$-neighborhood of $U_*(\boldsymbol{b})$ is connected. If no such constant exists we put $\phi_n(\boldsymbol{b}) = e_C$, and in the sequel we assume that $C_{\boldsymbol{b}}$ does exist.

Let $U = U^{r_1}(\boldsymbol{b})$ be the $C_{\boldsymbol{b}}$-neighborhood of U_*. For any $a \in A$ put

$$U_*^{\mathrm{cut}}(a) = U \setminus B(a, r_2(l_A(a))).$$

Note that $e_A \in U_*^{\mathrm{cut}}(a)$ since $r_2(n) < n$ for all n. Let $U^{\mathrm{cut}}(a)$ be the connected component of e_A in $U_*^{\mathrm{cut}} a$. Let $F_{\mathrm{lamp}}(a) = F_{\mathrm{lamp}}(\boldsymbol{b}, a)$ be the cardinality of the intersection of the support of $\boldsymbol{b}$ with $U^{\mathrm{cut}}(a)$.

Now for each n consider all a in the support of $\boldsymbol{b}$ such that $F_{\mathrm{lamp}}(a) \leq C_{\mathrm{lamp}} n$ and choose among them an element $\overline{Y}_n$ for which $F_{\mathrm{lamp}}(a)$ is maximal. If no such a exists, we put again $\phi_n(\boldsymbol{b}) = e_C$. If several a with maximal $F_{\mathrm{lamp}}(a)$ exist, one can take any of them. To make the constructed map measurable, one can enumerate once and for all elements of the group by the natural numbers, and each time choose the minimal element among all possible choices.

Previous steps of the construction: the sets $U_*(\boldsymbol{b})$, $U(\boldsymbol{b})$, $U_*^{\mathrm{cut}}(a)$, the constant $C_{\boldsymbol{b}}$ and the element $\bar{Y}_n$ depend on the support of $\boldsymbol{b}$ and do not depend on the values of $\boldsymbol{b}$. The following final step does depend not only on the support, but also on the values of the configuration $\boldsymbol{b}$. Consider $f_n^a \colon A \to B$ such that $f_n(x) = \boldsymbol{b}(x)$ for $x \in U^{\mathrm{cut}}(a)$ and $f(x) = e_A$ otherwise. Let $\bar{f}_n = f_n^{\bar{Y}_n}$. Observe that $\bar{f}_n$ has finite support (as far as $\bar{Y}_n$ above does exist).

Put

$$\phi_n(\boldsymbol{b}) = (\bar{Y}_n, \bar{f}_n).$$

Our goal is to choose functions $r_1(l)$ and $r_2(l)$ in such a way that the constructed mappings ϕ_n satisfy one of two assumptions of the ray approximation criterion (Proposition 1).

Remark. If the projected random walk has zero entropy, then the choice of $\bar{Y}_n$ is not important. This is the case under the assumption of Theorem 1, where $A = \mathbb{Z}^d$. However, even in this case it seems natural to consider $\bar{Y}_n$ as defined above.

Main steps of the proof of Theorem 1 are formulated in the lemmas below. In these lemmas (2.2–2.4) we are under the assumption of Theorem 1 and we assume that $r_1(l) = l^{1-\epsilon_1}$, $r_2(l) = l^{1-\epsilon_2}$, where $1 > \epsilon_1 > 3\epsilon_2 > 0$ and ϵ_1 is small enough.

Lemma 2.2. *With probability close to* 1 *the constant $C_{\boldsymbol{b}}$ and elements a do exist. For any $\epsilon > 0$, with probability close to* 1, *the constructed mapping $\phi_n(\boldsymbol{b}) = (\bar{Y}_n, \bar{f}_n)$ satisfies the following bound on the number of the lamps in the connected component of e_A:*

$$F_{\mathrm{lamp}}(\bar{Y}_n) \geq (C_{\mathrm{lamp}} - \epsilon)n.$$

Observe that by the construction we know also that

$$F_{\mathrm{lamp}}(\bar{Y}_n) \leq C_{\mathrm{lamp}}n \leq (C_{\mathrm{lamp}} + \epsilon)n.$$

Lemma 2.3. *For any sufficiently large fixed n with probability close to* 1 *the following holds. Consider functions $r_1' = r_1$ and $r_2' = r_2/2$. For the given n, the infinite trajectory $X_i = (Y_i, f_i)$ $(i = 1, 2 \dots)$ satisfies*

$$(C_{\mathrm{lamp}} - \epsilon)n \leq F'_{\mathrm{lamp}}(Y_n) \leq (C_{\mathrm{lamp}} + \epsilon)n.$$

Above the function F' is taken with respect to r_1' and r_2'.

We will see in Section 4 that Lemma 2.2 is essentially a corollary of Lemma 2.3.

If μ has finite support, (or, more generally, that μ has quasi-finite support – the support belongs to the product of $\mathbb{Z}^d$ with some finite set) one can prove that with probability close to 1 all points of the support of the limit configuration satisfying the property described in Lemma 2.3 lie at sublinear distance and give rise to elements of the wreath product that lie at sublinear distance from each other. That is, for any $\delta > 0$, $\epsilon_2 > 0$ there exists $\epsilon > 0$ such that with probability at least $1 - \epsilon_2$ for any sufficiently large n and for any points $a', a'' \subset \operatorname{supp} \boldsymbol{b}$ satisfying

$$(C_{\mathrm{lamp}} - \epsilon)n \le F_{\mathrm{lamp}}(a'), F_{\mathrm{lamp}}(a'') \le (C_{\mathrm{lamp}} + \epsilon)n,$$

it holds

$$\operatorname{dist}_A(a', a'') \le \delta n.$$

Moreover, the corresponding elements of the wreath product $c' = (a', f_n^{a'})$ and $c'' = (a'', f_n^{a''})$ satisfy

$$\operatorname{dist}_C(c', c'') \le \delta n.$$

This allows to use criterion A) of Proposition 1.

In order to treat the general case when the support of μ is not necessarily finite, it is easier to apply criterion B) of Proposition 1. To assure that we can apply condition B) we will prove the following lemma.

Lemma 2.4. *There exists a family of sets $Q(g, n, \epsilon)$ such that for some $C_1 > 0$ and any $g \in C = \mathbb{Z}^d \wr B$, any n and any $\epsilon > 0$ one has $\#Q(g, n, \epsilon) \le \exp(C_1 \epsilon n)$, and, further, such that with probability close to 1 the following holds: For any $\epsilon > 0$, all points Y'_n in the support of the limit configuration satisfying*

$$(C_{\mathrm{lamp}} - \epsilon)n \le F_{\mathrm{lamp}}(Y'_n) \le (C_{\mathrm{lamp}} + \epsilon)n$$

lie at sublinear distance from each other and the corresponding elements of the wreath products belong to the same Q-set.

The sets $Q(g, n, \epsilon)$ can be constructed in the following way. Take any ϵ_3, ϵ_4 such that $\epsilon_4 < \epsilon_3 < \epsilon_2$ and put $r_3(n) = n^{1-\epsilon_3}$, $r_4(n) = n^{1-\epsilon_4}$. For any g, n define $Z(g, n)$ to be the set containing all h such that the following holds:

(i) h and g has the same projection a in $\mathbb{Z}^d$, $h = (a, f_h)$, $g = (a, f_g)$.

(ii) f_h and f_g coincide outside the ball of radius $r_3(n)$ with the center at a.

(iii) The cardinality of the intersection of this ball with the support of f_h is at most $r_4(n)$.

(iv) The sum of the lengths of the values at these elements (at points of the support that lie inside our ball) is at most $r_4(n)$.

(Note that conditions (iii) and (iv) depend only on h, not on g.)

Now consider all elements $\dot{g}$ at distance at most ϵn from $Z(g, n)$, and consider the union of $Z(\dot{g}, n)$ over all such $\dot{g}$. This union we denote by $Q(g, n, \epsilon)$.

Central extensions. The construction of ϕ_n^{CE} is similar to that of ϕ_n in the case of wreath products. (In the notation above CE refers to “central extension”.) We define sets $U_{*\mathrm{CE}}$, $U_{*\mathrm{CE}}^{\mathrm{cut}}$, $U_{\mathrm{CE}}^{\mathrm{cut}}$, configurations $\bar{f}_n(a)$, $\bar{f}_n$, the map F_{lamp} and $\overline{Y}_n \in A$ in the same way as for wreath products, with the only difference that in all these definitions we consider configurations on the edges and not on the vertices of A. Now there is some difference in the construction of ϕ_n^{CE}. We cannot put $\phi_n^{\mathrm{CE}}(\mathbf{b}) = ((\overline{Y}_n, \bar{f}_n))$, since $((\overline{Y}_n, \bar{f}_n))$ need not be an element of our group (the condition on f_n about the sum of values on edges adjacent to a given vertex does not need to be satisfied). The fact that the configuration $((\overline{Y}_n, \bar{f}_n))$ does not correspond to an element of our group is not important. One can consider the space of all such pairs such that our group is a subspace of this space. One can show that the Ray Criterion (Proposition 1) is verified even if the approximation is constructed in a larger space. But to avoid an explanation of this we will do the following. We construct subsets of the set of configurations corresponding to the elements of our group) $Z^{\mathrm{CE}}(((\overline{Y}_n, \bar{f}_n)), n)$ and $Q^{\mathrm{CE}}(((\overline{Y}_n, \bar{f}_n)), n, \epsilon)$. The construction of these sets is analogous to the case of wreath products.

As before, we consider $r_1(l) = l^{1-\epsilon_1}$, $r_2(l) = l^{1-\epsilon_2}$, where $1 > \epsilon_1 > 3\epsilon_2 > 0$, ϵ_1 is small enough. We take ϵ_3, ϵ_4 such that $\epsilon_4 < \epsilon_3 < \epsilon_2$ and put $r_3(n) = n^{1-\epsilon_3}$, $r_4(n) = n^{1-\epsilon_4}$.

For any n and any configuration $((a_n, f_n))$ define $Z^{\mathrm{CE}}(((a_n, f_n)), n)$ to be the set of the configurations of the form $((a_n, h_n))$ (this condition is analogous to the assumption (i) in the case of wreath products) such that

(ii) h_n and f_n coincide outside the ball of radius $3r_3(n)$ with the center at a_n.

(iii) There exist at most $r_4(n)$ elements of the support of h_n that lie inside this ball.

(iv) The sum of the lengths of the values at these elements (at the points of the support that lie inside this ball) is at most $r_4(n)$.

Fix any $\epsilon > 0$ (we can choose ϵ arbitrarily small). Take any element $((\overline{Y}_n', \bar{f}_n'))$ of our group in the set set $Z^{\mathrm{CE}}(((\overline{Y}_n, \bar{f}_n)), n)$ (one checks that such element does exist). Put $\phi_n^{\mathrm{CE}}(\boldsymbol{b}) = ((\overline{Y}_n', \bar{f}_n'))$.

Now consider all elements of the metabelian group $\dot{g}$ at distance at most ϵn from some configuration of $Z^{\mathrm{CE}}(((\overline{Y}_n', \bar{f}_n')), n)$, corresponding to an element of our group. Consider the union of $Z^{\mathrm{CE}}(\dot{g}, n)$ over all such $\dot{g}$. This union we denote by $Q^{\mathrm{CE}}(((a_n', f_n')), n, \epsilon)$.

We will show that for any $\epsilon > 0$ with probability close to 1 the n-th step position of our random walk belongs to the set $Q^{\mathrm{CE}}(((\overline{Y}_n', \bar{f}_n')), n, \epsilon)$.

3. Auxiliary facts about random walks on A

Consider a random walk on A and let $p_i(y, x)$ be the i-th step transition probability from y to x. Let

$$\mathcal{G}(y, x) = p_1(y, x) + p_2(y, x) + p_2(y, x) + \cdots$$

be the Green kernel of a random walk on the group A and let $\mathcal{G}(x) = \mathcal{G}(e, x)$. Observe that $\mathcal{G}(e, y^{-1}x) = \mathcal{G}(y, x)$ is the expected number of visits of x for a random walk started at y. In particular, if the random walk is transient, then $\mathcal{G}(y, x)$ is finite for all x, y.

The lemma below is a version of a well-known fact from Potential Theory.

Lemma 3.1 (Green function). *Suppose that the random walk is transient, and that there exist non-increasing functions $\mathcal{G}_{\min}$ and $\mathcal{G}_{\max}$ such that for any $x \in A$,*

$$\mathcal{G}_{\min}(l_A(x)) \le \mathcal{G}(x) \le \mathcal{G}_{\max}(l_A(x)).$$

Then for any positive integers r, N with $2r \le N$ and any $c \in A$ with $l_A(c) = N$ the random walk starting at e visits the ball $B(c, r)$ (centered at c and of radius r) with probability at most

$$\mathcal{G}_{\max}(N/2)/\mathcal{G}_{\min}(r).$$

Proof. Consider the probability $u(z)$ to visit the ball at least once starting at some point z. Put $v = (I - P)a$, where P is the averaging operator of the random walk under consideration. It holds $v(z) = 0$ for all z outside the ball and $v(z) \ge 0$ for z inside the ball.

We have

$$u(z) = v(z) + \sum_{y \in A} \mathcal{G}(z, y)v(y).$$

If z is outside the ball, then

$$u(z) = \sum_{y \in B(c,r)} \mathcal{G}(z, y)v(y).$$

This implies that

$$u(e) \le \mathcal{G}_{\max}(N/2) \sum_{y \in B(c,r)} v(y),$$

since for any $y \in B(c, r)$ it holds $\mathcal{G}(e, y) \le \mathcal{G}_{\max}(l_A(y)) \le \mathcal{G}_{\max}(N/2)$.

Observe also that

$$u(c) = 1 = v(c) + \sum_{y \in B(c,r)} \mathcal{G}(c, y)v(y) \ge \sum_{y \in B(c,r)} \mathcal{G}(c, y)v(y).$$

We know that for all y inside the ball $\mathcal{G}(c, y) \ge \mathcal{G}_{\min}(r)$. Therefore, $\sum_{y \in B(c,r)} v(y) \le 1/\mathcal{G}(r)$, and this completes the proof of the lemma. □

Lemma 3.2 (Neighborhoods of trajectories in $\mathbb{Z}^d$). *Let μ be an adapted measure on $A = \mathbb{Z}^d$, $d \ge 5$, and assume that the third moment of μ is finite. Take $q < 1/2$ and*

$w < 1/2$, put $t(n) = n^{2w}$ and assume that $1 + q < 3w$. Let $Y_1, Y_2, \dots, Y_i, \dots$ be a trajectory of the random walk $(\mathbb{Z}^d, \mu)$.

i) *With probability close to* 1 *the distance between $\{Y_1, Y_2, \dots, Y_n\}$ and $\{Y_{n+[t(n)]}, Y_{n+[t(n)]+1}, Y_{n+[t(n)]+2}, \dots\}$ is greater than n^q.*

Indeed, even more is true.

ii) *Let*

$$U^q_{\mathrm{beg}}(n) = \bigcup_{i=1}^{n} B(X_i, n^q), \quad U^q_{\mathrm{mid}}(k,m) = \bigcup_{i=k}^{m} B(Y_i, i^q)$$

and $U^q_{\mathrm{end}}(n) = U^q_{\mathrm{mid}}(n,\infty)$. Then with probability close to 1 *the sets $U^q_{\mathrm{beg}}(n)$ and $U^q_{\mathrm{end}}(n + t(n))$ do not intersect.*

iii) *Let*

$$V^q_{\mathrm{mid}}(k,m) = \bigcup_{i=k}^{m} B(Y_i, l_A(Y_i)^{2q}),$$

$V^q_{\mathrm{beg}}(n) = V^q_{\mathrm{mid}}(1,n)$ and $V^q_{\mathrm{end}}(n) = V^q_{\mathrm{mid}}(n,\infty)$. Then with probability close to 1 *the sets $V^q_{\mathrm{beg}}(n)$ and $V^q_{\mathrm{end}}(n + t(n))$ do not intersect.*

Proof. First observe that if the measure μ is not centered, then the claims of the lemma follow from the Central Limit Theorem for this random walk. Therefore, below in the proof we will assume that μ is centered.

i) First suppose that $d = 5$.

One knows that if a centered adapted measure on $\mathbb{Z}^5$ has a finite third moment, then the Green function $\mathcal{G}(x)$ is asymptotically equivalent to $1/\|x\|^3$, where $\|x\|$ is the Euclidean length of $x \in \mathbb{Z}^5$, see [55].

By the Central Limit Theorem, for any $w' < w$ the distance between Y_n and $Y_{n+[t(n)]}$ is at least $n^{w'}$ with probability close to 1. Chose $w' < w$ and $q' < w'$ in such a way that $1 - 2q + 3q' - 3w' < 0$. Since $1 + q - 3w < 0$, the latter inequality is verified whenever q' is close enough to q and w' is close enough to w.

Applying Lemma 3.1 to the reflected random walk $(\mathbb{Z}^d, \check{\mu})$, we see that the distance between $Y_{n+[t(n)]}$ and the beginning of the trajectory $Y_1, Y_2, \dots, Y_n$ is at least $n^{q'}$ with probability at least $1 - (n^{q'})^3/(n^{w'}/2)^3$. Since $q' < w'$, this probability is close to 1.

Among the points visited up to the instant n we now consider the following ones: $X_{i(1+[n^{2q}])}$, where $1 \le i \le n/(1 + [n^{2q}])$. Observe that with probability close to 1 any among the first n points of the trajectory lies at distance at most $n^{q'}$ from one of the chosen points. For each chosen point place a ball of radius $2n^{q'}$ centered at this point. Then with probability close to 1 any point of the n^q-neighborhood of the first n points of the trajectory lies inside one of this balls.

Therefore it suffices to estimate from above the probability that the tail $\{Y_{n+[t(n)]}, Y_{n+[t(n)]+1}, Y_{n+[t(n)]+2}, \dots\}$ of the trajectory has a non-empty intersection with the unions of these balls.

By Lemma 3.1 applied to $(\mathbb{Z}^d, \mu)$ we see that for any of these balls the probability that the tail has a non-empty intersection with this ball is at most $(2n^{q'}/n^{w'})^3$. The events, consisting in that the tail does not intersect one given ball, are not independent; nevertheless we can always claim that the probability that at least one of these events occurs is not greater than the sum of the probabilities of these events. Therefore, the probability that the tail has a non-empty intersection with the unions of our balls is at most $n^{1-2q}(2n^{q'}/n^{w'})^3$. Observe that $n^{1-2q}(2n^{q'}/n^{w'})^3 = 8n^{1-2q+3q'-3w'}$. By our assumption on q' and w' we have $1-2q+3q'-3w' < 0$, and thus $n^{1-2q+3q'-3w'}$ tends to 0 as n tends to ∞.

This completes the proof of statement i) for $d = 5$. If $d > 5$ take the projection of the random walk to the first k coordinates. We know already that for the projection of the trajectory the conclusion of the lemma is true. Since the projection does not increase distances, we see that the conclusion is also true for the original trajectory in $\mathbb{Z}^d$.

ii) Take $q' > q$ such that $q' < 1/2$ and $1 + q' < 3w$. Then q', w satisfy the assumption of the lemma, and hence from the already proven first part of the lemma we known that with probability close to 1 the distance between $\{Y_1, Y_2, \dots, Y_n\}$ and $\{Y_{n+t(n)}, Y_{n+t(n)+1}, Y_{n+t(n)+2}, \dots\}$ is greater than $n^{q'}$. Take $y > 1$ such that $y < q'/q$. Note that

$$U^q_{\mathrm{end}}(n + t(n)) = U^q_{\mathrm{mid}}(n + t(n), \infty) = U^q_{\mathrm{mid}}(n + t(n), n^y) \cup U^q_{\mathrm{mid}}(n^y, \infty).$$

Observe that for $i \le n^y$ it holds $i^q < n^{q'}$, and thus with probability close to 1 the sets $U^q_{\mathrm{beg}}(n)$ and $U^q_{\mathrm{mid}}(n + t(n), n^y)$ do not intersect. By well-known estimates on the inner and outer radii (also called lower and upper classes) of the random walk we know that for any $\epsilon > 0$ an infinite trajectory Y satisfies with probability 1 that

$$n^{1/2-\epsilon} \le l_{\mathbb{Z}^d}(Y^n) \le n^{1/2+\epsilon}$$

for any sufficiently large n. Here the upper bound follows from the law of the iterated logarithm for each of the coordinates in $\mathbb{Z}^d$ (which is true for any variable with finite variance, that is, in our notation, for any μ with finite second moment, see e.g. Theorem 3.52 in [9]. For the lower bound see [17] in the case of simple random walks in $\mathbb{Z}^d$, $d \ge 3$, and [54], [29] for random walks with finite second moment.

This implies that the probability that $U^q_{\mathrm{beg}}(n)$ and $U^q_{\mathrm{end}}(n^y) = U^q_{\mathrm{mid}}(n^y, \infty)$ have a non-empty intersection, is close to 0. And thus we conclude that the probability that $U^q_{\mathrm{beg}}(n)$ and $U^q_{\mathrm{end}}(n + t(n))$ have a non-empty intersection is close to 0.

iii) By the above mentioned estimates of the inner and outer radii of the random walk we see that for any $q' > q$ with probability close to 1,

$$V^q_{\mathrm{beg}}(n) \subset U^{q'}_{\mathrm{beg}}(n) \text{ and } V^q_{\mathrm{end}}(n+t(n)) \subset U^{q'}_{\mathrm{end}}(n+t(n)),$$

and thus iii) follows from ii). □

4. Wreath products. Proof of Theorem 1

Proof of Lemma 1.1. We have

$$X_{i+1} = (Y_{i+1}, f_{i+1}) = (Y_i, f_i)(A_i, F_i) = X_i(A_i, F_i),$$

where $(A_i, F_i) \in \operatorname{supp} \mu$, so that $f_{i+1}(a_*) = f_i(a_*)F_i(Y_i^{-1}a_*)$. Thus

$$\mathrm{P}[F_i(Y_i^{-1}a_*) \neq e_B] = \sum_{a\in A} \mu_A^{(*)i}(a) \sum_{(x,f)\in C} \big(1-\delta(f(a^{-1}a_*))\big)\,\mu(x,f),$$

where μ_A is the projection of μ on A and $\delta(t) = 0$ if $t \neq 0$ and 1 if $t = 0$.

Therefore,

$$\begin{aligned}\sum_i \mathrm{P}[f_i(a_*) \neq f_{i+1}(a_*)] &= \sum_i \mathrm{P}[F_i(Y_i^{-1}a_*) \neq e_B] \\ &\leq \sum_{a\in A} (\mathcal{G}(a)+1) \sum_{(x,f)\in C} \big(1-\delta(f(a^{-1}a_*))\big)\,\mu(x,f),\end{aligned}$$

where $\mathcal{G}(a) = \sum_{i=1}^{\infty} \mu_A^{*i}(a)$ is the Green function of the projection of the random walk on A. Here and in the sequel, μ^{*i} denotes the i-th convolution of μ. Since the random walk is transient, $\mathcal{G}(a) < \infty$ for all a and also $\mathcal{G}(a) \leq \mathcal{G}(e_A) + 1$ for all $a \in A$. Therefore, the expression is at most

$$\begin{aligned}&\mathrm{Const} \sum_{(x,f)\in C} \mu(x,f) \sum_{a\in A} \big(1-\delta(f(a^{-1}a_*))\big) \\ &= \mathrm{Const} \sum_{(x,f)\in C} \mu(x,f)\#\operatorname{supp} f \\ &\leq \mathrm{Const} \sum_{(x,f)\in C} \mu(x,f)\, l_C((x,f)) < \infty,\end{aligned}$$

since the first moment of μ is finite. We have shown that

$$\sum_i \mathrm{P}[f_i(a_*) \neq f_{i+1}(a^*)] < \infty,$$

and hence for all a_*, we have $f_i(a_*) \neq f_{i+1}(a^*)$ for a finite number of i's. □

Lemma 4.1. *Let $C = \mathbb{Z}^d \wr B$, $d \geq 1$, B has at least 2 elements. Consider an adapted measure μ on C, and assume that the projection of this measure to $\mathbb{Z}^d$ is centered. Then there exist $K \geq 1$ and $g_1, g_2 \in \operatorname{supp} \mu^{*K}$ such that $g_1 \neq g_2$ and the projection of both of these elements on $\mathbb{Z}^d$ is the zero element $e_{\mathbb{Z}^d}$ of $\mathbb{Z}^d$.*

Proof. The claim of the lemma is obvious, if the support of μ generates $\mathbb{Z}^d \wr B$ as a semi-group.

We know that the projection of μ on $\mathbb{Z}^d$ is adapted and centered, therefore there exist K_1 and $g \in \operatorname{supp} \mu^{*K_1}$ such that g has zero projection to $\mathbb{Z}^d$, that is, $g = (e, f)$. If $f(h) \neq e_B$ for some $h \in A$, then we can put $g_1 = g$, $g_2 = g^2$. We have $g_2 = (e, f^2)$. We see that g_1 and g_2 have zero projection on $\mathbb{Z}^d$. Note that $f(h) \neq f(h)f(h)$ for any h such that $f(h) \neq e_B$, and this implies that $g_1 \neq g_2$.

Therefore, it is sufficient to consider the case where $f \equiv e_B$, that is, we can assume that $e_{\mathbb{Z}^d \wr B}$ is in the support of μ^{*K_1} for some $K_1 \geq 1$.

Let $C^+ \subset C$ be the sub-semigroup generated by the support of μ, let $C^{+,K_1} \subset C$ be the sub-semigroup generated by the support of μ^{*K_1} and $C^{K_1} \subset C$ be the subgroup generated by the support of μ^{*K_1}. The support of μ^{*K_1} is $(\operatorname{supp} \mu)^{K_1}$. Since μ is adapted and C is finitely generated, there exists a finite generating set $\mathcal{S}_0$ of C such that $\mathcal{S}_0 \subset \operatorname{supp} \mu$. This implies that the index of C^{K_1} in C is at most $(2\#\mathcal{S}_0)^{K_1-1}$, and, in particular, C^{K_1} is a finite index subgroup of C. In particular, C^{K_1} is not Abelian, not all elements of μ^{*K_1} commute, and this implies that C^{+,K_1} is not Abelian. Therefore, there exist $h_1, h_2 \in C^{+,K_1}$ having the same projection a to $\mathbb{Z}^d$ and $h_1 \neq h_2$. Our assumption that the projection of μ is centered implies that there exists $h \in C^{+,K_1}$, such that its projection to $\mathbb{Z}^d$ is equal to $-a$. Put $g_1 = h_1 h$, $g_2 = h_2 h$. It is clear that $g_1 \neq g_2$ and that these two elements have the same projection to $\mathbb{Z}^d$. Note that since $e \in \operatorname{supp} \mu^{*K_1}$, the supports of $\mu^{*K_1 N}$ form an increasing sequence of sets. Therefore, there exists N such that g_1 and g_2 belong to $\mu^{*K_1 N}$, and this completes the proof of the lemma. □

For the proof of Lemma 4.3 below we will need the following simple combinatorial lemma.

Lemma 4.2. *Fix a positive constant K_{ball}. Consider $[K_{\mathrm{ball}} N]$ unordered balls put in $N + 1$ ordered boxes and suppose that each configuration has equal weight. There exist $K_0, K_{\mathrm{full}} > 0$, depending on K_{ball}, such that f the probability that the number of non-empty boxes is not greater than $K_{\mathrm{full}} N$ is smaller than $\exp(-K_0 N)$.*

For the proof of this lemma see for example Lemma 5.1 in [22].

For proving the following lemma, the assumption of centeredness is not relevant but assumed in order to simplify the proof.

Lemma 4.3 (Connectivity properties of the set $U_*^{r_1}(\boldsymbol{b})$). *Let $C = A \wr B$, $A = \mathbb{Z}^d$, $d \geq 3$, B has at least 2 elements. Let μ be an adapted measure on C, such that the support of its projection on A is centered. Suppose that either* i) *the third moment of μ is finite and $r_1(l) \geq l^{1-\epsilon_1}$, $\epsilon_1 < 2/3$, or* ii) *μ has a finite support and $r_1(l) > K \log(l)$ for sufficiently large constant K.*

A) *Take an infinite trajectory $X = X_1, X_2, X_3, \ldots$ of the random walk (C, μ). Let $Y = Y_1, Y_2, Y_3, \ldots$ be its projection on A. Let $\boldsymbol{b}$ be the limit configuration of X, and let R be the set of points of A visited by the projection Y. With probability* 1 *on the space of infinite trajectories for all but finitely many points in $a_1 \in \operatorname{supp} \boldsymbol{b}$ there exists $a_2 \in R$ such that*

$$\operatorname{dist}_A(a_1, a_2) < \frac{1}{2} r_1(l_A(a_1)).$$

Moreover, $\mathbb{P}$-almost surely for all but finitely many points in $a_3 \in R$ there exists $a_4 \in \operatorname{supp} \boldsymbol{b}$ such that

$$\operatorname{dist}_A(a_3, a_4) < \frac{1}{2} r_1(l_A(a_3)).$$

B) *With probability* 1 *the set $U_*^{r_1}(\boldsymbol{b})$ has one infinite connected component and finitely many finite connected component. In particular, there exists $C_{\boldsymbol{b}} > 0$ such that the $C_{\boldsymbol{b}}$-neighborhood of $U_*^{r_1}(\boldsymbol{b})$ is connected.*

Proof. First assume that i) holds. Since the third moment is finite and since $2/3 - \epsilon_1 > 0$, with probability 1 the trajectory makes only a finite number of random walk's increments $Z_n = (A_n, F_n)$ such that the length of Z_n is larger than $r_1'(n) = n^{1/3+1/2(2/3-\epsilon_1)}$.

Define the set $W_*^{r_1,\epsilon}$ by

$$W_*^{r_1,\epsilon}(X) = \bigcup_{i \in \mathbb{N}} B(Y_i, r_1(i^{1/2-\epsilon})),$$

and put

$$V_*^{r_1,\epsilon}(X) = \bigcup_{i \in \mathbb{N}} B(Y_i, r_1(l_A(Y_i)^{1-\epsilon})).$$

Observe that with probability 1 the set $W_*^{r_1/2,\epsilon}(X)$ has one infinite connected component and finitely many finite connected components whenever ϵ is small enough. Observe also that, since for any positive δ and any sufficiently large n it holds $n^{1/2-\delta} < l_A(Y_i) < n^{1/2+\delta}$, the same is true for $V_*^{r_1,\epsilon}(X)$: for any sufficiently small ϵ, $V_*^{r_1,\epsilon}(X)$ has one infinite connected component and finitely many finite connected components almost surely.

Our goal is to show that for some $\epsilon > 0$ with probability 1 on the space of infinite trajectories the set $U_*^{r_1}(\boldsymbol{b})$ contains all but a finite number of points of $V_*^{r_1,\epsilon}(X)$, where $\boldsymbol{b}$ is the limit configuration of X.

By Lemma 4.1, replacing, if necessary, μ by a suitable convolution power, we can choose $c = (e_A, f_c)$ such that f_c is not the identity (there exist $a \in A$ such that $f_c(a) \neq e_B$) and assume that c and e_C belong to the support of μ. A convolution power of μ has finite third moment if μ has finite third moment, it is finitely supported if μ is finitely supported, and its projection to $\mathbb{Z}^d$ is centered if μ has the property. A convolution power of μ is not necessary adapted, but we needed the adaptedness of μ only to apply Lemma 4.1, and we do not use this property in the rest of the proof.

For any finite second moment random walk on $\mathbb{Z}^d$, such that supp $\mu \neq e_{\mathbb{Z}^d}$ and $d \geq 1$, the expectation of $\exp(-R_n)$ is small or equal to $\exp(-Kn^{1/3})$, where R_n is the number of distinct points visited until the instant n [16]. (In fact, the condition that the second moment is finite is not important for the inequality above, but we do not need this.)

This implies that there exist $K_1, K_3 > 0$ such that for any $K_2 > 0$ and for any sufficiently large n there exist at least $K_2 \log(n)$ different points of $A = \mathbb{Z}^d$ visited between instants n and $n + K_1K_2\log^3(n)$ with probability at least $1 - \exp(-K_3K_2\log(n))$.

With probability at least $1 - \exp(-K_4K_6\log(n))$ there are at least $K_5K_6\log(n)$ multiplications by c or e among any $K_6\log(n)$ consecutive steps of our random walk. Observe that for given K_6, K_8 and sufficiently large n it holds $K_8\log^3(n) > K_6\log(n)$, and therefore with probability at least $1-\exp(-K_4K_6\log(n))$ there are at least $K_5K_6\log(n)$ multiplications by c or e between instants n and $n + K_7\log^3(n)$.

Put $N = [K_6\log(n)] - 1$, $K_{\text{ball}} = K_5$ and apply Lemma 4.2. We see that with probability at least $1 - \exp(-K_0N)$ there are at least $K_{\text{full}}N$ multiplications by c or e that have occurred at distinct elements of the base group A, between instants n and $n + K_7\log^3(n)$. Therefore, there exist $K_8, K_9 > 0$ such that with probability at least $1 - \exp(-K_8\log(n))$ there are at least $K_9\log(n)$ multiplications by c or e that have occurred at distinct elements of the base group A, in the time interval between n and $n + K_7\log^3(n)$. This implies the second inequality in A) (under the assumption i)).

To prove B) observe that in view of A) we know that with probability 1 the set $U_*^{r_1}(bb)$ contains all but a finite number of points of the projected trajectory. Moreover, with probability 1 this set contains all but a finite number of points of $V_*^{r_1/2}$. Finally, note that a union of balls (or any other connected sets) that contains a connected set is connected, and this completes the proof of B) under the assumption i).

Now let us assume that ii) holds. Under the assumption of ii) the measure μ has finite support. This implies that for any $a_1 \in \operatorname{supp} \boldsymbol{b}$ there exists $a_2 \in R$ such that $\operatorname{dist}_A(a_1, a_2)$ is not greater than D, where D is a positive constant depending on the support of μ. This proves the first inequality in A).

Now we want to prove B) and the second inequality in A). As before, we replace μ by a suitable convolution power, so that we can assume $g_1, g_2 \in \operatorname{supp}\mu$, where g_1 and g_2 have zero projection to $\mathbb{Z}^d$. Let $\mathcal{S}_A$ be the projection of the support of μ to $\mathbb{Z}^d$. In the Cayley graph of $(\mathbb{Z}^d, \mathcal{S}_A)$ the sets of points visited by the projection of the trajectory of our random walk is connected. Let $a_0 \in \mathbb{Z}^d$ be a point, visited by the projection.

With probability 1 the set of points visited by the projection of the trajectory is infinite. Observe that an intersection of any infinite connected set containing a_0 with the ball of radius r, centered at a_0, has cardinality at least r. As before, we see that with probability at least $1 - (\exp(-K_{11}r))$ there exists at least one point in the ball $B_{S_A}(a_0, r)$, where a multiplication by g_1 or g_2 had happened, at some moment $n \geq 0$. Therefore, with probability at least $1 - (\exp(-K_{12}r))$ there exists at least one point in the ball $B_{S_A}(a_0, r)$ such that the value of $\boldsymbol{b}$ at this point is non-trivial. □

Lemma 4.4. *Under the assumptions of Theorem* 1 *consider a point* $X_n = (Y_n, f_n)$ *visited at the* n*-th step of the random walk under consideration. Let* ϵ_1, ϵ_2, ϵ_3 *be such that* $0 < \epsilon_2 < 1/3$ *and* $0 < \epsilon_3 < \epsilon_2$. *Consider the points of the support of* f_n *that lie inside the ball centered at* Y_n *and of radius* $n^{1-\epsilon_2}$. *The following holds with probability close to* 1:

i) *the number of these points is at most* $n^{1-\epsilon_3}$,

ii) *the sum of the lengths* (*in* B) *of the values of* f_n *at these points is at most* $n^{1-\epsilon_3}$.

Proof. Replacing the defining measure μ by its reflection $\check{\mu}$, we can reduce the problem to the situation, where the center of the ball is at e, not at x_n. Consider ϵ_4 such that $\epsilon_3 < \epsilon_4 < \epsilon_2$. Since the random walk has the third moment and since $\epsilon_3 < 1/3$, we know that with probability close to 1 there are no random walk's increments that have the projection on the base group longer than $n^{1-\epsilon_3}$, and there are no random walk's increments such that the corresponding configurations have elements of length greater than $n^{1-\epsilon_3}$ in their supports.

Combining this with the fact that for any $\epsilon > 0$ an infinite trajectory Y satisfies, with probability 1,

$$n^{1/2-\epsilon} \leq l_{\mathbb{Z}^d}(Y^n) \leq n^{1/2+\epsilon}$$

for any sufficiently large n, we see that with probability close to 1 all the changes of the values of the configuration at points that lie inside our ball have occurred at the time interval between 1 and $n^{1-\epsilon_4}$.

Since the first moment of the random walk on the wreath product is finite, this implies i) and ii). □

Lemma 4.5. *Under the assumptions of Theorem* 1 *consider a point* $X_n = (Y_n, f_n)$ *visited at the* n*-th step of the random walk under consideration. Let* ϵ_2, ϵ_3, ϵ_4 *be such that* $1/3 > \epsilon_2 > 0$ *and* $0 < \epsilon_4 < \epsilon_3\epsilon_2$. *Let* $\boldsymbol{b}$ *be the limiting configuration for our infinite trajectory, and let* f_n' *be the configuration which coincides with* $\boldsymbol{b}$ *in the connected component of the identity, after removing the ball of radius* $n^{1-\epsilon_2}$ *from the set* $U^{r_1}(\boldsymbol{b})$ *and which is equal to the identity outside this connected component.*

Then with probability close to 1 *the elements* (Y_n, f_n) *and* (Y_n, f'_n) *have the following properties.*

(ii) *The functions* f_n *and* f'_n *coincide outside the ball of radius* $r_3(n) = n^{1-\epsilon_3}$ *with the center at* Y_n.

(iii) *The cardinality of the intersection of the supports of both* f_n *and* f'_n *with this ball* $(B(Y_n, n^{1-\epsilon_3}))$ *is at most* $r_4(n) = n^{1-\epsilon_4}$.

(iv) *Moreover, the sum of the lengths (in* B*) of values of* f_n*, as well as of* f'_n *at the points of the support inside the ball* $B(Y_n, n^{1-\epsilon_3})$ *is at most* $r_4(n) = n^{1-\epsilon_4}$.

Properties (ii)–(iv) correspond to the properties mentioned after Lemma 2.4.

Proof. By ii) of Lemma 3.2 we know that with probability close to 1 the ball of radius $n^{(1-\epsilon_2)/2}$ separates infinity from the identity in the set $U^{r_1}(\boldsymbol{b})$. Below we will assume that this is the case. This implies, in particular, that f'_n has finite support. From Lemma 4.4 we know already that statements (iii) and (iv) hold for f_n. Let us show that these statements hold also for f'_n. As the third moment of the measure μ that defines our random walk is finite, the random walks does not have increments of length greater that $n^{1/3+\epsilon_5}$, for any $\epsilon_5 > 0$ and for any sufficiently large n. We combine this with the estimates on the outer and inner radii for the projections of μ and $\check{\mu}$ to $\mathbb{Z}^d$: for any $\epsilon > 0$, with probability 1 the infinite trajectories Y, $\check{Y}$ satisfy

$$n^{1/2-\epsilon} \leq l_{\mathbb{Z}^d}(\check{Y}^n), l_{\mathbb{Z}^d}(Y^n) \leq n^{1/2+\epsilon},$$

for any sufficiently large n. We get that the values of $\boldsymbol{b}$ inside the ball of radius $r_3(n)$ with the center at Y_n could be changed only between the instants $n - r_5(n)$ and $n + r_5(n)$. Here $r_5(n) = n^{1-\epsilon_5}$ and we choose ϵ_5 in such a way that $\epsilon_4 < \epsilon_5 < \epsilon_3$. This implies (iii) and (iv).

Now let us prove (ii). Take some $x \in \mathbb{Z}^d$, which is in the support of f_n or f'_n and which does not belong to the connected component of e_A. Let us show that if the value at x undergoes a modification at the instant m, then $n - n^{1-\epsilon_4} < m < n + n^{1-\epsilon_4}$. Indeed, observe that with probability 1 the following holds for sufficiently large n. If m is such that the ball of radius $n^{1-\epsilon_4}$ with the center at Y_n is not visited in the time interval from 1 to m, then at time m our random walk is in the connected component of e_A. Moreover, all points where the value undergoes a change at time n also belong to the connected component of e_A. Analogously, if m is such that we have never visited the ball of radius $n^{1-\epsilon_4}$ with the center at Y_n between instants m and ∞, then at the instant m we cannot change the value in x. □

Proof of Lemma 2.3. Consider $r_1(l) = l^{1-\epsilon_1}$, $r_2(l) = l^{1-\epsilon_2}$ from the assumption of the lemma. Put $q = (1-\epsilon_1)/2$, $w = (1-\epsilon_2)/2$. By the assumption of the lemma $\epsilon_1 > 3\epsilon_2$. Therefore, $1+q < 3w$, and we can apply iii) of Lemma 3.2 to q, w and the projection of our random walk to $\mathbb{Z}^d$. We get that $V^q_{\mathrm{beg}}(n) = \bigcup_{i=1}^n B(Y_i, l_A(Y_i)^{2q})$

and $V^q_{\text{end}}(n+n^{2w}) = \bigcup_{i=n+n^{2w}}^{\infty} B(Y_i, l_A(Y_i)^{2q})$ do not intersect. Then we combine the conclusion of (ii) and (iii) of Lemma 4.5 with that of Lemma 2.1. We get that

$$(C_{\text{lamp}} - \epsilon)n \le F_{\text{lamp}}(Y_n) \le (C_{\text{lamp}} + \epsilon)n.$$

We know that with probability close to 1 the ball $B(Y_n, r_2(l_A(Y_n))/2)$ separates infinity from the identity. In this case, the connected component of the identity, corresponding to $r_2' = r_2/2$ contains the connected component of the identity, corresponding to r_2. It implies that the function F', taken with respect to $r_1' = r_1$ and $r_2' = r_2/2$, satisfies $F'_{\text{lamp}}(Y_n) > F_{\text{lamp}}(Y_n)$, and therefore

$$(C_{\text{lamp}} - \epsilon)n \le F'_{\text{lamp}}(Y_n).$$

By construction of F' we also know that

$$F'_{\text{lamp}}(Y_n) \le n \le (C_{\text{lamp}} + \epsilon)n. \qquad \square$$

Proof of Lemma 2.2. Consider $n' = (1-\epsilon')n$, $\epsilon' < \epsilon$, choose ϵ_2 such that

$$(1-\epsilon')(C_{\text{lamp}} + \epsilon_2) \le C_{\text{lamp}} + \epsilon \quad \text{and} \quad (1-\epsilon')(C_{\text{lamp}} - \epsilon_2) \ge C_{\text{lamp}} - \epsilon,$$

and apply Lemma 2.3 to n' and ϵ_2. We see that there exists a sequence $\overline{Y}_n^0$ in A such that for any sufficiently large n

$$F'_{\text{lamp}}(\overline{Y}_n^0) \ge (C_{\text{lamp}} - \epsilon)n.$$

Now let $\overline{Y}_n^1$ be an element of the support of $\boldsymbol{b}$, which is closest to the element $\overline{Y}_n^0$. By Lemma 4.3 we know that the distance between $\overline{Y}_n^1$ and $\overline{Y}_n^0$ is at most $r_1(n)/2 \le r_2(n)/2$ with probability close to 1. This implies that the ball $B(\overline{Y}_1, r_2(n))$ contains the ball $B(\overline{Y}_0, r_2(n)/2)$, and hence $B(\overline{Y}_1, r_2(n))$ separates the identity and infinity if $B(\overline{Y}_0, r_2(n)/2)$ has this property. By Lemma 4.4 we know that the cardinality of the intersection of the support of $\boldsymbol{b}$ with $B(\overline{Y}_0, 3r_2(n))$ is smaller than ϵn for any positive ϵ and any sufficiently large n. In view of this, we conclude that for any $\epsilon > 0$ with probability close to 1 the constructed element $\overline{Y}_n^1$ satisfies

$$F_{\text{lamp}}(\overline{Y}_n^1) \ge (C_{\text{lamp}} - \epsilon)n.$$

Here F_{lamp} is defined with respect to the functions r_1 and r_2. $\square$

Proof of Lemma 2.4. We consider the family of sets $Q(g, n, \epsilon)$ described after Lemma 2.4. We fix $\epsilon > 0$ and consider Y_n' in the support of the limit configuration $\boldsymbol{b}$ satisfying

$$(C_{\text{lamp}} - \epsilon)n \le F_{\text{lamp}}(Y_n') \le (C_{\text{lamp}} + \epsilon)n.$$

We want to prove that with probability close to 1 all such points lie at sublinear distance from each other and the corresponding elements of the wreath products belong to the same Q. We are going to show that with probability close to 1 any Y_n' as above lies at sublinear distance from Y_n, and that the corresponding elements of the wreath product belong to $Q(X_n, n, \epsilon)$.

First we will assume that Y_n' belongs to the trajectory of the projection of our random walk to $\mathbb{Z}^d$: there exists N such that $Y_n' = Y_N$.

Observe that the random walk we consider is adapted and its projection to $\mathbb{Z}^d$ is transient, therefore $C_{\mathrm{lamp}} > 0$. Take any $\epsilon_1 > 0$ and apply Lemma 3.2 and Lemma 2.3 to $n_1 = n - (\epsilon_1/C_{\mathrm{lamp}})n$, $n_2 = n$ and $n_3 = n + (\epsilon_1/C_{\mathrm{lamp}})n$ and $r_1'(l) = l^{q'}$ and $r_2(l) = l^w$, where $q' > q$ is such that the inequality from the formulation of Lemma 3.2) holds also for q' and w: $1 + q' < 3w$.

By Lemma 3.2 we know that the balls with center at Y_{n_1}, Y_{n_2} and Y_{n_3} and of radius $l_A(Y_{n_j})^w$ ($j = 1, 2$ or 3) separate the identity from infinity with probability close to 1. Therefore, with probability close to 1 these three events happen simultaneously. Below we assume that each of the balls $B(Y_{n_j}, l_A^w(Y_{n_j}))$ separates the identity from infinity, $j = 1, 2$ and 3.

First suppose that N is such that $N < n_1 - (\epsilon_1/C_{\mathrm{lamp}})n = n(1 - 2\epsilon_1/C_{\mathrm{lamp}})$. Let us show that if the ball with center at the point Y_N separates the identity from infinity, then the number of elements of the support in the corresponding connected component of the identity is less then $(1 - \epsilon_1/2)n$, with probability 1 for sufficiently large n (independent of all other assumptions). Indeed, there are two possible cases. In the first case, this ball intersects with the corresponding ball with the center in Y_{n_2}. But the probability of this event is not greater than the probability that the random walk visits the ball of radius $r_2\left(l_A(Y_{n_2})\right)$ after time $\epsilon_1 n$. For any $\epsilon_4 > 0$ with probability 1 it holds $l_A(Y_{n_2}) < n^{1/2+\epsilon_4}$, and therefore $r_2(l_A(a_{n_2}) < n_2^{(1/2+\epsilon_4)(1-\epsilon_2))})$. Taking ϵ_4 small enough, we see that the probability that our balls have non-empty intersection is close to 0.

The second case is when the ball has no intersection with the corresponding ball with the center in Y_{n_2}. Let us consider connected components after removing the ball with the center in Y_{n_2}. There is a connected component of the identity, of infinity and possible some more finite connected components. Our ball cannot lie in those last mentioned components, as in this case it cannot cut the identity from infinity. By Lemma 3.2 we know that with probability close to 1, the ball does not lie inside the connected component of the identity e_A. Therefore, we can assume that our ball lies in the connected component of infinity.

Then the number of points in the connected component of the identity after removing our ball is smaller than in the connected component corresponding to n_2.

Now take $N > n_3 + (\epsilon_1/C_{\mathrm{lamp}})n = n(1 + 2\epsilon_1/C_{\mathrm{lamp}})$.

By iii) of Lemma 3.2 we know that the following holds with probability close to one (where "with probability close to one" refers to n). The ball $B(Y_N, l_A(Y_N)^{q'})$ has empty intersection with the connected component of the identity in the set, which we get from the neighborhood under consideration by removing the ball

$B(Y_{n_2}, l_A(Y_{n_2})^{q'})$. This implies that with probability close to 1 (referring to n) the connected component of the identity after removing the ball $B(Y_N, l_A(Y_N)^{q'})$ is larger than the connected component of the identity, after removing the ball $B(Y_{n_2}, l_A(Y_{n_2})^{q'})$. In this case the cardinality of the set of points of the support of $\boldsymbol{b}$ in the first mentioned connected components is not less than the cardinality of the intersection of the support of $\boldsymbol{b}$ with the connected component of the identity, after removing the ball $B(Y_{n_2}, l_A(Y_{n_2})^{q'})$. This shows that with probability close to 1 (again "probability close to 1" refers to n, not to N) the number of elements of the intersection of the support of $\boldsymbol{b}$ with the connected component of the identity after removing the ball $B(Y_N, l_A(Y_N)^{q'})$ is at least $(C_{\mathrm{lamp}} + \epsilon_1/2)n$ (for all N such that $N > n(1 + 2\epsilon_1/C_{\mathrm{lamp}})$).

This completes the proof in the case when Y_n' belongs to the trajectory of the projection of our random walk to $\mathbb{Z}^d$.

Finally, observe that in view of Lemma 4.3 it is sufficient to consider the above mentioned case: if Y_n' does not belong to the point of the projected trajectory, we can consider an element Y_n'', closest to Y_n' in $\mathbb{Z}^d$.

It remains to show the bound on the cardinality of the sets $Q(g, n, \epsilon)$: we want to show that for any $g \in C = \mathbb{Z}^d \wr B$, any n and any $\epsilon > 0$ one has $\#Q(g, n, \epsilon) \leq \exp(C_1 \epsilon n)$. First observe that the set $Z(g, n)$ defined in Section 2 has subexponential cardinality, that is, $\log(\#Z(g, n))/n \to 0$ when $n \to \infty$. Indeed, if B is finite, then the cardinality of this set is at most $B^{K_2 r_3(n)} C_{n^d}^{r_3(n)}$, which is less than $\exp(K_3 r_3(n) + K_3 \log(n) r_3(n))$, where K_2, K_3, K_4 are positive constants.

Now for an arbitrary B the cardinality of $Z(g, n)$ is at most $2^{K_2 r_3(n)} C_{n^d}^{r_3(n)}$ multiplied by $\sum_{n_1 + \cdots + n_j = r_3(n)} \Pi v_B(K n_i)$, where $v_B(n)$ is the growth function of B with respect to some generating set of this group. This sum is at most

$$\sum_{n_1 + \cdots + n_j = r_3(n)} \exp(C_1 r_3(n)),$$

that is, at most $\exp(C_2 r_3(n))$ (for some positive constants C_1 and C_2).

Note also that the union of a subexponential number of uniformly subexponential sets (the cardinality of an n-th set in each family is not greater than $F_{\mathrm{maj}}(n)$, where the function $F_{\mathrm{maj}}(n)$ satisfies $\log(F_{\mathrm{maj}}(n))/n \to 0$ as $n \to \infty$) has subexponential cardinality, and this implies the desired upper bound on the cardinality of $Q(g, n, \epsilon)$. □

Lemmas 2.2 and 2.4 show that we can apply the Ray Criterion (B of Proposition 1), and this completes the proof of the Theorem 1. □

Remark. In the theorem we have considered $A = \mathbb{Z}^d$. A similar argument can be applied (under an appropriate assumption of the measure) to A of growth at least n^5, but we do not provide details here. The statement was known previously in the case when the base is a tree (see the already mentioned result from [47]).

4.1. Remarks about the exchangeability algebra. As we have already mentioned, James and Peres has shown in [32] that the number of visits of points of the base provides a complete description of the Poisson–Furstenberg boundary of a certain measure on $\mathbb{Z}^d \wr \mathbb{Z}^+$. The Poisson boundary of particular random walks on wreath products of A with $\mathbb{Z}^+$ is equivalent to the exchangeability boundary of the projection random walk on A ([37]). To explain this connection we recall below the definitions of the tail boundary and of exchangeability boundary.

Tail boundary. Let A_n^∞ be the σ-algebra of measurable subsets of the trajectory space G^∞ that are determined by the coordinates $Y_n, Y_{n+1}, Y_{n+2} \dots$ of the trajectory Y. The intersection $A_\infty = \cap_n A_n^\infty$ is called the exit σ-algebra of the random walk. The corresponding G-space with probability measure is called *the tail boundary* of the random walk.

The difference between this definition and the definition of the exit (Poisson–Furstenberg) boundary is that here we identify trajectories that coincide after some instant, while in the definition of the exit (Poisson–Furstenberg) boundary we identify trajectories that coincide after some instant and after some shift in time.

For random walks on graphs the tail boundary does not need to be equal to the Poisson boundary (see Example 2 in [8], for more on this see [41]). However, for groups it is always the case [44].

Exchangeability boundary. Consider a random walk on A. We say that two trajectories $X_1, X_2, \dots$ and $Y_1, Y_2, \dots$ are equivalent if there exists N such that $X_i = Y_i$ for any $i > N$ and $X_1, X_2, \dots, X_N$ can be obtained by a permutation of $Y_1, Y_2, \dots, Y_N$. We consider a measurable hull of this equivalence relation in the space of infinite trajectories, and the corresponding quotient is called *exchangeability boundary*.

Note that if two infinite trajectories X and Y satisfy the condition in the definition above, then for $i > N$ and any $a \in A$ the number of visits of a until the instant i by the first trajectory is equal to the number of visits of this point by the second trajectory.

And, conversely, if X and Y coincide after some time N and for $i > N$ and any $a \in A$ the point a is visited until the instant i the same number of times by X and by Y, then these two infinite trajectories satisfy the condition in the definition of the exchangeability algebra.

Consider the following random walk on $A \wr \mathbb{Z}^+ \subset A \wr \mathbb{Z}$. We walk on A and at each visited element we add $+1$ in the corresponding subgroup $\mathbb{Z}$. It is clear that if A is a group, this random walk is a random walk in a group sense, that is, it is given by convolutions of some probability measure on $A \wr \mathbb{Z}$. From the definitions of tail and exchangeability boundaries it is clear that the exchangeability boundary of the random walk on A is equal to the tail boundary of the constructed random walk on the wreath product. If A is a group, it was already mentioned that the tail boundary is equal to the Poisson boundary.

In [32] it has been observed that the existence of infinitely many cut-points in the base group implies that the number of visits provides the complete description of the exchangeable σ-algebra. Furthermore, one knows that for a simple random walk on a transient group there are infinitely many cut points with probability 1. (For all groups except finite extensions of the nilpotent group of upper-triangular 2×2 matrices this is proved in [32] and for that latter group it is proved by Blachère in [6].)

Even in this special case of random walks on the wreath product $\mathbb{Z}^d \wr \mathbb{Z}^+$ (corresponding to the exchangeability algebra of a random walk on $\mathbb{Z}^d$), Theorem 1 provides some new information, since we consider random walks defined by probability measures that need not have a finite support. (For $d \geq 5$; for quickly decreasing measures on $\mathbb{Z}^d \wr \mathbb{Z}^+$ a similar argument works for $d \geq 4$ for this particular case.)

5. Free metabelian groups. Proof of Theorem 2

Proof of Lemma 1.2. The proof is analogous to the proof of Lemma 1.1, only that we consider visits of the edges and not on the vertices. Let $\mu_A^i(E)$ be the probability to visit the edge E on the i-th step for the projection of the random walk on A and $\mathcal{G}_{\text{edge}}(E) = \sum_{i=1}^{\infty} \mu^{*i}(E)$ be the "edge Green function" of the projection of the random walk on A. Note that $\mathcal{G}_{\text{edge}}(E)$ is not greater then the sum of the values of the Green function at the vertices adjacent to E, and, in particular, for some Const > 0 it holds $\mathcal{G}_{\text{edge}}(E) <$ Const for all edges E. Then in the same way as in the proof of Lemma 1.1 we observe that the sum $\sum_{i=1}^{\infty} \mathrm{P}[f_i(E_*) \neq f_i(E_*)]$ is at most

$$\begin{aligned}
&\text{Const} \sum_{((x,f))\in C_A} \mu(((x,f))) \sum_{\substack{a \text{ is an} \\ \text{edge of } A}} (1-\delta(f(a))) \\
&\quad = \text{Const} \sum_{((x,f))\in C_A} \mu(x,f)\#\operatorname{supp} f \\
&\quad \leq \text{Const} \sum_{((x,f))\in C_A} \mu(((x,f)))l_C(((x,f))) < \infty.
\end{aligned}$$

Therefore $f_i(E_*)$ can change its value only for a finite number of i's □

Now we prove Theorem 2. An analog of Lemma 4.3 and Lemmas 2.3, 2.4 (where instead of configuration on vertices we consider configurations on edges) is proved in exactly the same way. Now, as it was already mentioned in Section 2, ϕ_n^{CE} is defined as follows. Consider $Q(((\bar{Y}_n, \bar{f}_n)), n, \epsilon)$ (defined in the space of all configurations.) Take any element of our group in this set $((a'_n, f'_n))$ and put $\phi_n^{\mathrm{CE}}(\boldsymbol{b}) = ((\bar{Y}'_n, \bar{f}'_n))$.

Similarly to the wreath product case we argue that a union of $\exp(\epsilon' n)$ sets of cardinality at most $\exp(\epsilon'' n)$ has cardinality at most $\exp((\epsilon' + \epsilon'')n)$, and hence the union of $Q(h, n, \epsilon)$, where h is in $Q(g, n, \epsilon)$, has arbitrary small exponential growth if ϵ is small enough (for fixed g). This implies that we can apply B) of Proposition 1 (Ray Criterion), and completes the proof of Theorem 2. □

6. Non-standard boundaries for slowly decaying measures

Lemma 6.1. i) *Let $A = \mathbb{Z}^d$, $d \geq 1$, $\#B \geq 2$, and let B be finite. There exists a non-degenerate finite entropy symmetric measure μ on $C = A \wr B$ with the following property: for any finite set $V \subset \mathbb{Z}^d$ and any $f_V \colon V \to B$ with probability* 1 *any infinite trajectory of the random walk (C, μ) visits points, such that the restriction of their configuration to V is equal to f_V.*

ii) *Moreover, the measure μ can be chosen in such a way that it has finite entropy, and the projection of μ to $\mathbb{Z}^d$ defines a transient random walk.*

Proof. i) Let $W_{B,d}(c)$ be the subset of $\mathbb{Z}^d \wr B$ consisting of all the elements of the form $(0, f)$, where $\operatorname{supp} f \subset [-c, c]^d$. Given a non-decreasing sequence c_i, $0 = c_1 \leq c_2 \leq c_3 \leq \cdots$ and a sequence d_i, $i \geq 0$, such that $\sum_{i\geq 0} d_i = 1$, consider the following probability measure on C:

$$\mu = \mu_{c_i,d_i} = d_0\mu_0 + d_1\mu_1 + d_2\mu_2 + \cdots,$$

where μ_0 is a non-degenerate measure on $\mathbb{Z}^d$ and for $i \geq 1$ the measure μ_i is the uniform probability measure on $W_{B,d}(c_i)$. The projection of the random walk on $\mathbb{Z}^d$ is $(\mathbb{Z}^d, \mu_0)$.

For any μ_0 and any sequence d_i we can choose a rapidly growing sequence c_i, in such a way that the constructed measure μ satisfies the first claim of the lemma. Assume that c_j is an increasing sequence of integers. Let μ_+ be the probability measure on $\mathbb{R}^+$ such that $\mu_+(c_j) = d_j$ and $\mu_+(i) = 0$ if there is no j such that $i = c_j$. Consider a sequence of independent random variables Z_j, with distribution μ_+. Observe that if the sequence c_i grows rapidly enough, then with probability 1 there exist infinitely many i such that $Z_i > i$.

Assume that μ_0 in the construction of μ is chosen in such a way that for the random walk Y_i, defined by μ_0, with probability 1 for sufficiently large i it holds that $|Y_i| < i/2$. (This is for example the case if $(\mathbb{Z}^d, \mu_0)$ is a simple random walk on $\mathbb{Z}^d$.)

In this case infinitely many times the random walk $X_i = (Y_i, f_i)$ has an increment at the instant i that comes from some μ_j such that $c_j > l_{\mathbb{Z}^d}(Y_i) + D_V + 1$. Here D_V is such that the ball of radius $D_V/2$, centered at the identity of $\mathbb{Z}^d$, contains the set V. Let us subdivide the space of infinite trajectories into conditional events in the following way. We fix Y_i (the projection of the trajectory to $A = \mathbb{Z}^d$) and, moreover, for each i we fix j_i such that the i-th increment of the random walk comes from μ_{j_i}:

this increment is an element of the set $W_{B,d}(c_{j_i})$, and all elements of this set have the same probability to be an increment of our random walk, so far we have fixed j_i.

Given this information on the infinite trajectory of the random walk, we observe that after each time $i+1$, such that $c_{j_i} > l_{\mathbb{Z}^d}(Y_i) + D_V + 1$, the distribution of the values $f(x)$, $x \in V$ is the uniform distribution on the set, isomorphic to $B^{\#V}$. We know that with probability 1 there are infinitely many such i's, and this completes the proof of the first part of the lemma.

ii) We provide a construction of μ which is more general than the construction given in the proof of i). For integers b, c such that $b \le c$, let $W_{B,d}(c,b)$ be the subset of $\mathbb{Z}^d \wr B$ containing elements of the form $(0, f)$, where $\operatorname{supp} f \subset [-c,c]^d$ and such that $f(x_1, x_2, \dots, x_d) = f(y_1, y_2, \dots, y_d)$ whenever $x_k - y_k$ is divisible by b for $k = 1, \dots, d$. In other words, $W_{B,d}(c,b)$ consists of b-periodic functions of $W_{B,d}(c)$. Note that the cardinality of $W_{B,d}(c,b)$ is $\#B^{b^d}$.

Given non-decreasing sequences b_i, c_i, satisfying $b_i \le c_i$, $b_1 = c_1 = 0$, and a sequence d_i, $i \ge 0$, such that $\sum_{i \ge 0} d_i = 1$ consider the following probability measure on C:

$$\mu = \mu_{c_i, b_i, d_i} = d_0 \mu_0 + d_1 \mu_1 + d_2 \mu_2 + \cdots,$$

where μ_0 is a non-degenerate measure on $\mathbb{Z}^d$ and for $i \ge 1$ the measure μ_i is the uniform probability measure on $W_{B,d}(c_i, b_i)$.

With the same argument as in the proof of i) we see that μ satisfies the claim of i) whenever c_i (chosen depending on μ_0 and d_i) grows rapidly enough and $b_i \to \infty$ when $i \to \infty$. Observe that the entropy of μ is at most

$$\begin{aligned} H(\mu) &\le H(\mu_0) + \sum_{i \ge 1} d_i \log(\#W_{B,d}(c_i, b_i) d_i) \\ &\le H(\mu_0) + \sum_{i \ge 1} d_i \log(d_i) + d \sum_{i \ge 1} d_i \log(b_i). \end{aligned}$$

We see that if μ_0 has finite entropy, $\sum_{i \ge 1} d_i \log d_i < \infty$. and if b_i grows slowly enough, then the entropy of μ is finite.

Finally, recall that for $d \ge 1$ there exist finite entropy transient random walks on $\mathbb{Z}^d$ (if $d \ge 3$ any non-degenerate random walk is transient). □

Kaimanovich has observed that there exist random walks on a wreath product, such that the boundary is non-trivial, but for any point in the base the value of the configuration at this point does not stabilize along trajectories. He constructs an example non-symmetric random walks with these properties on $\mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z}$, where $f(1) - f(0)$ stabilizes (see Proposition 1.1 in [33]). We show that there are random walks on wreath products having non-trivial boundary, such that there is no functional defined by a finite set which stabilizes along infinite trajectories.

Let $C = \mathbb{Z}^d \wr B$, B is finite and has at least 2 elements. There exist symmetric random walks on C such that the Poisson boundary is non-trivial and such that for any finite set $V \subset \mathbb{Z}^d$ and any $f_V : V \to B$ with probability 1 any infinite trajectory of the random walk (C, μ) visits points such that the restriction of their configuration to V is equal to f_V. Indeed, if $d \geq 3$ any non-degenerate symmetric random walk with finite entropy has non-trivial Poisson boundary ([20]). Therefore, any non-degenerate finite entropy symmetric measure on $\mathbb{Z}^d \wr B$, $d \geq 3$, for which the conclusion of i) in Lemma 6.1 holds, provides such an example. For $d = 1$ or 2 take a measure from ii) of 6.1. Again, as it is a finite entropy measure with a transient projection to $\mathbb{Z}^d$, the Poisson boundary is non-trivial ([20]).

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