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Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **88 (2013)**

PDF erstellt am: **17.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-515635

Comment. Math. Helv. 88 (2013), 185–203
DOI 10.4171/CMH/282

Commentarii Mathematici Helvetici

Gehring–Hayman Theorem for conformal deformations

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Abstract. We study conformal deformations of a uniform space that satisfies the Ahlfors Q-regularity condition on balls of Whitney type. We verify the Gehring–Hayman Theorem by using a Whitney covering of the space.

Mathematics Subject Classification (2010). 30C65.

Keywords. Conformal deformations, uniform space, Whitney covering.

1. Introduction

Given $x, y \in B^2(0,1)$, the hyperbolic geodesic $[x, y]$ is essentially the shortest curve joining x to y in $B^2(0,1)$. More precisely

$$\ell([x, y]) \leq \frac{\pi}{2}\ell(\gamma)$$

whenever γ is a path that joins x to y in $B^2(0,1)$. This simple fact is an instance of a theorem of Gehring and Hayman in [GH]: If $f : B^2(0,1) \to \Omega \subset \mathbb{C}$ is a conformal mapping and γ is a path joining points x and y, then

$$\int_{[x,y]} |f'(z)|\, ds \leq C \int_{\gamma} |f'(z)|\, ds, \tag{1.1}$$

where $C \geq 1$ is an absolute constant. The density $\rho(z) = |f'(z)|$ satisfies a Harnack inequality

$$\frac{\rho(z)}{A} \leq \rho(w) \leq A\rho(z)$$

whenever $z \in B^2(0,1)$ and $w \in B(z, (1-|z|)/2)$. It also satisfies the area growth estimate

$$\int_{B\rho(z,r)} \rho^2\, dA \leq \pi r^2,$$

*Both authors were supported by the Academy of Finland, grant no. 120972.

where $B_\rho(z,r)$ refers to the ball with centre z and radius r in the path metric

$$d_\rho(x,y) = \inf \int_\gamma \rho\, ds,$$

where the infimum is taken over all curves γ joining points x and y.

In [BKR] the Gehring–Hayman inequality (1.1) was extended to $B^n(0,1)$, $n \geq 2$, for conformal deformations of the Euclidean metric. By a conformal deformation (a conformal density) ρ we mean a continuous function $\rho\colon B^n(0,1) \to (0,\infty)$ that satisfies a Harnack inequality with a constant $A \geq 1$,

$$\frac{\rho(z)}{A} \leq \rho(w) \leq A\rho(z) \quad \text{for all } w \in B(z,(1-|z|)/2) \text{ and all } z \in B^n(0,1),$$

and a volume growth condition with a constant $B > 0$,

$$\int_{B_\rho(z,r)} \rho^n\, dm_n \leq Br^n \quad \text{for all } z \in B^n(0,1) \text{ and all } r > 0,$$

with respect to n-dimensional Lebesgue measure m_n.

Subsequently, Herron showed in [H1] that $B^n(0,1)$ can be replaced by any uniform space (Ω, d) of bounded geometry. In this setting conformal densities are defined by conditions analogous to those given above – see Section 2 for details. Here uniformity is a substitute for the "roundness" of $B^n(0,1)$. The assumption of bounded geometry includes two conditions. First, it requires that Ω carries a Borel regular measure μ that satisfies the *(Ahlfors) Q-regularity condition on balls of Whitney type* for some $Q > 1$. That is, there is a constant $C_1 \geq 1$ such that if $r \leq d(z,\partial\Omega)/2$, then

$$C_1^{-1} r^Q \leq \mu(B(z,r)) \leq C_1 r^Q.$$

Secondly, it requires that balls $B(z, d(z,\partial\Omega)/2)$ allow for nice lower bounds for the Q-modulus (see e.g. [HK], [BHK]). In fact, the Q-regularity condition on balls of Whitney type is not explicitly stated in [H1] but it follows from the other assumptions. The precise definition of a uniform space is given in Section 2 below. This concept, introduced in [BHK], generalizes the notion of a uniform domain introduced by Jones [Jo] and Martio and Sarvas [MaSa], see also [GO]. The volume growth condition for ρ then refers to integrals of ρ^Q with respect to the measure μ. For predecessors of the results in [H1], see [HN], [HR]. For connections to Gromov hyperbolicity, see [Gr], [BHK] and [BB].

In this paper we show that, surprisingly, lower bounds on the Q-modulus are not needed to prove the Gehring–Hayman inequality.

Theorem 1.1 (Gehring–Hayman Theorem). *Let $Q > 1$ and let (Ω, d, μ) be a non-complete uniform space equipped with a measure that is Q-regular on balls* *of Whitney type. If* $\rho\colon \Omega \to (0,\infty)$ *is a conformal density on* Ω*, then there is a constant* $C \geq 1$ *that depends only on the data associated with* Ω *and* ρ *such that*

$$\ell_\rho([x,y]) \leq C\ell_\rho(\gamma),$$

whenever $[x,y]$ *is a quasihyperbolic geodesic and* γ *is a curve joining* x *to* y *in* Ω*.*

The definition of a quasihyperbolic geodesic is given in Section 2 and the proof of the theorem is in Section 4. Especially Subcase D of the proof is the novelty, that allows us to avoid the use of lower bounds for the Q-modulus. The previous arguments [BKR], [H1], [HN] and [HR] rely on modulus estimates.

The Gehring–Hayman Theorem was a central tool in [BHR], [BKR], [H1] and [H2]. We expect that Theorem 1.1 will allow one to remove the use of modulus bounds in [BHR], [BKR], [H1] and [H2] and thus extend large parts of those papers to a much more general setting. A very simple example of a space that satisfies the assumptions of Theorem 1.1 but does not support lower bounds for the Q-modulus is

$$\Omega = \{(x,y) \in \mathbb{R}^2 : |y| \leq |x|,\ -1 < x < 1\}$$

equipped with the path metric and Lebesgue measure.

2. Preliminaries

Let (Ω, d) be a metric space. A *curve* means a continuous map $\gamma\colon [a,b] \to \Omega$ from an interval $[a,b] \subset \mathbb{R}$ to Ω. We also denote the image set $\gamma([a,b])$ of γ by γ. The *length* $\ell_d(\gamma)$ of γ with respect to the metric d is defined as

$$\ell_d(\gamma) = \sup \sum_{i=0}^{m-1} d(\gamma(t_i), \gamma(t_{i+1})),$$

where the supremum is taken over all partitions $a = t_0 < t_1 < \cdots < t_m = b$ of the interval $[a,b]$. If $\ell_d(\gamma) < \infty$, then γ is said to be a *rectifiable curve.* When the parameter interval is open or half-open, we set

$$\ell_d(\gamma) = \sup \ell_d(\gamma|_{[c,d]}),$$

where the supremum is taken over all compact subintervals $[c,d]$. For a rectifiable curve γ we define the *arc length* $s\colon [a,b] \to [0,\infty)$ along γ by

$$s(t) = \ell_d(\gamma|_{[a,t]}).$$

Next, let us assume that $\rho\colon \Omega \to [0,\infty]$ is a Borel function. For each rectifiable curve $\gamma\colon [a,b] \to \Omega$ we define the *ρ-length* $\ell_\rho(\gamma)$ of γ by

$$\ell_\rho(\gamma) = \int_\gamma \rho\, ds = \int_a^b \rho(\gamma(t))\, ds(t).$$

If Ω is *rectifiably connected* – that is, every pair of points in Ω can be joined by a rectifiable curve – then ρ determines a distance function

$$d_\rho(x, y) = \inf \ell_\rho(\gamma),$$

where the infimum is taken over all rectifiable curves γ joining $x, y \in \Omega$. In general, the distance function d_ρ need not be a metric. However, it is a metric – called a *ρ-metric* – if ρ is positive and continuous. If $\rho \equiv 1$, then $\ell_\rho(\gamma) = \ell_d(\gamma)$ is the length of the curve γ with respect to the metric d. Furthermore, if $\ell_d(\gamma) = d(x, y)$ for some curve γ joining points $x, y \in \Omega$, then γ is said to be a *geodesic*. If every pair of points in Ω can be joined by a geodesic, then (Ω, d) is called a *geodesic space.*

Let (Ω, d) be a locally compact, rectifiably connected and non-complete metric space and denote by $\overline{\Omega}$ its metric completion. Then the *boundary* $\partial\Omega := \overline{\Omega} \setminus \Omega$ is nonempty. We write

$$d(z) = \operatorname{dist}_d(z, \partial\Omega) = \inf\{d(z, x) : x \in \partial\Omega\}$$

for $z \in \Omega$. If we choose

$$\rho(z) = \frac{1}{d(z)},$$

we obtain the *quasihyperbolic metric* k in Ω. In this special case we denote the metric d_ρ by k and the quasihyperbolic length of the curve γ by $\ell_k(\gamma)$. That $\ell_k(\gamma) = \ell_\rho(\gamma)$ is shown in [BHK], Appendix. Moreover, $[x, y]$ refers to a quasihyperbolic geodesic joining points x and y in Ω.

Given a real number $D \geq 1$, a curve $\gamma\colon [a, b] \to (\Omega, d)$ is called a *D-uniform curve* if it is *quasiconvex:*

$$\ell_d(\gamma) \leq D d(\gamma(a), \gamma(b)), \tag{2.1}$$

and

$$\min\{\ell_d(\gamma|_{[a,t]}), \ell_d(\gamma|_{[t,b]})\} \leq D d(\gamma(t)) \tag{2.2}$$

for every $t \in [a, b]$. A metric space (Ω, d) is called a *D-uniform space* if every pair of points in it can be joined by a D-uniform curve.

If (Ω, d) is a uniform space, then by Proposition 2.8 and Theorem 2.10 of [BHK] the quasihyperbolic space (Ω, k) is complete, proper (closed balls are compact), and geodesic. Furthermore, each quasihyperbolic geodesic $[x, y]$ is a D'-uniform curve for every $x, y \in \Omega$, where $D' = D'(D) \geq 1$. Quasihyperbolic geodesics are also *locally D'-uniform curves* – that is, every subcurve $[u, v] \subset [x, y]$ is a D'-uniform curve – because $[u, v]$ is a quasihyperbolic geodesic as well. We also have an estimate for a quasihyperbolic distance of every pair of points x and y in the D-uniform space (Ω, d) (see [BHK], Lemma 2.13):

$$k(x, y) \leq 4D^2 \log\left(1 + \frac{d(x, y)}{\min\{d(x), d(y)\}}\right). \tag{2.3}$$

Let us consider a continuous function $\rho\colon \Omega \to (0,\infty)$, called a *density*. The metric d_ρ is then well defined. We use the subscript ρ for metric notations which refer to d_ρ, and similarly for k and d. For example, $B_\rho(a,r)$, $B_k(a,r)$ and $B_d(a,r)$ are open balls with centre a and radius r in metrics d_ρ, k and d. Furthermore, we abbreviate the "Whitney ball" $B_d(z,\frac{1}{2}d(z))$ to B_z.

Let μ be a Borel regular measure on (Ω,d) with dense support. We call ρ a *conformal density* provided it satisfies both a *Harnack type inequality*, HI(A), for some constant $A \geq 1$:

$$\frac{1}{A} \leq \frac{\rho(x)}{\rho(y)} \leq A \quad \text{for all } x, y \in B_z \text{ and all } z \in \Omega, \tag{HI(A)}$$

and a *volume growth condition*, VG(B), for some constant $B > 0$:

$$\mu_\rho(B_\rho(z,r)) \leq Br^Q \quad \text{for all } z \in \Omega \text{ and } r > 0. \tag{VG(B)}$$

Here μ_ρ is the Borel measure on Ω defined by

$$\mu_\rho(E) = \int_E \rho^Q \, d\mu \quad \text{for a Borel set } E \subset \Omega,$$

and Q is a positive real number. Generally Q will be the Hausdorff dimension of our space (Ω,d).

We defined in the introduction the concept of Q-regularity on balls of Whitney type. The immediate consequence is that the measure μ is also *doubling on balls of Whitney type:* there exists a constant $C_2 \geq 1$ such that

$$\mu(B_d(z,2r)) \leq C_2\mu(B_d(z,r)) \tag{2.4}$$

for every $z \in \Omega$ and every $0 < r \leq \frac{1}{4}d(z)$.

3. Whitney covering

In this section we assume that (Ω,d,μ) is a locally compact, rectifiably connected, and non-complete metric measure space such that the measure μ is doubling on balls of Whitney type. Let $r(z) = d(z)/50$. From the family of balls $\{B_d(z,r(z))\}_{z\in\Omega}$ we select a maximal (countable) subfamily $\{B_d(z_i,r(z_i)/5)\}_{i\in I}$ of pairwise disjoint balls. Let $\mathcal{B} = \{B_i\}_{i\in I}$, where $B_i = B_d(z_i,r_i)$ and $r_i = r(z_i)$. We call the family $\mathcal{B}$ the *Whitney covering* of Ω. Let us list a few facts concerning the Whitney covering. The last property is a consequence of the doubling on balls of Whitney type property of the measure μ. For more properties of the Whitney covering, see e.g. Theorem III.1.3 of [CW], Lemma 2.9 of [MaSe], Lemma 7 of [HKT], and [BS], Theorem 5.3 and Lemma 5.5.

Lemma 3.1. *There is* $N \in \mathbb{N}$ *such that*

(i) *the balls* $B_d(z_i, r_i/5)$ *are pairwise disjoint,*

(ii) $\Omega = \bigcup_{i \in I} B_d(z_i, r_i)$,

(iii) $B_d(z_i, 5r_i) \subset \Omega$,

(iv) $\sum_{i=1}^{\infty} \chi_{B_d(z_i, 5r_i)}(x) \leq N$ *for all* $x \in \Omega$.

The family $\mathcal{B}$ has the same kind of properties as the usual Whitney decomposition $\mathcal{W}$ of a domain $\Omega \subset \mathbb{R}^n$ and next we prove a couple of them. In addition to the assumptions above, we assume that for each pair of points in $B \in \mathcal{B}$ for every $B \in \mathcal{B}$ can be joined by a D-uniform curve in Ω.

Lemma 3.2. *Let* $x, y \in (\Omega, d, \mu)$ *and* $d(x, y) \geq d(x)/2$. *There is a constant* $C = C(C_2, D) > 0$ *such that*

$$C^{-1} N(x, y) \leq k(x, y) \leq C N(x, y),$$

where $N(x, y)$ *is the number of balls* $B \in \mathcal{B}$ *intersecting a quasihyperbolic geodesic* $[x, y]$.

Proof. Let $x, y \in \Omega$ be points so that $d(x, y) \geq d(x)/2$. Since $24 \operatorname{diam}_d(B) \leq d(z)$ for every $B \in \mathcal{B}$ and for every $z \in B$, then the basic estimate (2.3) implies

$$\operatorname{diam}_k(B) \leq 4D^2 \log\left(1 + \frac{\operatorname{diam}_d(B)}{24 \operatorname{diam}_d(B)}\right) = 4D^2 \log \frac{25}{24}.$$

Thus

$$N(x, y) \geq \frac{k(x, y)}{4D^2 \log \frac{25}{24}}.$$

Lemma 3.1 (iv) says that there are only N balls $B \in \mathcal{B}$ that contain x. Fix one of them and denote it by B_1. A *neighbour* of the ball B_1 is a ball $B \in \mathcal{B}$ which intersects the ball $5B_1 = B_d(z_1, 5r_1) = B_d(z_1, d(z_1)/10)$. Because the measure μ is doubling in every ball $B_d(z, r)$ with radius $0 < r \leq d(z)/4$, the ball B_1 has a uniformly bounded number of neighbours. Let this number be $N' \in \mathbb{N}$ and let $y_1 \in [x, y]$ be the first point such that y_1 does not belong to any neighbour of B_1. This choice is possible because $d(x, y) \geq d(x)/2$. The geodesic $[x, y_1]$ intersects at most N' balls $B \in \mathcal{B}$ and

$$\begin{aligned} k(x, y_1) &= \int_{[x, y_1]} \frac{1}{d(z)}\, ds \geq \int_{5B_1 \cap [x, y_1]} \frac{10}{11 d(z_1)}\, ds \\ &\geq \frac{10}{11 d(z_1)} \left(\frac{d(z_1)}{10} - \frac{d(z_1)}{50}\right) = \frac{4}{55}. \end{aligned} \tag{3.1}$$

Let $B_2 \in \mathcal{B}$ be a ball such that $y_1 \in B_2$ and $B_2 \cap B \neq \emptyset$ for some neighbour $B \in \mathcal{B}$ of B_1. Again there are only N' balls $B \in \mathcal{B}$ which are neighbours of B_2. Let $y_2 \in [x, y]$ be the first point so that y_2 does not belong to any neighbour of B_2. Then the geodesic $[y_1, y_2]$ intersects at most N' balls $B \in \mathcal{B}$ and $k(y_1, y_2) \geq \frac{4}{55}$, by the same way than in inequality (3.1). We continue this process until we end up with a ball B_m whose neighbours contain $[y_{m-1}, y]$. This process really ends and $m < \infty$, because $[x, y]$ is compact. We may start doing this process from every ball B that contains x. Thus we obtain the upper bound to the number of balls that intersects the quasihyperbolic geodesic $[x, y]$:

$$N(x, y) \leq \frac{55}{4} N N' k(x, y). \qquad \square$$

Fix a ball B_0 from the Whitney covering $\mathcal{B}$ and let z_0 be its centre point. For each $B_i \in \mathcal{B}$ we fix a geodesic $[z_0, z_i]$. Furthermore, for each $B_i \in \mathcal{B}$ we set $P(B_i) = \{B \in \mathcal{B} : B \cap [z_0, z_i] \neq \emptyset\}$ and define the *shadow* $S(B)$ of a ball $B \in \mathcal{B}$ by

$$S(B) = \bigcup_{\substack{B_i \in \mathcal{B} \\ B \in P(B_i)}} B_i.$$

For $n \in \mathbb{N}$ we set

$$\mathcal{B}_n = \{B_i \in \mathcal{B} : n \leq k(z_0, z_i) < n + 1\}.$$

The next two lemmas are metric space analogues of [KL], Lemma 2.1 and Lemma 2.2.

Lemma 3.3. *Let γ be a quasihyperbolic geodesic in Ω starting at the point z_0. Then there is a constant $C = C(C_2, D) > 0$ such that, for each $n \in \mathbb{N}$,*

$$\#\{B \in \mathcal{B}_n : B \cap \gamma \neq \emptyset\} \leq C.$$

Proof. Put

$$a_n := \#\{B \in \mathcal{B}_n : B \cap \gamma \neq \emptyset\} < \infty.$$

Let $B_1, \ldots, B_{a_n} \in \mathcal{B}_n$ be the balls intersecting γ, ordered so that if $k < l$, then there exists $x_k \in B_k \cap \gamma$ such that for every $z \in B_l \cap \gamma$, we have $k(z_0, x_k) \leq k(z_0, z)$. We may assume that $d(x_1, x_{a_n}) \geq d(x_1)/2$, otherwise $x_{a_n} \in B_{x_1}$ and we get the result by doubling on balls of Whitney type. Thus by Lemma 3.2, $k(x_1, x_{a_n}) \geq \frac{a_n}{C}$. Since $k(z_i, x_i) \leq \frac{1}{49} < 1$ for all $i = 1, \ldots, a_n$, we may compute

$$\begin{aligned} \frac{a_n}{C} &\leq k(x_1, x_{a_n}) = k(z_0, x_{a_n}) - k(z_0, x_1) \\ &\leq k(z_0, z_{a_n}) + k(z_{a_n}, x_{a_n}) - (k(z_0, z_1) - k(x_1, z_1)) \\ &\leq (n + 1) + 1 - n + 1 = 3. \end{aligned}$$

Hence $a_n \leq 3C$. $\square$

Lemma 3.4. *There is a constant $C = C(C_2, D) > 0$ such that, for each $n \in \mathbb{N}$,*

$$\sum_{B \in \mathcal{B}_n} \chi_{S(B)}(x) \leq C$$

whenever $x \in \Omega$.

Proof. Let $x \in \Omega$. The number of balls $B \in \mathcal{B}$ containing x is bounded, so we may assume that there is a unique ball, denote it by B_1, in $\mathcal{B}$ such that $x \in B_1$. Let $[z_0, z_1]$ be the fixed geodesic joining z_0 to z_1. Then $x \in S(B)$ for $B \in \mathcal{B}_n$ if and only if $[z_0, z_1] \cap B \neq \emptyset$. By Lemma 3.3, the number of balls $B \in \mathcal{B}_n$ is bounded by a constant that is independent of n. $\square$

4. Gehring–Hayman Theorem

We begin with *Frostman's Lemma.* First we recall the definitions of the Hausdorff measure and the weighted Hausdorff measure.

Let (X, d) be a compact metric space. Let $0 \leq s < \infty$ and $0 < \delta \leq \infty$. We set

$$\lambda^s_\delta(X) = \inf\left\{\textstyle\sum_{i=1}^\infty c_i \operatorname{diam}_d(E_i)^s : \chi_X \leq \sum_i c_i \chi_{E_i},\ c_i > 0,\ \operatorname{diam}_d(E_i) \leq \delta\right\}.$$

The *weighted Hausdorff s-measure* of X is

$$\lambda^s(X) = \lim_{\delta \to 0} \lambda^s_\delta(X).$$

In the special case, where $c_i = 1$ for every $i = 1, 2, \ldots$, we set $\mathcal{H}^s_\delta(X) = \lambda^s_\delta(X)$, and we obtain the *Hausdorff s-measure*

$$\mathcal{H}^s(X) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(X).$$

The *Hausdorff s-content* of X is

$$\mathcal{H}^s_\infty(X) = \inf\left\{\textstyle\sum_{i=1}^\infty \operatorname{diam}_d(E_i)^s : X \subset \bigcup_{i=1}^\infty E_i\right\}.$$

By Lemma 8.16 of [Ma] we know that $\mathcal{H}^s(X) \leq 30^s \lambda^s(X)$, but in fact from the proof of that lemma one obtains that

$$\mathcal{H}^s_{30\delta}(X) \leq 30^s \lambda^s_\delta(X) \quad \text{for every } 0 < \delta \leq \infty.$$

In particular

$$\mathcal{H}^s_\infty(X) \leq 30^s \lambda^s_\infty(X).$$

The following formulation of Frostman's Lemma (cf. [Ma], Theorem 8.17, and [BO], Theorem 2) is suitable for our purposes.

Theorem 4.1 (Frostman's Lemma). *For any $s \geq 0$ there is a Radon measure ω on X such that*

$$\omega(X) = \lambda^s_\infty(X)$$

and

$$\omega(E) \leq \operatorname{diam}_d(E)^s \quad \text{for all } E \subset X.$$

In particular, when $s = 1$ and X is connected, we obtain

$$\omega(X) \geq \frac{1}{30}\mathcal{H}^1_\infty(X) \geq \frac{\operatorname{diam}_d(X)}{60}.$$

In this paper we apply the version of Frostman's Lemma, where X is connected and $s = 1$.

For the rest of the paper we assume that (Ω, d, μ) is a locally compact, non-complete and D-uniform metric measure space such that the measure μ is Q-regular on balls of Whitney type for some $Q > 1$. Let ρ be a conformal density such that the number Q in the definition VG(B) coincides with the previous $Q > 1$.

Proof of Theorem 1.1. Let x and y be points in $\overline{\Omega}$ and let $[x, y]$ be a quasihyperbolic geodesic in Ω joining points x and y. Because quasihyperbolic geodesics are D'-uniform curves, $[x, y]$ is rectifiable in the metric d.

Let γ be another rectifiable curve in Ω joining points x and y. Let $a \in [x, y]$ be the point such that $\ell_d([x,a]) = \ell_d([a, y])$, and write $p = d(x,a)$. Moreover, for each $j = 0, 1, 2, \ldots$, write $A_j = (\overline{B}_d(x, 2^{-j}p) \setminus B_d(x, 2^{-(j+1)}p)) \cap \Omega$. Let $[x_{j+1}, x_j] \subset [x,a] \subset [x, y]$ be a subcurve, where x_{j+1} is the last point of $[x, y]$ in $\overline{B}(x, 2^{-(j+1)}p)$ and x_j is the last point of $[x, y]$ in $\overline{B}(x, 2^{-j}p)$, and set $\gamma_j = \gamma \cap A_j$. We may clearly assume that γ_j is connected. By summing and symmetry it suffices to prove that

$$\ell_\rho([x_{j+1}, x_j]) \leq C\ell_\rho(\gamma_j) \tag{4.1}$$

for every $j = 0, 1, 2, \ldots$.

Let $j = 0, 1, 2, \ldots$. From the definition of the curve γ_j it follows that

$$\ell_d(\gamma_j) \geq 2^{-(j+1)}p. \tag{4.2}$$

From the definition of the quasihyperbolic geodesic $[x_{j+1}, x_j]$ and from the local D'-uniformity of the curve $[x, y]$, we have that

$$\ell_d([x_{j+1}, x_j]) \leq D'd(x_{j+1}, x_j) \leq D'2^{-j+1}p, \tag{4.3}$$

$$2^{-(j+1)}p \leq \ell_d([x, z]) \leq D'd(z) \quad \text{for every } z \in [x_{j+1}, x_j], \tag{4.4}$$

and

$$k(x_{j+1}, x_j) = \int_{[x_{j+1}, x_j]} \frac{1}{d(z)}\, ds \le \frac{D'}{p} 2^{j+1} \ell_d([x_{j+1}, x_j]) \le 4D'^2. \tag{4.5}$$

The proof consists of two parts: the "easy part", Case A, and the "hard part", Case B. Furthermore, Case B is divided into two parts, Subcase C and Subcase D. Here Subcase D is the hardest part and the novelty of our proof.

Case A. We first prove that inequality (4.1) holds when the curves $[x_{j+1}, x_j]$ and γ_j are "close" to each other in the quasihyperbolic metric k. Let

$$M > \max\left\{4D^2 \frac{\log(4D'^2)}{\log 2} + 1, 4D^2 \frac{\log(B(2 + A^2/6)^Q/c_1)}{\log 2}\right\},$$

where $c_1 > 0$ is a sufficiently small constant depending on A, C_1, D and Q, and let us assume that $\operatorname{dist}_k([x_{j+1}, x_j], \gamma_j) \le M$. Let $y_j \in [x_{j+1}, x_j]$ and $\tilde{y}_j \in \gamma_j$ be points such that $k(y_j, \tilde{y}_j) \le M$. Let us show that we may estimate the ρ-length of the quasihyperbolic geodesic $[x_{j+1}, x_j]$ from above by $2^{-j} p \rho(y_j)$ in the following way

$$\ell_\rho([x_{j+1}, x_j]) \le A^b D' \rho(y_j) 2^{-j+1} p, \tag{4.6}$$

where $b = 4c_2 D'^2$ and $c_2 = c_2(C_1, D) > 0$ is the constant from Lemma 3.2.

If there exists $z \in [x_{j+1}, x_j]$ such that $[x_{j+1}, x_j] \subset B_z = B_d(z, d(z)/2)$, we obtain from HI($A$) and (4.3)

$$\ell_\rho([x_{j+1}, x_j]) \le A \rho(y_j) \ell_d([x_{j+1}, x_j]) \le A D' \rho(y_j) 2^{-j+1} p.$$

Otherwise we may assume that $d(x_{j+1}, x_j) \ge d(x_{j+1})/2$. From Lemma 3.2 and inequality (4.5), it follows that

$$N(x_{j+1}, x_j) \le 4c_2 D'^2 =: b,$$

where the constant $c_2 = c_2(C_1, D) > 0$ is the constant from Lemma 3.2. Then by HI(A), every $z \in [x_{j+1}, x_j]$ satisfies

$$\rho(z) \le A^b \rho(y_j).$$

This with (4.3) gives us inequality (4.6)

$$\begin{aligned} \ell_\rho([x_{j+1}, x_j]) &\le A^b \rho(y_j) \ell_d([x_{j+1}, x_j]) \\ &\le A^b D' \rho(y_j) 2^{-j+1} p. \end{aligned}$$

Next we estimate the ρ-length of the curve γ_j from below by $2^{-j} p \rho(y_j)$. If $[x_{j+1}, x_j] \cap B_{\tilde{y}_j} \ne \emptyset$, we easily get from HI($A$) an estimate for $\ell_\rho(\gamma_j)$:

$$\ell_\rho(\gamma_j) \ge \frac{1}{A^{b+1}} \rho(y_j) \ell_d(\gamma_j \cap B_{\tilde{y}_j}). \tag{4.7}$$

Furthermore, for every $z \in [x_{j+1}, x_j] \cap B_{\tilde{y}_j}$, using inequalities (4.2) and (4.4) it holds that

$$\ell_d(\gamma_j \cap B_{\tilde{y}_j}) \geq \begin{cases} 2^{-(j+1)}p & \text{if } \gamma_j \subset B_{\tilde{y}_j}, \\ \frac{1}{2}d(\tilde{y}_j) \geq \frac{1}{2}\big(\frac{3}{2}d(z)\big) \geq \frac{3}{4D'}2^{-(j+1)}p & \text{if } \gamma_j \not\subset B_{\tilde{y}_j}. \end{cases} \tag{4.8}$$

In this case, combining (4.6), (4.7) and (4.8) we obtain the desired result (4.1)

$$\ell_\rho([x_{j+1}, x_j]) \leq \frac{16}{3}A^{2b+1}D'^2\ell_\rho(\gamma_j).$$

Therefore we may assume that $[x_{j+1}, x_j] \cap B_{\tilde{y}_j} = \emptyset$. This implies that $d(y_j, \tilde{y}_j) \geq d(\tilde{y}_j)/2$. By Lemma 3.2 there are at most $h := Mc_2$ balls in the Whitney covering $\mathcal{B}$ that intersect $[y_j, \tilde{y}_j]$ and hence, by HI(A),

$$\rho(y_j) \leq A^h\rho(\tilde{y}_j). \tag{4.9}$$

On the other hand, by HI(A) and (4.2),

$$\ell_\rho(\gamma_j) \geq \frac{1}{A}\rho(\tilde{y}_j)\ell_d(\gamma_j \cap B_{\tilde{y}_j}) \geq \begin{cases} \frac{1}{A}\rho(\tilde{y}_j)2^{-(j+1)}p & \text{if } \gamma_j \subset B_{\tilde{y}_j}, \\ \frac{1}{2A}\rho(\tilde{y}_j)d(\tilde{y}_j) & \text{if } \gamma_j \not\subset B_{\tilde{y}_j}. \end{cases} \tag{4.10}$$

If $\gamma_j \subset B_{\tilde{y}_j}$, again we obtain the desired inequality (4.1) by combining inequalities (4.6), (4.9) and (4.10). If $\gamma_j \not\subset B_{\tilde{y}_j}$, then (4.10) with (4.9) gives

$$\rho(y_j) \leq A^{h+1}\frac{2}{d(\tilde{y}_j)}\ell_\rho(\gamma_j). \tag{4.11}$$

By elementary inequalities in [GP], Lemma 2.1, and [BHK], Inequality (2.4), we obtain

$$\log\left(1 + \frac{d(y_j, \tilde{y}_j)}{\min\{d(y_j), d(\tilde{y}_j)\}}\right) \leq k(y_j, \tilde{y}_j) \leq M$$

and further,

$$\frac{1}{d(\tilde{y}_j)} \leq \frac{e^M - 1}{d(y_j, \tilde{y}_j)}. \tag{4.12}$$

Moreover, the assumption $d(y_j, \tilde{y}_j) \geq d(\tilde{y}_j)/2$ gives us

$$d(y_j) \leq d(y_j, \tilde{y}_j) + d(\tilde{y}_j) \leq 3d(y_j, \tilde{y}_j).$$

This, along with inequalities (4.11), (4.12) and (4.4), yields an estimate for the ρ-length of γ_j:

$$\begin{aligned} \rho(y_j) &\leq 2A^{h+1}\frac{e^M - 1}{d(y_j, \tilde{y}_j)}\ell_\rho(\gamma_j) \leq 6A^{h+1}\frac{e^M - 1}{d(y_j)}\ell_\rho(\gamma_j) \\ &\leq 6A^{h+1}(e^M - 1)\frac{D'}{p}2^{j+1}\ell_\rho(\gamma_j). \end{aligned} \tag{4.13}$$

Now combining (4.6) and (4.13) we obtain

$$\ell_\rho([x_{j+1}, x_j]) \le 24(e^M - 1)A^{b+h+1}D'^2\ell_\rho(\gamma_j).$$

Thus (4.1) is proven when the curves $[x_{j+1}, x_j]$ and γ_j are "close" to each other in the quasihyperbolic metric.

Case B. By Case A we may assume that $\mathrm{dist}_k([x_{j+1}, x_j], \gamma_j) > M$. Let $w_j \in [x_{j+1}, x_j]$ satisfy $d(x, w_j) = 3 \cdot 2^{-(j+2)}p$. Let $r := \ell_\rho(\gamma_j)$ and let $w \in \gamma_j$. Let us consider the ρ-ball $B_\rho(w, 2r)$.

Subcase C. If $\mathrm{dist}_k(w_j, B_\rho(w, 2r)) < M$, there exists $u \in B_\rho(w, 2r)$ such that $k(w_j, u) \le M$ and hence $\rho(w_j) \le A^h\rho(u)$ (cf. inequality (4.9)). We may assume that $\gamma_j \cap B_u = \emptyset$. Otherwise $\mathrm{dist}_k([x_{j+1}, x_j], \gamma_j) \le M + 1$ and replacing M with $M + 1$ we obtain the result by the case A. As we have assumed $\gamma_j \cap B_u = \emptyset$,

$$\begin{aligned} 2\ell_\rho(\gamma_j) = 2r &> \mathrm{dist}_\rho(u, \gamma_j) \\ &\overset{\mathrm{HI}(A)}{\ge} \frac{1}{2A}\rho(u)d(u) \\ &\overset{(4.9)}{\ge} \frac{1}{2A^{h+1}}\rho(w_j)d(u) \\ &\overset{(*)}{\ge} \frac{1}{2A^{h+1}e^M}\rho(w_j)d(w_j) \\ &\overset{(4.4)}{\ge} \frac{2^{-(j+1)}p}{2A^{h+1}D'e^M}\rho(w_j) \\ &\overset{(4.6)}{\ge} \frac{1}{8A^{b+h+1}D'^2e^M}\ell_\rho([x_{j+1}, x_j]). \end{aligned}$$

The inequality $(*)$ above follows from the elementary estimate ([GP], Lemma 2.1, [BHK], Inequality (2.3))

$$\left|\log\frac{d(w_j)}{d(u)}\right| \le k(w_j, u) \le M.$$

Again we find a constant $C \ge 1$ such that $\ell_\rho([x_{j+1}, x_j]) \le C\ell_\rho(\gamma_j)$. So (4.1) is satisfied.

Subcase D. By Subcase C we may assume that the ρ-ball $B_\rho(w, 2r)$ is "far away" from the quasihyperbolic geodesic $[x_{j+1}, x_j]$. More precisely, we may assume that $\mathrm{dist}_k(w_j, B_\rho(w, 2r)) \ge M$. Our plan is to prove that the volume growth condition VG(B) does not hold for such a ρ-ball. This is done by considering subcurves of ρ-length r of quasihyperbolic geodesics $[z, w_j]$ with $z \in \gamma_j$ and "averaging over γ_j" with respect to a suitable Frostman measure.

Let for every $z \in \gamma_j$, $[z, w_j]$ be a quasihyperbolic geodesic which joins z and w_j. Cover $[z, w_j]$ with balls $\{B_1, \dots, B_{n(z)}\} \subset \mathcal{B}$ ordered so that if $m < n$, then there exists $z_m \in B_m \cap [z, w_j]$ such that for every $\tilde{z} \in B_n \cap [z, w_j]$, we have $k(z, z_m) \le k(z, \tilde{z})$. Recall that $n(z) < \infty$.

Let $[z, w_z] \subset [z, w_j]$, where w_z is the first point which does not belong to $B_\rho(w, 2r)$. Thus $\ell_\rho([z, w_z]) \ge r$. Let $\{B_1, \dots, B_{n_r(z)}\} \subset \{B_1, \dots, B_{n(z)}\}$ be those balls which cover $[z, w_z]$. So by HI(A) and by the local D'-uniformity (quasiconvexity) of quasihyperbolic geodesics we obtain

$$\begin{aligned} r \le \ell_\rho([z, w_z]) &\le \sum_{i=1}^{n_r(z)} A\rho(z_i)\ell_d([z, w_z] \cap B_i) \\ &\le AD' \sum_{i=1}^{n_r(z)} \rho(z_i)\,\mathrm{diam}_d(B_i). \end{aligned} \tag{4.14}$$

We next provide a tool that will be used to estimate the μ_ρ-measure of the ρ-ball $B_\rho(w, 2r)$. We claim that if $B \in \mathcal{B}$ intersects $B_\rho(w, 2r)$, then $B \subset B_\rho(w, (2+\frac{A^2}{6})r)$. To show this, it suffices to prove that if $B \in \mathcal{B}$ intersects $B_\rho(w, 2r)$ then

$$\mathrm{diam}_\rho(B) \le \frac{A^2}{6} r. \tag{4.15}$$

Consider such a ball $B \in \mathcal{B}$. It follows from HI(A) that

$$\mathrm{diam}_\rho(B) \le A\rho(z_B)\,\mathrm{diam}_d(B) = \frac{A}{25}\rho(z_B)d(z_B)$$

for each $B \in \mathcal{B}$, where z_B is the centre of B. Hence it actually suffices to prove that

$$\rho(z_B)d(z_B) \le \frac{25}{6} Ar. \tag{4.16}$$

Let $y \in B \cap B_\rho(w, 2r)$. If $w \notin B_{z_B}$, then there exists a curve γ, which joins points w and y and

$$\begin{aligned} 2r &\ge \int_\gamma \rho(z)\, ds \ge \frac{1}{A}\rho(z_B)\ell_d(\gamma \cap B_{z_B}) \\ &\ge \Big(\frac{1}{2} - \frac{1}{50}\Big)\frac{1}{A}\rho(z_B)d(z_B) = \frac{12}{25A}\rho(z_B)d(z_B), \end{aligned}$$

and the inequality (4.16) is proven.

Let us assume that $w \in B_{z_B}$. The elementary estimate (2.3) implies

$$M \le k(w_j, w) \le 4D^2 \log\left(1 + \frac{d(w_j, w)}{\min\{d(w_j), d(w)\}}\right).$$

Along with the assumption that $M > 4D^2 \frac{\log(4D'^2)}{\log 2} + 1$, we see that

$$\min\{d(w_j), d(w)\} \le \frac{d(w_j, w)}{e^{M/4D^2} - 1} \le 2^{-j+1-(M-1)/4D^2} p. \tag{4.17}$$

The assumption $M > 4D^2 \frac{\log(4D'^2)}{\log 2} + 1$ and (4.4) give us

$$\begin{aligned} d(w_j) &\geq \frac{p}{D'} 2^{-(j+1)} = 2^{-j+1-(M-1)/4D^2} p \frac{2^{(M-1)/4D^2}}{2^2 D'} \\ &\geq 2^{-j+1-(M-1)/4D^2} p. \end{aligned} \tag{4.18}$$

Thus it follows from inequality (4.17) that

$$d(w) \leq 2^{-j+1-(M-1)/4D^2} p \leq 2^{-(j+1)} p.$$

Hence, from the definition of the curve γ_j and inequality (4.2) we know that γ_j cannot be a subset of B_w. Then by HI(A)

$$r = \int_{\gamma_j} \rho(z)\, ds \geq \frac{1}{2A} \rho(z_B) d(w) \geq \frac{1}{4A} \rho(z_B) d(z_B),$$

and (4.16) is proven.

Now we know that if $B \in \mathcal{B}$ intersects $B_\rho(w, 2r)$, then $B \subset B_\rho(w, (2 + \frac{1}{6}A^2)r)$. Then by HI($A$), Lemma 3.1 (iv) and Q-regularity on balls of Whitney type, we have

$$\begin{aligned} \mu_\rho(B_\rho(w, (2 + \tfrac{1}{6}A^2)r)) &= \int_{B\rho(w,(2+\frac{1}{6}A^2)r)} \rho^Q \, d\mu \\ &\geq \sum_{\substack{B \in \mathcal{B} \\ B \cap B_\rho(w,2r) \neq \emptyset}} \frac{1}{NA^Q} \rho(z_B)^Q \mu(B) \\ &\geq \sum_{\substack{B \in \mathcal{B} \\ B \cap B_\rho(w,2r) \neq \emptyset}} c_3 \rho(z_B)^Q \left(\frac{\operatorname{diam}_d(B)}{2} \right)^Q, \end{aligned} \tag{4.19}$$

where $c_3 = \dfrac{1}{NC_1 A^Q}$.

Let us choose the basepoint z_0 to be w_j. According to Frostman's Lemma (Theorem 4.1) there is a Radon measure ω supported on γ_j such that $\omega(\gamma_j) \geq \frac{\operatorname{diam}_d(\gamma_j)}{60}$ and $\omega(E) \leq \operatorname{diam}_d(E)$ for every $E \subset \gamma_j$. Then with (4.14) we obtain (a version of Fubini's theorem)

$$\begin{aligned} \omega(\gamma_j) r &\leq AD' \int_{\gamma_j} \sum_{i=1}^{n_r(z)} \rho(z_i) \operatorname{diam}_d(B_i)\, d\omega(z) \\ &\leq AD' \sum_{n=M-1}^{\infty} \sum_{\substack{B \in \mathcal{B}_n \\ B \cap [z, w_z] \neq \emptyset \\ z \in \gamma_j}} \rho(z_B) \operatorname{diam}_d(B) \omega(S(B) \cap \gamma_j). \end{aligned} \tag{4.20}$$

By Hölder's inequality we obtain that

$$\begin{aligned}
&\sum_{n=M-1}^{\infty} \sum_{\substack{B\in\mathcal{B}_n \\ B\cap[z,w_z]\neq\emptyset \\ z\in\gamma_j}} \rho(z_B)\operatorname{diam}_d(B)\omega(S(B)\cap\gamma_j) \\
&\leq \Big(\sum_{n=M-1}^{\infty} \sum_{\substack{B\in\mathcal{B}_n \\ B\cap[z,w_z]\neq\emptyset \\ z\in\gamma_j}} \rho(z_B)^Q \operatorname{diam}_d(B)^Q\Big)^{\frac{1}{Q}} \\
&\quad \Big(\sum_{n=M-1}^{\infty} \sum_{\substack{B\in\mathcal{B}_n \\ B\cap[z,w_z]\neq\emptyset \\ z\in\gamma_j}} \omega(S(B)\cap\gamma_j)^{\frac{Q}{Q-1}}\Big)^{\frac{Q-1}{Q}}.
\end{aligned}$$

Combining this with (4.20), (4.19) and the assumption $\operatorname{dist}_k(w_j, B_\rho(w,2r)) \geq M$ we obtain the estimate

$$\begin{aligned}
\omega(\gamma_j)r &\leq AD'\Big(\frac{2^Q}{c_3}\mu_\rho\big(B_\rho\big(w,\big(2+\tfrac{1}{6}A^2\big)r\big)\big)\Big)^{\frac{1}{Q}} \\
&\quad \Big(\sum_{n=M-1}^{\infty} \sum_{\substack{B\in\mathcal{B}_n \\ B\cap[z,w_z]\neq\emptyset \\ z\in\gamma_j}} \omega(S(B)\cap\gamma_j)^{\frac{Q}{Q-1}}\Big)^{\frac{Q-1}{Q}} \\
&= c_4\big(\mu_\rho\big(B_\rho\big(w,\big(2+\tfrac{1}{6}A^2\big)r\big)\big)\big)^{\frac{1}{Q}}\Big(\sum_{n=M-1}^{\infty} \sum_{\substack{B\in\mathcal{B}_n \\ B\cap[z,w_z]\neq\emptyset \\ z\in\gamma_j}} \omega(S(B)\cap\gamma_j)^{\frac{Q}{Q-1}}\Big)^{\frac{Q-1}{Q}},
\end{aligned} \tag{4.21}$$

where $c_4 = 2AD'c_3^{-\frac{1}{Q}} = 2(NC_1)^{\frac{1}{Q}}A^2D'$.

In order to estimate the measure of the shadow of the ball $B \in \mathcal{B}_n$, let us make a couple of preliminary estimates. For every $v \in B \cap [z, w_j]$, where $B \in \mathcal{B}$ and $z \in \gamma_j$, we have by uniformity (quasiconvexity) and inequality (4.3) that

$$d(w_j, v) \leq \ell_d([w_j, v]) \leq \ell_d([w_j, z]) \leq D'd(w_j, z) \leq 2^{-j+1}pD'.$$

In the same way as in inequalities (4.17) and (4.18), we obtain from inequality (4.4) and the assumption $n \geq M-1 \geq 4D^2\frac{\log(4D'^2)}{\log 2}$ that for every $v \in B\cap[z, w_j]$, where $B \in \mathcal{B}_n$ and $z \in \gamma_j$, it holds that

$$d(v) \leq 2^{-j+1-n/4D^2}pD'.$$

Furthermore, for every centre point $z_B \in B \in \mathcal{B}_n$, such that $B \cap [z, w_j] \neq \emptyset$ for some $z \in \gamma_j$, it holds that

$$d(z_B) \leq \frac{50}{49} d(v) \leq 2^{-j+1-n/4D^2} p \frac{50D'}{49}. \tag{4.22}$$

Also from the uniformity of the space (Ω, d) and inequality (4.22) it follows that there exist a constant $c_5 = c_5(C_1, D) \geq 1$ such that for every $B \in \mathcal{B}_n$, so that $B \cap [z, w_j] \neq \emptyset$ for some $z \in \gamma_j$, it holds

$$\operatorname{diam}_d(S(B)) \leq c_5 \operatorname{diam}_d(B) \leq 2^{-j+2-n/4D^2} p c_5 \frac{50D'}{49}. \tag{4.23}$$

Now for every $n \geq M - 1$ it holds by Lemma 3.4, Frostman's Lemma and inequality (4.23) that

$$\begin{aligned}
&\sum_{\substack{B \in \mathcal{B}_n \\ B \cap [z, w_z] \neq \emptyset \\ z \in \gamma_j}} \omega(S(B) \cap \gamma_j)^{\frac{Q}{Q-1}} \\
&\leq \max_{\substack{B \in \mathcal{B}_n \\ B \cap [z, w_z] \neq \emptyset \\ z \in \gamma_j}} \omega(S(B) \cap \gamma_j)^{\frac{1}{Q-1}} \sum_{\substack{B \in \mathcal{B}_n \\ B \cap [z, w_z] \neq \emptyset \\ z \in \gamma_j}} \omega(S(B) \cap \gamma_j) \\
&\leq c_6 \omega(\gamma_j) \max_{\substack{B \in \mathcal{B}_n \\ B \cap [z, w_z] \neq \emptyset \\ z \in \gamma_j}} \omega(S(B) \cap \gamma_j)^{\frac{1}{Q-1}} \\
&\leq c_6 \omega(\gamma_j) \max_{\substack{B \in \mathcal{B}_n \\ B \cap [z, w_z] \neq \emptyset \\ z \in \gamma_j}} \operatorname{diam}_d(S(B) \cap \gamma_j)^{\frac{1}{Q-1}} \\
&\leq c_6 \Big(\frac{200D' c_5}{49}\Big)^{\frac{1}{Q-1}} \omega(\gamma_j) (2^{-j-n/4D^2} p)^{\frac{1}{Q-1}},
\end{aligned}$$

where $c_6 = c_6(C_1, D)$ is from Lemma 3.4. Furthermore, using this we may compute that

$$\begin{aligned}
&\sum_{n=M-1}^{\infty} \sum_{\substack{B \in \mathcal{B}_n \\ B \cap [z, w_z] \neq \emptyset \\ z \in \gamma_j}} \omega(S(B) \cap \gamma_j)^{\frac{Q}{Q-1}} \\
&\leq c_6 \Big(\frac{200D' c_5}{49}\Big)^{\frac{1}{Q-1}} \omega(\gamma_j) \sum_{n=M-1}^{\infty} (2^{-j-n/4D^2} p)^{\frac{1}{Q-1}} \\
&\leq c_7 \omega(\gamma_j) p^{\frac{1}{Q-1}} 2^{\frac{-j}{Q-1}} 2^{\frac{-M}{4D^2(Q-1)}},
\end{aligned}$$

where $c_7 = c_6\big(\frac{200D'c_5}{49}\big)^{\frac{1}{Q-1}} \dfrac{2^{\frac{2}{4D^2(Q-1)}}}{2^{\frac{1}{4D^2(Q-1)}} - 1}$. Thus with (4.21) we have

$$\omega(\gamma_j)^Q r^Q \le c_4^Q c_7^{Q-1} \mu_\rho(B_\rho(w, (2 + \tfrac{1}{6}A^2)r))\omega(\gamma_j)^{Q-1} 2^{-j-\frac{M}{4D^2}} p.$$

Furthermore $\omega(\gamma_j) \ge \frac{\operatorname{diam}_d(\gamma_j)}{60}$, and this gives us

$$\begin{aligned}\mu_\rho(B_\rho(w, (2 + \tfrac{1}{6}A^2)r)) &\ge \omega(\gamma_j) \frac{1}{c_4^Q c_7^{Q-1}} \frac{2^{j+\frac{M}{4D^2}}}{p} r^Q \\ &\ge \frac{2^{-j-1}p}{60} \frac{1}{c_4^Q c_7^{Q-1}} \frac{2^{j+\frac{M}{4D^2}}}{p} r^Q \\ &= 2^{\frac{M}{4D^2}} c_1 r^Q,\end{aligned}$$

where $c_1 = \dfrac{49 \cdot 2^{\frac{-2}{4D^2}-1}\big(2^{\frac{1}{4D^2(Q-1)}} - 1\big)^{Q-1}}{12000 c_5 N C_1 (2A^2)^Q D'^{Q+1} c_6^{Q-1}}$.

This is a contradiction because when M is sufficiently big, the volume growth condition VG(B) will not hold. Consequently, if $k([x_{j+1}, x_j], \gamma_j) > M$ then our ρ-ball is in the quasihyperbolic metric k so big that $\operatorname{dist}_k(w_j, B_\rho(w, 2r)) \le M$. Thus the conclusion is that $\ell_\rho([x_{j+1}, x_j]) \le C\ell_\rho(\gamma_j)$, where $C = C(A, B, C_1, D, Q)$. □

There is nothing special about the constant $\frac{1}{2}$ in condition HI(A) and the constants $\frac{1}{50}$ and 5 in Whitney covering. The only restriction in the Whitney covering is that if $\lambda_1 B_d(z_1, d(z_1)/\lambda_2) \cap \lambda_1 B_d(z_2, d(z_2)/\lambda_2) \ne \emptyset$, then $\lambda_1 B_d(z_1, d(z_1)/\lambda_2)$ must be included in some ball $B_d(z_2, d(z_2)/\lambda_3)$ on which the measure μ is doubling. Otherwise one can choose the constants as desired.

Acknowledgement. The results of this paper were obtained while the authors were visiting the Centre de Recerca Matemàtica, Universitat Autònoma de Barcelona in April 2009, during the research programme "Harmonic Analysis, Geometric Measure Theory and Quasiconformal Mappings". They thank CRM for its hospitality. The authors also thank Kevin Wildrick for carefully reading the manuscript.

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Received January 8, 2010

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