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A characterization of Inoue surfaces

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Abstract. We give a characterization of Inoue surfaces in terms of automorphic pluriharmonic functions on a cyclic covering. Together with results of Chiose and Toma, this completes the classification of compact complex surfaces of Kähler rank one.

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In this paper we shall solve a problem proposed in [C-T]:

Theorem 1. *Let S be a compact connected complex surface of algebraic dimension 0. Suppose that there exists an infinite cyclic covering $\widetilde{S} \xrightarrow{\pi} S$ (with covering transformations generated by $\varphi \in \mathrm{Aut}(\widetilde{S})$) and a nonconstant positive pluriharmonic function F on $\widetilde{S}$ such that*

$$F \circ \varphi = \lambda \cdot F$$

for some positive real λ. Then S is a (possibly blown up) Inoue surface.

The class of Inoue surfaces was discovered by Inoue (and independently Bombieri) around 1972 [Ino], [Nak]. They are special (and explicit) compact quotients of $\mathbb{H} \times \mathbb{C}$, and they enjoy the following properties:

– the first Betti number is 1, the second Betti number is 0;

– they admit holomorphic foliations;

– they do not contain compact complex curves.

Conversely, Inoue proved in [Ino] that any compact connected complex surface with the above properties is an Inoue surface.

Our proof of Theorem 1 will be ultimately a reduction to Inoue's theorem. The pluriharmonic function F naturally induces a holomorphic (and possibly singular) foliation $\mathcal{F}$ on S. By a "topological" study of such a foliation we will be able to understand some topological structure of S, and in particular to show that $c_2(S_{\min}) = 0$ or $c_1^2(S_{\min}) = 0$ (where $S_{\min}$ denotes the minimal model of S). From this vanishing of Chern numbers, and results of Kodaira and Inoue, the conclusion will be immediate. Remark that, conversely and by construction, every Inoue surface satisfies the hypotheses of Theorem 1, which therefore gives a precise characterization of Inoue surfaces.

Together with the results of [C-T], Theorem 1 allows to complete the classification of compact surfaces of Kähler rank one. Recall [H-L], [C-T] that a compact connected complex surface S has *Kähler rank one* if it is not Kählerian but it admits a closed semipositive $(1,1)$-form, not identically vanishing (this is not the original definition of [H-L], but it is equivalent to it by the results of [C-T], see also [Lam] and [Tom]).

Corollary 2 ([C-T] and Theorem 1). *The only compact connected complex surfaces of Kähler rank one are*

(1) *non-Kählerian elliptic fibrations;*

(2) *certain Hopf surfaces, and their blow-ups;*

(3) *Inoue surfaces, and their blow-ups.*

In the case of an Inoue surface, a closed semipositive $(1,1)$-form is given by $(dF/F) \wedge (d^cF/F)$, with F as in Theorem 1. The closedness of that $(1,1)$-form is a consequence of the pluriharmonicity of F, $dd^cF \equiv 0$.

1. Geometric preliminaries

Let S be a surface as in Theorem 1. Without loss of generality, we may assume that S is minimal, since the hypotheses are clearly bimeromorphically invariant. The assumption $a(S) = 0$ implies, by Enriques–Kodaira classification [BPV], p. 188, that S is either a torus or a K3 surface or a surface of class $\mathrm{VII}_\circ$, that is $b_1(S) = 1$ and $kod(S) = -\infty$. However, the existence of a positive nonconstant pluriharmonic function on some covering of S, and therefore on its universal covering, excludes the case of tori, by Liouville theorem, and the case of K3 surfaces, which are simply connected. Thus S is of class $\mathrm{VII}_\circ$. For the same reason (Liouville theorem), S cannot be a Hopf surface, whose universal covering is $\mathbb{C}^2 \setminus (0,0)$.

We claim that, in order to prove Theorem 1, it is sufficient to prove that

$$c_2(S) = 0$$

or

$$c_1^2(S) = 0.$$

Indeed, we firstly observe that these two conditions are equivalent, by Noether formula and $\chi(\mathcal{O}_S) = 0$ (which follows from $S \in \mathrm{VII}_\circ$). Then, $c_2(S) = 0$ and $b_1(S) = 1$ imply $b_2(S) = 0$. By a classical result of Kodaira ([Nak], Theorem 2.4) S contains no compact complex curve, otherwise it would be a Hopf surface. Since S also admits a holomorphic foliation (see below), all the hypotheses of Inoue's theorem [Ino] are satisfied and we get that S is an Inoue surface.

The automorphic function F on $\widetilde{S}$ induces a real analytic map

$$f = \log F : S \longrightarrow \mathbb{S}^1 = \mathbb{R}/[\mathbb{Z} \cdot \log(\lambda)].$$

The regular fibers of f are smooth (possibly disconnected) Levi-flat hypersurfaces in S, because F is pluriharmonic. However, f could have also some singular fibers, corresponding to critical points of F. In fact, our aim is precisely to show that these singular fibers do not exist at all, since this is clearly equivalent to the vanishing of the Euler characteristic $c_2(S)$.

The holomorphic 1-form $\omega = \partial F \in \Omega^1(\widetilde{S})$ descends to S to a holomorphic section (still denoted by ω) of $\Omega^1(S) \otimes L$, where L is a flat line bundle (the one defined by the cocycle $\lambda \in \mathbb{R}^+ \subset \mathbb{C}^* = H^1(S, \mathbb{C}^*)$). This twisted closed holomorphic 1-form induces a holomorphic foliation $\mathcal{F}$ on S, which is tangent to the fibers of f.

In the following it will be important to distinguish between the singularities of $\mathcal{F}$, $\mathrm{Sing}(\mathcal{F})$, and the zeroes of ω, $\mathrm{Z}(\omega)$. The former are only isolated points, since (as customary) we like to deal with "saturated" foliations. The latter, on the contrary, may contain some compact complex curves. Remark also that $\mathrm{Z}(\omega)$ coincides with the set of critical points of f, $\mathrm{Crit}(f)$.

The foliation $\mathcal{F}$ has a normal bundle $N_{\mathcal{F}}$ and a tangent bundle $T_{\mathcal{F}}$ [Br1], which are related to the canonical bundle K_S of S by the adjunction type relation

$$N_{\mathcal{F}} \otimes T_{\mathcal{F}} = K_S^{-1}.$$

Because $\mathcal{F}$ is generated by $\omega \in \Omega^1(S) \otimes L$, we have [Br1]

$$N_{\mathcal{F}} = L \otimes \mathcal{O}\big(-\textstyle\sum m_j C_j\big)$$

where $\{C_j\}$ are the curves contained in $\mathrm{Z}(\omega)$ (if any) and $\{m_j\}$ are the respective vanishing orders.

We shall prove below that $\mathrm{Z}(\omega)$ is at most composed by isolated points, giving by the previous formula the flatness of $N_{\mathcal{F}} = L$. Then we shall prove that either $c_2(S) = 0$ or $T_{\mathcal{F}}$ is also flat. But in this second case we therefore get that K_S is flat too, hence $c_1^2(S) = 0$.

2. The structure of the smooth fibers

For every $\vartheta \in \mathbb{S}^1$, let $M_\vartheta = f^{-1}(\vartheta)$ be the fiber of f over ϑ, and denote by $M_{\vartheta,j}$, $j = 1, \dots, \ell$, its connected components (with ℓ possibly depending on ϑ). In this section we consider the smooth components, that is the components around which f has no critical point, and we prove that they have the expected structure.

Proposition 2.1. *Let $M_{\vartheta,j}$ be a smooth connected component of a fiber M_ϑ. Then the leaves of $\mathcal{F}|_{M_{\vartheta,j}}$ are either all isomorphic to $\mathbb{C}$, or all isomorphic to $\mathbb{C}^*$. In the first case, $M_{\vartheta,j}$ is diffeomorphic to $\mathbb{T}^3$, and $\mathcal{F}|_{M_{\vartheta,j}}$ is a linear totally irrational foliation. In the second case, $M_{\vartheta,j}$ is a $\mathbb{S}^1$-bundle over $\mathbb{T}^2$, and $\mathcal{F}|_{M_{\vartheta,j}}$ is the pull-back of a linear irrational foliation on $\mathbb{T}^2$.*

The first case will lead to Inoue surface of type S_M, and the second case to those of type $S^{(+)}_{N,p,q,r;t}$ or $S^{(-)}_{N,p,q,r}$ [Ino].

Proof. The foliation $\mathcal{H} = \mathcal{F}|_{M_{\vartheta,j}}$ is defined by the closed and nonsingular 1-form $\beta = d^cF|_{M_{\vartheta,j}}$ (which is well defined on a neighbourhood of any fiber, up to a multiplicative constant). We may use some classical results of Tischler [God], I.4, concerning the structure of (real) codimension one foliations defined by closed 1-forms. According to those results, the foliation can be smoothly perturbed to a fiber bundle over the circle with fiber Σ_g, the (real) oriented compact surface of genus $g \geq 1$. Note that, since $a(S) = 0$, the leaves of $\mathcal{H}$ cannot be all compact, and so they are all dense in $M_{\vartheta,j}$ [God], I.4.3. Moreover, by using the flow of a smooth vector field v on $M_{\vartheta,j}$ such that $\beta(v) \equiv 1$, and the closedness of β, we see that the leaves are all diffeomorphic to the same abelian covering of Σ_g [God], I.4.2 and I.4.6.

The above flow of v sends leaves to leaves, but of course it does not need to preserve the complex structure of the leaves, that is it does not need to realize a conformal diffeomorphism between the leaves. However, the compactness of $M_{\vartheta,j}$ implies, at least, that such a diffeomorphism is quasi-conformal [Ahl]. More precisely, if ϕ_t is the flow of v at time t, then there exists a constant $k_t < 1$ such that, for every $p \in M_{\vartheta,j}$, the complex dilatation of $d\phi_t$ acting from $T_p\mathcal{H}$ to $T_{\phi_t(p)}\mathcal{H}$ is bounded by k_t (the complex dilatation is the quotient between the antiholomorphic and the holomorphic part of $d\phi_t$, and it is at each point less than 1 because ϕ_t is orientation preserving between the leaves; the compactness of $M_{\vartheta,j}$ gives a uniform bound). Thus, ϕ_t realizes a k_t-quasi-conformal diffeomorphism between any leaf L and its image leaf $\phi_t(L)$, and therefore a k_t-quasi-conformal diffeomorphism between their respective universal coverings.

In particular, since $\mathbb{C}$ and $\mathbb{D}$ are *not* quasi-conformal, we obtain that all the leaves of $\mathcal{H}$ have the same (conformal) universal covering: either they are all parabolic, uniformised by $\mathbb{C}$, or all hyperbolic, uniformised by $\mathbb{D}$.

For our purposes, it is sufficient to prove that the leaves are parabolic: since they are abelian coverings of Σ_g, this implies that $g = 1$, i.e. $M_{\vartheta,j}$ is a torus bundle over the circle. The rest of the statement is a standard fact, see I.4, IV.2.23 and IV.2.24 in [God].

We can associate to $\mathcal{H}$ a closed positive current $\Phi \in A^{1,1}(S)'$, by integration along the leaves against the transverse measure defined by β [Ghy]: if $\eta \in A^2(S)$, we define

$$\Phi(\eta) = \int_{M_{\vartheta,j}} \beta \wedge \eta.$$

Remark that this is indeed a $(1,1)$-current: if η is of bidegree $(2,0)$ or $(0,2)$ then η identically vanishes when restricted to the leaves, hence the 3-form $\beta \wedge \eta$ on $M_{\vartheta,j}$ is identically zero. The hypersurface $M_{\vartheta,j}$ is cooriented, and hence oriented, by its defining function F, and this gives also the positivity of Φ: if $\eta \geq 0$ then $dF \wedge d^cF \wedge \eta \geq 0$ too, and so the integral of $d^cF \wedge \eta$ on a level set of F is nonnegative. Finally, Φ is closed because β is closed.

Obviously this current does not charge compact complex curves, since there are no such curves at all in $M_{\vartheta,j}$, hence by a result of Lamari [Lam], [Tom], Remark 8, it is an *exact* positive current. Actually, in our case the proof of such a fact is very simple. The current Φ, supported on $M_{\vartheta,j}$, can be approximated by a current Φ' supported on a nearby fiber component $M_{\vartheta',j}$, and the disjointness of supports gives $[\Phi]\cdot[\Phi'] = 0$, i.e. $[\Phi]^2 = 0$ (here $[\cdot]$ denotes the De Rham cohomology class). Now, on a class $\mathrm{VII}_\circ$ surface the intersection form is negative definite [BPV], p. 120, and the vanishing of the selfintersection implies the vanishing of the cohomology class.

As a consequence of this, the De Rham cohomology class $[\Phi]$ (which is zero!) has vanishing product with the Chern class of $T_{\mathcal{F}}$:

$$c_1(T_{\mathcal{F}}) \cdot [\Phi] = 0.$$

Let us show that this implies the parabolicity of the leaves. This is a particularly simple instance of the foliated Gauss-Bonnet theorem, see [Ghy]. In the opposite case, we may put on the leaves of $\mathcal{H}$ their Poincaré metric g, which can be seen as a hermitian metric on $T_{\mathcal{F}}|_{M_{\vartheta,j}}$. It is a continuous metric [Ghy], §5.2, and it can be regularized by a smooth hermitian metric on $T_{\mathcal{F}}|_{M_{\vartheta,j}}$ whose curvature along the leaves is still strictly negative. For instance, this can be done with the help of the flow of the above vector field v. Indeed, the leafwise riemannian metric $\phi_t^*(g)$ induces, by symmetrization, a leafwise hermitian metric $\phi_t^*(g)^h$, and, for t small, the leafwise curvature of $\phi_t^*(g)^h$ is strictly negative; a convolution of these leafwise metrics produces the desired result. We then extend this smooth hermitian metric on $T_{\mathcal{F}}|_{M_{\vartheta,j}}$ to the full $T_{\mathcal{F}}$, on the full S, in any smooth way. The curvature form $\Theta \in A^{1,1}(S)$ is negative on the leaves of $\mathcal{H}$, and therefore

$$\Phi(\Theta) = \int_{M_{\vartheta,j}} \beta \wedge \Theta < 0.$$

This is in contradiction with the vanishing of $c_1(T_{\mathcal{F}}) \cdot [\Phi]$. □

Remark 2.2. Let us stress a subtle detail of the previous proof. The current Φ can be also considered as a current on the real threefold $M_{\vartheta,j}$, or more precisely as the direct image of a current Φ_0 on $M_{\vartheta,j}$ under the inclusion map $M_{\vartheta,j} \to S$. This current Φ_0, however, is *not* exact in $M_{\vartheta,j}$ (i.e., any current Ψ on S with $d\Psi = \Phi$ cannot have support contained in $M_{\vartheta,j}$) . Thus, in order to get the vanishing of $c_1(T_{\mathcal{F}}) \cdot [\Phi]$, we used also the fact that the tangent bundle $T_{\mathcal{H}}$ extends to the full S, or more precisely that its Chern class in $H^2(M_{\vartheta,j}, \mathbb{R})$ extends to S, which is obvious in our case since we have a global foliation on S. Now, one can imagine a more general situation, in which we have a Levi-flat hypersurface M in a class VII$_\circ$ surface, such that the Levi foliation is given by a closed 1-form (or, more generally, admits a transverse measure invariant by holonomy). Is it still true that the leaves of this Levi foliation are parabolic?

3. The structure of the singularities

In order to study Sing($\mathcal{F}$) and Z(ω), we need a general lemma on critical points.

Let $\widetilde{U}$ be a smooth complex surface and let $D \subset \widetilde{U}$ be a compact connected curve (with possibly several irreducible components). Suppose that the intersection form on D is negative definite, so that D is contractible to one point [BPV], p. 72. After contraction, we get a normal surface U and a point $q \in U$, image of D; we do not exclude that q is a smooth point. Let now $\widetilde{H}$ be a holomorphic function on $\widetilde{U}$, vanishing on D, such that

$$\mathrm{Crit}(\widetilde{H}) = D.$$

After contraction, we thus get a holomorphic function H on U with (at most) an isolated critical point at q. If B is a small ball centered at q, then $H_0 = H^{-1}(0) \cap B$ is a collection of k discs $H_0^1, \dots, H_0^k$ passing through q, whereas $H_\varepsilon = H^{-1}(\varepsilon) \cap B$ (ε small and not zero) is a connected curve with k boundary components. The topological type of H_ε does not depend on ε (small and not zero), it is the so-called Milnor fiber of H at q.

Lemma 3.1. *Under the previous notation, suppose that the genus of the Milnor fiber of H at q is zero. Then:*

(1) *q is a smooth point of U;*

(2) *H has a Morse type critical point at q.*

Proof. The hypothesis means that the Milnor fiber is a sphere with k holes. By a standard construction (see the figure below), we may glue to $W = \cup_{|\varepsilon|<r} H_\varepsilon$ ($r > 0$ small) a collection of k bidiscs in such a way that we obtain a normal complex surface V and a proper holomorphic map

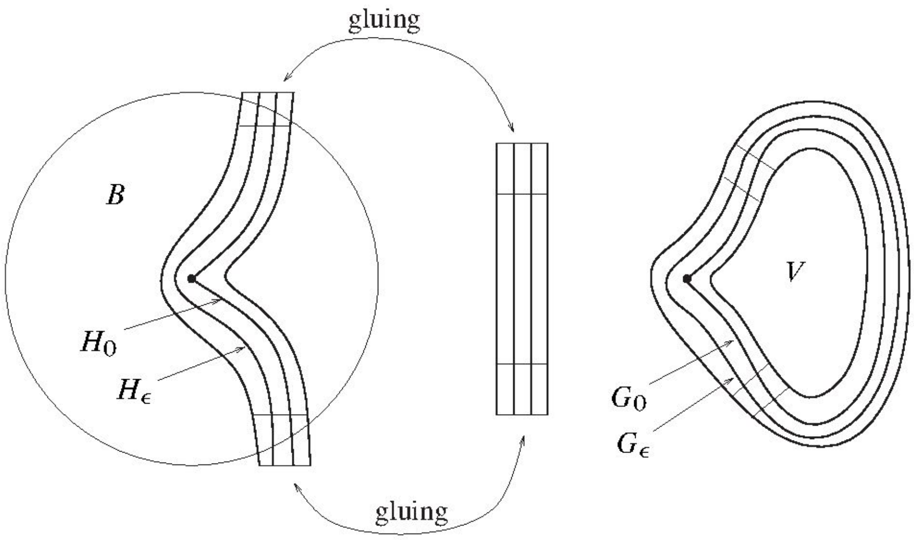
$$G \colon V \longrightarrow \mathbb{D}(r)$$

such that:

(i) $W \subset V$ and $G|_W = H$;

(ii) $G_\varepsilon = G^{-1}(\varepsilon)$ is a smooth rational curve for every nonzero $\varepsilon \in \mathbb{D}(r)$;

(iii) $G_0 = G^{-1}(0)$ is a collection of k rational curves $G_0^1, \dots, G_0^k$ passing through q, with $G_0^j \cap W = H_0^j$ for every j.



Remark that all the components G_0^j of G_0 have multiplicity 1, i.e. G vanishes along $G_0^j \setminus \{q\}$ at first order only. On the other hand, we may blow-up q to the original D, and we get a smooth complex surface $\widetilde{V}$ and a map

$$\widetilde{G}\colon \widetilde{V} \longrightarrow \mathbb{D}(r)$$

whose fiber over 0 is $\widehat{G}_0^1 \cup \ldots \cup \widehat{G}_0^k \cup D$, with $\widehat{G}_0^j$ the strict transform of G_0^j in $\widetilde{V}$. By construction, we have

$$\operatorname{mult}(\widehat{G}_0^j) = 1$$

for every j and

$$\operatorname{mult}(C) \geq 2$$

for every irreducible component C of D, since $\operatorname{Crit}(\widetilde{H}) = D$.

Recall now [BPV], p. 142, that such a $\widetilde{V}$ can be also blow-down to the trivial fibration $\mathbb{D}(r) \times \mathbb{C}P^1$, in such a way that the singular fiber of $\widetilde{G}$ is sent to the regular fiber $\{0\} \times \mathbb{C}P^1$. In other words, that singular fiber is obtained from a regular fiber by a sequence of monoidal transformations. It is then easy to see that D necessarily contains a (-1)-curve: the reason is that a monoidal transformation at a point belonging to an irreducible component of multiplicity m creates a new irreducible component whose multiplicity will be not less than m. By iterating this principle, we see that D contracts to a regular point, whence the first part of the lemma.

Moreover, after this contraction the singular fiber becomes a curve (the fiber G_0 in the now smooth surface V) still dominating a regular fiber, hence in particular it has only normal crossings. Since all the components of G_0 pass through q, we get $k = 1$ (G_0 is a single smooth rational curve of selfintersection 0) or $k = 2$ (G_0 is a pair of two smooth rational curves of selfintersection -1). In this second case, H has at q a Morse type critical point. The first case cannot occur: H would be regular at q, but then at least one of the components of D (which contracts to q) would be of multiplicity 1, i.e. would be not contained in $\mathrm{Crit}(\widetilde{H})$. □

Remark 3.2. If $H\colon U \to \mathbb{C}$, $U \subset \mathbb{C}^2$, has an isolated critical point whose Milnor fiber has genus zero, then the critical point is of Morse type: it is a particular case of the previous lemma, but it is also a consequence of classical formulae estimating the genus of the Milnor fiber. However, some care is needed when $\mathbb{C}^2$ is replaced by a singular surface. For instance, take the function zw on $\mathbb{C}^2$ and quotient by the involution $(z, w) \mapsto (-z, -w)$. We get a normal surface U and a holomorphic function H on U with an isolated critical point whose Milnor fiber has genus zero. This kind of examples (and more complicated ones) do not appear in Lemma 3.1 because, when we take the resolution $\widetilde{U} \to U$, the critical set of $\widetilde{H}$ is not the *full* exceptional divisor D.

We can now return to our compact complex surface S.

Proposition 3.3. *The zero set* $\mathrm{Z}(\omega)$ *is composed only by isolated points, all of Morse type. In particular, the normal bundle* $N_{\mathcal{F}}$ *coincides with the flat line bundle* L.

Proof. Let D be a connected component of $\mathrm{Z}(\omega)$. If it is a curve, then it is a tree of rational curves with negative definite intersection form: this follows from results of Nakamura on the possible configurations of curves on $\mathrm{VII}_\circ$ surfaces [Nak], and the absence of elliptic curves and cycles of rational curves [Tom], [C-T]. In particular, D is simply connected, and so the (twisted) closed 1-form ω is exact on a neighbourhood $\widetilde{U}$ of D: $\omega = d\widetilde{H}$ and $\mathrm{Crit}(\widetilde{H}) = D$. We therefore are in the setting of Lemma 3.1, and we have just to verify the genus zero hypothesis.

Now, D is contained in a singular fiber M_{ϑ_0}, which can be approximated by regular ones, on which we already know that the foliation has leaves $\mathbb{C}$ or $\mathbb{C}^*$. It follows obviously that the Milnor fiber has genus zero, and so by Lemma 3.1 the contraction of D produces a smooth point. But we are also assuming since the beginning that S is minimal, hence such a contraction cannot exist and so $\mathrm{Z}(\omega)$ is composed only by isolated points.

By a similar argument, and again Lemma 3.1 (or its particular case explained in Remark 3.2), all these points are of Morse type. □

4. The structure of the singular fibers

The fact that f has only isolated critical points, all of Morse type, allows to describe the structure of the foliation also on the singular components of the fibers of f. Basically, everything in this section is already contained in [Mil].

Let us firstly observe that, since f is the logarithm of a pluriharmonic function F, all its critical points have index 2 [Mil], p. 39. Around such a point p we may choose holomorphic coordinates (z, w) such that $F(z, w) = \mathrm{Re}(zw) + c$, $c = F(p)$. It follows that each connected component of any fiber M_ϑ is a real analytic subvariety of S of dimension 3, with isolated singularities, each one being topologically a cone over $\mathbb{T}^2$. Moreover, when we cross a critical value the number of connected components of the fiber does not change (even locally, around a critical point). It follows that there exists a finite cyclic covering $S' \overset{q}{\to} S$ such that $f' = f \circ q \colon S' \to \mathbb{S}^1$ has connected fibers.

Since the class of Inoue surfaces is invariant by finite coverings (by Inoue's results), this means that, without loss of generality, we may suppose in the following that the fibers of f are *connected*.

Let $\mathrm{Cv}(f) \subset \mathbb{S}^1$ be the set of critical values of f, and let $J \subset \mathbb{S}^1$ be a connected component of $\mathbb{S}^1 \setminus \mathrm{Cv}(f)$. Set $\mathcal{F}_\vartheta = \mathcal{F}|_{M_\vartheta}$, and recall that, for every $\vartheta \in J$, $(M_\vartheta, \mathcal{F}_\vartheta)$ is described by Proposition 2.1.

Lemma 4.1. *The differentiable type of $(M_\vartheta, \mathcal{F}_\vartheta)$ does not depend on $\vartheta \in J$.*

Proof. On $f^{-1}(J)$ we have a real codimension one foliation $\mathcal{G}$ given by the integrable nonsingular 1-form $d^c F/F$ (or, locally, by the closed 1-form $d^c F$). The foliation $\mathcal{G}$ is transverse to the fibers M_ϑ, $\vartheta \in J$, and its trace on M_ϑ is precisely the foliation $\mathcal{F}_\vartheta$. Thus, the foliations $\mathcal{F}_\vartheta$ are *integrably homotopic*, and it is a standard fact [God], I.3.8, to check that they are isotopically conjugate. Let us anyway recall the argument, since it will be useful later in a slightly more general context. We can easily construct a smooth vector field v on $f^{-1}(J)$ such that: (i) v is tangent to the leaves of $\mathcal{G}$; (ii) $df(v) \equiv 1$. Then the flow of v sends fibers of f to fibers of f, by (ii), and it conjugates the corresponding foliations, by (i). □

Take now a singular fiber M_ϑ, and set

$$M_\vartheta^\circ = M_\vartheta \setminus \mathrm{Sing}(M_\vartheta), \quad \mathcal{F}_\vartheta^\circ = \mathcal{F}|_{M_\vartheta^\circ}.$$

Call J_1 and J_2 the intervals of $\mathbb{S}^1 \setminus \mathrm{Cv}(f)$ adjacent to ϑ (with $J_1 = J_2$ if f has only one critical value) and take $\vartheta_1 \in J_1$, $\vartheta_2 \in J_2$. Set n equal to the cardinality of $\mathrm{Sing}(M_\vartheta)$.

Proposition 4.2. *For every $k = 1, 2$, there exists n disjoint smooth closed curves*

$$\gamma_1^k, \dots, \gamma_n^k \subset M_{\vartheta_k},$$

all tangent to the foliation $\mathcal{F}_{\vartheta_k}$, such that:

(1) *if each γ_j^k, $j = 1, \dots, n$, is collapsed to one point, we get a topological space homeomorphic to M_ϑ, with a foliation homeomorphic to $\mathcal{F}_\vartheta$;*

(2) *$(M_\vartheta^\circ, \mathcal{F}_\vartheta^\circ)$ is diffeomorphic to $(M_{\vartheta_k}, \mathcal{F}_{\vartheta_k})$ with the circles $\gamma_1^k, \dots, \gamma_n^k$ removed.*

Proof. The idea is the same as in Lemma 4.1, but of course the singularities give troubles.

On a neighbourhood of M_ϑ we still have a real codimension one foliation $\mathcal{G}$, given by the Kernel of d^cF, but now it is singular at $\mathrm{Sing}(M_\vartheta)$ (and only there). Outside those singularities, $\mathcal{G}$ is transverse to the fibers of f. Around a singular point, we may choose local holomorphic coordinates (z, w) such that

$$F(z, w) = \mathrm{Re}(zw) + c,$$

so that $\mathcal{G}$ is given by the level sets of

$$G(z, w) = \mathrm{Im}(zw),$$

since $d^cF = dG$. Write $z = x + \mathrm{i}y$, $w = s + \mathrm{i}t$, so that $F = xs - yt + c$ and $G = xt + ys$. Let v_0 be the (euclidean) gradient of F:

$$v_0 = s\frac{\partial}{\partial x} + x\frac{\partial}{\partial s} - t\frac{\partial}{\partial y} - y\frac{\partial}{\partial t}$$

and note that $dG(v_0) \equiv 0$, that is v_0 is tangent to $\mathcal{G}$.

This vector field has an hyperbolic behaviour: there is a stable manifold $W^s = \{s = -x, t = y\}$, corresponding to the eigenspace of v_0 of eigenvalue -1, and an unstable manifold $W^u = \{s = x, t = -y\}$, corresponding to the eigenspace of eigenvalue $+1$. The trajectories of v_0 on W^s (resp. on W^u) converge to the origin 0 when the time tends to $+\infty$ (resp. to $-\infty$); all the other trajectories stay far from 0. Remark that $F|_{W^s}$ has a maximum point at 0, whereas $F|_{W^u}$ has a minimum point. If $\varepsilon > 0$ (small), then $F^{-1}(c + \varepsilon)$ is disjoint from W^s and intersects W^u on a closed curve γ_ε^u. Similarly, $F^{-1}(c - \varepsilon)$ is disjoint from W^u and intersects W^s along a closed curve γ_ε^s. Remark also that G is identically zero on $W^s \cup W^u$, so that the curves γ_ε^u and γ_ε^s are tangent to $\mathcal{F}$.

Consider now the normalized vector field $v = v_0/||v_0||^2$, which is smooth outside the origin, and note that $dF(v) \equiv 1$ and $dG(v) \equiv 0$, so that the local flow of v sends F-fibers to F-fibers by respecting the G-foliations on them. By the previous analysis, for $\varepsilon > 0$ small the flow ϕ_ε is well defined on $F^{-1}(c-\varepsilon) \setminus \gamma_\varepsilon^s$, with values in $F^{-1}(c) \setminus \{0\}$, and it extends continuously to $F^{-1}(c-\varepsilon)$ by sending γ_ε^s to 0. Similarly, $\phi_{-\varepsilon}$ extends to a continuous map from $F^{-1}(c + \varepsilon)$ to $F^{-1}(c)$ which collapses γ_ε^u to 0.

This local construction gives the desired result locally, on a neighbourhood of a singular point. By using a partition of unity, we can find a smooth vector field v

on $f^{-1}(J_1 \cup J_2) \cup M_\vartheta^\circ$, tangent to $\mathcal{G}$, satisfying $df(v) \equiv 1$, and such that on a neighbourhood of any point in $\mathrm{Sing}(M_\vartheta)$ it has the form described above. The flow of this vector field gives then the global result. □

Remark 4.3. The above proof furnishes also an explicit diffeomorphism from $M_{\vartheta_1} \setminus \{\gamma_1^1, \dots, \gamma_n^1\}$ to $M_{\vartheta_2} \setminus \{\gamma_1^2, \dots, \gamma_n^2\}$, which conjugates the foliations. It is not difficult to see that this corresponds to a Dehn surgery from $(M_{\vartheta_1}, \gamma_1^1, \dots, \gamma_n^1)$ to $(M_{\vartheta_2}, \gamma_1^2, \dots, \gamma_n^2)$.

5. The planar case

Let us say that a smooth fiber M_ϑ is *of type* $\mathbb{C}$ if all the leaves of $\mathcal{F}_\vartheta$ are isomorphic to $\mathbb{C}$, and *of type* $\mathbb{C}^*$ if they are all isomorphic to $\mathbb{C}^*$.

Proposition 5.1. *If there exists a smooth fiber of type $\mathbb{C}$, then all the fibers are smooth and of type $\mathbb{C}$, and S is an Inoue surface of type S_M.*

Proof. Suppose, by contradiction, that there exists a singular fiber M_ϑ, which can be chosen so that M_{ϑ_1} (notation as in Proposition 4.2) is of type $\mathbb{C}$. Hence, $\mathcal{F}_\vartheta$ is obtained from $\mathcal{F}_{\vartheta_1}$ by collapsing some circles contained in some leaves. Since the leaves of $\mathcal{F}_{\vartheta_1}$ are simply connected, we see that at least one of these circles (call it γ) bounds on the corresponding leaf a disc D which does not contain any other circle. When we collapse the circles, and in particular γ to $p \in \mathrm{Sing}(M_\vartheta)$, this disc becomes a leaf L of $\mathcal{F}_\vartheta^\circ$, simply connected and accumulating only to p. The union

$$C = L \cup \{p\}$$

is then a smooth rational curve in S, invariant by $\mathcal{F}$, and over which $\mathcal{F}$ has only one singularity, the point p.

We can compute the selfintersection of C by using Camacho–Sad formula [Br1]. For a Morse type singular point, the Camacho–Sad residue along a separatrix is -1. Hence we get

$$C \cdot C = -1.$$

But this is in contradiction with the minimality of S.

Therefore, f has no critical point, and $f : S \to \mathbb{S}^1$ is a smooth $\mathbb{T}^3$-bundle, in particular $c_2(S) = 0$. It follows from [Ino] that S is an Inoue surface of type S_M. □

Remark 5.2. It is worth observing that, in this quite special context, the proof of Inoue's theorem can be highly simplified. Indeed, by the previous results, on the universal covering $\widehat{S}$ of S the foliation is given by a submersion $\pi : \widehat{S} \to \mathbb{H}$ (with

$\mathrm{Im}(\pi)$ coming from the pluriharmonic function F on $\widetilde{S}$) all of whose fibers are isomorphic to $\mathbb{C}$. The key point is to prove that such a universal covering is a product:

$$\widehat{S} = \mathbb{H} \times \mathbb{C}.$$

Indeed, once we know this fact, it remains to study the action of $\Gamma = \pi_1(S)$ on $\mathbb{H} \times \mathbb{C}$. But we already know a lot of properties of such an action (for instance, Γ is a semidirect product of $\mathbb{Z}$ and $\mathbb{Z}^3$, which acts on the $\mathbb{H}$-factor in a special affine way, etc.), and using that knowledge it is easy to conclude that S is an Inoue surface of type S_M.

In order to prove that $\widehat{S}$ is a product, it is sufficient to show that π is a locally trivial fibration, i.e. that every $z \in \mathbb{H}$ has a neighbourhood U_z such that $\pi^{-1}(U_z) = U_z \times \mathbb{C}$. By a classical theorem of Nishino [Nis], this is equivalent to show that $V_z = \pi^{-1}(U_z)$ is Stein. By an argument of Ohsawa [Ohs], the Steinness of V_z follows from the existence of a smooth (not holomorphic!) foliation $\mathcal{H}$ on V_z whose leaves are holomorphic sections of π over U_z (i.e., V_z is trivialisable by a smooth foliation with holomorphic leaves).

Now, in our case such a foliation $\mathcal{H}$ is easy to construct. On any fiber M_ϑ we can take a real analytic foliation by real curves transverse to $\mathcal{F}|_{M_\vartheta}$. By complexifying, we get, on a neighbourhood of M_ϑ, a real analytic foliation by complex curves, transverse to $\mathcal{F}$. Using the special form of $\mathcal{F}$, it is easy to see that this foliation, lifted to $\widehat{S}$, as the required property (here U_z is an horizontal strip in $\mathbb{H}$).

6. The cylindrical case

We will now suppose that all the smooth fibers of f are of type $\mathbb{C}^*$.

Lemma 6.1. *Let M_ϑ be a singular fiber. Then every leaf of $\mathcal{F}_\vartheta^\circ$ is isomorphic to $\mathbb{C}^*$.*

Proof. By the same argument of Proposition 5.1, we see that $\mathcal{F}_\vartheta$ is obtained from $\mathcal{F}_{\vartheta_1}$ by collapsing circles in leaves which are not homotopic to zero. Since the leaves of $\mathcal{F}_{\vartheta_1}$ are all isomorphic to $\mathbb{C}^*$, we deduce that every leaf of $\mathcal{F}_\vartheta^\circ$ is, topologically, a cylinder. More precisely, and using also the density of the leaves of $\mathcal{F}_{\vartheta_1}$, we get three possibilities for any leaf L of $\mathcal{F}_\vartheta^\circ$:

(i) L is a cylinder with both ends converging to singular points p_1, p_2;

(ii) L is a cylinder with one end converging to a singular point p and the other end dense in M_ϑ;

(iii) L is a cylinder with both ends dense in M_ϑ.

In the first case, we obviously have $L \simeq \mathbb{C}^*$. More precisely, the union $C = L \cup \{p_1, p_2\}$ is a smooth rational curve, and by using Camacho–Sad formula we get

$C \cdot C = -2$. Remark that this case occurs only when two (or more) collapsed circles belong to the same leaf.

In the second case, the end of L converging to p is obviously of parabolic type, and we need to prove the same property also for the dense end. Take a relatively compact annulus $A \subset L$, not homotopic to zero. Since L has no holonomy, A can be smoothly deformed in nearby leaves of $\mathcal{F}^{\circ}_{\vartheta}$, where it remains not homotopic to zero (this is an immediate consequence of the analogous property of $\mathcal{F}_{\vartheta_1}$). Even if this deformation is perhaps not conformal, it is at least quasi-conformal (compare with the proof of Proposition 2.1). In particular, and since L accumulates to itself, we get infinitely many disjoint annuli $A_n \subset L$, $n \in \mathbb{Z}$, all nonhomotopic to zero, necessarily diverging toward the dense end of L. Moreover, these annuli are all k-quasi-conformally equivalent, for some $k < 1$ independent on n. Equivalently, the moduli $\mu_n \in (0,1)$ of $A_n \simeq \{z \in \mathbb{C} \mid \mu_n < |z| < 1\}$ stay in some compact subset of $(0,1)$. It follows from these properties that the dense end of L is of parabolic type [Ahl], and hence $L \simeq \mathbb{C}^*$.

The third case is completely analogous. □

Example 6.2. Before continuing the proof, it may be useful to see an example showing that singular fibers of type $\mathbb{C}^*$ cannot be excluded by some "local" argument, as it was done in the planar case (local = working in a neighbourhood of a singular fiber).

Set $B = \{z \in \mathbb{C} \mid 1 < |z| < 3\}$, and let $g: W \to B$ be the fibration obtained by pulling back the fibration $zw: \bar{\mathbb{D}}^2 \to \bar{\mathbb{D}}$ under an embedding $i: B \to \mathbb{D}$ which sends 2 to 0. Thus, every fiber $g^{-1}(z)$, $z \neq 2$, is a closed annulus, and the fiber $g^{-1}(2)$ is a pair of closed discs intersecting at a Morse critical point. The boundary ∂W has two connected components N_1 and N_2, both CR-isomorphic to $B \times \mathbb{S}^1$, and the fibration is holomorphically trivial (a product) around each component. We can glue together N_1 and N_2 so that we obtain a complex surface V and an elliptic fibration $\hat{g}: V \to B$, with $\hat{g}^{-1}(2)$ a rational nodal curve. The function $F = r \circ \hat{g}: V \to (1,3)$ ($r(z) = |z|$) is pluriharmonic, $F^{-1}(t)$ is smoothly foliated by elliptic curves for $t \neq 2$, $F^{-1}(2)$ is singularly foliated by elliptic curves plus a rational nodal curve.

Now, we can modify the (trivial) gluing of N_1 and N_2 by inserting a rotation $\rho_\alpha: B \times \mathbb{S}^1 \to B \times \mathbb{S}^1$, $\rho_\alpha(z,s) = (e^{2\pi i\alpha} z, s)$. The resulting surface V_α has no more a fibration over B, but still we have a pluriharmonic function $F_\alpha: V_\alpha \to (1,3)$. If α is irrational, the smooth fibers $F_\alpha^{-1}(t)$, $t \neq 2$, are foliated by dense copies of $\mathbb{C}^*$, and the singular fiber $F_\alpha^{-1}(2)$ is foliated by dense copies of $\mathbb{C}^*$ plus a singular point.

The previous lemma has the following important consequence.

Lemma 6.3. *The line bundle $T_{\mathcal{F}}^{\otimes 2}$ admits a continuous section on $S \setminus \mathrm{Sing}(\mathcal{F})$ which is nowhere vanishing.*

Proof. The complex curve $\mathbb{C}^*$ admits a "almost canonical" holomorphic vector field: the vector field $z\frac{\partial}{\partial z}$, which can be almost uniquely characterized as a complete holomorphic vector field whose flow is 2πi-periodic. There is however a minor ambiguity, since also the vector field $-z\frac{\partial}{\partial z}$ (conjugate to the previous one by the inversion $z \mapsto 1/z$, which exchanges the two ends) is complete and 2πi-periodic. This ambiguity can be removed when we take the square: $(z\frac{\partial}{\partial z})^{\otimes 2} = (-z\frac{\partial}{\partial z})^{\otimes 2}$. This means that, given any foliation $\mathcal{F}$ with leaves isomorphic to $\mathbb{C}^*$, we get a *canonical* nonvanishing section of $T_{\mathcal{F}}^{\otimes 2}$ on $S \setminus \mathrm{Sing}(\mathcal{F})$, by the previous recipe. The point to be proved is that such a section is (at least) continuous.

This is equivalent to prove the following. Let $T \subset S$ be a local transversal to $\mathcal{F}$, isomorphic to a disc, and let V_T be the corresponding holonomy tube [Br2], p. 734. Remark that, as already observed in the course of the proof of Lemma 6.1, the foliation $\mathcal{F}$ has no "vanishing cycles" in the sense of [Br2]. The holonomy tube V_T is then a complex surface, homeomorphic to $T\times\mathbb{C}^*$, equipped with a holomorphic submersion $Q_T : V_T \to T$, all of whose fibers are isomorphic to $\mathbb{C}^*$, and a holomorphic section $q_T : T \to V_T$. For every $t \in T$ we have a unique isomorphism i_t from $Q_T^{-1}(t)$ to $\mathbb{C}^*$, sending $q_T(t)$ to 1 (really, there is again a $\mathbb{Z}_2$-ambiguity, which however can be easily removed by prescribing an homotopy class). Therefore we get a canonical trivialising map

$$u\colon V_T \longrightarrow T \times \mathbb{C}^*, \quad u|_{Q_T^{-1}(t)} = (t, i_t)$$

and the continuity of the above canonical section of $T_{\mathcal{F}}^{\otimes 2}$ is clearly equivalent to the continuity of u (for every transversal T).

As shown in [Ghy], p. 78 (see also I.2 in [Nis]), the continuity of u readily follows from Koebe's Theorem. Let us recall the argument, for completeness.

Take a compact $K \subset Q_T^{-1}(t_0)$ and an exhaustion of $Q_T^{-1}(t_0)$ by relatively compact open subsets $\{\Omega_n\}_{n\in\mathbb{N}}$. By a standard argument (e.g. Royden's Lemma), each Ω_n can be holomorphically deformed to the nearby fibers $Q_T^{-1}(t)$, $t \in U_n$ = a neighbourhood of t_0 in T. Thus, the maps i_t, $t \in U_n$, can be seen as all defined on the same domain Ω_n. By Koebe's Theorem, the distorsion of i_t on $K \subset \Omega_n$ is uniformly bounded by a constant which tends to zero as $n \to \infty$, since $Q_T^{-1}(t_0)$ is parabolic. We get in this way that $i_t|_K$ uniformly converge to $i_{t_0}|_K$ as $t \to t_0$, and since K was arbitrary we get the continuity of u. □

Remark that, a posteriori, the above map u will be even holomorphic, as well as the canonical section of $T_{\mathcal{F}}^{\otimes 2}$.

Remark 6.4. The geometrical meaning of Lemma 6.3, or more precisely of its proof, is the following: if we take in each leaf $L \simeq \mathbb{C}^*$ its canonical fibration by circles, then we get in $S \setminus \mathrm{Sing}(\mathcal{F})$ a *continuous* fibration by circles. Due to the particular structure of the fibers of f, provided by Propositions 2.1 and 4.2, one could try to construct directly a fibration by circles on $S \setminus \mathrm{Sing}(\mathcal{F})$, tangent to $\mathcal{F}$, by using only "topological" arguments. However, due to the possibly nontrivial monodromy of f, in order to do so one should prove a fact which is not so trivial nor so evident, namely: the space of fibrations by circles on M_ϑ, tangent to $\mathcal{F}_\vartheta$, is connected. Our proof, which works with "canonical" objects, avoids this type of difficulty.

It is now easy to complete the proof of Theorem 1.

Proposition 6.5. *If there exists a smooth fiber of type $\mathbb{C}^*$, then all the fibers are smooth and of type $\mathbb{C}^*$, and S is an Inoue surface of type $S^{(+)}_{N,p,q,r;t}$ or $S^{(-)}_{N,p,q,r}$.*

Proof. By Lemma 6.3, the line bundle $T_{\mathcal{F}}^{\otimes 2}$ is topologically trivial, i.e. it is flat: indeed, the nonvanishing section on $S \setminus \mathrm{Sing}(\mathcal{F})$ gives the topological triviality there, and hence everywhere since $\mathrm{Sing}(\mathcal{F})$ has (real) codimension 4. From Proposition 3.3 and $K_S^{-1} = N_{\mathcal{F}} \otimes T_{\mathcal{F}}$ it follows that K_S is flat too, and so $c_1^2(S) = 0$. As explained at the beginning, this is the same as $c_2(S) = 0$, the foliation is nonsingular, and S is an Inoue surface (of the claimed type). □

As in the planar case, also in the cylindrical case we do not need the full strength of Inoue's theorem, since we can directly prove that a covering of S is isomorphic to $\mathbb{H} \times \mathbb{C}^*$.

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