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Abstract commensurators of lattices in Lie groups

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Abstract. Let Γ be a lattice in a simply-connected solvable Lie group. We construct a $\mathbb{Q}$-defined algebraic group $\mathcal{A}$ such that the abstract commensurator of Γ is isomorphic to $\mathcal{A}(\mathbb{Q})$ and $\mathrm{Aut}(\Gamma)$ is commensurable with $\mathcal{A}(\mathbb{Z})$. Our proof uses the algebraic hull construction, due to Mostow, to define an algebraic group $\mathbf{H}$ so that commensurations of Γ extend to $\mathbb{Q}$-defined automorphisms of $\mathbf{H}$. We prove an analogous result for lattices in connected linear Lie groups whose semisimple quotient satisfies superrigidity.

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Contents

1. Introduction

Given a group Γ, its *abstract commensurator* $\mathrm{Comm}(\Gamma)$ is the set of equivalence classes of isomorphisms between finite index subgroups of Γ, where two isomorphisms are equivalent if they agree on a finite index subgroup of Γ. Elements of $\mathrm{Comm}(\Gamma)$ are called *commensurations* of Γ. The abstract commensurator forms a group under composition.

The computation of $\mathrm{Comm}(\Gamma)$ is a fundamental problem. Commensurations play an important role in the study of rigidity, see e.g. [35] and [21]. Commensurations also arise in classification problems in geometry and topology, e.g. [25], [13], [14], [19], and [2].

The structure of $\mathrm{Comm}(\Gamma)$ is often much richer than that of $\mathrm{Aut}(\Gamma)$. For example, $\mathrm{Aut}(\mathbb{Z}^n) \cong \mathrm{GL}_n(\mathbb{Z})$ while $\mathrm{Comm}(\mathbb{Z}^n) \cong \mathrm{GL}_n(\mathbb{Q})$. There are a few notable exceptions, which include the cases that Γ is a higher genus mapping class group, that $\Gamma = \mathrm{Out}(F_n)$ for $n \geq 4$, or that Γ is a nonarithmetic lattice in a semisimple Lie group without compact factors and not locally isomorphic to $\mathrm{PSL}_2(\mathbb{R})$. In these cases, $\mathrm{Comm}(\Gamma)$ is virtually isomorphic to Γ; see [18], [10], and [21], respectively.

This paper is motivated by the following problem.

Problem. *Let G be a (connected, linear, real) Lie group and let $\Gamma \leq G$ be a lattice. Compute* $\mathrm{Comm}(\Gamma)$.

Standing Assumption. *Unless otherwise noted, in this paper every Lie group is assumed to be real and connected, and to admit a faithful continuous linear representation. In particular, semisimple Lie groups have finite center.*

Every Lie group G satisfies a short exact sequence

$$1 \to \mathrm{Rad}(G) \to G \to G^{ss} \to 1,$$

where $\mathrm{Rad}(G)$ is the maximal connected solvable normal Lie subgroup of G, and G^{ss} is semisimple. The study of Lie groups therefore roughly splits into three pieces: one for solvable groups, one for semisimple groups, and a final piece to combine the previous two. Our computation of $\mathrm{Comm}(\Gamma)$ follows this outline.

Semisimple G. Suppose G is a connected semisimple Lie group, not locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$, and $\Gamma \leq G$ is an irreducible lattice. Then the computation of $\mathrm{Comm}(\Gamma)$ is a result of work by Borel, Mostow, Prasad, and Margulis. Recall that the *relative commensurator* of Γ in G is defined as

$$\mathrm{Comm}_G(\Gamma) = \left\{ g \in G \;\middle|\; \Gamma \cap g\Gamma g^{-1} \text{ is of finite index in } \Gamma \text{ and } g\Gamma g^{-1} \right\}.$$

- If Γ is abstractly commensurable to $\mathbf{G}(\mathbb{Z})$ for some $\mathbb{Q}$-defined, adjoint semisimple algebraic group $\mathbf{G}$ with no $\mathbb{Q}$-defined normal subgroup $\mathbf{N}$ such that $\mathbf{N}(\mathbb{R})$ is compact, then $\mathrm{Comm}_{\mathbf{G}}(\Gamma) = \mathbf{G}(\mathbb{Q})$ [7]. (Such a lattice Γ is called *arithmetic*.) For example, if $\Gamma = \mathrm{PSL}_n(\mathbb{Z})$ for $n \geq 2$, then Γ is abstractly commensurable with the group $\mathbf{G}(\mathbb{Z})$, where $\mathbf{G}$ is the semisimple algebraic group $\mathbf{G} = \mathrm{PGL}_n$, and so $\mathrm{Comm}_{\mathbf{G}}(\Gamma) \cong \mathbf{G}(\mathbb{Q})$; see §7.3 for details.
- A major theorem of Margulis [21] says that Γ is arithmetic if and only if $[\mathrm{Comm}_G(\Gamma) : \Gamma] = \infty$, which occurs if and only if $\mathrm{Comm}_G(\Gamma)$ is dense in G.
- If G has no center and no compact factors, then every commensuration of Γ extends to an automorphism of G by Mostow–Prasad–Margulis rigidity [24].

- The inner automorphisms of a semisimple real Lie group are finite index in the automorphism group. Therefore

$$\mathrm{Comm}(\Gamma) \doteq \begin{cases} \mathbf{G}(\mathbb{Q}) & \text{if } \Gamma \text{ is arithmetic} \\ \Gamma & \text{if } \Gamma \text{ is non-arithmetic,} \end{cases}$$

where $H \doteq K$ if and only if H and K are abstractly commensurable, i.e. contain isomorphic finite index subgroups. See Theorem 7.5 for a more precise statement.

Remark 1.1. In the case $G = \mathrm{PSL}_2(\mathbb{R})$, every lattice is either virtually free or virtually the fundamental group of a closed surface. In either case, the abstract commensurator is not linear; see Proposition 7.6. The abstract commensurator of a surface group has been studied in [26] and [6], and may be described as a certain subgroup of the mapping class group of the universal 2-dimensional hyperbolic solenoid.

Solvable G. Suppose G is a connected, simply-connected solvable real Lie group and $\Gamma \leq G$ is a lattice. In contrast with the semisimple case, $\mathrm{Aut}(\Gamma)$ is not typically abstractly commensurable with Γ. On the other hand, the fact that $\mathrm{Aut}(\Gamma)$ is commensurable with the $\mathbb{Z}$-points of a $\mathbb{Q}$-defined algebraic group holds for both arithmetic lattices in higher rank semisimple groups and lattices in simply-connected solvable groups, at least on passage to a subgroup of finite index in Γ; see [4, 1.12] and [30, Ch 8].

In the case that Γ is a lattice in a simply-connected nilpotent group, arithmeticity of $\mathrm{Aut}(\Gamma)$ is a classical result of Baumslag and Auslander. Merzljakov [22] showed that $\mathrm{Aut}(\Gamma)$ embeds in some $\mathrm{GL}_n(\mathbb{Z})$ for any polycyclic group Γ, and this was extended to virtually polycyclic groups by Wehrfritz [33]. For more history and a detailed discussion of arithmeticity results see [4], whose Theorem 1.3 provides a deeper statement on the structure of $\mathrm{Aut}(\Gamma)$ for virtually polycyclic Γ.

This similarity between arithmetic semisimple lattices and solvable lattices is reflected in their abstract commensurators. For example, consider $G = \mathbb{R}^n$ and $\Gamma = \mathbb{Z}^n$. Then $\mathrm{Aut}(\mathbb{Z}^n) = \mathrm{GL}_n(\mathbb{Z})$ is arithmetic in the $\mathbb{Q}$-defined real algebraic group $\mathrm{Aut}(\mathbb{R}^n) = \mathrm{GL}_n(\mathbb{R})$, and $\mathrm{Comm}(\Gamma) = \mathrm{GL}_n(\mathbb{Q})$. Our first main theorem extends this to lattices in arbitrary simply-connected solvable groups, following techniques of [4].

Theorem 1.2. *Let Γ be a lattice in a connected, simply-connected solvable Lie group G. Then there is some $\mathbb{Q}$-defined algebraic group $\mathcal{A}_\Gamma$ such that*

$$\mathrm{Comm}(\Gamma) \cong \mathcal{A}_\Gamma(\mathbb{Q})$$

and the image of $\mathrm{Aut}(\Gamma)$ *in* $\mathcal{A}_\Gamma(\mathbb{Q})$ *is commensurable with* $\mathcal{A}_\Gamma(\mathbb{Z})$.

Remark 1.3. If G is 'sufficiently nice' then $\mathcal{A}_\Gamma(\mathbb{R}) = \mathrm{Aut}(G)$. This is proved in Theorem 4.2 in the case that G is nilpotent. See Proposition 6.4 for a more general result.

Remark 1.4. Any virtually polycyclic group contains a subgroup of finite index that embeds as a lattice in a connected, simply-connected solvable Lie group. Therefore Theorem 1.2 describes $\mathrm{Comm}(\Gamma)$ for any virtually polycyclic group Γ.

A fundamental difficulty in dealing with lattices in solvable groups is lack of rigidity; automorphisms of a lattice may not extend to automorphisms of its ambient Lie group, even virtually. There are a number of results addressing this to some extent, most notably [34]. Instead of applying results providing rigidity in the ambient Lie group, our proof of Theorem 1.2 uses methods developed by Baues and Grunewald in [4], following work of Grunewald and Platonov [16, 15].

Our proof utilizes the *virtual algebraic hull*, a connected solvable $\mathbb{Q}$-defined algebraic group $\mathbf{H}$ in which Γ virtually embeds as a Zariski-dense subgroup. The construction of the virtual algebraic hull is due to Mostow [23]. (See [29, §4] for an alternate construction.) There is a natural map

$$\xi : \mathrm{Comm}(\Gamma) \to \mathrm{Aut}(\mathbf{H})$$

such that $\xi([\phi])$ is $\mathbb{Q}$-defined for each $[\phi] \in \mathrm{Comm}(\Gamma)$. The automorphism group $\mathrm{Aut}(\mathbf{H})$ naturally has the structure of a $\mathbb{Q}$-defined algebraic group, and we set $\mathcal{A}_\Gamma$ equal to the Zariski-closure of $\xi(\mathrm{Comm}(\Gamma))$ in $\mathrm{Aut}(\mathbf{H})$. Note that our map ξ extends the map $\mathrm{Aut}(\tilde{\Gamma}) \to \mathrm{Aut}(\mathbf{H})$ defined in [4, §4.1] for some subgroup $\tilde{\Gamma} \leq \Gamma$ of finite index.

Remark 1.5. Baues extends Mostow's algebraic hull construction to certain virtually polycyclic groups Γ in [3], and this hull is applied in [4] to describe $\mathrm{Aut}(\Gamma)$ and $\mathrm{Out}(\Gamma)$. Though our proof of Theorem 1.2 is heavily based on the techniques in [4], we use only the identity component of the algebraic hull. This is because $\mathrm{Comm}(\Gamma)$ only depends on Γ up to commensurability.

Remark 1.6. Though the group $\mathcal{A}_\Gamma$ of Theorem 1.2 is defined abstractly, a finite index subgroup of $\mathrm{Comm}(\Gamma)$ can be understood fairly concretely. There is a unique maximal normal nilpotent subgroup $\mathrm{Fitt}(\Gamma) \leq \Gamma$. Let $\mathbf{F}$ denote the Zariski-closure of $\mathrm{Fitt}(\Gamma)$ in $\mathbf{H}$. Define $\mathrm{Comm}_{\mathbf{H}|\mathbf{F}}(\Gamma)$ to be the group of commensurations trivial on $\Gamma/\mathrm{Fitt}(\Gamma)$. By rigidity of tori, $\mathrm{Comm}_{\mathbf{H}|\mathbf{F}}$ is of finite index in $\mathrm{Comm}(\Gamma)$. The group $\mathrm{Comm}_{\mathbf{H}|\mathbf{F}}$ decomposes as the product of the group of commensurations arising from conjugation by elements of $\mathbf{F}(\mathbb{Q})$ and the group of commensurations fixing a maximal $\mathbb{Q}$-defined torus $\mathbf{T} \leq \mathbf{H}$. See §5.5 and §6 for details.

General G. When G is not necessarily either semisimple or solvable, we prove:

Theorem 1.7. *Suppose G is a connected, linear Lie group with connected, simply-connected solvable radical. Suppose $\Gamma \leq G$ is a lattice with the property that there* *is no surjection* $\phi : G \to H$ *to any group* H *locally isomorphic to any* $\mathrm{SO}(1,n)$ *or* $\mathrm{SU}(1,n)$ *so that* $\phi(\Gamma)$ *is a lattice in* H*. Then*

(1) Γ *virtually embeds in the group of* $\mathbb{Q}$*-points of a* $\mathbb{Q}$*-defined algebraic group* $\mathbf{G}$ *with Zariski-dense image so that every commensuration* $[\phi] \in \mathrm{Comm}(\Gamma)$ *induces a unique* $\mathbb{Q}$*-defined automorphism of* $\mathbf{G}$ *virtually extending* ϕ*.*

(2) *There is a* $\mathbb{Q}$*-defined algebraic group* $\mathcal{B}$ *so that*

$$\mathrm{Comm}(\Gamma) \cong \mathcal{B}(\mathbb{Q})$$

and the image of $\mathrm{Aut}(\Gamma)$ *in* $\mathcal{B}$ *is commensurable with* $\mathcal{B}(\mathbb{Z})$*.*

The group $\mathbf{G}$ of Theorem 1.7 is, roughly speaking, constructed as the semidirect product of the virtual algebraic hull $\mathbf{H}$ of the "solvable part" of Γ and a semisimple group $\mathbf{S}$ such that $\mathbf{S}(\mathbb{Z})$ is commensurable with the "semisimple part" of Γ. The technical work comes first in making this precise, and second in constructing an action of $\mathbf{S}$ on $\mathbf{H}$ compatible with the group structure of Γ.

The hypothesis that Γ does not surject to a lattice in either $\mathrm{SO}(1,n)$ or $\mathrm{SU}(1,n)$ is used to apply the superrigidity results of Margulis and Corlette, which are used to extend commensurations of Γ to automorphisms of $\mathbf{G}$. In the case that Γ surjects to a non-superrigid lattice, our construction may fail to produce a candidate group $\mathbf{G}$. Even in the presence of such a candidate group $\mathbf{G}$, commensurations do not generally extend to automorphisms of $\mathbf{G}$. Additional commensurations arise from the nontriviality of $H^1(\Gamma,\mathbb{Q})$; see the remark at the end of §8.

Remark 1.8. If $\mathbf{A}$ is a $\mathbb{Q}$-defined algebraic group, then there is a natural map $\Xi : \mathrm{Aut}_{\mathbb{Q}}(\mathbf{A}) \to \mathrm{Comm}(\mathbf{A}(\mathbb{Z}))$. If $\mathbf{A}$ is unipotent, or if $\mathbf{A}$ is $\mathbb{Q}$-simple, semisimple, and such that $\mathbf{A}(\mathbb{R})$ is not compact and has no factor isogenous to $\mathrm{PSL}_2(\mathbb{R})$, then Ξ is injective because $\mathbf{A}(\mathbb{Z})$ is Zariski-dense in $\mathbf{A}$, and Ξ is surjective because $\mathbf{A}(\mathbb{Z})$ is strongly rigid in $\mathbf{A}$ by results of Malcev and Mostow–Prasad–Margulis (see Theorems 4.2 and 7.3). See [16] for analogous results in the case that $\mathbf{A}$ is solvable.

The difficulty in proving our results comes from the fact that lattices in solvable Lie groups need not be commensurable with the $\mathbb{Z}$-points of any algebraic group; see [30] for an example. When Γ is a lattice in a simply-connected solvable group, the algebraic hull construction provides an algebraic group $\mathbf{H}$ so that Γ virtually embeds in $\mathbf{H}(\mathbb{Z})$ as a Zariski-dense subgroup, but in general the image of this embedding may be of infinite index. Despite this, automorphisms of $\mathbf{H}$ extending commensurations of Γ may be understood in terms of the algebraic structure of $\mathbf{H}$.

Outline. We review basic results in the theory of linear algebraic groups in §2. We define and review basic properties of the abstract commensurator in §3, including definitions of commensuristic and strongly commensuristic subgroups.

In §4 we prove Theorem 1.2 for nilpotent G using classical rigidity of nilpotent lattices. In §5, we review the basic theory of polycyclic groups and the definition of the algebraic hull. Our exposition largely follows [4]. We define the unipotent shadow and discuss the algebraic structure of Aut(**H**). In §6 we prove Theorem 1.2.

In §7 we review results on commensurations of lattices in semisimple Lie groups, which are due primarily to Borel, Mostow, Prasad, and Margulis. In §8 we combine the solvable and semisimple cases to prove Theorem 1.7.

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2. Notation and preliminaries

If g, h are elements of a group, their commutator is written $[g,h] = ghg^{-1}h^{-1}$. A group Γ *virtually* has a property P if there is a finite index subgroup of Γ with P. In particular, if $\Gamma \leq G$, say that a homomorphism $\phi : \Gamma \to H$ virtually extends to a homomorphism $\Phi : G \to H$ if there is a finite index subgroup $\Gamma_0 \leq \Gamma$ so that $\phi|_{\Gamma_0} = \Phi|_{\Gamma_0}$.

2.1. Algebraic groups. We use the basic theory of linear algebraic groups. A good general reference is [8]. Our preliminaries overlap with those in [4].

Let $K \subseteq \mathbb{C}$ be a subfield. A *linear algebraic group* $\mathbf{A}$ is a subgroup of $\mathrm{GL}_n(\mathbb{C})$ for some natural number n that is closed in the Zariski topology. An algebraic group $\mathbf{A}$ is *K-defined* if it is closed in the Zariski topology with closed subsets those defined by polynomials with coefficients in a subfield K of $\mathbb{C}$. A K-defined algebraic group is called a K-group. A K-group is *K-simple* if it has no connected normal K-defined subgroup, and *absolutely simple* if it has no connected normal subgroup defined over $\mathbb{C}$. (Such groups are sometimes called "almost K-simple" or "absolutely almost simple", respectively.)

If R is a subring of $\mathbb{C}$, then define $\mathbf{A}(R) = \mathbf{A} \cap \mathrm{GL}_n(R) \subseteq \mathrm{GL}_n(\mathbb{C})$. If V is a complex vector space with a fixed basis, then $V(R)$ denotes the collection of R-linear combinations of basis vectors. Every algebraic group has finitely many Zariski-connected components. The connected component of the identity $\mathbf{A}^0$ is a finite index subgroup of $\mathbf{A}$.

Proposition 2.1 (cf. [8, 1.3]). *If* $\mathbf{A}$ *is* K*-defined and* $\Gamma \leq \mathbf{A}(K)$ *is a subgroup, then the Zariski-closure of* Γ *is a* K*-defined subgroup.*

Proposition 2.2 (cf. [8, 18.3]). *If* $\mathbf{A}$ *is a connected* K*-defined algebraic group, then* $\mathbf{A}(K)$ *is Zariski-dense in* $\mathbf{A}$.

A *homomorphism of algebraic groups* is a group homomorphism that is also a morphism of the underlying affine algebraic varieties. If both varieties are K-defined and the variety morphism is defined over K, then we say that the homomorphism of algebraic groups is K*-defined.* A K*-defined isomorphism* is a K-defined morphism of algebraic groups with an inverse that is also K-defined. Let $\mathrm{Aut}(\mathbf{A})$ denote the group of automorphisms of $\mathbf{A}$ as an algebraic group, and $\mathrm{Aut}_K(\mathbf{A})$ denote the group of K-defined automorphisms of $\mathbf{A}$.

Quotients and semi-direct products of K-defined algebraic groups exist:

Lemma 2.3 (cf. [8, 6.8]). *Suppose* G *is a* K*-defined algebraic group and* $H \leq G$ *is a normal, closed,* K*-defined subgroup. Then* G/H *is a* K*-defined algebraic group, and the quotient map* $\pi : G \to G/H$ *is* K*-defined.*

Lemma 2.4 (cf. [8, 1.11]). *Suppose* G *and* H *are* K*-defined algebraic groups. Suppose* G *acts on* H*, and the action map* $\alpha : G \times H \to H$ *is* K*-defined. Then the semi-direct product* $H \rtimes G$ *naturally has the structure of a* K*-defined algebraic group.*

A *torus* is an algebraic group isomorphic to $(\mathbb{C}^*)^n$ for some n. Because the automorphism group of a torus is discrete, we have:

Lemma 2.5 (cf. [8, 8.10]). *Let* $\mathbf{T}$ *be any torus and* $\mathbf{A}$ *any algebraic group acting on* $\mathbf{T}$ *by homomorphisms, so that the map* $\mathbf{A} \times \mathbf{T} \to \mathbf{T}$ *is a morphism of varieties. Then* $\mathbf{A}^0$ *acts trivially on* $\mathbf{T}$.

Let $\mathbf{A}$ be a K-defined algebraic group. The *unipotent radical* $\mathbf{U_A}$ of $\mathbf{A}$ is the unique maximal closed unipotent normal subgroup of $\mathbf{A}$. The *solvable radical* $\mathrm{Rad}(\mathbf{A})$ of $\mathbf{A}$ is unique maximal connected closed solvable normal subgroup of $\mathbf{A}$. Both $\mathbf{U_A}$ and $\mathrm{Rad}(\mathbf{A})$ are K-defined subgroups of $\mathbf{A}$. Say $\mathbf{A}$ is *reductive* if $\mathbf{U_A}$ is trivial, and *semisimple* if $\mathrm{Rad}(\mathbf{A})$ is trivial. A *Levi subgroup* is a connected reductive subgroup $\mathbf{L} \leq \mathbf{A}$ so that $\mathbf{A} = \mathbf{U_A} \rtimes \mathbf{L}$.

Theorem 2.6 (Mostow, see [27, Theorem 2.3]). *For any* K*-defined algebraic group* $\mathbf{A}$*, there is a* K*-defined Levi subgroup* $\mathbf{L}$*. Moreover, any reductive* K*-defined subgroup is conjugate by an element of* $\mathbf{U_A}(K)$ *into* $\mathbf{L}$.

The following summarizes some standard results concerning solvable algebraic groups.

Proposition 2.7 (cf. [8, 10.6]). *Let* $\mathbf{H}$ *be a* $\mathbb{Q}$*-defined connected solvable algebraic group. Then:*

(1) $\mathbf{U_H}$ *consists of all unipotent elements of* $\mathbf{H}$.

(2) $[\mathbf{H}, \mathbf{H}] \subseteq \mathbf{U_H}$.

(3) *There is a* $\mathbb{Q}$*-defined maximal torus* $\mathbf{T} \leq \mathbf{H}$.

(4) *Any two maximal $\mathbb{Q}$-defined tori are conjugate by an element of* $[\mathbf{H},\mathbf{H}](\mathbb{Q})$.

(5) *If* $\mathbf{T}$ *is a* $\mathbb{Q}$*-defined maximal torus, then* $\mathbf{H}$ *is a semidirect product* $\mathbf{H} = \mathbf{U_H} \rtimes \mathbf{T}$.

(6) *If* $\mathbf{D}$ *is the centralizer of a maximal torus and* $\mathbf{F} \leq \mathbf{U_H}$ *is any normal subgroup containing* $[\mathbf{H},\mathbf{H}]$*, then* $\mathbf{H} = \mathbf{F} \cdot \mathbf{D}$.

2.2. Semisimple Lie and algebraic groups. A general reference for the theory of semisimple algebraic groups used here is [21, Chapter 1].

If $\mathbf{A}$ is an $\mathbb{R}$-defined algebraic group, then $\mathbf{A}(\mathbb{R})$ is a real Lie group with finitely many connected components. We always consider $\mathbf{A}(\mathbb{R})$ with its topology as a Lie group. In particular, $\mathbf{A}(\mathbb{R})^0$ denotes the connected component of the identity in the Lie group topology. Every connected semisimple Lie group with trivial center is of the form $\mathbf{S}(\mathbb{R})^0$ for some $\mathbb{Q}$-defined semisimple algebraic group $\mathbf{S}$; for proof see [35, 3.1.6].

An *isogeny* of algebraic groups is a surjective morphism with finite kernel. An isogeny is *central* if its kernel is central. A connected semisimple algebraic group $\mathbf{S}$ is *simply-connected* if every central isogeny $\Phi : \mathbf{S}' \to \mathbf{S}$ is an isomorphism. For every connected K-defined semisimple algebraic group $\mathbf{S}$, there is a unique simply-connected K-defined semisimple algebraic group $\tilde{\mathbf{S}}$ and central K-defined isogeny $p : \tilde{\mathbf{S}} \to \mathbf{S}$. Every simply-connected semisimple K-group decomposes uniquely into a product of K-simple simply-connected K-groups.

Proposition 2.8 (cf. [21, I.2.6.5]). *Suppose* $\mathbf{A}$ *is an* $\mathbb{R}$*-defined algebraic group, and* $\mathbf{S}$ *is a simply-connected semisimple* $\mathbb{R}$*-defined algebraic group. Let* $\rho : \mathbf{S}(\mathbb{R})^0 \to \mathbf{A}(\mathbb{R})$ *be a continuous representation. Then* ρ *extends to an* $\mathbb{R}$*-defined morphism* $\tilde{\rho} : \mathbf{S} \to \mathbf{A}$.

A $\mathbb{Q}$-defined semisimple algebraic group $\mathbf{S}$ is *without* $\mathbb{Q}$*-compact factors* if there is no nontrivial $\mathbb{Q}$-defined connected normal subgroup $\mathbf{N} \leq \mathbf{S}$ such that $\mathbf{N}(\mathbb{R})$ is compact. (This terminology is not standard.)

Theorem 2.9 (Borel Density Theorem [7]). *Suppose* $\mathbf{S}$ *is a connected,* $\mathbb{Q}$*-defined semisimple algebraic group without* $\mathbb{Q}$*-compact factors. Then* $\mathbf{S}(\mathbb{Z})$ *is Zariski-dense in* $\mathbf{S}$.

Definition 2.10. Let $\mathbf{S}$ be an $\mathbb{R}$-defined semisimple algebraic group. The *real rank* of $\mathbf{S}$, denoted $\operatorname{rank}_{\mathbb{R}}(\mathbf{S})$, is the maximal dimension of an abelian $\mathbb{R}$-defined subgroup diagonalizable over $\mathbb{R}$. If S is a connected semisimple Lie group with finite center, define $\operatorname{rank}_{\mathbb{R}}(S)$ to be the real rank of the $\mathbb{Q}$-defined algebraic group $\mathbf{S}$ satisfying $\mathbf{S}(\mathbb{R})^0 = S/Z(S)$.

Our results use strong rigidity of Mostow, Prasad, and Margulis, and superrigidity results of Margulis, Corlette, and Gromov–Schoen. The following statement is an immediate corollary of [15, 2.6].

Theorem 2.11 (cf. [15]). *Suppose* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *are connected, simply-connected,* $\mathbb{Q}$-*defined,* $\mathbb{Q}$-*simple semisimple algebraic groups with* $\operatorname{rank}_{\mathbb{R}}(\mathbf{S}_1) > 0$ *and* $\operatorname{rank}_{\mathbb{R}}(\mathbf{S}_2) > 0$. *Suppose* Γ_1 *and* Γ_2 *are finite index subgroups of* $\mathbf{S}_1(\mathbb{Z})$ *and* $\mathbf{S}_2(\mathbb{Z})$, *respectively. Assume that* $\mathbf{S}_1(\mathbb{R})^0$ *has no simple factor locally isomorphic to* $\mathrm{SL}_2(\mathbb{R})$ *such that the projection of* $\Gamma_1 \cap \mathbf{S}_1(\mathbb{R})^0$ *into this factor is discrete. Then every isomorphism* $\Gamma_1 \to \Gamma_2$ *virtually extends to a* $\mathbb{Q}$-*defined isomorphism of algebraic groups* $\mathbf{S}_1 \to \mathbf{S}_2$.

3. The abstract commensurator

Let Γ be an abstract group. In this section we will define the abstract commensurator $\operatorname{Comm}(\Gamma)$ and review its basic properties.

A *partial automorphism* of Γ is an isomorphism $\phi : \Gamma_1 \to \Gamma_2$ where Γ_1 and Γ_2 are finite index subgroups of Γ. Two partial automorphisms ϕ and ϕ' of Γ are *equivalent* if there is some finite index subgroup $\Gamma_3 \leq \Gamma$ so that ϕ and ϕ' are both defined on Γ_3 and $\phi|_{\Gamma_3} = \phi'|_{\Gamma_3}$. If $\phi : \Gamma_1 \to \Gamma_2$ is a partial automorphism of Γ, its equivalence class $[\phi]$ is called a *commensuration* of Γ. There is a natural composition of commensurations. If $\phi : \Gamma_1 \to \Gamma_2$ and $\phi' : \Gamma_1' \to \Gamma_2'$ are partial automorphisms of Γ, then we define

$$[\phi'] \circ [\phi] = \left[\phi' \circ \phi|_{\phi^{-1}(\Gamma_2 \cap \Gamma_1')}\right].$$

This definition is independent of choice of representatives of equivalence classes $[\phi]$ and $[\phi']$.

Definition 3.1. Given a group Γ, the *abstract commensurator* $\operatorname{Comm}(\Gamma)$ is the group of commensurations of Γ under composition.

Example 3.2. $\operatorname{Comm}(\mathbb{Z}^n) \cong \mathrm{GL}_n(\mathbb{Q})$

Two subgroups $\Delta_1, \Delta_2 \leq \Gamma$ are *commensurable* if $[\Delta_1 : \Delta_1 \cap \Delta_2] < \infty$ and $[\Delta_2 : \Delta_1 \cap \Delta_2] < \infty$. Define an equivalence relation on the set of subgroups of Γ by $\Delta_1 \sim \Delta_2$ if and only if Δ_1 and Δ_2 are commensurable. Let $[\Delta]$ denote the equivalence class of a subgroup $\Delta \leq \Gamma$ under this relation. The abstract commensurator $\operatorname{Comm}(\Gamma)$ acts on the set of commensurability classes of subgroups of Γ in an obvious way; given a partial automorphism $\phi : \Gamma_1 \to \Gamma_2$ of Γ, define

$$[\phi] \cdot [\Delta] = [\phi(\Delta \cap \Gamma_1)].$$

Clearly this is independent of choice of representatives ϕ and Δ.

Definition 3.3 (Commensuristic subgroup). A subgroup $\Delta \leq \Gamma$ is *commensuristic* if $[\phi] \cdot [\Delta] = [\Delta]$ for every $[\phi] \in \operatorname{Comm}(\Gamma)$. A subgroup $\Lambda \leq \Gamma$ is *strongly commensuristic* if, for every partial automorphism $\phi : \Gamma_1 \to \Gamma_2$ of Γ,

$$\phi(\Gamma_1 \cap \Lambda) = \Gamma_2 \cap \Lambda.$$

Every strongly commensuristic subgroup is both characteristic and commensuristic. Neither converse holds.

Example 3.4. Consider the group

$$\Gamma = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, x, y \in \mathbb{Z} \text{ and } z \in \frac{1}{2}\mathbb{Z} \right\} \leq \mathrm{GL}_3(\mathbb{Q}).$$

Note that Γ is a lattice in the real Heisenberg group. Denote elements of Γ by triples (x, y, z) where x, y, and z are as above. The center $Z(\Gamma)$ is infinite cyclic, generated by $(0, 0, \frac{1}{2})$, and contains the commutator subgroup $[\Gamma, \Gamma]$ with index 2. By Proposition 4.1, the center $Z(\Gamma)$ is strongly commensuristic and the commutator subgroup $[\Gamma, \Gamma]$ is commensuristic. Further, $[\Gamma, \Gamma]$ is evidently characteristic.

Now consider the subgroup $\Gamma_2 \leq \Gamma$ generated by $(2, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 2)$. Then the map $\phi : \Gamma \to \Gamma_2$ defined by $\phi(x, y, z) = (2x, 2y, 4z)$ is a partial automorphism of Γ. But ϕ takes $[\Gamma, \Gamma]$ to $[\Gamma_2, \Gamma_2]$, which is the infinite cyclic group generated by $(0, 0, 4)$. Therefore $[\Gamma, \Gamma]$ is not strongly commensuristic. □

Question. *Let Γ be a finitely generated group. Is every characteristic subgroup of Γ commensuristic? Is every commensuristic subgroup of Γ commensurable with a characteristic subgroup? Is every commensuristic subgroup of Γ commensurable with a strongly commensuristic subgroup?*

The notions of 'commensuristic' and 'strongly commensuristic' are motivated by the following lemma.

Lemma 3.5. *If $\Delta \leq \Gamma$ is commensuristic, then restriction induces a homomorphism*

$$\mathrm{Comm}(\Gamma) \to \mathrm{Comm}(\Delta).$$

If Δ is normal in Γ and strongly commensuristic, then there is a homomorphism

$$\mathrm{Comm}(\Gamma) \to \mathrm{Comm}(\Gamma/\Delta).$$

Proof. Suppose $\Delta \leq \Gamma$ is commensuristic. Let $\phi : \Gamma_1 \to \Gamma_2$ be a partial automorphism of Γ. Then $\phi(\Delta \cap \Gamma_1)$ is commensurable with Δ, and so $\Delta_1 = \phi^{-1}(\Delta \cap \phi(\Delta \cap \Gamma_1))$ is a finite index subgroup of Δ. The restriction of ϕ to Δ_1 defines a partial automorphism of Δ. Restriction clearly respects the equivalence relation on partial automorphisms and is compatible with composition, so this determines a well-defined homomorphism $\mathrm{Comm}(\Gamma) \to \mathrm{Comm}(\Delta)$.

Suppose now that $\Delta \leq \Gamma$ is strongly commensuristic and normal, and let $\phi : \Gamma_1 \to \Gamma_2$ be a partial automorphism of Γ. Then ϕ descends a map $\hat{\phi} : \Gamma_1 \to \Gamma_2/(\Gamma_2 \cap \Delta)$. Because Δ is strongly commensuristic, the kernel of this map is precisely $\Gamma_1 \cap \Delta$. There is then an isomorphism

$$\phi_* : \Gamma_1/(\Gamma_1 \cap \Delta) \to \Gamma_2/(\Gamma_2 \cap \Delta).$$

The map ϕ_* is a partial automorphism of Γ/Δ. If ϕ_1 and ϕ_2 are equivalent partial automorphisms, then $\hat{\phi}_1$ and $\hat{\phi}_2$ agree on some finite index subgroup of Γ_1. It follows that $(\phi_1)_*$ and $(\phi_2)_*$ are equivalent partial automorphisms of Γ/Δ. Therefore there is a well-defined map $\mathrm{Comm}(\Gamma) \to \mathrm{Comm}(\Gamma/\Delta)$, which is obviously a homomorphism. □

Remark 3.6. Lemma 3.5 is inspired by the methods of [20], where the result is applied with $\Gamma = B_n$, the braid group on n strands, and $\Delta = Z(B_n)$ as a step in the computation of $\mathrm{Comm}(B_n)$ for $n \geq 4$.

We will often use the following corollaries implicitly in this paper. Two groups Γ and Λ are called *abstractly commensurable*, written $\Gamma \doteq \Lambda$, if there are finite index subgroups $\Gamma_1 \leq \Gamma$ and $\Lambda_1 \leq \Lambda$ such that $\Gamma_1 \cong \Lambda_1$.

Corollary 3.7. *If* $[\Gamma : \Gamma'] < \infty$ *then* $\mathrm{Comm}(\Gamma') \cong \mathrm{Comm}(\Gamma)$.

Corollary 3.8. *If* $\Gamma \doteq \Lambda$ *then* $\mathrm{Comm}(\Gamma) \cong \mathrm{Comm}(\Lambda)$.

There is a weaker notion of equivalence similar to that of abstract commensurability. Define a relation on groups by $\Gamma_1 \sim \Gamma_2$ if there is a homomorphism $\phi : \Gamma_1 \to \Gamma_2$ with finite index image and finite kernel. Say that Γ_1 and Γ_2 are *commensurable up to finite kernels* if they lie in the same equivalence class of the equivalence relation generated by $\sim$. In general, groups which are commensurable up to finite kernels need not be abstractly commensurable.

Recall that a group Γ is *residually finite* if the intersection of all finite index subgroups is trivial. It is a theorem of Malcev that finitely generated linear groups are residually finite. The following is an easy exercise that will be used in §7 and §8; see [9] for proof.

Proposition 3.9. *Two residually finite groups are abstractly commensurable if and only if they are commensurable up to finite kernels.*

4. Commensurations of lattices in nilpotent groups

4.1. Example: the Heisenberg group. Consider the $(2n + 1)$-dimensional Heisenberg group

$$\mathcal{H}^{2n+1} = \left\{ \begin{pmatrix} 1 & \mathbf{x} & z \\ 0 & I_n & \mathbf{y}^t \\ 0 & 0 & 1 \end{pmatrix} \middle| \; \mathbf{x}, \mathbf{y} \in \mathbb{C}^n \text{ and } z \in \mathbb{C} \right\} \leq \mathrm{GL}_{n+2}(\mathbb{C}).$$

Then $N = \mathcal{H}^{2n+1}(\mathbb{R})$ is a simply-connected, 2–step nilpotent Lie group in which $\Gamma = \mathcal{H}^{2n+1}(\mathbb{Z})$ is a lattice. Let $Z = Z(N)$ denote the center of N; note that $Z \cong \mathbb{R}$ and that $N/Z \cong \mathbb{R}^{2n}$. The group commutator induces a map $N/Z \to Z$ by $[\mathbf{x}, \mathbf{y}] = \omega(\mathbf{x}, \mathbf{y})$, where ω is the standard symplectic form on $\mathbb{R}^{2n}$.

Suppose $\phi : \Gamma_1 \to \Gamma_2$ is a partial automorphism of Γ. We will see that $\phi(\Gamma_1 \cap Z) = \Gamma_2 \cap Z$, and so $[\phi]$ induces a commensuration $[\bar{\phi}]$ of $\Gamma/Z(\Gamma) \cong \mathbb{Z}^{2n}$. The induced map $\bar{\phi} \in \mathrm{GL}_{2n}(\mathbb{Q})$ has image in the general symplectic group $\mathrm{GSp}_{2n}(\mathbb{Q})$, defined as

$$\mathrm{GSp}_{2n}(\mathbb{Q}) = \{A \in \mathrm{GL}_n(\mathbb{Q}) \mid \omega(Au, Av) = \alpha\omega(u, v) \text{ for some } \alpha \in \mathbb{Q}^*\}.$$

In fact the induced map $\Theta : \mathrm{Comm}(\Gamma) \to \mathrm{GSp}_{2n}(\mathbb{Q})$ is surjective. Each partial automorphism $\phi : \Gamma_1 \to \Gamma_2$ such that $[\phi] \in \ker(\Theta)$ is trivial on Z, hence is determined by an element of $H^1(\pi(\Gamma_1), \mathbb{Z})$, where $\pi : \Gamma \to \Gamma/(Z \cap \Gamma)$ denotes the natural projection. One can check that

$$\ker(\Theta) \cong \varprojlim_{[\Gamma:H]<\infty} H^1(\pi(H), \mathbb{Z}) \cong H^1(\pi(\Gamma), \mathbb{Q}) \cong \mathbb{Q}^{2n}.$$

Therefore $\mathrm{Comm}(\Gamma)$ satisfies the short exact sequence

$$1 \to \mathbb{Q}^{2n} \to \mathrm{Comm}(\Gamma) \to \mathrm{GSp}_{2n}(\mathbb{Q}) \to 1.$$

The action of $\mathrm{GSp}_{2n}(\mathbb{Q})$ on $\mathbb{Q}^{2n}$ is the tensor product of the dual representation with the 1–dimensional representation $\mu : \mathrm{GSp}_{2n}(\mathbb{Q}) \to \mathbb{Q}^*$ defined by $\omega(Au, Av) = \mu(A)\omega(u, v)$.

4.2. Commensuristic subgroups. Lattices in simply-connected nilpotent Lie groups provide a source of examples of commensuristic and strongly commensuristic subgroups. Recall that the upper central series $\gamma^i(G)$ and lower central series $\gamma_i(G)$ of a group G are defined inductively as follows. Let $\gamma^0(G) = 1$. Suppose that $\gamma^i(G)$ is a normal subgroup of G, and let $\pi : G \to G/\gamma^i(G)$. Define $\gamma^{i+1}(G) = \pi^{-1}(Z(G/\gamma^i(G)))$. Now let $\gamma_0(G) = G$. Supposing $\gamma_i(G)$ is defined, set $\gamma_{i+1}(G) = [G, \gamma_i(G)]$.

Proposition 4.1. *Let $\Gamma \leq N$ be a lattice in a simply-connected nilpotent Lie group. The upper central series of Γ is strongly commensuristic in Γ. The lower central series of Γ is commensuristic.*

Proof. A discrete subgroup $\Delta \leq N$ is a lattice in N if and only if Δ is Zariski-dense in some (equivalently, any) faithful unipotent representation of N into $\mathrm{GL}_n(\mathbb{R})$; see [29] for a proof. Using this it is easy to show by induction that $\gamma^k(\Gamma) = \Gamma \cap \gamma^k(N)$ for all k. Now suppose $\phi : \Gamma_1 \to \Gamma_2$ is a partial automorphism of Γ. Both Γ_1 and Γ_2 are lattices in N, so $\gamma^k(\Gamma_j) = \Gamma_j \cap \gamma^k(N)$ for $j = 1, 2$. It follows that

$$\gamma^k(\Gamma_j) = \Gamma_j \cap \gamma^k(\Gamma) \text{ for } j = 1, 2.$$

Clearly, $\phi(\gamma^k(\Gamma_1)) = \gamma^k(\Gamma_2)$ for all k, from which it follows that $\gamma^k(\Gamma)$ is strongly commensuristic for all k.

Consider the lower central series $\gamma_k(\Gamma)$. Then $\gamma_k(\Gamma)$ is Zariski-dense in $\gamma_k(N)$ for all k by [8, 2.4]. Now suppose $\phi : \Gamma_1 \to \Gamma_2$ is a partial automorphism of Γ. Then $\gamma_k(\Gamma_j)$ is a lattice in $\gamma_k(N)$ for all k for $j = 1, 2$. Since $\gamma_k(\Gamma_j) \leq \gamma_k(\Gamma)$, it follows that $\gamma_k(\Gamma_j) \leq \gamma_k(\Gamma)$ is of finite index for all k for $j = 1, 2$. Since ϕ clearly takes $\gamma_k(\Gamma_1)$ to $\gamma_k(\Gamma_2)$ for all k, it follows that $[\phi] \cdot [\gamma_k(\Gamma)] = [\gamma_k(\Gamma)]$ for all k. □

4.3. Commensurations are rational. Let N be a simply-connected nilpotent Lie group containing a lattice Γ. Let $\mathfrak{n}$ denote the Lie algebra of N. Then $\mathfrak{n}$ admits rational structure constants by [29, 2.12]. It follows that $\mathfrak{n}$ admits a basis, unique up to $\mathbb{Q}$-defined isomorphism, so that $\log(\Gamma) \subseteq \mathfrak{n}(\mathbb{Q})$. Further, $\mathrm{Aut}(\mathfrak{n})$ is identified with $\mathbf{A}(\mathbb{R})$ for a $\mathbb{Q}$-defined algebraic subgroup $\mathbf{A} \leq \mathrm{GL}(\mathfrak{n} \otimes \mathbb{C})$ unique up to $\mathbb{Q}$-defined isomorphism. It is a standard fact of Lie theory that the exponential map identifies $\mathrm{Aut}(N)$ with $\mathrm{Aut}(\mathfrak{n})$. This identifies $\mathrm{Aut}(N)$ with the real points of a $\mathbb{Q}$-defined algebraic group $\mathbf{A}$. By abuse of notation, we write $\mathrm{Aut}(N)(\mathbb{Q})$ for the subgroup of $\mathrm{Aut}(N)$ corresponding to $\mathbf{A}(\mathbb{Q})$. The group $\mathrm{Aut}(N)(\mathbb{Q})$ depends only on N and Γ.

Theorem 4.2. *Let $\Gamma \leq N$ be a lattice in a simply-connected nilpotent Lie group. Identify* $\mathrm{Aut}(N)$ *with the real points of a $\mathbb{Q}$-defined algebraic group as above. Then there is an isomorphism*

$$\xi : \mathrm{Comm}(\Gamma) \to \mathrm{Aut}(N)(\mathbb{Q}).$$

To prove this theorem we will use the fact, due to Malcev, that lattices in nilpotent groups are strongly rigid:

Theorem 4.3 ([29, 2.11]). *Let N_1 and N_2 be two simply-connected nilpotent Lie groups, with lattices $\Gamma_1 \leq N_1$ and $\Gamma_2 \leq N_2$. Then every isomorphism $\Gamma_1 \to \Gamma_2$ extends uniquely to an isomorphism $N_1 \to N_2$.*

Proof of Theorem 4.2: Suppose $\phi : \Gamma_1 \to \Gamma_2$ is a partial automorphism of Γ. Then ϕ extends to an automorphism $\Phi \in \mathrm{Aut}(N)$ by Theorem 4.3. Since $\log(\Gamma)$ is contained in $\mathfrak{n}(\mathbb{Q})$, this extension is in $\mathrm{Aut}(N)(\mathbb{Q})$. This gives an injective homomorphism

$$\xi : \mathrm{Comm}(\Gamma) \to \mathrm{Aut}(N)(\mathbb{Q}).$$

Now suppose $\Phi \in \mathrm{Aut}(N)(\mathbb{Q})$. It is well-known (for example, see [29, Chapter 2]) that there is a $\mathbb{Q}$-defined unipotent algebraic group $\mathbf{U}$ so that $N \cong \mathbf{U}(\mathbb{R})$ and Γ is commensurable with $\mathbf{U}(\mathbb{Z})$. Then Φ extends to a $\mathbb{Q}$-defined automorphism of $\mathbf{U}$. By [29, 10.14] the group $\Phi(\Gamma)$ is commensurable with Γ, hence Φ induces a commensuration of Γ. It follows that ξ is surjective. □

5. The algebraic hull of a polycyclic group

5.1. Polycyclic groups. We briefly review the general theory of lattices in solvable Lie groups. See [29] for a general reference, and [30] for the theory of polycyclic groups.

Definition 5.1. A group Γ is *polycyclic* if there is a subnormal series

$$1 \lhd \Gamma_1 \lhd \Gamma_2 \lhd \cdots \lhd \Gamma \tag{5.1}$$

so that Γ_i / Γ_{i-1} is cyclic for each i.

The *Hirsch number* of Γ, denoted $\operatorname{rank}(\Gamma)$, is the number of i such that Γ_i / Γ_{i-1} is infinite cyclic. Hirsch number is independent of choice of such subnormal series, and is invariant under passage to finite index subgroups. Every polycyclic group contains a finite index subgroup admitting a subnormal series (5.1) such that each Γ_i / Γ_{i-1} is infinite cyclic. Such a group is called *strongly polycyclic*. It is well-known that every lattice in a connected, simply-connected solvable Lie group is strongly polycyclic.

Every polycyclic group Γ admits a unique maximal normal nilpotent subgroup, called the *Fitting subgroup*, denoted $\operatorname{Fitt}(\Gamma)$. If Γ is a strongly polycyclic group, then $\operatorname{Fitt}(\Gamma)$ is isomorphic to a lattice in a simply-connected nilpotent Lie group N. By Theorem 4.3 conjugation extends to a representation $\tilde{\sigma} : \Gamma \to \operatorname{Aut}(N)$. If $\mathfrak{n}$ is the Lie algebra of N, then by identifying $\operatorname{Aut}(N)$ with $\operatorname{Aut}(\mathfrak{n}) \subseteq \operatorname{GL}(\mathfrak{n})$ we have a representation $\sigma : \Gamma \to \operatorname{GL}(\mathfrak{n})$.

Proposition 5.2 ([29, 4.10]). *Let Γ be strongly polycyclic, and $\sigma : \Gamma \to \operatorname{GL}(\mathfrak{n})$ as above. Then*

$$\operatorname{Fitt}(\Gamma) = \{\gamma \in \Gamma \mid \sigma(\gamma) \text{ is unipotent}\}.$$

Lemma 5.3. *Let Γ be a strongly polycyclic group with $\Gamma_1 \leq \Gamma$ a subgroup of finite index. Then* $\operatorname{Fitt}(\Gamma_1) = \operatorname{Fitt}(\Gamma) \cap \Gamma_1$.

Proof. It is clear that $\operatorname{Fitt}(\Gamma_1) \cap \Gamma \leq \operatorname{Fitt}(\Gamma)$, so we have only to show that $\operatorname{Fitt}(\Gamma_1) \leq \operatorname{Fitt}(\Gamma)$. Let N be the Lie group containing $\operatorname{Fitt}(\Gamma)$ as a lattice, and let N_1 be the Lie group containing $\operatorname{Fitt}(\Gamma_1)$ as a lattice. Then $\Gamma_1 \cap \operatorname{Fitt}(\Gamma)$ is a lattice in N. It follows that the inclusion $\Gamma_1 \cap \operatorname{Fitt}(\Gamma) \to \operatorname{Fitt}(\Gamma_1)$ extends to an embedding of Lie groups $i : N \to N_1$ by [29, 2.11]. This gives an embedding of $\mathfrak{n}$ as a Lie subalgebra of $\mathfrak{n}_1$. Let $\sigma : \Gamma \to \operatorname{GL}(\mathfrak{n})$ and $\sigma_1 : \Gamma_1 \to \operatorname{GL}(\mathfrak{n}_1)$ be as above. Then $\mathfrak{n}$ is invariant under $\sigma_1(\Gamma_1)$ because $\operatorname{Fitt}(\Gamma)$ is normal in Γ. Suppose $\gamma \in \operatorname{Fitt}(\Gamma_1)$. Then by Proposition 5.2, $\sigma_1(\gamma)$ is unipotent. It follows that $\sigma(\gamma)$ is unipotent, and so $\gamma \in \operatorname{Fitt}(\Gamma)$ by Proposition 5.2. □

Corollary 5.4. *If Γ is strongly polycyclic then* $\operatorname{Fitt}(\Gamma)$ *is strongly commensuristic in Γ.*

5.2. Algebraic hulls. Our main tool for understanding commensurations of a polycyclic group Γ will be its algebraic hull. Roughly speaking, the algebraic hull is algebraic group in which Γ embeds Zariski-densely that has minimal torus while having maximal unipotent radical. The extremality conditions are important for the extension of commensurations to algebraic automorphisms. The original construction is due to Mostow [23], with an alternate construction in [29]. More recently, algebraic hulls have been constructed for certain virtually polycyclic groups by Baues in [3]. We will need only the classical algebraic hull.

Definition 5.5 (Algebraic hull). Suppose Γ is a strongly polycyclic group. An *algebraic hull* of Γ is a $\mathbb{Q}$-defined algebraic group $\mathbf{H}$ with an embedding $i : \Gamma \to \mathbf{H}(\mathbb{Q})$ so that

(H1) $i(\Gamma)$ is Zariski-dense in $\mathbf{H}$,

(H2) $Z_{\mathbf{H}}(\mathbf{U_H}) \leq \mathbf{U_H}$, where $\mathbf{U_H}$ is the unipotent radical of $\mathbf{H}$,

(H3) $\dim(\mathbf{U_H}) = \mathrm{rank}(\Gamma)$, and

(H4) $i(\Gamma) \cap \mathbf{H}(\mathbb{Z})$ is of finite index in $i(\Gamma)$.

Algebraic hulls exist for all strongly polycyclic groups; see [29] for a construction. The importance of the algebraic hull is its uniqueness:

Lemma 5.6 ([29, 4.41]). *Suppose Γ_1 and Γ_2 are two strongly polycyclic groups, and $\phi : \Gamma_1 \to \Gamma_2$ is an isomorphism. Let $i_1 : \Gamma_1 \to \mathbf{H}_1$ and $i_2 : \Gamma_2 \to \mathbf{H}_2$ be algebraic hulls for Γ_1 and Γ_2, respectively. Then ϕ extends to a $\mathbb{Q}$-defined isomorphism $\Phi : \mathbf{H}_1 \to \mathbf{H}_2$.*

We wish to use rigidity of the algebraic hull to construct an embedding of $\mathrm{Comm}(\Gamma)$ into $\mathrm{Aut}_{\mathbb{Q}}(\mathbf{H})$ analogous to the use of Malcev rigidity in Theorem 4.2. For this, the natural setting is the Zariski-connected component of the identity of the algebraic hull.

Definition 5.7 (Virtual algebraic hull). Let Γ be a virtually polycyclic group. A *virtual algebraic hull* of Γ is a triple $(\mathbf{H}, \Delta, i)$, where $\mathbf{H}$ is a $\mathbb{Q}$-defined algebraic group, Δ is a finite index subgroup of Γ, and $i : \Delta \to \mathbf{H}(\mathbb{Q})$ is an injective homomorphism so that

(1) $\mathbf{H}$ is connected, and

(2) $\mathbf{H}$ with the embedding i is an algebraic hull of Δ.

Lemma 5.8. *Every virtually polycyclic group has a virtual algebraic hull.*

Proof. Suppose Γ is virtually polycyclic. Let $\widetilde{\Gamma} \leq \Gamma$ be any finite index strongly polycyclic subgroup. Let $\widetilde{\mathbf{H}}$ be an algebraic hull for $\widetilde{\Gamma}$. Then the identity component $\widetilde{\mathbf{H}}^0$ is of finite index in $\widetilde{\mathbf{H}}$. Let $\Delta = \widetilde{\Gamma} \cap \widetilde{\mathbf{H}}^0$ and $\mathbf{H} = \widetilde{\mathbf{H}}^0$. It is easy to verify that $\mathbf{H}$ with the given inclusion of Δ in $\mathbf{H}(\mathbb{Q})$ is a virtual algebraic hull of Γ. □

We will often abuse notation and refer to the algebraic group $\mathbf{H}$ of Definition 5.7 as the virtual algebraic hull of Γ, leaving the subgroup Δ and the inclusion $i : \Delta \to \mathbf{H}(\mathbb{Q})$ implicit.

Lemma 5.9. *Let Γ be virtually polycyclic with virtual algebraic hull $(\mathbf{H}, \Delta, i)$. The algebraic group $\mathbf{H}$ is unique up to $\mathbb{Q}$-defined isomorphism.*

Proof. Suppose $(\mathbf{H}_1, \Gamma_1, i_1)$ and $(\mathbf{H}_2, \Gamma_2, i_2)$ are two virtual algebraic hulls of Γ. Then $\Gamma_1 \cap \Gamma_2$ is of finite index in both Γ_1 and Γ_2. Because $\mathbf{H}_1$ and $\mathbf{H}_2$ are connected, both $i_1|_{\Gamma_1\cap\Gamma_2}$ and $i_2|_{\Gamma_1\cap\Gamma_2}$ satisfy (H1)–(H4) for $\Gamma_1 \cap \Gamma_2$ in place of Γ. It follows from Lemma 5.6 that there is a $\mathbb{Q}$-defined isomorphism $\Phi : \mathbf{H}_1 \to \mathbf{H}_2$ extending $i_2 \circ i_1|_{\Gamma_1\cap\Gamma_2}^{-1}$. □

Definition 5.10 (Fitting subgroup). Suppose $(\mathbf{H}, \Delta, i)$ is a virtual algebraic hull of a virtually polycyclic group Γ. Define Fitt$(\mathbf{H})$, the *Fitting subgroup* of $\mathbf{H}$, to be the Zariski-closure of Fitt(Δ) in $\mathbf{H}$. Note that Fitt$(\mathbf{H})$ depends the inclusion $i : \Delta \to \mathbf{H}(\mathbb{Q})$; we suppress this dependence from our notation as the embedding i is implicit in the choice of virtual algebraic hull $\mathbf{H}$.

Note that $[\mathbf{H}, \mathbf{H}] \leq \mathrm{Fitt}(\mathbf{H})$, by [4, 4.6].

Lemma 5.11. *Suppose Γ is virtually polycyclic with virtual algebraic hull $(\mathbf{H}, \Delta, i)$. There is an embedding*

$$\xi : \mathrm{Comm}(\Gamma) \to \mathrm{Aut}_{\mathbb{Q}}(\mathbf{H}). \tag{5.2}$$

Proof. Suppose $\phi : \Delta_1 \to \Delta_2$ is a partial automorphism of Δ. Then $\mathbf{H}$ is an algebraic hull for both Δ_1 and Δ_2 by connectedness, so ϕ extends to $\Phi \in \mathrm{Aut}_{\mathbb{Q}}(\mathbf{H})$. Equivalent partial automorphisms clearly give rise to equal extensions. The assignment $\phi \mapsto \Phi$ gives an injective homomorphism $\mathrm{Comm}(\Delta) \to \mathrm{Aut}_{\mathbb{Q}}(\mathbf{H})$ by density of Δ_1 and Δ_2. The proof is complete since $\mathrm{Comm}(\Gamma) \cong \mathrm{Comm}(\Delta)$. □

There is an analogous construction of algebraic hulls for simply-connected solvable Lie groups G, though they are only $\mathbb{R}$-defined rather than $\mathbb{Q}$-defined.

Definition 5.12 (Algebraic hull). Suppose G is a simply-connected solvable Lie group. A *real algebraic hull* of G is an $\mathbb{R}$-defined algebraic group $\mathbf{H}$ with an embedding $i : G \to \mathbf{H}(\mathbb{R})$ so that

(1) $i(G)$ is Zariski-dense in $\mathbf{H}$,

(2) $Z_{\mathbf{H}}(\mathbf{U}_{\mathbf{H}}) \leq \mathbf{U}_{\mathbf{H}}$, where $\mathbf{U}_{\mathbf{H}}$ is the unipotent radical of $\mathbf{H}$, and

(3) $\dim(\mathbf{U}_{\mathbf{H}}) = \dim(G)$.

The real algebraic hull of the group G may be strictly larger than the algebraic hull of a lattice $\Gamma \leq G$. See [5] for a detailed discussion of the relationship between the algebraic hull of a lattice and the real algebraic hull of the ambient Lie group. We use this theory in §8.

5.3. Unipotent shadow.

Much of the theory of lattices in solvable Lie groups builds on the much easier theory of lattices in nilpotent Lie group. A common tool is the *unipotent shadow*. The following proposition summarizes the theory of unipotent shadows of strongly polycyclic groups in algebraic hulls, as explained in Sections 5 and 7 of [4]. For the reader's convenience we include a sketch of a proof.

Proposition 5.13 ([4]). *Suppose* Γ *is a virtually polycyclic group with virtual algebraic hull* $\mathbf{H}$. *Let* $\mathbf{F}$ *be the Fitting subgroup of* $\mathbf{H}$. *There is a strongly polycyclic subgroup* $\Lambda \leq \mathbf{H}(\mathbb{Q})$ *abstractly commensurable with* Γ *so that:*

(1) *There is a nilpotent subgroup* $C \leq \Lambda$ *so that* $\Lambda = \mathrm{Fitt}(\Lambda) \cdot C$.

(2) *There is a* $\mathbb{Q}$*-defined maximal torus* $\mathbf{T} \leq \mathbf{H}$ *with centralizer* $\mathbf{D} \leq \mathbf{H}$ *so that* $C = \Lambda \cap \mathbf{D}$ *and* C *is Zariski-dense in* $\mathbf{D}$.

(3) *The subgroup* $\theta \leq \mathbf{U_H}(\mathbb{Q})$ *generated by* $\mathrm{Fitt}(\Lambda)$ *and* C_u, *the group of unipotent parts of elements of* C, *is a finitely generated subgroup Zariski-dense in* $\mathbf{U_H}$, *such that* $\mathrm{Fitt}(\Lambda) = \theta \cap \mathbf{F}$.

Sketch of proof. Let Δ be a strongly polycyclic subgroup of Γ so that $\mathbf{H}$ is an algebraic hull of Δ. Fix any maximal $\mathbb{Q}$-defined torus $\mathbf{T} \leq \mathbf{H}$, and let $\mathbf{D}$ be the normalizer of $\mathbf{T}$ in $\mathbf{H}$. Then $\mathbf{D}$ is a connected nilpotent $\mathbb{Q}$-defined subgroup of $\mathbf{H}$ that centralizes $\mathbf{T}$. By replacing $\mathrm{Fitt}(\Delta)$ with a finite index supergroup, we obtain a strongly polycyclic group $\Lambda \leq \mathbf{H}(\mathbb{Q})$ commensurable with Δ for which the group $C = \Lambda \cap \mathbf{D}$ is Zariski-dense in $\mathbf{D}$ and satisfies $\Lambda = \mathrm{Fitt}(\Lambda) \cdot C$. The group Λ is called a *thickening* of Δ, and C is called a *nilpotent supplement* in Λ.

We now want to construct the group θ by taking the unipotent parts of elements of Λ. For every $c \in \mathbf{D}$, let c_s and c_u denote the semisimple and unipotent parts, respectively, of its Jordan decomposition in $\mathbf{D}$. Because $\mathbf{D}$ centralizes $\mathbf{T}$, the map $c \mapsto c_u$ is a homomorphism $\mathbf{D} \to \mathbf{U_H}$. Define θ to be the subgroup of $\mathbf{U_H}(\mathbb{Q})$ generated by $\mathrm{Fitt}(\Lambda)$ and C_u. By replacing Λ with a further thickening, we can guarantee that $\theta \cap \mathbf{F} = \mathrm{Fitt}(\Lambda)$. Such a group θ is called a *good unipotent shadow*. □

5.4. Algebraic structure of $\mathrm{Aut}(\mathbf{H})$. Suppose $\Gamma \leq G$ is a lattice in a simply-connected solvable Lie group, and let $\mathbf{H}$ be its virtual algebraic hull. We recall the structure of $\mathrm{Aut}_{\mathbb{Q}}(\mathbf{H})$ explained in Section 3 of [4]. Let $\mathbf{U}$ be the unipotent radical of $\mathbf{H}$. Fix a $\mathbb{Q}$-defined maximal torus $\mathbf{T} \leq \mathbf{H}$. There is a decomposition $\mathbf{H} = \mathbf{U} \rtimes \mathbf{T}$. Define

$$\mathrm{Aut}(\mathbf{H})_{\mathbf{T}} = \{\Phi \in \mathrm{Aut}(\mathbf{H}) \mid \Phi(\mathbf{T}) = \mathbf{T}\}. \tag{5.3}$$

By property (H2) of the algebraic hull, the restriction map $\mathrm{Aut}(\mathbf{H})_{\mathbf{T}} \to \mathrm{Aut}(\mathbf{U})$ is injective. Its image is a $\mathbb{Q}$-defined closed subgroup of $\mathrm{Aut}(\mathbf{U})$. The map

$$\begin{aligned} \Theta : \mathbf{U} \rtimes \mathrm{Aut}(\mathbf{H})_{\mathbf{T}} &\to \mathrm{Aut}(\mathbf{H}) \\ (u, \Phi) &\mapsto \mathrm{Inn}_u \circ \Phi \end{aligned} \tag{5.4}$$

is a surjection with $\mathbb{Q}$-defined kernel. The quotient $\mathbf{U} \rtimes \mathrm{Aut}(\mathbf{H})_{\mathbf{T}} / \ker(\Theta)$ is a $\mathbb{Q}$-defined algebraic group, which gives $\mathrm{Aut}(\mathbf{H})$ the structure of a $\mathbb{Q}$-defined algebraic group. Because $\ker(\Theta)$ is unipotent, it follows from the discussion of [27, 2.2.3] (see also [4, 3.6]) that there is a group isomorphism

$$\mathrm{Aut}_{\mathbb{Q}}(\mathbf{H}) \cong \mathbf{U}(\mathbb{Q}) \rtimes \mathrm{Aut}(\mathbf{H})_{\mathbf{T}}(\mathbb{Q}) / (\ker \Theta)(\mathbb{Q}). \tag{5.5}$$

Thus the algebraic structure of $\mathrm{Aut}(\mathbf{H})$ is such that $\mathrm{Aut}_{\mathbb{Q}}(\mathbf{H}) = \mathrm{Aut}(\mathbf{H})(\mathbb{Q})$.

5.5. A finite index subgroup of $\mathrm{Comm}(\Gamma)$**.** Let Γ, $\mathbf{H}$, and $\mathbf{U}$ be as above. Let $\mathbf{F} = \mathrm{Fitt}(\mathbf{H})$. Define

$$\mathcal{A}_{\mathbf{H}|\mathbf{U}} = \left\{ \Phi \in \mathrm{Aut}(\mathbf{H}) \;\middle|\; \Phi\big|_{\mathbf{H}/\mathbf{U}} = \mathrm{Id}_{\mathbf{H}/\mathbf{U}} \right\}. \tag{5.6}$$

Lemma 5.14. *The subgroup* $\mathcal{A}_{\mathbf{H}|\mathbf{U}} \leq \mathrm{Aut}(\mathbf{H})$ *is of finite index.*

Proof. The quotient $\mathbf{H}/\mathbf{U}$ is a $\mathbb{Q}$-defined torus. By Lemma 2.5, the identity component $\mathrm{Aut}(\mathbf{H})^0$ acts trivially on the torus $\mathbf{H}/\mathbf{U}$, and so $\mathrm{Aut}(\mathbf{H})^0 \leq \mathcal{A}_{\mathbf{H}|\mathbf{U}}$. The claim follows since $[\mathrm{Aut}(\mathbf{H})^0 : \mathrm{Aut}(\mathbf{H})] < \infty$. □

Let $N_{\mathrm{Aut}(\mathbf{H})}(\mathbf{F})$ denote the subgroup of $\mathrm{Aut}(\mathbf{H})$ preserving $\mathbf{F}$. Define

$$\mathcal{A}_{\mathbf{H}|\mathbf{F}} = \left\{ \Phi \in N_{\mathrm{Aut}(\mathbf{H})}(\mathbf{F}) \;\middle|\; \Phi\big|_{\mathbf{H}/\mathbf{F}} = \mathrm{Id}_{\mathbf{H}/\mathbf{F}} \right\}. \tag{5.7}$$

By Corollary 5.4, the image of the map $\xi : \mathrm{Comm}(\Gamma) \to \mathrm{Aut}(\mathbf{H})$ preserves $\mathbf{F}$. Define

$$\mathrm{Comm}_{\mathbf{H}|\mathbf{F}}(\Gamma) = \xi^{-1}(\mathcal{A}_{\mathbf{H}|\mathbf{F}}). \tag{5.8}$$

Lemma 5.15. $[\mathrm{Comm}(\Gamma) : \mathrm{Comm}_{\mathbf{H}|\mathbf{F}}(\Gamma)] < \infty$.

Proof. By Lemma 5.14, it suffices to show that $\mathrm{Comm}_{\mathbf{H}|\mathbf{F}}(\Gamma) = \mathrm{Comm}(\Gamma) \cap \mathcal{A}_{\mathbf{H}|\mathbf{U}}$. Since $\mathcal{A}_{\mathbf{H}|\mathbf{F}} \leq \mathcal{A}_{\mathbf{H}|\mathbf{U}}$, it is clear that $\mathrm{Comm}_{\mathbf{H}|\mathbf{F}}(\Gamma) \leq \mathrm{Comm}(\Gamma) \cap \mathcal{A}_{\mathbf{H}|\mathbf{U}}$. On the other hand, suppose that $[\phi] \in \mathrm{Comm}(\Gamma) \cap \mathcal{A}_{\mathbf{H}|\mathbf{U}}$. Without loss of generality, assume that ϕ is a partial automorphism of a finite index subgroup $\Delta \leq \Gamma$ for which $\mathbf{H}$ is an algebraic hull. By Proposition 5.2, we have that $\Delta \cap \mathbf{U} = \mathrm{Fitt}(\Delta)$. It follows that if $\phi(\gamma)\gamma^{-1} \in \mathbf{U}$ for some $\gamma \in \Delta$, then $\phi(\gamma)\gamma^{-1} \in \mathrm{Fitt}(\Delta)$. Therefore $[\phi] \in \mathrm{Comm}_{\mathbf{H}|\mathbf{F}}(\Gamma)$. □

The structure of $\mathcal{A}_{\mathbf{H}|\mathbf{F}}$ can be made more explicit, following Section 3.3 of [4]. Let $\mathbf{T}$ denote a maximal $\mathbb{Q}$-defined torus in $\mathbf{H}$. Define

$$\mathcal{A}^1_{\mathbf{T}} = \left\{ \Phi \in \mathcal{A}_{\mathbf{H}|\mathbf{F}} \mid \Phi(\mathbf{T}) = \mathbf{T},\ \Phi\big|_{\mathbf{T}} = \mathrm{id}_{\mathbf{T}} \right\}, \tag{5.9}$$

$$\mathrm{Inn}^{\mathbf{H}}_{\mathbf{F}} = \left\{ \Phi \in \mathrm{Aut}(\mathbf{H}) \mid \Phi(x) = fxf^{-1} \text{ for some } f \in \mathbf{F} \right\}. \tag{5.10}$$

Clearly $\mathrm{Inn}^{\mathbf{H}}_{\mathbf{F}}$ and $\mathcal{A}^1_{\mathbf{T}}$ are both $\mathbb{Q}$-defined subgroups of $\mathcal{A}_{\mathbf{H}|\mathbf{F}}$, and $\mathrm{Inn}^{\mathbf{H}}_{\mathbf{F}}$ is normal in $\mathcal{A}_{\mathbf{H}|\mathbf{F}}$. Let $(\mathcal{A}_{\mathbf{H}|\mathbf{F}})_{\mathbb{Q}}$ denote the group of $\mathbb{Q}$-defined automorphisms in $\mathcal{A}_{\mathbf{H}|\mathbf{F}}$. Because any two maximal $\mathbb{Q}$-defined tori are conjugate by an element of $[\mathbf{H},\mathbf{H}](\mathbb{Q}) \leq \mathbf{F}(\mathbb{Q})$, we have

Lemma 5.16. $\mathcal{A}_{\mathbf{H}|\mathbf{F}} = \mathrm{Inn}^{\mathbf{H}}_{\mathbf{F}} \cdot \mathcal{A}^1_{\mathbf{T}}$. *Moreover,* $(\mathcal{A}_{\mathbf{H}|\mathbf{F}})_{\mathbb{Q}} = \mathrm{Inn}^{\mathbf{H}}_{\mathbf{F}}(\mathbb{Q}) \cdot \mathcal{A}^1_{\mathbf{T}}(\mathbb{Q})$.

Proof. See [4, 3.13]. The latter statement follows from equation (5.5); cf. [4, 3.6]. □

6. Commensurations of lattices in solvable groups

6.1. Example: Sol lattice. Let $\psi : \mathbb{Z}^2 \to \mathbb{Z}^2$ be the automorphism defined by $\psi(1,0) = (2,1)$ and $\psi(0,1) = (1,1)$. Let C be the infinite cyclic group generated by ψ, and define $\Gamma = \mathbb{Z}^2 \rtimes C$. Note that Γ is a lattice in 3-dimensional Sol geometry. We have that $\mathrm{Fitt}(\Gamma) = \mathbb{Z}^2$, so there are induced maps

$$r : \mathrm{Comm}(\Gamma) \to \mathrm{Comm}(\mathbb{Z}^2) \cong \mathrm{GL}_2(\mathbb{Q})$$

and

$$\pi : \mathrm{Comm}(\Gamma) \to \mathrm{Comm}(C) \cong \mathbb{Q}^*.$$

Suppose $\phi : \Gamma_1 \to \Gamma_2$ is a partial automorphism of Γ. There are nonzero p, q so that $\pi(\phi)[\psi^q] = [\psi^p]$. Using the fact that ϕ is an isomorphism, we have

$$\phi(\psi^q(v)) = \psi^p(\phi(v)) \tag{6.1}$$

for all $v \in \Gamma_1 \cap \mathbb{Z}^2$. Since $\Gamma_1 \cap \mathbb{Z}^2$ spans $\mathbb{Z}^2 \otimes \mathbb{Q}$, it follows that $r(\phi)$ conjugates ψ^q to ψ^p in $\mathrm{GL}_2(\mathbb{Q})$. Therefore $p = \pm q$ since ψ has an eigenvalue not on the unit circle. It follows that there is an index 2 subgroup $\mathrm{Comm}^+(\Gamma)$ so that π is trivial when restricted to $\mathrm{Comm}^+(\Gamma)$.

Let $Z_{\mathrm{GL}_2(\mathbb{Q})}(\psi)$ denote the centralizer of ψ in $\mathrm{GL}_2(\mathbb{Q})$. From (6.1) we see that $r(\phi) \in Z_{\mathrm{GL}_2(\mathbb{Q})}(\psi)$ for all $\phi \in \mathrm{Comm}^+(\Gamma)$. Moreover, it is clear that the induced map $\bar{r} : \mathrm{Comm}^+(\Gamma) \to Z_{\mathrm{GL}_2(\mathbb{Q})}(\psi)$ is surjective. Let $K = \ker(\bar{r})$. Every $\phi \in K$ is of the form $\phi(v, \psi^p) = (v + \rho(\psi^p), \psi^p)$ for a cocycle $\rho : H \to \mathbb{Z}^2$ defined on some finite index subgroup $H \leq C$. One can show that

$$K = \varprojlim_{[C:H]<\infty} H^1(H, \mathbb{Z}^2) \cong H^1(C, \mathbb{Q}^2) \cong \mathbb{Q}^2.$$

Therefore $\mathrm{Comm}^+(\Gamma)$ satisfies the short exact sequence

$$1 \to \mathbb{Q}^2 \to \mathrm{Comm}^+(\Gamma) \to Z_{\mathrm{GL}_2(\mathbb{Q})}(\psi) \to 1.$$

This sequence splits, and the action of $Z_{\mathrm{GL}_2(\mathbb{Q})}(\psi)$ on $\mathbb{Q}^2$ is the standard action.

6.2. Commensurations of solvable lattices are rational. We continue to use the notation developed in §5. Given a lattice Γ in a connected, simply-connected solvable Lie group, let $\mathbf{H}$ denote its virtual algebraic hull with Fitting subgroup $\mathbf{F}$ and $\mathbb{Q}$-defined maximal torus $\mathbf{T}$. Then $\mathcal{A}_{\mathbf{H}|\mathbf{F}}$ denotes the group of automorphisms of $\mathbf{H}$ preserving $\mathbf{F}$ and trivial on $\mathbf{H}/\mathbf{F}$. Let $\mathrm{Inn}_{\mathbf{F}}^{\mathbf{H}}$ denote the group of automorphisms of $\mathbf{H}$ induced by conjugation by elements of $\mathbf{F}$, and $\mathcal{A}_{\mathbf{T}}^1$ denote the group of automorphisms fixing $\mathbf{T}$.

Theorem 6.1. *Let Γ be a virtually polycyclic group. Let* $\mathbf{H}$ *be the virtual algebraic hull of* Γ*, with* $\mathbf{F} = \mathrm{Fitt}(\mathbf{H})$*. The map* $\xi : \mathrm{Comm}(\Gamma) \to \mathrm{Aut}(\mathbf{H})$ *induces an isomorphism of groups*

$$\mathrm{Comm}_{\mathbf{H}|\mathbf{F}}(\Gamma) \cong (\mathcal{A}_{\mathbf{H}|\mathbf{F}})_{\mathbb{Q}}.$$

The proof of the theorem is in two steps. First we show that $\mathrm{Inn}_{\mathbf{F}}^{\mathbf{H}}(\mathbb{Q}) \leq \xi(\mathrm{Comm}(\Gamma))$, and second that $\mathcal{A}_{\mathbf{T}}^{1}(\mathbb{Q}) \leq \xi(\mathrm{Comm}(\Gamma))$. The unipotent shadow will be our main tool. Before we start the proof of Theorem 6.1, we note the following technical lemma, which will be used again in §8.

Lemma 6.2. *Let* $\mathbf{U}$ *be a* $\mathbb{Q}$*-defined unipotent algebraic group and* $\theta' \leq \mathbf{U}(\mathbb{Q})$ *be a finitely generated, Zariski-dense subgroup. Let* P *be a group acting on* $\mathbf{U}$ *by algebraic group automorphisms preserving* θ'*. Suppose* $f \in \mathbf{U}(\mathbb{Q})$*. There is some finite index subgroup* $P'' \leq P$ *so that* $f(p \cdot f^{-1}) \in \theta'$ *for all* $p \in P''$*.*

Proof. Let Λ be the group generated by θ' and f. Then Λ is commensurable with $\mathbf{U}(\mathbb{Z})$, hence contains θ' as a subgroup of finite index d. It is not hard to see that there are finitely many subgroups of $\mathbf{U}(\mathbb{Q})$ containing θ' with index d (see [30, Ch 6] and [4, 6.3]). The group P permutes these subgroups, hence there is a subgroup $P' \leq P$ preserving Λ. Because θ' has finite index in Λ and is preserved by P, there is a further finite index subgroup $P'' \leq P$ that acts trivially on the coset space Λ/θ'. This completes the proof. □

Proof of Theorem 6.1. Given Γ and $\mathbf{H}$ as in the theorem, let $\mathbf{U} = \mathbf{U}_{\mathbf{H}}$. Find a strongly polycyclic subgroup $\Lambda \leq \mathbf{H}(\mathbb{Q})$ abstractly commensurable with Γ, along with $\mathbf{T}$, $\mathbf{D}$, C, and θ as in Proposition 5.13. That is, $\mathbf{T}$ is a maximal $\mathbb{Q}$-defined torus, $\mathbf{D}$ is the centralizer of $\mathbf{T}$ containing $C = \Lambda \cap \mathbf{D}$ as a Zariski-dense subgroup, and $\theta \leq \mathbf{U}(\mathbb{Q})$ is a good unipotent shadow of Λ.

By Lemma 5.11 there is an embedding

$$\xi : \mathrm{Comm}(\Gamma) \to \mathrm{Aut}_{\mathbb{Q}}(\mathbf{H}).$$

By definition of $\mathrm{Comm}_{\mathbf{H}|\mathbf{F}}(\Gamma)$, this restricts to an embedding

$$\hat{\xi} : \mathrm{Comm}_{\mathbf{H}|\mathbf{F}}(\Gamma) \to (\mathcal{A}_{\mathbf{H}|\mathbf{F}})_{\mathbb{Q}}.$$

There is a decomposition $(\mathcal{A}_{\mathbf{H}|\mathbf{F}})_{\mathbb{Q}} = \mathrm{Inn}_{\mathbf{F}}^{\mathbf{H}}(\mathbb{Q}) \cdot \mathcal{A}_{\mathbf{T}}^{1}(\mathbb{Q})$ by Lemma 5.16. We have only to show that both $\mathrm{Inn}_{\mathbf{F}}^{\mathbf{H}}(\mathbb{Q})$ and $\mathcal{A}_{\mathbf{T}}^{1}(\mathbb{Q})$ are in the image of $\hat{\xi}$.

Claim 1. $\mathrm{Inn}_{\mathbf{F}}^{\mathbf{H}}(\mathbb{Q}) \leq \xi(\mathrm{Comm}(\Gamma))$.

Proof of Claim 1. Suppose $\Phi \in \mathrm{Inn}_{\mathbf{F}}^{\mathbf{H}}(\mathbb{Q})$. Then there is some $f \in \mathbf{F}(\mathbb{Q})$ so that $\Phi(x) = fxf^{-1}$ for all $x \in \mathbf{H}$. Because θ is Zariski-dense in $\mathbf{U}$, conjugation by f induces a commensuration of θ by Theorem 4.2. Let θ_1 and θ_2 be finite index subgroups of θ so that $\Phi(\theta_1) = \theta_2$. Let $C' \leq C$ be a finite index subgroup normalizing both θ_1 and θ_2. By Lemma 6.2, applied with $\theta' = \theta_1 \cap \theta_2$ and $P = C'$, there is some finite index subgroup $C'' \leq C'$ so that

$$fcf^{-1}c^{-1} \in \theta_1 \cap \theta_2 \tag{6.2}$$

for all $c \in C''$. Because $\mathbf{F}$ is normal in $\mathbf{U}$, for all $c \in C''$ we have $fcf^{-1}c^{-1} \in \mathbf{F}$. By (6.2) and the fact that $\theta \cap \mathbf{F} = \mathrm{Fitt}(\Lambda)$, for all $c \in C''$ we have

$$fcf^{-1}c^{-1} \in \mathrm{Fitt}(\Lambda) \cap \theta_1 \cap \theta_2. \tag{6.3}$$

Let $F_1 = \theta_1 \cap \mathrm{Fitt}(\Lambda)$ and $F_2 = \theta_2 \cap \mathrm{Fitt}(\Lambda)$. Then Φ induces an isomorphism $F_1 \to F_2$. Because C'' normalizes both F_1 and F_2, we may form subgroups $\Lambda_1 = F_1C''$ and $\Lambda_2 = F_2C''$, both of which are of finite index in Λ. We claim that Φ induces an isomorphism $\Lambda_1 \to \Lambda_2$. Suppose $f_1 \in F_1$ and $c_1 \in C''$. Then $ff_1f^{-1} \in F_2$ by definition of θ_1 and θ_2, and $fc_1f^{-1} = f_2c_1$ for some $f_2 \in F_2$ by (6.3). Therefore

$$ff_1c_1f^{-1} = ff_1f^{-1}fc_1f^{-1} \in F_2C''.$$

It follows that Φ induces an injection $\Lambda_1 \to \Lambda_2$. Note that (6.3) holds for all $c \in C''$ with f replaced by f^{-1}. Similar reasoning then gives that Φ^{-1} induces an injection $\Lambda_2 \to \Lambda_1$. Thus Φ induces a partial automorphism $\Lambda_1 \to \Lambda_2$ of Λ, and so induces a commensuration of Γ. This completes the proof of Claim 1. □

Claim 2. $\mathcal{A}^1_{\mathbf{T}}(\mathbb{Q}) \leq \xi(\mathrm{Comm}(\Gamma))$.

Proof of Claim 2. Suppose $\Phi \in \mathcal{A}^1_{\mathbf{T}}(\mathbb{Q})$. Then Φ corresponds to a $\mathbb{Q}$-defined map under the restriction $\mathcal{A}^1_{\mathbf{T}} \to \mathrm{Aut}(\mathbf{U})$, so Φ induces a partial automorphism $\theta_1 \to \theta_2$ of θ by Theorem 4.2. The map $C \to C_u$ is a homomorphism. Define a finite index subgroup

$$C_1 = \{c \in C \mid c_u \in \theta_1\} \leq C.$$

Take any $c_1 \in C_1$, and write $c_1 = u_1s$ for $u_1 \in \theta_1$ and $s \in \mathbf{T}$. Since $\Phi \in \mathcal{A}_{\mathbf{H}|\mathbf{F}}$, there is some $f \in \mathbf{F}(\mathbb{Q})$ so that $\Phi(u_1) = fu_1$. Since $\Phi \in \mathcal{A}^1_{\mathbf{T}}$, we have

$$\Phi(c_1) = \Phi(u_1)\Phi(s) = fu_1s = fc_1.$$

Both $\Phi(u_1)$ and u_1 are in θ, so $f \in \theta \cap \mathbf{F} = \mathrm{Fitt}(\Lambda)$. Therefore $\Phi(c_1) \in \Lambda$. Since Φ preserves $\mathbf{T}$, it also preserves $\mathbf{D}$. Therefore $\Phi(C_1) \leq C$ since $\Lambda \cap \mathbf{D} = C$.

Define

$$C_2 = \{c \in C \mid c_u \in \theta_2\} \leq C.$$

It is evident from the definitions of θ_1 and θ_2 that $\Phi(C_1) \leq C_2$. Applying the same logic as above to Φ^{-1}, we conclude that $\Phi(C_1) = C_2$. Therefore Φ induces a partial automorphism $C_1 \to C_2$ of C.

Since Φ preserves $\mathbf{F}$, it induces a partial automorphism $F_1 \to F_2$ of $\mathrm{Fitt}(\Lambda)$. Without loss of generality, suppose F_1 is characteristic in $\mathrm{Fitt}(\Lambda)$. Then F_1C_1 and

F_2C_2 are both finite index subgroups of Λ. So Φ induces a partial automorphism $F_1C_2 \to F_2C_2$ of Λ, and hence a commensuration of Γ. This completes the proof of Claim 2. □

Claims 1 and 2 show that $\hat{\xi}$ is surjective, and therefore $\hat{\xi}$ exhibits an isomorphism $\mathrm{Comm}_{\mathbf{H}|\mathbf{F}}(\Gamma) \cong (\mathcal{A}_{\mathbf{H}|\mathbf{F}})_{\mathbb{Q}}$. This completes the proof of Theorem 6.1. □

Proof of Theorem 1.2. Let $\mathbf{H}$ be the virtual algebraic hull of Γ. By Lemma 5.11 there is an embedding

$$\xi : \mathrm{Comm}(\Gamma) \to \mathrm{Aut}(\mathbf{H})(\mathbb{Q}),$$

where $\mathrm{Aut}(\mathbf{H})$ has the structure of an algebraic group as described in Section 5.4. Let $\mathcal{A}_\Gamma$ be the Zariski-closure of $\xi(\mathrm{Comm}(\Gamma))$ in $\mathrm{Aut}(\mathbf{H})$. Then $\mathcal{A}_\Gamma$ is a $\mathbb{Q}$-defined algebraic group by Proposition 2.1. Now take any $\Psi \in \mathcal{A}_\Gamma(\mathbb{Q})$. Take any element $\Phi \in \xi(\mathrm{Comm}(\Gamma))$ so that $\Psi \circ \Phi^{-1} \in \mathcal{A}_\Gamma^0(\mathbb{Q})$. We have $\mathcal{A}_\Gamma^0 \leq \mathcal{A}_{\mathbf{H}|\mathbf{U}}$ by Lemma 5.14 and then $\mathcal{A}_\Gamma^0 \leq \mathcal{A}_{\mathbf{H}|\mathbf{F}}$ by Lemma 5.15. Therefore $\Psi \circ \Phi^{-1} \in \mathcal{A}_{\mathbf{H}|\mathbf{F}}(\mathbb{Q})$. It follows from Theorem 6.1 that $\Psi \in \xi(\mathrm{Comm}(\Gamma))$, hence the isomorphism

$$\mathrm{Comm}(\Gamma) \cong \mathcal{A}_\Gamma(\mathbb{Q}).$$

We have only to show that the image of $\mathrm{Aut}(\Gamma)$ in $\mathrm{Aut}(\mathbf{H})$ is commensurable with $\mathcal{A}_\Gamma(\mathbb{Z})$. Let $F = \mathrm{Fitt}(\Gamma)$ and define

$$A_{\Gamma|F} = \left\{\phi \in \mathrm{Aut}(\Gamma) \;\middle|\; \phi\big|_{\Gamma/F} = \mathrm{Id}\big|_{\Gamma/F}\right\}.$$

The proof of Lemma 5.15 shows that $A_{\Gamma|F}$ is finite index in $\mathrm{Aut}(\Gamma)$; see also [4, 9.1]. The group $A_{\Gamma|F}$ is commensurable with $\mathcal{A}_{\mathbf{H}|\mathbf{F}}(\mathbb{Z})$ by [4, 8.9], so the result follows. □

We conclude this section with a result relating the structure of $\mathcal{A}_\Gamma$ to that of $\mathrm{Aut}(G)$ for certain solvable groups G. This strengthens the analogy with semisimple groups; compare with Theorem 7.5 below.

Definition 6.3. Let $\mathrm{Nil}(G)$ denote the maximal normal nilpotent subgroup of G. A solvable Lie group G is *unipotently connected* if $\mathrm{Nil}(G)$ is connected.

Proposition 6.4. *Suppose G is a connected, simply-connected, unipotently connected solvable Lie group. Let $\Gamma \leq G$ be a Zariski-dense lattice and let $\mathcal{A}_\Gamma$ be the group such that $\mathcal{A}_\Gamma(\mathbb{Q}) \cong \mathrm{Comm}(\Gamma)$. Then*

$$\mathcal{A}_\Gamma(\mathbb{R}) \doteq \mathrm{Aut}(G).$$

Proof. Let $\mathbf{H}$ be the real algebraic hull of G. By [5, 3.11] the group $\mathbf{H}$ is also an algebraic hull for Γ. It further follows that $\mathbf{F}(\mathbb{R}) = \mathrm{Nil}(G)$ by [5, 5.4]. For any $\Phi \in \mathcal{A}_{\mathbf{H}|\mathbf{F}}(\mathbb{R})$, there is some $f \in \mathbf{F}(\mathbb{R})$ such that $\Phi(g) = fg$ for all $g \in \mathbf{H}$. Therefore $\mathcal{A}_{\mathbf{H}|\mathbf{F}}(\mathbb{R})$ preserves $G \leq \mathbf{H}(\mathbb{R})$, and so $\mathcal{A}_{\mathbf{H}|\mathbf{F}}(\mathbb{R}) \leq \mathrm{Aut}(G)$. In fact, $[\mathrm{Aut}(G) : \mathcal{A}_{\mathbf{H}|\mathbf{F}}(\mathbb{R})] < \infty$ by [5, 6.9]. The result follows because $\mathcal{A}_{\mathbf{H}|\mathbf{F}}$ is a subgroup of finite index in $\mathcal{A}_\Gamma$. □

Every lattice in a connected, simply-connected solvable Lie group virtually embeds as a Zariski-dense lattice in a connected, simply-connected, unipotently connected solvable Lie group G' (cf. [5, 5.3]). Therefore we have:

Corollary 6.5. *Let Γ be a lattice in a connected, simply-connected Lie group G. Let $\mathcal{A}_\Gamma$ denote the algebraic group such that $\mathcal{A}_\Gamma(\mathbb{Q}) = \mathrm{Comm}(\Gamma)$. Then Γ virtually embeds as a lattice in a Lie group G' such that $\mathcal{A}_\Gamma(\mathbb{R}) \doteq \mathrm{Aut}(G')$.*

7. Commensurations of lattices in semisimple groups

Abstract commensurators of lattices in semisimple Lie groups not isogenous to $\mathrm{PSL}_2(\mathbb{R})$ are fairly well understood, by work of Borel, Mostow, and Margulis. For example, see the first section of [1]. We recall the basic results here for completeness.

7.1. Arithmetic lattices in semisimple groups.

Definition 7.1. Suppose $\Gamma \leq S$ is a lattice in a semisimple Lie group with trivial center and no compact factors. We say that Γ is *arithmetic* if there is a $\mathbb{Q}$-defined semisimple algebraic group $\mathbf{S}$ and a surjective homomorphism $f : \mathbf{S}(\mathbb{R})^0 \to S$ with compact kernel such that $f(\mathbf{S}(\mathbb{Z}) \cap \mathbf{S}(\mathbb{R})^0)$ and Γ are commensurable.

Note that $\mathbf{S}$ may be chosen to be simply-connected, and that $\Gamma \doteq \mathbf{S}(\mathbb{Z})$ by Proposition 3.9. Hence, to compute the abstract commensurators of arithmetic lattices in semisimple Lie groups, it suffices to consider groups of the form $\mathbf{S}(\mathbb{Z})$ for a simply-connected $\mathbb{Q}$-defined semisimple algebraic group $\mathbf{S}$.

Recall that a $\mathbb{Q}$-defined, connected, semisimple algebraic group $\mathbf{S}$ is *without $\mathbb{Q}$-compact factors* if there is no nontrivial, $\mathbb{Q}$-defined, connected, normal subgroup $\mathbf{N} \leq \mathbf{S}$ such that $\mathbf{N}(\mathbb{R})$ is compact. Note that given any $\mathbb{Q}$-defined connected, simply-connected, semisimple algebraic group, there is a $\mathbb{Q}$-defined, connected, simply-connected, semisimple algebraic group $\mathbf{S}'$ without $\mathbb{Q}$-compact factors such that $\mathbf{S}(\mathbb{Z})$ and $\mathbf{S}'(\mathbb{Z})$ are abstractly commensurable.

If $\mathbf{S}$ is a $\mathbb{Q}$-defined semisimple algebraic group, then $\mathrm{Aut}(\mathbf{S})$, the group of automorphisms of $\mathbf{S}$ as an algebraic group, has the structure of a $\mathbb{Q}$-defined algebraic group such that $\mathrm{Aut}(\mathbf{S})_{\mathbb{Q}} \cong \mathrm{Aut}(\mathbf{S})(\mathbb{Q})$; see [32, 5.7.2].

Proposition 7.2. *Suppose $\mathbf{S}$ is a $\mathbb{Q}$-defined, connected, simply-connected, semisimple algebraic group without $\mathbb{Q}$-compact factors. Then there is a canonical inclusion*

$$\Xi : \mathrm{Aut}(\mathbf{S})(\mathbb{Q}) \hookrightarrow \mathrm{Comm}(\mathbf{S}(\mathbb{Z})).$$

Proof. If $\Phi \in \mathrm{Aut}(\mathbf{S})(\mathbb{Q})$, then Φ is a $\mathbb{Q}$-defined automorphism of $\mathbf{S}$. Arithmetic groups are mapped to arithmetic groups under $\mathbb{Q}$-defined isomorphism of algebraic groups (see e.g. [29, 10.14]), so Φ induces a commensuration of $\mathbf{S}(\mathbb{Z})$. Because $\mathbf{S}(\mathbb{Z})$ is Zariski-dense in $\mathbf{S}$ by Theorem 2.9, the induced map $\Xi : \mathrm{Aut}(\mathbf{S})(\mathbb{Q}) \to \mathrm{Comm}(\mathbf{S}(\mathbb{Z}))$ is injective. □

The following consequence of Mostow–Prasad–Margulis rigidity is likely known to experts. We include a proof, having found no reference in the literature, using the techniques of [15].

Theorem 7.3. *Let* $\mathbf{S}$ *be a* $\mathbb{Q}$*-defined, connected, simply-connected, semisimple algebraic group without* $\mathbb{Q}$*-compact factors. Suppose that if* F *is a factor of* $\mathbf{S}(\mathbb{R})^0$ *locally isomorphic to* $\mathrm{PSL}_2(\mathbb{R})$ *then* $\mathbf{S}(\mathbb{Z})$ *projects to a non-discrete subgroup of* F*. Then the inclusion*

$$\Xi : \mathrm{Aut}(\mathbf{S})(\mathbb{Q}) \to \mathrm{Comm}(\mathbf{S}(\mathbb{Z}))$$

is an isomorphism.

Proof. Let $\mathbf{S}_1, \dots, \mathbf{S}_n$ be the $\mathbb{Q}$-simple factors of $\mathbf{S}$, so that

$$\mathbf{S} = \mathbf{S}_1 \cdot \mathbf{S}_2 \cdot \dots \cdot \mathbf{S}_{n-1} \cdot \mathbf{S}_n.$$

For each j, let $\pi_j : \mathbf{S} \to \mathbf{S}_j$ be the canonical projection.

Suppose $[\phi] \in \mathrm{Comm}(\mathbf{S}(\mathbb{Z}))$. Without loss of generality, we may assume that $\phi : \Gamma_1 \to \Gamma_2$ is a partial isomorphism of $\mathbf{S}(\mathbb{Z})$ where

$$\Gamma_1 = (\Gamma_1 \cap \mathbf{S}_1) \cdot (\Gamma_1 \cap \mathbf{S}_2) \cdot \dots \cdot (\Gamma_1 \cap \mathbf{S}_n).$$

Let

$$\Gamma_{1,i} = \Gamma_1 \cap \mathbf{S}_i \quad \text{and} \quad \Gamma_1^i = \Gamma_{1,1} \cdot \dots \cdot \Gamma_{1,i-1} \cdot \Gamma_{1,i+1} \cdot \dots \cdot \Gamma_{1,n}.$$

Note that each $\Gamma_{1,i}$ is of finite index in $\mathbf{S}_i(\mathbb{Z})$.

Given any i, choose some j such that $\pi_j(\phi(\Gamma_{1,i}))$ is infinite. Let $\mathbf{A}_1$ be the Zariski closure of $\pi_j(\phi(\Gamma_{1,i}))$ in $\mathbf{S}_j$, and $\mathbf{A}_2$ be the Zariski closure of $\pi_j(\phi(\Gamma_1^i))$ in $\mathbf{S}_j$. Replacing Γ_1 with a finite index subgroup if necessary, we may assume both $\mathbf{A}_1$ and $\mathbf{A}_2$ are connected. Then $\mathbf{A}_1$ commutes with $\mathbf{A}_2$ because $\Gamma_{1,i}$ commutes with Γ_1^i. Note that $\pi_j(\phi(\Gamma_{1,i})) \cdot \pi_j(\phi(\Gamma_1^i))$ is commensurable with $\mathbf{S}_j(\mathbb{Z})$, hence Zariski-dense in $\mathbf{S}_j$ by Theorem 2.9. Therefore $\mathbf{A}_1 \cdot \mathbf{A}_2 = \mathbf{S}_j$. Since $\pi_j(\phi(\Gamma_{1,i}))$ is infinite and $\mathbb{Q}$-defined, and $\mathbf{S}_j$ is $\mathbb{Q}$-simple, it must be that $\mathbf{A}_1 = \mathbf{S}_j$. Since $\mathbf{A}_1$ commutes with $\mathbf{A}_2$ and $\mathbf{A}_2$ is connected, it follows that $\mathbf{A}_2$ is trivial. Therefore $\pi_j(\phi(\Gamma_1^i))$ must be trivial.

It follows that, after replacing Γ_1 with a subgroup of finite index, for each i there is exactly one j so that $\pi_j(\phi(\Gamma_{1,i}))$ is nontrivial. Therefore for each i there is exactly one j so that the image of $\Gamma_{1,i}$ under ϕ is a subgroup of $\mathbf{S}_j$ of finite index in $\mathbf{S}_j(\mathbb{Z})$. It follows from Theorem 2.11 that $\phi|_{\Gamma_{1,i}}$ virtually extends to an isomorphism $\Phi_i : \mathbf{S}_i \to \mathbf{S}_j$. The map $\Phi : \mathbf{S} \to \mathbf{S}$ defined by $\Phi|_{\mathbf{S}_i} = \Phi_i$ is a $\mathbb{Q}$-defined automorphism virtually extending ϕ, and so Ξ is surjective. □

7.2. More general lattices in semisimple groups. A lattice Γ in a connected semisimple Lie group S with finite center is *irreducible* if the projection of Γ to S/N is dense for every nontrivial connected normal subgroup $N \leq S$. Let $\Gamma \leq S$ be an irreducible lattice in a connected semisimple Lie group with trivial center and no compact factors. The relative commensurator $\mathrm{Comm}_S(\Gamma)$ satisfies a dichotomy (see [35]): either $\mathrm{Comm}_S(\Gamma)$ contains Γ as a subgroup of finite index, or $\mathrm{Comm}_S(\Gamma)$ is dense in S. In fact, it is a celebrated theorem of Margulis that this is precisely the dichotomy of arithmeticity versus non-arithmeticity.

Theorem 7.4 (Margulis, see [35], [21]). *Let $\Gamma \leq S$ be an irreducible lattice in a connected semisimple Lie group with trivial center and no compact factors. Then* $\mathrm{Comm}_S(\Gamma)$ *is dense in S if and only if Γ is arithmetic.*

We summarize the above results:

Theorem 7.5. *Let Γ be an irreducible lattice in a noncompact connected semisimple Lie group S. Assume that S is not locally isomorphic to* $\mathrm{PSL}_2(\mathbb{R})$*. One of the following holds:*

(1) *Γ is arithmetic and there is a $\mathbb{Q}$-defined, connected, simply-connected, $\mathbb{Q}$-simple, semisimple algebraic group $\mathbf{S}$ so that*

$$\mathrm{Comm}(\Gamma) \cong \mathrm{Aut}(\mathbf{S})(\mathbb{Q}).$$

Moreover, the group $\mathrm{Aut}(\Gamma)$ *is commensurable with* $\mathrm{Aut}(\mathbf{S})(\mathbb{Z})$.

(2) *Γ is not arithmetic and* $\mathrm{Comm}(\Gamma) \doteq \Gamma$.

Proof. Suppose Γ is arithmetic. Then there is a $\mathbb{Q}$-defined, connected, simply-connected, semisimple algebraic group $\mathbf{S}$ without $\mathbb{Q}$-compact factors so that $\Gamma \doteq \mathbf{S}(\mathbb{Z})$. Since Γ is irreducible in S, the group $\mathbf{S}$ is $\mathbb{Q}$-simple. The isomorphism $\mathrm{Comm}(\Gamma) \cong \mathrm{Aut}(\mathbf{S})(\mathbb{Q})$ follows from Theorem 7.3. Since $\mathrm{Aut}(\Gamma)$ is commensurable with Γ and Γ is commensurable with $\mathbf{S}(\mathbb{Z})$, the result follows since $\mathbf{S}(\mathbb{Z})$ is commensurable with $\mathrm{Aut}(\mathbf{S})(\mathbb{Z})$.

Now suppose Γ is not arithmetic. Let $S' = S/Z(S)$ and $\pi : S \to S'$ the canonical projection. There is a finite index subgroup of Γ taken faithfully to a lattice $\Gamma' \leq S'$. Let N be the maximal compact factor of S' and $S'' = S'/N$. Then Γ' contains a finite index subgroup Γ'' mapping isomorphically to a lattice $\Gamma'' \leq S''$. By Mostow–Prasad–Margulis rigidity (cf. [24]), every commensuration of Γ'' extends to an automorphism of S''. Since $[\mathrm{Aut}(S'') : \mathrm{Inn}(S'')] < \infty$, where $\mathrm{Inn}(S'')$ is the group of inner automorphisms of S'', it follows that $[\mathrm{Comm}(\Gamma'') : \mathrm{Comm}_{S''}(\Gamma'')] < \infty$, and hence $[\mathrm{Comm}(\Gamma'') : \Gamma''] < \infty$ by Theorem 7.4. Since Γ'' is of finite index in Γ, the result follows. □

The case that $S = \mathrm{PSL}_2(\mathbb{R})$ is dramatically different.

Proposition 7.6. *Suppose S is locally isomorphic to* $\mathrm{PSL}_2(\mathbb{R})$ *and $\Gamma \leq S$ is a lattice. Then there is no faithful embedding* $\mathrm{Comm}(\Gamma) \to \mathrm{GL}_N(\mathbb{C})$ *for any N.*

Proof. Γ is either virtually free or virtually the fundamental group of a closed surface. All finitely generated free groups are abstractly commensurable to each other, as are all closed surface groups. Therefore we have that $\mathrm{Comm}(\Gamma)$ is isomorphic either to $\mathrm{Comm}(F_2)$ or to $\mathrm{Comm}(\pi_1(\Sigma_2))$, where F_n is the free group on n letters and Σ_g is a closed surface of genus g.

A group G has the *unique root property* if $x^k = y^k$ implies $x = y$ for all $x, y \in G$ and nonzero k. If G has the unique root property and $H \leq G$ is a finite index subgroup, then the natural map $\mathrm{Aut}(H) \to \mathrm{Comm}(G)$ is faithful (see [26]). It is easy to see that free groups and closed surface groups have the unique root property. Therefore $\mathrm{Aut}(F_n) \leq \mathrm{Comm}(F_2)$ for all $n \geq 2$, and $\mathrm{Aut}(\pi_1(\Sigma_g)) \leq \mathrm{Comm}(\pi_1(\Sigma_2))$ for all $g \geq 2$.

In [12] it is shown that $\mathrm{Aut}(F_n)$ is not linear for any $n \geq 3$. Therefore $\mathrm{Comm}(F_2)$ cannot be linear. On the other hand, the proof of [11, 1.6] shows that for each N there is some g_0 so that if $g \geq g_0$ then $\mathrm{Mod}^{\pm}(\Sigma_{g,1})$, the extended mapping class group of the punctured surface of genus g, has no faithful complex linear representation of dimension less than or equal to N. Since $\mathrm{Mod}^{\pm}(\Sigma_{g,1}) \cong \mathrm{Aut}(\pi_1(\Sigma_g))$, it follows that $\mathrm{Comm}(\pi_1(\Sigma_2))$ is not linear. □

Nonarithmetic irreducible lattices can occur only in groups isogenous to $\mathrm{SO}(1,n)$ or $\mathrm{SU}(1,n)$ up to compact factors. We will use this fact in §8.

Theorem 7.7 (see [21],[17]). *Let S be a connected semisimple Lie group with trivial center and no compact factors. Suppose either $S = \mathrm{Sp}(1,n)$ for $n \geq 2$, or $S = F_4^{-20}$, or $\mathrm{rank}_{\mathbb{R}}(S) \geq 2$. Then every irreducible lattice in S is arithmetic.*

7.3. Example: $\mathrm{PGL}_n(\mathbb{Z})$.

Consider the algebraic group PGL_n for $n \geq 3$. The group $\mathrm{PGL}_n(\mathbb{R})^0$ is a semisimple Lie group, containing $\mathrm{PGL}_n(\mathbb{Z}) \cap \mathrm{PGL}_n(\mathbb{R})^0$ as a lattice. By Theorem 7.3 we have

$$\mathrm{Comm}(\mathrm{PGL}_n(\mathbb{Z})) \cong \mathrm{Aut}(\mathrm{PGL}_n)(\mathbb{Q}).$$

Let $\tau : \mathrm{PGL}_n \to \mathrm{PGL}_n$ be the automorphism given by $\tau(A) = (A^{-1})^t$. Then PGL_n acts on itself faithfully by conjugation, and there is a decomposition

$$\mathrm{Aut}(\mathrm{PGL}_n) = \mathrm{PGL}_n \rtimes \langle \tau \rangle .$$

Since τ preserves $\mathrm{PGL}_n(\mathbb{Z})$, there is an isomorphism

$$\mathrm{Comm}(\mathrm{PGL}_n(\mathbb{Z})) \cong \mathrm{PGL}_n(\mathbb{Q}) \rtimes \langle \tau \rangle . \tag{7.1}$$

Remark 7.8. Note that $\mathrm{PSL}_n(\mathbb{R}) = \mathrm{PGL}_n(\mathbb{R})^0$ and $\mathrm{PSL}_n(\mathbb{Z}) \doteq \mathrm{PGL}_n(\mathbb{Z})$, so it follows from equation (7.1) the above that

$$\mathrm{Comm}(\mathrm{PSL}_n(\mathbb{Z})) \doteq \mathrm{PGL}_n(\mathbb{Q}).$$

In particular, $\mathrm{Comm}(\mathrm{PSL}_n(\mathbb{Z}))$ is *not* commensurable with the group

$$\mathrm{PSL}_n(\mathbb{Q}) = \mathrm{SL}_n(\mathbb{Q})/Z(\mathrm{SL}_n(\mathbb{Q})).$$

To understand this precisely, consider the $\mathbb{Q}$-defined surjection of algebraic groups $\pi : \mathrm{SL}_n \to \mathrm{PGL}_n$. The kernel of π is isomorphic to the multiplicative group of order n, denoted μ_n. By definition, $\mathrm{PSL}_n(\mathbb{Q}) = \pi(\mathrm{SL}_n(\mathbb{Q}))$. As in [27, 2.2.3], the exact sequence of $\mathbb{Q}$-defined algebraic groups

$$1 \to \mu_n \to \mathrm{SL}_n \to \mathrm{PGL}_n \to 1$$

gives rise to a long exact sequence of cohomology groups

$$1 \to \mu_n(\mathbb{Q}) \to \mathrm{SL}_n(\mathbb{Q}) \to \mathrm{PGL}_n(\mathbb{Q}) \to H^1(\overline{\mathbb{Q}}/\mathbb{Q}, \mu_n) \to 1.$$

There is an isomorphism $H^1(\overline{\mathbb{Q}}/\mathbb{Q}, \mu_n) \cong \mathbb{Q}^*/(\mathbb{Q}^*)^n$. This is infinitely generated for $n \geq 2$, hence $[\mathrm{PGL}_n(\mathbb{Q}) : \mathrm{PSL}_n(\mathbb{Q})] = \infty$.

8. Commensurations of general lattices

Suppose Γ is a lattice in a connected Lie group G which is not necessarily either solvable or semisimple. Our main result is:

Theorem 1.7. *Suppose G is a connected, linear Lie group with connected, simply-connected solvable radical. Suppose $\Gamma \leq G$ is a lattice with the property that there is no surjection $\phi : G \to H$ to any group H locally isomorphic to any* $\mathrm{SO}(1,n)$ *or* $\mathrm{SU}(1,n)$ *so that $\phi(\Gamma)$ is a lattice in H. Then:*

(1) *Γ virtually embeds in a $\mathbb{Q}$-defined algebraic group* **G** *with Zariski-dense image so that every commensuration $[\phi] \in \mathrm{Comm}(\Gamma)$ induces a unique $\mathbb{Q}$-defined automorphism of* **G** *virtually extending ϕ.*

(2) *There is a $\mathbb{Q}$-defined algebraic group $\mathcal{B}$ so that*

$$\mathrm{Comm}(\Gamma) \cong \mathcal{B}(\mathbb{Q})$$

and the image of $\mathrm{Aut}(\Gamma)$ *in $\mathcal{B}$ is commensurable with $\mathcal{B}(\mathbb{Z})$.*

The proof of Theorem 1.7 proceeds in four steps:

(1) Construct the algebraic group **G**, called the *virtual algebraic hull* of Γ, such that Γ virtually embeds in **G** with Zariski-dense image.

(2) Show that commensurations of Γ induce $\mathbb{Q}$-defined automorphisms of **G**.

(3) Show that $\mathrm{Aut}(\mathbf{G})$ has the structure of an algebraic group, and that $\mathrm{Comm}(\Gamma)$ is realized as the $\mathbb{Q}$-points of a $\mathbb{Q}$-defined subgroup of $\mathrm{Aut}(\mathbf{G})$.

(4) Show that the image of $\mathrm{Aut}(\Gamma)$ in $\mathrm{Aut}(\mathbf{G})$ is commensurable with $\mathcal{B}(\mathbb{Z})$.

Proof of Theorem 1.7. Let Γ be as in the theorem. Let R be the solvable radical of G.

Step 1: (Construction of virtual algebraic hull). We will construct $\mathbf{G}$ as the semidirect product of a solvable group $\mathbf{H}$ with a semisimple group $\mathbf{S}$. Roughly speaking, $\mathbf{H}$ is the virtual algebraic hull of the "solvable part" of Γ, while $\mathbf{S}$ is a $\mathbb{Q}$-defined semisimple group without $\mathbb{Q}$-compact factors such that the "semisimple part" of Γ is abstractly commensurable with $\mathbf{S}(\mathbb{Z})$. To make this precise, we modify the Lie group G and lattice Γ as follows.

Because G is linear, there is a connected semisimple subgroup $S \leq G$ so that $G = R \rtimes S$. Let $\mathbf{S}'$ be a $\mathbb{Q}$-defined linear algebraic group so that $S = \mathbf{S}'(\mathbb{R})^0$. There is a simply-connected algebraic group $\tilde{\mathbf{S}}'$ and a surjection $\pi : \tilde{\mathbf{S}}' \to \mathbf{S}'$ with finite central kernel. Let $\tilde{S} = \tilde{\mathbf{S}}'(\mathbb{R})^0$. Then $\pi : \tilde{S} \to S$ is a finite covering map with central kernel. The lattice $\Gamma \leq R \rtimes S$ lifts to a lattice $\tilde{\Gamma} \leq R \rtimes \tilde{S}$, which is commensurable with Γ by Proposition 3.9. Replacing $R \rtimes S$ by $R \rtimes \tilde{S}$ and Γ by $\tilde{\Gamma}$, we may assume that no finite cover of the semisimple quotient of G has a linear representation, i.e. that G is *algebraically simply-connected* (cf. [34, 9.4]).

Let K be the maximal compact quotient of S such that Γ projects to a finite subgroup of K. Because G is algebraically simply-connected, K may be identified with a subgroup of S, and there is a subgroup $S' \leq S$ so that $S = S' \times K$. Then $\Gamma \cap (R \rtimes S')$ is of finite index in Γ, so we may replace S by S' and assume that Γ projects densely into the maximal compact factor of S. It follows by [31, 4.5] that, passing to a finite index subgroup of Γ, we have chosen $S \leq G$ so that $\Gamma = (\Gamma \cap R)(\Gamma \cap S)$. Let $\Gamma_r = \Gamma \cap R$ and $\Gamma_s = \Gamma \cap S$. This makes precise our notions of "solvable" and "semisimple" parts of Γ.

We now want to find a $\mathbb{Q}$-defined algebraic group $\mathbf{S}$ without $\mathbb{Q}$-compact factors so that Γ_s is abstractly commensurable with $\mathbf{S}(\mathbb{Z})$. Because S is algebraically simply-connected, there is a decomposition $S = S_1 \times \cdots \times S_k$ so that Γ_s virtually decomposes as $\Gamma_{s,1} \times \cdots \times \Gamma_{s,k}$, where $\Gamma_{s,i} \leq S_i$ is an irreducible lattice for each i. Since each $\Gamma_{s,i}$ does not project to a lattice in $\mathrm{SO}(1,n)$ or $\mathrm{SU}(1,n)$, it follows from Theorem 7.7 that for each i there is a connected $\mathbb{Q}$-defined semisimple algebraic group $\mathbf{S}_i$ and a surjection $\pi_i : \mathbf{S}_i(\mathbb{R})^0 \to S_i$ with compact kernel so that $\pi_i(\mathbf{S}_i(\mathbb{Z}) \cap \mathbf{S}_i(\mathbb{R})^0)$ is commensurable with $\Gamma_{s,i}$. Set

$$\mathbf{S} = \mathbf{S}_1 \times \cdots \times \mathbf{S}_k \quad \text{and} \quad \Gamma_s' = \prod_{i=1}^{k} \mathbf{S}_i(\mathbb{Z}) \cap \mathbf{S}_i(\mathbb{R})^0.$$

Each $\mathbf{S}_i$ is $\mathbb{Q}$-simple and $\mathbf{S}_i(\mathbb{R})^0$ is not compact, so $\mathbf{S}$ is without $\mathbb{Q}$-compact factors.

Our next goal is to define an action of $\mathbf{S}$ on the virtual algebraic hull of Γ_r. To do this, we use the fact that the virtual algebraic hull of Γ_r is a real algebraic hull for any unipotently connected, simply-connected solvable Lie group R containing Γ_r as a Zariski-dense lattice. A classical construction may be used to produce a simply-connected solvable Lie group R' so that Γ_r is Zariski-dense in R' and R' is unipotently connected. To ensure that we can apply this construction while respecting the action of S, we present a proof based on ideas in [5].

Lemma 8.1. *Suppose $G = R \rtimes S$ is a connected linear Lie group with R simply-connected solvable and S semisimple. Let $\Gamma = (\Gamma \cap R)(\Gamma \cap S)$ be a lattice, and set $\Gamma_r = \Gamma \cap R$ and $\Gamma_s = \Gamma \cap S$. There is a finite index subgroup $\Gamma' \leq \Gamma$ of the form $\Gamma' = \Gamma_r' \rtimes \Gamma_s$ and a simply-connected solvable Lie group R' so that Γ' is a lattice in $R' \rtimes S$ with the property that Γ_r' is Zariski-dense in R' and R' is unipotently connected.*

Proof. Let $\mathbf{H}_R$ be the real algebraic hull of R and $\mathbf{H}_\Gamma$ the virtual algebraic hull of Γ_r. There is a finite index characteristic subgroup $\Gamma_r' \leq \Gamma_r$ so that $\mathbf{H}_\Gamma$ is the algebraic hull of Γ_r'. By [5, 5.3] we may moreover assume that there is some simply-connected solvable Lie group R' that is unipotently connected and so that Γ_r' is Zariski-dense in R'. The algebraic group $\mathbf{H}_\Gamma$ is a real algebraic hull for R' by [5, 3.11]. In particular, we identify R' with a subgroup $R' \leq \mathbf{H}_\Gamma(\mathbb{R})$ containing Γ_r'.

By [5, 3.9], the inclusion $\Gamma_r' \leq R$ extends to an $\mathbb{R}$-defined embedding $\mathbf{H}_\Gamma \to \mathbf{H}_R$. The action of S on R extends to an action of S on $\mathbf{H}_R$ by $\mathbb{R}$-defined algebraic automorphisms. Let Φ be an $\mathbb{R}$-defined automorphism of $\mathbf{H}_R$ induced by some $s \in S$. We would like to show that Φ preserves R'.

Let N be the maximal connected nilpotent normal subgroup of R, and let $\mathbf{F}$ denote the Zariski-closure of $\mathrm{Fitt}(\Gamma)$ in $\mathbf{H}_R$. We have $N \leq \mathbf{F}$ by a classical result of Mostow. It follows from [5, 3.3] that $N \leq \mathbf{H}_R(\mathbb{R})$ is normal. Because S is connected, the action of S on R/N is trivial by [5, 6.9]. It follows that $\Phi(\mathbf{F}) = \mathbf{F}$. By density of $R \leq \mathbf{H}_R$, we conclude that Φ is trivial on the quotient $\mathbf{H}_R/\mathbf{F}$.

Let N' be the maximal normal nilpotent subgroup of R'. Then $\mathbf{F}(\mathbb{R}) = N'$ in $\mathbf{H}_\Gamma$ because R' is unipotently connected. It follows that $\Phi(R') \subseteq R'\mathbf{F}(\mathbb{R}) = R'$, and so Φ induces an automorphism of R'. This agrees with the given action of Γ_s on Γ_r', so we may form the semidirect product $G' = R' \rtimes S$ containing the lattice $\Gamma' = \Gamma_r' \rtimes \Gamma_s$. □

We may therefore assume that the radical R of G is unipotently connected and Γ_r is Zariski-dense in R. Let $\mathbf{H}$ be the virtual algebraic hull of Γ_r. Because R is unipotently connected and Γ_r is Zariski-dense in R, [5, 3.11] implies that $\mathbf{H}$ has the structure of a $\mathbb{R}$-defined connected algebraic hull of R. There is a representation $\rho : S \to \mathrm{Aut}_{\mathbb{R}}(\mathbf{H})$ by the automorphism extension property of the algebraic hull. Because $\mathbf{S}$ is simply-connected, ρ extends to an $\mathbb{R}$-defined representation $\rho : \mathbf{S} \to \mathrm{Aut}(\mathbf{H})$ by Proposition 2.8. Since Γ_s preserves Γ_r, we have that $\rho(\gamma)$ is $\mathbb{Q}$-defined for every $\gamma \in \Gamma_s$. Because $\mathbf{S}$ is without $\mathbb{Q}$-compact factors and connected, we know Γ_s is Zariski-dense in $\mathbf{S}$ by Theorem 2.9. It follows from a standard fact, e.g. [21, I.0.11], that the representation $\rho : \mathbf{S} \to \mathrm{Aut}(\mathbf{H})$ is $\mathbb{Q}$-defined.

The definition of the variety structure on Aut(**H**) implies that the action map $\alpha : \mathbf{H} \times \mathrm{Aut}(\mathbf{H}) \to \mathbf{H}$ is a $\mathbb{Q}$-defined map of varieties. It follows that the action map $\mathbf{H} \times \mathbf{S} \to \mathbf{H}$ is $\mathbb{Q}$-defined. The semidirect product of groups

$$\mathbf{G} = \mathbf{H} \rtimes \mathbf{S} \tag{8.1}$$

therefore has the structure of a $\mathbb{Q}$-defined algebraic group. It is evident from the construction that Γ embeds in $\mathbf{G}(\mathbb{Q})$ as a Zariski-dense subgroup. This concludes the first step of the proof.

Step 2: (Extension of commensurations). We now construct a map

$$\xi : \mathrm{Comm}(\Gamma) \to \mathrm{Aut}_{\mathbb{Q}}(\mathbf{G}).$$

Let Λ be a thickening of Γ_r in $\mathbf{H}$ with nilpotent supplement C and good unipotent shadow θ, as in Proposition 5.13. The action of Γ_s on Γ_r extends to an action on Λ. Then $\Lambda \rtimes \Gamma_s$ is a Zariski-dense subgroup of $\mathbf{G}(\mathbb{Q})$ containing Γ as a finite index subgroup.

Lemma 8.2. *Let* **U** *denote the unipotent radical of* **H**. *Suppose* $u \in \mathbf{U}(\mathbb{Q})$. *Then conjugation by* u *induces a commensuration of* Γ.

Proof. Suppose $u \in \mathbf{U}(\mathbb{Q})$. Let $\mathbf{F} = \mathrm{Fitt}(\mathbf{H})$. Conjugation by u induces two partial automorphisms: a partial automorphism $\phi_\theta : \theta_1 \to \theta_2$ of θ, and a partial automorphism $\phi_R : \Lambda_1 \to \Lambda_2$ of Γ_r by Theorem 6.1. As in the proof of Theorem 6.1, we may choose θ_1, θ_2, Λ_1, and Λ_2 so that $\theta_i \cap \mathbf{F} = \mathrm{Fitt}(\Lambda_i)$ for $i = 1, 2$. We want to find some finite index subgroup $\Gamma_s'' \leq \Gamma_s$ so that conjugation by u induces an isomorphism $\Lambda_1 \Gamma_s'' \to \Lambda_2 \Gamma_s''$.

Let N be the maximal connected, normal, nilpotent subgroup of R. Because S is connected, the action of S on R is trivial on R/N (see [5, 6.9]). Since we have assumed that R is unipotently connected, N is Zariski-dense in the Fitting subgroup $\mathbf{F} \leq \mathbf{H}$ by [5, 5.4], and so the induced action of Γ_s on $\mathbf{H}$ is trivial on the quotient $\mathbf{H}/\mathbf{F}$. Therefore for any $s \in \Gamma_s$ we have

$$sus^{-1}u^{-1} \in \mathbf{F}. \tag{8.2}$$

Restricting our attention to Λ, we see that for any $s \in \Gamma_s$ and $c \in C$, there is some $f \in \mathrm{Fitt}(\Lambda)$ so that $scs^{-1} = fc$. It follows that conjugation by $s \in \Gamma_s$ preserves θ. Let $\Gamma_s' \leq \Gamma_s$ be a finite index subgroup normalizing both Λ_1 and Λ_2. Then Γ_s' also normalizes both θ_1 and θ_2. By Lemma 6.2, there is a finite index subgroup $\Gamma_s'' \leq \Gamma_s'$ so that $usu^{-1}s^{-1} \in \theta_1 \cap \theta_2$ for all $s \in \Gamma_s''$. Combining this with (8.2), for all $s \in \Gamma_s''$ we have

$$usu^{-1}s^{-1} \in \mathrm{Fitt}(\Lambda_1) \cap \mathrm{Fitt}(\Lambda_2). \tag{8.3}$$

The same arguments as in Claim 1 of the proof of Theorem 6.1 show that conjugation by u induces a partial isomorphism $\Lambda_1 \Gamma_s'' \to \Lambda_2 \Gamma_s''$ of $\Lambda \rtimes \Gamma_s$. □

Proposition 8.3. *Every commensuration* $[\phi] \in \mathrm{Comm}(\Gamma)$ *induces a unique* $\mathbb{Q}$-*defined automorphism of* $\mathbf{G}$ *virtually extending* ϕ. *Hence there is an injective homomorphism*

$$\xi : \mathrm{Comm}(\Gamma) \to \mathrm{Aut}_{\mathbb{Q}}(\mathbf{G}).$$

Proof. Suppose there are finite index subgroups Γ_1 and Γ_2 of $\Lambda \rtimes \Gamma_s$ with $\phi : \Gamma_1 \to \Gamma_2$ a partial automorphism representing $[\phi]$. Passing to a finite index subgroup so that $\Gamma_s \cap Z(S)$ is trivial, we may assume that $\Gamma_i \cap \mathbf{H}$ is the unique maximal normal solvable subgroup of Γ_i for $i = 1, 2$ (cf. [28, Lemma 6]). It follows that $\phi(\Gamma_1 \cap \mathbf{H}(\mathbb{R})) = \Gamma_2 \cap \mathbf{H}(\mathbb{R})$, and so ϕ induces a commensuration $[\phi_R] \in \mathrm{Comm}(\Lambda)$ by Lemma 3.5. It follows from Lemma 5.11 that ϕ_R extends to an automorphism $\Phi_R \in \mathrm{Aut}_{\mathbb{Q}}(\mathbf{H})$.

Now let $\mathbf{L}$ be the Zariski-closure of $\phi(\Gamma_1 \cap \Gamma_s)$ in $\mathbf{G}$. Then $\mathbf{L}$ is $\mathbb{Q}$-defined, and is semisimple by [31, Theorem 2]. (Note that here we are using the assumption that Γ_s does not surject to a lattice in any $\mathrm{SU}(1,n)$ or $\mathrm{SO}(1,n)$.) There is some $u \in \mathbf{U}(\mathbb{Q})$ conjugating $\mathbf{L}$ into $\mathbf{S}$ by Theorem 2.6. It follows from Lemma 8.2 that $\mathrm{Inn}_u \circ \phi$ virtually restricts to a partial automorphism $\phi_S : \Delta_1 \to \Delta_2$ of Γ_s. The partial automorphism ϕ_S virtually extends to a $\mathbb{Q}$-defined automorphism $\Phi_S \in \mathrm{Aut}_{\mathbb{Q}}(\mathbf{S})$ by Theorem 7.3.

Define an automorphism $\Phi \in \mathrm{Aut}(\mathbf{G})$ by

$$\Phi(r,s) = \mathrm{Inn}_{u^{-1}} \left(\mathrm{Inn}_u \circ \Phi_R(r), \Phi_S(s)\right).$$

Then Φ virtually extends the partial automorphism ϕ. This extension is unique up to choice of $u \in \mathbf{U}(\mathbb{Q})$ conjugating $\mathbf{L}$ to $\mathbf{S}$. However, any two such u differ by an element of $\mathbf{U}(\mathbb{Q})$ centralized by $\mathbf{S}$, hence Φ is unique. □

Step 3: (Algebraic structure). We now show that the image of $\xi : \mathrm{Comm}(\Gamma) \to \mathrm{Aut}_{\mathbb{Q}}(\mathbf{G})$ has the structure of the $\mathbb{Q}$-rational points of a $\mathbb{Q}$-defined algebraic group. We first show that $\mathrm{Aut}(\mathbf{G})$ in fact has the structure of a $\mathbb{Q}$-defined algebraic group.

Definition 8.4. A pair of automorphisms $(\Phi_R, \Phi_S) \in \mathrm{Aut}(\mathbf{H}) \times \mathrm{Aut}(\mathbf{S})$ is *compatible* if there is some $\Phi \in \mathrm{Aut}(\mathbf{G})$ preserving $\mathbf{S}$ with $\Phi|_{\mathbf{H}} = \Phi_R$ and $\Phi|_{\mathbf{S}} = \Phi_S$. Let $C(\mathbf{G}) \subseteq \mathrm{Aut}(\mathbf{H}) \times \mathrm{Aut}(\mathbf{S})$ be the set of compatible pairs of automorphisms.

As both $\mathrm{Aut}(\mathbf{H})$ and $\mathrm{Aut}(\mathbf{S})$ have structures of $\mathbb{Q}$-defined algebraic groups, their product $\mathrm{Aut}(\mathbf{H}) \times \mathrm{Aut}(\mathbf{S})$ is a $\mathbb{Q}$-defined algebraic group.

Lemma 8.5. $C(\mathbf{G})$ *is a* $\mathbb{Q}$-*defined subgroup of* $\mathrm{Aut}(\mathbf{H}) \times \mathrm{Aut}(\mathbf{S})$.

Proof. Let $\rho : \mathbf{S} \to \mathrm{Aut}(\mathbf{H})$ be the $\mathbb{Q}$-defined representation by conjugation. Any automorphism $\Phi \in \mathrm{Aut}(\mathbf{G})$ preserving $\mathbf{S}$ must satisfy

$$[\Phi \circ \rho(s)](r) = \Phi(srs^{-1}) = \Phi(s)\Phi(r)\Phi(s)^{-1} = [\rho(\Phi(s)) \circ \Phi](r)$$

for all $r \in \mathbf{H}$ and all $s \in \mathbf{S}$.

From this it is clear that any $(\Phi_R, \Phi_S) \in C(\mathbf{G})$ satisfies

$$\Phi_R \circ \rho(s) \circ \Phi_R^{-1} \circ \rho(\Phi_S(s))^{-1} = \mathrm{Id} \in \mathrm{Aut}(\mathbf{H}) \tag{8.4}$$

for all $s \in \mathbf{S}$. Conversely, suppose a pair $(\Phi_R, \Phi_S) \in \mathrm{Aut}(\mathbf{H}) \times \mathrm{Aut}(\mathbf{S})$ satisfies (8.4) for all $s \in \mathbf{S}$. Then the function $\Phi : \mathbf{G} \to \mathbf{G}$ defined by $\Phi(r,s) = \Phi_R(r)\Phi_S(s)$ is an automorphism of $\mathbf{G}$, and so $(\Phi_R, \Phi_S) \in C(\mathbf{G})$. Thus $C(\mathbf{G})$ is equal to the set of pairs (Φ_R, Φ_S) satisfying (8.4) for all $s \in \mathbf{S}(\mathbb{Q})$. For a fixed element $s \in \mathbf{S}$, the solution set of equation (8.4) is a $\mathbb{Q}$-defined closed subset of $\mathrm{Aut}(\mathbf{H}) \times \mathrm{Aut}(\mathbf{S})$. It follows that $C(\mathbf{G})$ is a $\mathbb{Q}$-defined subgroup. □

Lemma 8.6. *The map*

$$\begin{aligned} \Theta : \mathbf{U} \rtimes C(\mathbf{G}) &\to \mathrm{Aut}(\mathbf{G}) \\ (u, \Phi_R, \Phi_S) &\mapsto \mathrm{Inn}_u \circ \Phi_R \circ \Phi_S \end{aligned} \tag{8.5}$$

is a surjective group homomorphism with $\mathbb{Q}$-defined unipotent kernel. Hence $\mathrm{Aut}(\mathbf{G})$ *has the structure of a $\mathbb{Q}$-defined algebraic group, such that*

$$\mathrm{Aut}_{\mathbb{Q}}(\mathbf{G}) \cong \mathrm{Aut}(\mathbf{G})(\mathbb{Q}) \cong \mathbf{U}(\mathbb{Q}) \rtimes C(\mathbf{G})(\mathbb{Q})/(\ker\Theta)(\mathbb{Q}). \tag{8.6}$$

Proof. This follows from standard arguments. Compare to §5.4 and [4, §3.1], for example. □

We will now show that the image of

$$\xi : \mathrm{Comm}(\Gamma) \to \mathrm{Aut}(\mathbf{G})$$

is equal to the $\mathbb{Q}$-points of a $\mathbb{Q}$-defined subgroup of $\mathrm{Aut}(\mathbf{G})$. Let $\mathcal{A}_{\Gamma_r} \leq \mathrm{Aut}(\mathbf{H})$ be the $\mathbb{Q}$-defined subgroup such that $\mathcal{A}_{\Gamma_r}(\mathbb{Q}) \cong \mathrm{Comm}(\Gamma_r)$, as in Theorem 1.2. Define

$$\mathcal{B} = \left\{\Phi \in \mathrm{Aut}(\mathbf{G}) \;\middle|\; \Phi\big|_{\mathbf{H}} \in \mathcal{A}_{\Gamma_r}\right\}.$$

Then $\mathcal{B}$ is evidently a $\mathbb{Q}$-defined subgroup of $\mathrm{Aut}(\mathbf{G})$. It is clear that $\xi(\mathrm{Comm}(\Gamma)) \leq \mathcal{B}(\mathbb{Q})$.

Proposition 8.7. *The map $\xi : \mathrm{Comm}(\Gamma) \to \mathcal{B}(\mathbb{Q})$ is an isomorphism.*

Proof. Clearly ξ is injective. Suppose $\Phi \in \mathcal{B}(\mathbb{Q})$. By Theorem 2.6 there is some $u \in \mathbf{U}(\mathbb{Q})$ such that $\mathrm{Inn}_u \circ\Phi$ preserves $\mathbf{S}$. Since $\mathrm{Inn}_u \in \mathcal{A}_{\Gamma_r}$, it follows that $\mathrm{Inn}_u \circ\Phi \in \mathcal{B}(\mathbb{Q})$. Therefore there are $\Phi_R \in \mathcal{A}_{\Gamma_r}(\mathbb{Q})$ and $\Phi_S \in \mathrm{Aut}(\mathbf{S})(\mathbb{Q})$ such that $\mathrm{Inn}_u \circ\Phi = \Phi_R \circ \Phi_S$.

We have that Φ_R induces a partial automorphism $\phi_R : \Lambda_1 \to \Lambda_2$ of Λ by Theorem 1.2, and Φ_S induces a partial automorphism $\phi_S : \Gamma_{s,1} \to \Gamma_{s,2}$ of Γ_s by Proposition 7.2. We may choose Λ_1 to be characteristic in Λ, and then choose $\Gamma_{s,2}$ to normalize $\Lambda_2 \leq \Lambda$. It follows that there is a well-defined isomorphism $\phi : \Lambda_1\Gamma_{s,1} \to \Lambda_2\Gamma_{s,2}$ defined by $\phi(r,s) = \Phi_R(r)\Phi_S(s)$, which clearly satisfies $\xi([\phi]) = \Phi_R \circ \Phi_S$. Since $\mathrm{Inn}_u \in \xi(\mathrm{Comm}(\Gamma))$ by Lemma 8.2, it follows that $\Phi \in \xi(\mathrm{Comm}(\Gamma))$. □

Step 4: ($\operatorname{Aut}(\Gamma)$ **commensurable with** $\mathcal{B}(\mathbb{Z})$**).** It remains only to show that $\operatorname{Aut}(\Gamma)$ is commensurable with $\mathcal{B}(\mathbb{Z})$. For this, we first show that the element $u \in \mathbf{U}(\mathbb{Q})$ arising in the proof of Proposition 8.3 can be chosen in a controlled way. Given a vector space V of finite dimension over a field of characteristic 0, we say that a subset $L \subseteq V$ is a *vector space lattice* if L is a finitely generated $\mathbb{Z}$-submodule of $V(\mathbb{Q})$ spanning V.

Lemma 8.8. *Let P be any group acting nontrivially and irreducibly on a vector space $V \cong \mathbb{R}^n$. Suppose P preserves a vector space lattice $L' \subseteq V(\mathbb{Q})$. Then there is a vector space lattice $L \subseteq V(\mathbb{Q})$ such that if $v \in V(\mathbb{Q})$ satisfies $v - p \cdot v \in L'$ for all $p \in P$ then $v \in L$.*

Proof. The action of P descends to an action of P on the torus V/L'. It suffices to show that this action has finitely many fixed points, as the fixed points of $V(\mathbb{Q})/L'$ lift to the desired vector space lattice $L \subseteq V$. To see this, simply note that the fixed point set X of the action of P is a closed, hence compact, Lie subgroup of V/L'. The dimension of X must be zero by the assumption that P acts irreducibly and nontrivially on V. Therefore X is finite. □

Lemma 8.9. *There is a subgroup $\Lambda \leq \mathbf{U}(\mathbb{Q})$ commensurable with $\mathbf{U}(\mathbb{Z})$ such that if $\phi \in \operatorname{Aut}(\Gamma)$ virtually extends to $\Phi \in \operatorname{Aut}(\mathbf{G})$ then there is some $u \in \Lambda$ such that*

$$(\operatorname{Inn}_u \circ \Phi)(\mathbf{S}) \subseteq \mathbf{S}.$$

Proof. Let $\mathfrak{u}$ denote the Lie algebra of $\mathbf{U}$. The action of Γ_s on $\mathbf{U}$ induces a linear action of Γ_s on $\mathfrak{u}$. Let θ be a good unipotent shadow of Γ_r. Fix a vector space lattice $L' \subseteq \mathfrak{u}(\mathbb{Q})$ containing $\log(\theta)$ preserved by the action of Γ_s on $\mathfrak{u}$.

Suppose $\phi \in \operatorname{Aut}(\Gamma)$ virtually extends to $\Phi \in \operatorname{Aut}(\mathbf{G})$. By Theorem 2.6, there is some $u \in \mathbf{U}(\mathbb{Q})$ so that

$$(\operatorname{Inn}_u \circ \Phi)(\mathbf{S}) \subseteq \mathbf{S}. \tag{8.7}$$

Define $\phi_1 : \Gamma_s \to \Gamma_r$ and $\phi_2 : \Gamma_s \to \Gamma_s$ by

$$\phi(0, \gamma_s) = (\phi_1(\gamma_s), \phi_2(\gamma_s)).$$

Take any $\gamma_s \in \Gamma_s$. It follows from equation (8.7) that $\phi_1(\gamma_s) \in \mathbf{U} \cap \Gamma_r$, and so $\phi_1(\gamma_s) \in \theta$. From this we conclude that

$$u(\gamma_s \cdot u^{-1}) \in \theta,$$

and therefore

$$\log(u) - \gamma_s \cdot \log(u) \in L'.$$

Because $\mathbf{S}$ is semisimple, the action of Γ_s on $\mathfrak{u}$ is completely reducible. Applying Lemma 8.8 to each irreducible component of this representation of Γ_s, we find a vector space lattice $L \subseteq \mathfrak{u}(\mathbb{Q})$ with the property that any $u \in \mathbf{U}(\mathbb{Q})$ satisfying equation (8.7) satisfies $\log(u) \in L$. Let $\Lambda \leq \mathbf{U}(\mathbb{Q})$ be any subgroup such that $\log(\Lambda)$ is a vector space lattice containing L with finite index. Such a subgroup exists by the methods of [30, §6B]. The fact that Λ is commensurable with $\mathbf{U}(\mathbb{Z})$ is immediate from the fact that $\log(\Lambda) \subseteq \mathfrak{u}(\mathbb{Q})$ is a vector space lattice. □

Now let

$$A_{\Lambda,\mathbf{H}} = \left\{\Phi \in \mathcal{A}_{\mathbf{H}|\mathbf{F}} \;\middle|\; \Phi(\Lambda) \subseteq \Lambda\right\}.$$

Then $A_{\Lambda,\mathbf{H}}$ is commensurable with $\mathcal{A}_{\mathbf{H}|\mathbf{F}}(\mathbb{Z})$ by [4, 8.1], hence is commensurable with $\mathrm{Aut}(\Gamma_r)$. Define a $\mathbb{Q}$-defined subgroup of $C(\mathbf{G})$ by

$$C_\Gamma(\mathbf{G}) = \{(\Phi_R, \Phi_S) \in C(\mathbf{G}) \mid \Phi_R \in \mathcal{A}_{\Gamma_r}\},$$

and

$$A_\Lambda = \{(\Phi_R, \Phi_S) \in C_\Gamma(\mathbf{G}) \mid \Phi_R \in A_{\Lambda,\mathbf{H}} \text{ and } \Phi_S(\Gamma_s) = \Gamma_s\}.$$

Then A_Λ is commensurable with $C_\Gamma(\mathbf{G})(\mathbb{Z})$. Note that the map Θ of Lemma 8.6 descends to a map

$$\bar{\Theta} : \mathbf{U} \rtimes C_\Gamma(\mathbf{G}) \to \mathrm{Aut}(\mathbf{G}),$$

and there is an isomorphism of algebraic groups

$$\mathcal{B} \cong \mathbf{U} \rtimes C_\Gamma(\mathbf{G}) / \ker(\bar{\Theta}).$$

Let

$$\mathrm{Aut}_\Lambda(\Gamma) = \left\{\phi \in \mathrm{Aut}(\Gamma) \;\middle|\; \phi|_{\Gamma_r} \in A_{\Lambda,\mathbf{H}}\right\}.$$

Note that $[\mathrm{Aut}(\Gamma) : \mathrm{Aut}_\Lambda(\Gamma)] < \infty$. By Lemma 8.9 there is a map

$$\xi : \mathrm{Aut}_\Lambda(\Gamma) \to \Lambda \rtimes A_\Lambda / \ker(\bar{\Theta}).$$

This map is clearly injective, and the preceding discussion shows that its image is of finite index. Therefore the image of $\mathrm{Aut}(\Gamma)$ in $\mathcal{B}$ is commensurable with $\mathcal{B}(\mathbb{Z})$. This completes the proof. □

Remark 8.10. The assumption that the lattice Γ is superrigid in S cannot be removed from Theorem 1.7. Consider for example $S = \mathrm{SO}(1, n)$ for $n \geq 2$ with a lattice $\Gamma \leq S$ such that $\Gamma/[\Gamma, \Gamma]$ is infinite. Let $\tau : \Gamma \to \mathbb{Z}$ be any nontrivial homomorphism. Then $\phi_\tau : \mathbb{Z} \times \Gamma \to \mathbb{Z} \times \Gamma$ defined by

$$\phi_\tau(t, \gamma) = (t + \tau(\gamma), \gamma)$$

is an automorphism of $\mathbb{Z} \times \Gamma$, which is a lattice in $\mathbb{R} \times S$. However, ϕ_τ neither is induced by conjugation by an element of $\mathbb{Q} \subseteq \mathbb{R}$ nor preserves S in any sense, and ϕ_τ cannot be extended to an automorphism of $\mathbb{R} \times S$.

Automorphisms of the form ϕ_τ as above are in one-to-one correspondence with elements of $H^1(\Gamma, \mathbb{Z})$. If $\Delta \leq \Gamma$ is a finite index subgroup and $\sigma \in H^1(\Delta, \mathbb{Z})$, then ϕ_σ defines a partial automorphism of $\mathbb{Z} \times \Gamma$. In this way we identify the inverse limit

$$\mathcal{C} = \varprojlim \{H^1(\Delta, \mathbb{Z}) \mid [\Gamma : \Delta] < \infty\}$$

with a subgroup of $\mathrm{Comm}(\mathbb{Z} \times \Gamma)$. Nontrivial commensurations in $\mathcal{C}$ do not virtually extend to automorphisms of $\mathbb{R} \times S$. For any finite index subgroup $\Delta \leq \Gamma$, we may identify $H^1(\Delta, \mathbb{Q})$ as a subgroup of $\mathcal{C}$. In this way, the virtual first rational Betti number of the semisimple quotient of a lattice may be seen as an obstruction to the realization of commensurations as automorphisms of an algebraic group.

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